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To cite this version:
Toufic El Arwadi, Sérêna Dib, Toni Sayah. A PRIORI AND A POSTERIORI OF A LINEAR ELLIPTIC PROBLEM WITH DYNAMICAL BOUNDARY CONDITION.. 2014. <hal-01054311v1>

HAL Id: hal-01054311
https://hal.archives-ouvertes.fr/hal-01054311v1
Submitted on 6 Aug 2014 (v1), last revised 11 Jun 2015 (v2)

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A PRIORI AND A POSTERIORI OF A LINEAR ELLIPTIC PROBLEM WITH DYNAMICAL BOUNDARY CONDITION.

TOUFIC EL ARWADI†, SÉRÉNA DIB‡, AND TONI SAYAH‡

ABSTRACT. In this paper, we study the time dependent linear elliptic problem with dynamical boundary condition. The problem is discretized by the backward Euler’s scheme in time and finite elements in space. In this work, an optimal a priori error estimate is established and an optimal a posteriori errors with two types of computable error indicators is proved. The first one being linked to the time discretization and the second one to the space discretization. Using these a posteriori errors estimates, an adaptative algorithm for computing the solution is proposed. Finally, numerical experiments are presented to show the efectiveness of the obtained error estimators and the proposed adaptive algorithm.

KEYWORDS. Dynamic boundary condition, finite element method, a posteriori analysis.

1. Introduction.

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded simply-connected open domain in \( \mathbb{R}^2 \), with a Lipschitz-continuous connected boundary \( \Gamma \), and let \([0, T]\) denotes an interval in \( \mathbb{R} \) where \( T \in (0, +\infty) \) is a fixed final time. We denote by \( \mathbf{n}(x) \) the unit outward normal vector at \( x \in \Gamma \). We intend to work with the following time dependent linear elliptic problem with dynamical boundary condition:

\[
\begin{align*}
-\Delta u(t, x) &= 0 \text{ in } [0, T] \times \Omega, \\
\frac{\partial u}{\partial t}(t, x) + \beta \mathbf{n}(x) \cdot \nabla u(t, x) &= 0 \text{ on } [0, T] \times \Gamma, \\
u(0, x) &= u_0 \text{ on } \Gamma,
\end{align*}
\]  

(1.1)

where \( \beta \) is a positive constant. The unknown is \( u \) and \( u_0 \) is the initial condition at time \( t = 0 \).

The solution of the problem (1.1) can be represented on the boundary by the Dirichlet-to-Neumann semigroup defined as : \( (S(t)f)(x) := u(t, x)|_\Gamma \). For the existence and uniqueness of this solution see [17]. In a particular case, where \( \Omega = B(0, 1) \) the unit ball of \( \mathbb{R}^2 \), In his book [14], P.Lax showed that the Dirichlet-to-Neumann semigroup had a simple explicit representation. In [9], it is shown that the Lax representation can not be generalized if \( \Omega \) is not the unit ball of \( \mathbb{R}^2 \). This motivates the authors of [9] and [7] to introduce a semi discrete explicit and implicit Euler’s scheme in order to approximate the Dirichlet-to-Neumann semigroup numerically. The convergence of these semi discrete schemes is based on the Chernoff’s product formula. For the discretisation of the problem (1.1), the others of [9] show simple numerical experiments. The aim of this work is to show optimal a priori and a posteriori estimates and some numerical investigations.

For the finite element method, the a posteriori error estimates have been developed since about thirteen years ago. The idea turns about an upper bounds of the error between the exact solution and numerical one with a sum of a local indicators expressed in each element of the mesh. To get more precision and to minimize the error, the goal is to decrease this indicators by using the adapt mesh technics which consists

August 5, 2014.

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to refine or coarse some regions of the mesh. The \textit{a posteriori} error estimates is optimal if we can bound each one of these indicators by the local error of the solution around the corresponding element. We refer for example to the books Verfürth [16] or Ainsworth et Oden [1]. For the time dependent problems, we have two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization. We have to handle the two kind of indicators and some times, we change the time step and in an other times, we adapt the mesh. A lot amount of work has been made concerning the \textit{a posteriori} errors, we can cite for example, Ladevèze [12] for constitutive relation error estimators for time-dependent nonlinear FE analysis, Verfürth [15] for the heat equation, Bernardi and Verfürth [5] for the time dependent Stokes equations, Bernardi and E. Süli [4] for the time and space adaptivity for the second–order wave equation, Bergam, Bernardi and Mghazli [6] for some parabolic equations, Ern and Vohralík [10] for estimation based on potential and flux reconstruction for the heat equation and Bernardi and Sayah [3] for the time dependent Stokes equations with mixed boundary conditions, . . . .

In this project, the time derivative appears in the boundary condition and the data of the problem is the initial condition of the unknown at the boundary. We propose a very standard low cost discretization relying on the Euler’s implicit scheme in time combined with finite elements in space, and prove optimal \textit{a priori} and \textit{a posteriori} error estimates for the discrete problem. Finally, some numerical simulations are presented based on the proposed algorithm, and which we made easy thanks to the FreeFem++ software.

The outline of the paper is as follows:

- Section 2 is devoted to the study of the continuous problem.
- In section 3, we introduce the discrete problem and we recall its main properties.
- In section 4, we study the \textit{a priori} errors and derive optimal estimates.
- In section 5, we study the \textit{a posteriori} errors and derive optimal estimates.
- In section 6, we show numerical results of validation.

2. Analysis of the model

In order to write the variational formulation of the problem (1.1), we introduce the Sobolev spaces:

\[
H^m(\Omega) = \{ v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega), \quad \forall |\alpha| \leq m \},
\]

equipped with the following semi-norm and norm:

\[
| v |_{m, \Omega} = \left\{ \sum_{|\alpha| = m} \int_{\Omega} |\partial^\alpha v(x)|^2 \, dx \right\}^{1/2} \quad \text{and} \quad \| v \|_{m, \Omega} = \left\{ \sum_{|\alpha| = m} \| v \|_{2,k,\Omega}^2 \right\}^{1/2}.
\]

As usual, we denote by \((\cdot, \cdot)\) the scalar product of \(L^2(\Omega)\).

For handling time-dependent problems, it is convenient to consider functions defined on a time interval \([a, b]\) with values in a separable functional space, say \(Y\). In the following, \(f(t)\) represents the function \(f(t, \cdot)\). Let \(\| \cdot \|_Y\) denote the norm of \(Y\); then for any \(r, 1 \leq r \leq \infty\), we define

\[
L^r(a, b; Y) = \left\{ f \text{ measurable in } [a, b]; \int_a^b \| f(t) \|_Y^r \, dt < \infty \right\},
\]

equipped with the norm

\[
\| f \|_{L^r(a, b; Y)} = \left( \int_a^b \| f(t) \|_Y^r \, dt \right)^{1/r},
\]

with the usual modifications if \(r = \infty\). It is a Banach space if \(Y\) is a Banach space.

By the same way, for any integer \(k\), we define

\[
C^k(a, b; Y) = \left\{ f \text{ measurable in } [a, b]; \sup_{t \in [a, b], |\alpha| \leq k} \| f^{(\alpha)}(t, \cdot) \|_Y < \infty \right\}.
\]

For the existence and the uniqueness of the solution of problem (1.1), we refer to the theorem 2.1, page 169 in the book [17].
Furthermore, we have the following bound:

$$\beta \| \nabla u \|^2_{L^2([0, +\infty); L^2(\Omega))} \leq \frac{1}{2} \| u_0 \|^2_{L^2(\Gamma)};$$

(2.1)

If in addition, \( u_0 \in H^{1/2}(\Gamma) \), and the unique solution of the problem

$$-\Delta u = 0 \quad \text{in } \Omega$$
$$u = u_0 \quad \text{on } \Gamma$$

satisfies \( n.\n \n u \in L^2(\Gamma) \), then the solution \( u \) of the problem (1.1) satisfies

(1) \( u \in C^1([0, +\infty); H^1(\Omega)) \);
(2) \( u_T \in C^1([0, +\infty); L^2(\Gamma)) \);
(3) \( n.\n \n u \in C([0, +\infty); L^2(\Gamma)) \).

We suppose that \( u_0 \in H^{1/2}(\Gamma) \) and introduce the following variational problem :

Find \( u(t) \in H^1(\Omega) \) such that :

$$u(0, x) = u_0(x) \quad \text{on } \Gamma,$$
$$\beta \int_\Omega \nabla u(t, x) \nabla v(t, x) \, dx + \int_\Gamma \frac{\partial u}{\partial \nu}(t, s)v(t, s) \, ds = 0 \quad \forall v(t) \in H^1(\Omega).$$

(2.2)

Theorem 2.2. If \( u(t) \in H^2(\Omega) \), the problem (1.1) is equivalent to the variational one (2.2).

Proof. Let \( u \) be the solution of the problem (1.1). Multiplying the first equation of the problem (1.1) by \( v(t) \in H^1(\Omega) \), integrating over \( \Omega \), applying the Green’s formula and using the second equation of the problem (1.1) to obtain that \( u \) is also solution of the problem (2.2). Conversely, if \( u \) is a solution of the problem (2.2), it suffices to take \( v(t) \in D(\Omega) \) to get the first line of the problem (1.1). Next, multiplying the first equation of the problem (1.1) by \( v(t) \in H^1(\Omega) \), integrating on \( \Omega \), using the Green’s formula and comparing with the problem (2.2), get the second line of (1.1). \( \square \)

Proposition 2.3. The solution of the problem (2.2) satisfies the following bounds

$$\| u \|^2_{L^2(0, T; L^2(\Gamma))} \leq \| u_0 \|^2_{L^2(\Gamma)}$$
(2.3)

and

$$\| \nabla u \|^2_{L^2(0, T; L^2(\Omega))} \leq \frac{1}{2} \beta \| u_0 \|^2_{L^2(\Gamma)};$$
(2.4)

Proof. Let \( u \) be the unique solution of the problem (1.1). We take \( v = u \) and integrate in time between 0 and \( \tau \) to get the bound

$$\beta \| \nabla u \|^2_{L^2(0, \tau; L^2(\Omega))} + \frac{1}{2} \| u(\tau) \|^2_{L^2(\Omega)} \leq \frac{1}{2} \| u_0 \|^2_{L^2(\Gamma)}$$

which leads directly to the result. \( \square \)

3. The discrete problem

From now on, we assume that \( \Omega \) is a polyhedron. In order to describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval \([0, T]\) into subintervals \([t_{n-1}, t_n]\), \( 1 \leq n \leq N \), such that \( 0 = t_0 \leq t_1 \leq \cdots \leq t_N = T \). We denote by \( \tau_n \) the length of \([t_{n-1}, t_n]\), by \( \tau \) the N-tuple \((\tau_1, \ldots, \tau_N)\), by \( |\tau| \) the maximum of the \( \tau_n, 1 \leq n \leq N \), and by \( \sigma_{\tau} \) the regularity parameter

$$\sigma_{\tau} = \max_{2 \leq n \leq N} \frac{\tau_n}{\tau_{n-1}}.$$
We can easily check that the Lax-Milgram theorem, we have to show the coercivity of the bilinear form and the linear one.

Proof. We introduce the bilinear form:

We introduce an operator \( \pi_{\tau} \) by the next definition.

**Definition 3.1.** For any Banach space \( X \) and any function \( g \) continuous from \([0, T]\) into \( X \), \( \pi_{\tau}g \) denotes the step function which is constant and equal to \( g(t_n) \) on each interval \([t_{n-1}, t_n]\), \( 1 \leq n \leq N \). Similarly, with any sequence \( \{\phi_n\}_{1 \leq n \leq N} \) in \( X \), we associate the step function \( \pi_{\tau}\phi_{\tau} \) which is constant and equal to \( \phi_n \) on each interval \([t_{n-1}, t_n]\), \( 1 \leq n \leq N \).

Now, we describe the space discretization. For each \( n \), \( 0 \leq n \leq N \), a regular triangulation of \( \Omega \) \((\mathcal{T}_{nh})_n \) is a set of non degenerate elements which satisfies:

- for each \( h \), \( \mathcal{\bar{\Omega}} \) is the union of all elements of \( \mathcal{T}_{nh} \);
- the intersection of two distinct elements of \( \mathcal{T}_{nh} \), is either empty, a common vertex, or an entire common edge;
- the ratio of the diameter of an element \( \kappa \) in \( \mathcal{T}_{nh} \) to the diameter of its inscribed circle is bounded by a constant independent of \( n \) and \( h \).

As usual, \( h \) denotes the maximal diameter of the elements of all \( \mathcal{T}_{nh}, 0 \leq n \leq N \), while for each \( n \), \( h_n \) denotes the maximal diameter of the elements of \( \mathcal{T}_{nh} \). For each \( \kappa \) in \( \mathcal{T}_{nh} \), we denote by \( P_1(\kappa) \) the space of restrictions to \( \kappa \) of polynomials with two variables and total degree at most one.

In what follows, \( c, c', C, C', c_1, \ldots \) stand for generic constants which may vary from line to line but are always independent of \( h \) and \( n \). For a fixed \( n \in \mathbb{N} \) and a given triangulation \( \mathcal{T}_{nh} \), we define by \( X_{nh} \) a finite dimensional space of functions such that their restrictions to any element \( \kappa \) of \( \mathcal{T}_{nh} \) belong to a space of polynomials of degree one. In another words,

\[
X_{nh} = \{ v_h^n \in C^0(\bar{\Omega}), \ v_h^n|_\kappa \text{ is affine } \forall \kappa \in \mathcal{T}_{nh} \}
\]

We note that for each \( n \) and \( h \), \( X_{nh} \subset H^1(\Omega) \). There exists an approximation operator, \( I_h \in L(\mathcal{H}^2(\Omega) ; X_{nh}) \) such that for \( m = 0, 1 \)

\[
\forall v \in \mathcal{H}^2(\Omega), \ |I_h(v) - v|_{m, \Omega} \leq C h^{2-m} |v|_{2, \Omega}.
\]

The full discrete implicit scheme associated to the Problem (2.2) is: Given \( u_{h}^{n-1} \in X_{n-1} \), find \( u_{h}^{n} \) with values in \( X_{nh} \) solution of

\[
\forall v_h(t) \in X_{nh}, \quad \beta \int_\Omega \nabla u_{h}^{n} \nabla v_h(t)dx + \int_\Gamma \frac{1}{\tau_n} (u_{h}^{n} - u_{h}^{n-1}) v_h(t)d\sigma = 0. \tag{3.1}
\]

by assuming that \( u_h^0 \) is an approximation of \( u(0) \) in \( X_{0h} \).

**Theorem 3.2.** The problem (3.1) admits a unique solution in \( X_{nh} \).

**Proof.** We introduce the bilinear form:

\[
a(v, w) = \beta \tau_n \int_\Omega \nabla v \nabla w dx + \int_\Gamma v w d\sigma
\]

and the linear one

\[
l(w) = \int_\Gamma u_{h}^{n-1} w d\sigma.
\]

The problem (3.1) can be written as

\[
\forall v_h \in X_{nh}, \quad a(u_{h}^{n}, v_h) = l(u_{h}^{n-1}). \tag{3.2}
\]

We can easily check that \( a \) and \( l \) are continuous respectively in \( X_{nh} \times X_{nh} \) and \( X_{nh} \). In order to apply the Lax-Milgram theorem, we have to show the coercivity of the bilinear form \( a \) which will be proved by contradiction: that is, we suppose that \( a \) is not coercive, ie:

\[
\forall C > 0, \ \exists v \in X_{nh}; \ a(v, v) < C \|v\|_{0, \Omega}^2. \tag{3.3}
\]
Let $C_m = \frac{1}{m}$, there exists $v_m \in X_{nh}$ such that $a(v_m, v_m) < \frac{1}{m} \|v_m\|_{0,\Omega}^2$. We denote by $w_m = \frac{v_m}{\|v_m\|_{0,\Omega}}$ and we obtain $a(w_m, w_m) < \frac{1}{m}$. The definition of the bilinear form allow us to obtain

$$|w_m|_{1,\Omega}^2 < \frac{1}{m \tau_n} \beta \quad \text{as} \quad m \to +\infty$$

and

$$\int_{\Gamma} |w_m(s)|^2 \, ds < \frac{1}{m} \quad \text{as} \quad m \to +\infty. \quad (3.4)$$

Then we deduce that

$$\|w_m\|_{1,\Omega}^2 \leq \|w_m\|_{0,\Omega}^2 + |w_m|_{1,\Omega}^2 < 1 + \frac{1}{m \beta \tau_n} \quad \text{as} \quad m \to +\infty. \quad (3.5)$$

Then the sequence $(w_m)_m$ is bounded in $H^1(\Omega)$ hence, we can extract a subsequence $(w_{m_k})_k$ which converges to a limit $w$ in $L^2(\Omega)$, therefore in $D'(\Omega)$. As a consequence, the convergence of $\partial_t w_{m_k}$ to $\partial_t w$ holds in $D'(\Omega)$, which gives with (3.5) that $\partial_t w = 0$, for $i = 1, 2$, and then $w = C$ is a constant. The trace theorem allows us to get

$$\int_{\Gamma} |w_{m_k}(s) - w(s)|^2 \, ds \leq C(\|w_{m_k} - w\|_{0,\Omega}^2 + \|\nabla(w_{m_k} - w)\|_{0,\Omega}^2) \quad \text{as} \quad m_k \to +\infty. \quad (3.6)$$

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and hence, with the relation (3.4) we get $w = 0$. Finally, the subsequence $w_{m_k}$ converges to 0 in $L^2(\Omega)$ with $\|w_{m_k}\|_{0,\Omega}^2 = 1$, which leads to a contradiction. We deduce the coerciveness of the bilinear form. All the hypotheses of the Lax-Milgram theorem are satisfied, therefore there is one and only one solution of the problem (3.1).

**Theorem 3.3.** For each $m = 0, \ldots, N - 1$, the solution $u^n_h$ of the problem (3.1) satisfies the bound:

$$\|u^n_h\|_{0,\Gamma}^2 + \sum_{n=1}^m \tau_n |u^n_h|_{1,\Omega}^2 \leq \frac{1}{\min(1, 2\beta)} \|u^0_h\|_{0,\Gamma}^2. \quad (3.7)$$

**Proof.** For all $v_h \in X_{nh}$, let $u^n_h$ be the unique solution of the (3.1). Choosing $v_h(t_n) = u^n_h$ in (3.1), we find

$$\beta \tau_n |u^n_h|_{1,\Omega}^2 + \|u^n_h\|_{0,\Gamma}^2 = \int_{\Gamma} u^{n-1}_h u^n_h \, d\sigma. \quad (3.8)$$

By Applying the Hölder inequality and the relation $ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$, we have:

$$\beta \tau_n |u^n_h|_{1,\Omega}^2 + \|u^n_h\|_{0,\Gamma}^2 \leq \frac{1}{2} \|u^{n-1}_h\|_{0,\Gamma}^2 + \frac{1}{2} \|u^n_h\|_{0,\Gamma}^2. \quad (3.9)$$

We multiply the last relation by 2 and sum over $n$ from $n = 1$ to $m$ to get

$$2\beta \sum_{n=1}^m \tau_n |u^n_h|_{1,\Omega}^2 + \|u^n_h\|_{0,\Gamma}^2 \leq \|u^0_h\|_{0,\Gamma}^2. \quad (3.10)$$

The claim is proved. \hfill \Box

4. A priori error estimates

To get the *a priori* error estimates, we suppose that time step $\tau_n$ and the mesh $T_{nh}$ don’t change during time iterations. We denote by $k$ the time step, by $h$ the parameter of the mesh and by $X_h$ the discrete space.

In this section, the discrete variational formulation (3.1) taken in the time step $n + 1$, becomes

$$\forall v_h \in X_h, \quad \beta \int_{\Omega} \nabla u^{n+1}_h \cdot \nabla v_h dx + \int_{\Gamma} \frac{1}{\tau_n} (u^{n+1}_h - u^n_h) v_h d\sigma = 0. \quad (4.1)$$

To get the *a priori* error estimate, we need the following the classic Gronwall’s lemma.
Remark 4.1. \(\ll\) Gronwall’s lemma \(\gg\)
Let \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}\) and \((c_n)_{n \geq 0}\) three real positive sequences such that \((c_n)_{n \geq 0}\) is an increasing sequence. We suppose that we have:

(1) \[a_0 + b_0 \leq c_0,\] 
(2) there exists \(\lambda > 0\) such that:

\[\forall n \geq 1, a_n + b_n \leq c_n + \lambda \sum_{m=0}^{n-1} a_m.\] 

Then we have:

\[\forall n \geq 0, a_n + b_n \leq c_n e^{n\lambda}.\] 

In order to get the a priori error estimate, we begin with the next theorem.

**Theorem 4.2.** If \(u \in L^\infty(0,T,H^2(\Omega))\) and \(u' \in L^\infty(0,T,H^2(\Omega))\), and for all \(m = 0, ..., N - 1\), we have the bound:

\[\|I_h(u(t_{m+1})) - u_h^{m+1}\|^2_{0,\Omega} + 2k\beta \sum_{n=0}^{m} \|I_h(u(t_{n+1})) - u_h^{n+1}\|^2_{1,\Omega} \leq C(h^2 + k^2),\]

where \(C\) is a constant independent from \(h\) and \(k\).

**Proof.** We consider the equation (2.2) for \(t \in [t_n, t_{n+1}]\), take \(v = v_h^{n+1}\), integrate in time between \(t_n\) and \(t_{n+1}\), then take the difference with (4.1) to get

\[\beta \int_{t_n}^{t_{n+1}} \int_\Omega \nabla(u(t) - u_h^{n+1})(x)\nabla v_h^{n+1}(x) \, dx \, dt + \int_\Gamma ((u(t_{n+1}) - u(t_n)) - (u_h^{n+1} - u_h^{n})) v_h^{n+1}(s) \, ds = 0.\] 

We insert \(\pm \nabla(I_h(u(t_{n+1})))\) and \(\pm \nabla(u(t_{n+1}))\) in the first term, and \(\pm I_h(u(t_{n+1}))\) and \(\pm I_h(u(t_n))\) in the second term, we denote by \(a_n = I_h(u(t_n)) - u_h^n\) and we obtain

\[\int_\Gamma (a_{n+1} - a_n)(s) v_h^{n+1}(s) \, ds + k\beta |a_n|^2_{1,\Omega} = \int_\Gamma ((I_h(u(t_{n+1})) - u(t_{n+1})) - (I_h(u(t_n)) - u(t_n)))(s) v_h^{n+1}(s) \, ds \]

\[+ \beta \int_{t_n}^{t_{n+1}} \int_\Omega \nabla(u(t_{n+1}) - u(t))(x)\nabla v_h^{n+1}(x) \, dx \, dt \]

\[+ \beta \int_{t_n}^{t_{n+1}} \int_\Gamma \nabla(I_h(u(t_{n+1})) - u(t_{n+1}))\nabla v_h^{n+1}(x) \, dx \, dt.\]

We denote by \(T_1\) and \(T_2\) respectively the first and second terms of the left hand side, and \(T_3, T_4, T_5\) respectively the first, second and third terms of the right hand side of the equation (4.7). Then we choose \(v_h^n = a_n\).

The term \(T_1\) can be expressed as

\[T_1 = \frac{1}{2} \int_\Gamma a_{n+1}^2(s) \, ds - \frac{1}{2} \int_\Gamma a_n^2(s) \, ds + \frac{1}{2} \int_\Gamma (a_{n+1} - a_n)^2(s) \, ds.\]

The term \(T_3\) can be bounded as

\[T_3 = \int_\Gamma ((I_h(u(t_{n+1})) - u(t_{n+1})) - (I_h(u(t_n)) - u(t_n))(s) a_{n+1}(s) \, ds \]

\[= \int_{t_n}^{t_{n+1}} \int_\Gamma (I_h(u(\tau)) - u(\tau))(s) a_{n+1}(s) \, ds \, d\tau \]

\[\leq \int_{t_n}^{t_{n+1}} \|I_h(u'(\tau)) - u'(\tau)\|_{L^2(\Gamma)} \|a_{n+1}\|_{L^2(\Gamma)} d\tau \]

\[\leq C(hk\|u'\|_{L^\infty(0,T;H^2(\Omega))}) \|a_{n+1}\|_{L^2(\Gamma)} \]

\[\leq \frac{C^2 h^2 k}{2} \|u'\|^2_{L^\infty(0,T;H^2(\Omega))} + \frac{k \varepsilon_1}{2} \|a_{n+1}\|^2_{0,\Gamma}.\]
We consider the term $T_4$. We have
\[
T_4 = \beta \int_t^{t_{n+1}} \int_\Omega \nabla (u(t_{n+1}, x) - u(t, x)) \nabla a_{n+1}(x) \, dx \, dt
\leq \beta \int_t^{t_{n+1}} \int_\Omega \nabla u' (\tau, x) \nabla a_{n+1}(x) \, d\tau \, dt
\leq \beta k^2 \|u'\|_{L^\infty (0,T,H^1(\Omega))} |a_{n+1}|_{1,\Omega}
\leq \frac{k^3 \beta^2}{2 \epsilon_2} \|u'\|^2_{L^\infty (0,T,H^1(\Omega))} + \frac{k \epsilon_2}{2} |a_{n+1}|^2_{1,\Omega}.
\]

Finally, the term $T_5$ can be bounded as
\[
T_5 = \beta \int_t^{t_{n+1}} \int_\Omega \nabla (I_h(u(t_{n+1}))(x) - u(t_{n+1}, x)) \nabla a_{n+1}(x) \, dx \, dt
\leq \beta C_2 \int_t^{t_{n+1}} h \|u(t_{n+1})\|_{2,T} |a_{n+1}|_{1,\Omega} \, dt
\leq C_2 h \beta \sqrt{T} \|u\|_{L^\infty (0,T,H^2(\Omega))} \sqrt{T} |a_{n+1}|_{1,\Omega}
\leq \frac{C_2^2 h^2 k^2 \beta^2}{2 \epsilon_3} \|u\|^2_{L^\infty (0,T,H^2(\Omega))} + \frac{k \epsilon_3}{2} |a_{n+1}|^2_{1,\Omega}.
\]

Using the previous bounds, we get
\[
\frac{1}{2} \int_0^T a_{n+1}(s) \, ds - \frac{1}{2} \int_0^T a_n^2(s) \, ds + \frac{1}{2} \int_0^T (a_{n+1} - a_n)^2(s) \, ds + k \beta |a_{n+1}|^2_{1,\Omega}
= \frac{C_2^2 h^2 k^2}{2 \epsilon_1} \|u'\|^2_{L^\infty (0,T,H^2(\Omega))} + \frac{k \epsilon_1}{2} |a_{n+1}|^2_{0,\Gamma} + \frac{k^3 \beta^2}{2 \epsilon_2} \|u'\|^2_{L^\infty (0,T,H^1(\Omega))} + \frac{k \epsilon_2}{2} |a_{n+1}|^2_{1,\Omega}
(4.8)
= \frac{C_2^2 h^2 k^2 \beta^2}{2 \epsilon_3} \|u\|^2_{L^\infty (0,T,H^2(\Omega))} + \frac{k \epsilon_3}{2} |a_{n+1}|^2_{1,\Omega}.
\]

We choose $\epsilon_1 = \frac{1}{8 T}$, $\epsilon_2 = \frac{\beta}{2}$ and $\epsilon_3 = \frac{\beta}{2}$ to get the following bound
\[
\frac{1}{2} \|a_{m+1}\|^2_{0,\Gamma} + \frac{k \beta}{2} \sum_{n=0}^m |a_{n+1}|^2_{1,\Omega} \leq C_3 (h^2 + k^2) + \frac{k}{16 T} \sum_{n=0}^m \|a_{n+1}\|^2_{0,\Gamma}.
\]

We write the last term of the previous bound as
\[
\frac{k}{16 T} \sum_{n=0}^m \|a_{n+1}\|^2_{0,\Gamma} = \frac{k}{16 T} \sum_{n=0}^{m-1} \|a_{n+1}\|^2_{0,\Gamma} + \frac{k}{16 T} \|a_{n+1}\|^2_{0,\Gamma},
\]
we suppose that $\frac{k}{16 T} \leq \frac{1}{4}$ and then apply the classic Gronwall’s lemma to get the result. \qed

**Corollary 4.3.** If $u \in L^\infty (0,T,H^2(\Omega))$ and $u' \in L^\infty (0,T,H^2(\Omega))$, for all $m = 0,...,N-1$, we have the following bound:
\[
\|u(t_{m+1}) - u_h^{m+1}\|^2_{0,\Gamma} + 2 k \beta \sum_{n=0}^m \|u(t_{n+1}) - u_h^{n+1}\|^2_{1,\Omega} \leq C (h^2 + k^2),
\]
where $C$ is a constant independent of $h$ and $k$.

**Proof.** For all $m = 0,...,N-1$:
\[
\|u(t_{m+1}) - u_h^{m+1}\|^2_{0,\Gamma} + 2 k \beta \sum_{n=0}^m \|u(t_{n+1}) - u_h^{n+1}\|^2_{1,\Omega} \leq \|u(t_{m+1}) - I_h(u(t_{m+1}))\|^2_{0,\Gamma}
+ 2 k \beta \sum_{n=0}^m \|u(t_{n+1}) - I_h(u(t_{n+1}))\|^2_{1,\Omega} + \|I_h(u(t_{m+1})) - u_h^{m+1}\|^2_{0,\Gamma} + 2 k \beta \sum_{n=0}^m \|I_h(u(t_{n+1})) - u_h^{n+1}\|^2_{1,\Omega}.
\]
Based on the theorem 4.2, the second term of the inequality (4.11) can be bounded by \( C_1 \left( h^2 + k^2 \right) \), where \( C_1 \) is a constant independent of \( h \) and \( k \). The properties of \( I_h \) give the result.  

\[ \square \]

5. A POSTERIORI ERROR ESTIMATES

We now intend to prove a posteriori error estimates between the exact solution \( u \) of Problem (2.2) and the numerical solution \( u_h \) of Problem (3.1).

5.1. Construction of the error indicators.

In this section, we will introduce several notations and properties and we will define the indicators. For every element \( \kappa \) in \( T_{nh} \), we denote by

- \( \varepsilon_{\kappa} \) the set of edges of \( \kappa \) that are not contained in \( \partial \Omega \),
- \( \varepsilon_{\kappa}^m \) the set of edges of \( \kappa \) which are contained in \( \partial \Omega \),
- \( \Delta_{\kappa} \) the union of elements of \( T_{nh} \) that intersect \( \kappa \),
- \( \Delta_e \) the union of elements of \( T_{nh} \) that intersect the edge \( e \),
- \( h_{\kappa} \) the diameter of \( \kappa \) and \( h_e \) the diameter of the edge \( e \),
- and \( \left[ \cdot \right]_e \) the jump through \( e \) for each edge \( e \) in an \( \varepsilon_\kappa \) (making its sign precise is not necessary).

Also, \( n_\kappa \) stands for the unit outward normal vector to \( \kappa \) on \( \partial \kappa \).

For the demonstration of the next theorems, we introduce for an element \( \kappa \) of \( T_{nh} \), the bubble function \( \psi_\kappa \) (resp. \( \psi_e \) for the edge \( e \)) which is equal to the product of the 3 barycentric coordinates associated with the vertices of \( \kappa \). We also consider a lifting operator \( L_e \) defined on polynomials on \( e \) vanishing on \( \partial e \) into polynomials on the at most two elements \( \kappa \) containing \( e \) and vanishing on \( \partial \kappa \setminus e \), which is constructed by affine transformation from a fixed operator on the reference element. We recall the next results from [16, Lemma 3.3].

**Property 5.1.** Denoting by \( P_r(\kappa) \) the space of polynomials of degree smaller than \( r \) on \( \kappa \), we have

\[ \forall v \in P_r(\kappa), \quad \begin{cases} c ||v||_{0,\kappa} \leq ||v\psi_\kappa^{1/2}||_{0,\kappa} \leq c'||v||_{0,\kappa}, \\ ||v||_{1,\kappa} \leq ch_{\kappa}^{-1/2}||v||_{0,\kappa}. \end{cases} \]  

(5.1)

**Property 5.2.** Denoting by \( P_r(e) \) the space of polynomials of degree smaller than \( r \) on \( e \), we have

\[ \forall v \in P_r(e), \quad c ||v||_{0,e} \leq ||v\psi_e^{1/2}||_{0,e} \leq c'||v||_{0,e}. \]

and, for all polynomials \( v \) in \( P_r(e) \) vanishing on \( \partial e \), if \( \kappa \) is an element which contains \( e \),

\[ ||L_ev||_{0,\kappa} + h_e \mid L_ev \mid_{1,\kappa} \leq ch_e^{-1/2}||v||_{0,e}. \]

We also introduce a Clément type regularization operator \( C_{nh} \) [8] which has the following properties, see [2, Section IX.3]: For any function \( w \) in \( H^1(\Omega) \), \( C_{nh}w \) belongs to the space of continuous affine finite elements and satisfies for any \( \kappa \) in \( T_{nh} \) and \( e \) in \( \varepsilon_\kappa \),

\[ ||w - C_{nh}w||_{L^2(\kappa)} \leq ch_{\kappa}||w||_{1,\Delta_{\kappa}} \quad \text{and} \quad ||w - C_{nh}w||_{L^2(e)} \leq ch_e^{1/2}||w||_{1,\Delta_e}. \]

(5.2)

For the a posteriori error studies, we consider the piecewise affine function \( u_h \) which take in the interval \([t_{n-1}, t_n]\) the values

\[ u_h(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}(u_h^n - u_h^{n-1}) + u_h^{n-1}. \]
The solutions of Problems (2.2) and (3.1) verify for $t$ in $[t_{n-1}, t_n]$ and for all $v(t) \in H^1(\Omega)$ and $v_h(t) \in X_{nh}$:

$$
\beta \int_\Omega \nabla (u - u_h)(t, x) \nabla v(t, x) \, dx + \int_\Gamma \frac{\partial (u - u_h)}{\partial t}(t, s)v(t, s) \, ds \\
= \beta \int_\Omega \nabla u_h(t, x) \nabla v(t, x) \, dx - \int_\Gamma \frac{\partial u_h}{\partial t}(t, s)v(t, s) \, ds \\
= \beta \frac{t_n - t}{\tau_n} \int_\Omega \nabla (u^n_h - u^{n-1}_h)(x) \nabla v(t, x) \, dx \\
- \sum_{\kappa \in \mathcal{T}_{n,h}} \beta \int_\partial \kappa (\nabla u^n_h, n)(x)(v - v_h)(t, x) \, dx - \int_\Gamma \frac{u^n_h - u^{n-1}_h}{\tau_n} s(v - v_h)(t, s) \, ds.
$$

(5.3)

We introduce, for every edge $e$ of the mesh, the function

$$
\phi^e_{h,n} = \begin{cases} 
\frac{1}{2} \beta |\nabla u^n_h, n|_e & \text{si } e \in \mathcal{E}_\kappa, \\
\beta \nabla u^n_h, n + \frac{u^n_h - u^{n-1}_h}{\tau_n} & \text{si } e \in \mathcal{E}^m_\kappa,
\end{cases}
$$

(5.4)

Then, we get the equation

$$
\beta \int_\Omega \nabla (u - u_h)(t, x) \nabla v(t, x) \, dx + \int_\Gamma \frac{\partial (u - u_h)}{\partial t}(t, s)v(t, s) \, ds \\
= \beta \frac{t_n - t}{\tau_n} \int_\Omega \nabla (u^n_h - u^{n-1}_h)(x) \nabla v(t, x) \, dx - \beta \sum_{\kappa \in \mathcal{T}_{n,h}} \sum_{e \in \partial \kappa} \int_\partial \kappa \phi^e_{h,n}(x)(v - v_h)(t, x) \, dx.
$$

(5.5)

Since, we introduce the indicators: For each $\kappa$ in $\mathcal{T}_{n,h}$,

$$(\eta^\tau_{h,n})^2 = \tau_n \| \nabla (u^n_h - u^{n-1}_h) \|^2_{0,\kappa}$$

and

$$(\eta^\tau_{h,n})^2 = \sum_{e \in \partial \kappa} h_e \| \phi^e_{h,n} \|^2_{0,\kappa}.$$  

5.2. Upper bounds of the error.

We are now able to prove the upper bound.

**Theorem 5.3.** For all $m = 1, ..., N$, we have the following upper bound

$$
\beta \| \nabla (u - u_h) \|^2_{L^2(0, t_m; L^2(\Omega))} + \| u(t_m) - u^n_h \|^2_{0, \Gamma} \leq C \left[ \sum_{n=1}^{m} \sum_{\kappa \in \mathcal{T}_{n,h}} (\eta^\tau_{h,n})^2 + \sum_{n=1}^{m} \sum_{\kappa \in \mathcal{T}_{n,h}} \tau_n (\eta^\tau_{h,n})^2 + \| u_0 - u^0_h \|^2_{0, \Gamma} \right],
$$

(5.6)

where $C$ is a constant independent of $h_n$ and $\tau_n$.

**Proof.** We denote by $L(v)$ the right hand side of the equation (5.5) and we introduce the function $w(t, x) = e^{-t}(u - u_h)(t, x)$ which verify the equation

$$
\frac{\partial w}{\partial t}(t, x) + w(t, x) = e^{-t} \frac{\partial (u - u_h)}{\partial t}(t, x).
$$

(5.7)

We multiply $L(v)$ by $e^{-t}$ and take $v = w$ to obtain

$$
e^{-t}L(w) = \beta \int_0^t |\nabla w(t, x)|^2 \, dx + \int_\Gamma w^2(t, s) \, ds + \frac{1}{2} \int_\Gamma \frac{\partial w^2}{\partial t}(t, s) \, ds \\
\geq \beta \| \nabla w(t) \|^2_{0, \Omega} + \frac{1}{2} \int_\Gamma \frac{\partial w^2}{\partial t}(t, s) \, ds.
$$

(5.8)
By taking into account that $e^{-t} < 1$ and remark that $L(w) \leq L(u - u_h)$, we have

$$
\beta \| \nabla w(t) \|^2_{0, \Omega} + \frac{1}{2} \int_\Gamma \frac{\partial w^2}{\partial t}(t, s) \, ds \\
\leq \beta \int_\Omega (u - u_h)(t, x) \nabla (u - u_h)(t, x) \, dx + \int_\Gamma \frac{\partial (u - u_h)}{\partial t}(t, s)(u - u_h)(t, s) \, ds.
$$

(5.9)

We integrate the last relation in $[t_{n-1}, t_n]$, sum of $n$ from 1 to $m$, take into account the relation $e^{-2t} \geq e^{-2T}$ to get the following bound

$$
e^{-2T} \left[ \beta \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \nabla (u - u_h)(t) \|^2_{0, \Omega} \, dt + \frac{1}{2} \int_\Gamma |u - u_h|^2(t_m, s) \, ds \right] \\
\leq \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} L(u - u_h) \, dt + \frac{1}{2} \int_\Gamma |u - u_h|^2(0, s) \, ds.
$$

(5.10)

and then

$$
\beta \int_0^{t_m} \| \nabla(u(t) - u_h(t)) \|^2_{0, \Omega} \, dt + \frac{1}{2} \| u(t_m) - u_h \|^2_{0, \Gamma} \leq C\left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} L(u - u_h) \, dt + \| u_0 - u_h \|^2_{0, \Gamma} \right),
$$

(5.11)

where $C$ is a constant independent of $h_n$ and $\tau_n$.

Next, we have to bound the right hand side of the last inequality. In all the rest of the proof, we denote $v = u - u_h$ and we decompose $L(v) = L_1(v) + L_2(v)$ and we bound each one separately. First, we have

$$
L_1(v) = \beta \frac{t_n - t}{\tau_n} \sum_{\kappa \in \tau_{n, h-n}} \int_\kappa (u_h^n - u_h^{n-1})(x) \nabla v(t, x) \, dx \\
\leq \beta \frac{t_n - t}{\tau_n} \sum_{\kappa \in \tau_{n, h-n}} \| \nabla (u_h^n - u_h^{n-1}) \|_{0, \kappa} \| \nabla v(t) \|_{0, \kappa}.
$$

(5.12)

We integrate the last system in $[t_{n-1}, t_n]$ and we obtain

$$
\int_{t_{n-1}}^{t_n} L_1(v) \, dt \leq \sum_{\kappa \in \tau_{n, h-n}} \left( \beta^2 \| \nabla (u_h^n - u_h^{n-1}) \|^2_{0, \kappa} \int_{t_{n-1}}^{t_n} \left( t_n - t \right)^2 \, dt \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} \| \nabla v(t) \|^2_{0, \kappa} \, dt \right)^{1/2}. \\
\leq \frac{\beta}{\sqrt{3}} \left( \sum_{\kappa \in \tau_{n, h-n}} (\eta_{\kappa})^2 \right)^{1/2} \left( \sum_{\kappa \in \tau_{n, h-n}} \| \nabla v \|^2_{L^2(t_{n-1}, t_n, L^2(\kappa))} \right)^{1/2} \\
\leq C_1(\varepsilon_1) \sum_{\kappa \in \tau_{n, h-n}} (\eta_{\kappa})^2 + \frac{\varepsilon_1}{2} \| \nabla v \|^2_{L^2(t_{n-1}, t_n, L^2(\Omega))}.
$$

(5.13)

Next, we sum over $n$ from 1 to $m$ and get the bound

$$
\sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} L_1(u - u_h) \, dt \leq C_1(\varepsilon_1) \sum_{n=1}^{m} \sum_{\kappa \in \tau_{n, h-n}} (\eta_{\kappa})^2 + \frac{\varepsilon_1}{2} \| \nabla (u - u_h) \|^2_{L^2(0, t_m, L^2(\Omega))},
$$

(5.14)
where \( C_1(\varepsilon_1) \) is a constant independent of \( h_n \) and \( \tau_n \).

Next, by taking \( v_h(t) = R_{n,h}(v(t)) \), we have
\[
L_2(v) = -\beta \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \partial \kappa} \int_{\kappa} \phi_{h,n}(x)(v - v_h)(t, x) \, dx
\]
\[
\leq \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \partial \kappa} \|\phi_{h,n}\|_{0,e} \|v(t) - v_h(t)\|_{0,e}
\]
\[
\leq C_2 \left( \sum_{\kappa \in \mathcal{T}_h} \left( \sum_{e \in \partial \kappa} h_e \|\phi_{h,n}\|_{0,e} \right)^{\frac{1}{2}} \left( \sum_{e \in \partial \kappa} \|\nabla v(t)\|_{0,\Delta_e}^2 \right)^{\frac{1}{2}} \right)
\]
\[
\leq C_2 \left( \sum_{\kappa \in \mathcal{T}_h} (\eta^h_{n,\kappa})^2 \right)^{\frac{1}{2}} \left( \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \partial \kappa} \|\nabla v(t)\|_{0,\Delta_e}^2 \right)^{\frac{1}{2}}
\]
\[
\leq C_3 \left( \sum_{\kappa \in \mathcal{T}_h} (\eta^h_{n,\kappa})^2 \right)^{\frac{1}{2}} \|\nabla v(t)\|_{0,\Omega},
\]
where \( C_2 \) and \( C_3 \) are constants independent of \( h_n \) and \( \tau_n \).

We integrate the last system over \([t_{n-1}, t_n]\) and we have:
\[
\int_{t_{n-1}}^{t_n} L_2(v) \, dt \leq C_3 \left( \int_{t_{n-1}}^{t_n} \left( \sum_{\kappa \in \mathcal{T}_h} (\eta^h_{n,\kappa})^2 \right) dt \right) \left( \int_{t_{n-1}}^{t_n} \|\nabla v(t)\|_{0,\Omega}^2 \, dt \right)^{\frac{1}{2}}
\]
\[
\leq C_3 \left( \sum_{\kappa \in \mathcal{T}_h} \tau_n (\eta^h_{n,\kappa})^2 \right)^{\frac{1}{2}} \|\nabla v\|_{L^2(t_{n-1}, t_n, L^2(\Omega))},
\]
\[
\leq C_4(\varepsilon_2) \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_h} \tau_n (\eta^h_{n,\kappa})^2 + \frac{\varepsilon_2}{2} \|\nabla (u - u_h)\|_{L^2(0,t_m,L^2(\Omega))}^2,
\]
where \( C_4(\varepsilon_2) \) is a constant independent of \( h_n \) and \( \tau_n \).

The relations (5.11), (5.14) and (5.16) allow us to get the following bound
\[
\beta \|\nabla (u - u_h)\|_{L^2(0,t_m,L^2(\Omega))}^2 + \frac{1}{2} \|u(t_m) - u^m_h\|_{0,\Gamma}^2
\]
\[
\leq c \left[ \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_h} (\eta^h_{n,\kappa})^2 + \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_h} \tau_n (\eta^h_{n,\kappa})^2 + \|u - u^0_h\|_{0,\Gamma}^2 \right]
\]
\[
+ \frac{(\varepsilon_1 + \varepsilon_2)}{2} \|\nabla (u - u_h)\|_{L^2(0,t_m,L^2(\Omega))}^2,
\]
where \( c \) is a constant independent of \( h_n \) and \( \tau_n \).

By choosing \( \varepsilon_1 = \frac{\beta}{2} \) and \( \varepsilon_2 = \frac{\beta}{2} \), we get the desired upper bound. \( \square \)

Next, we will bound the term \( \|\frac{\partial(u - u_h)}{\partial t}\|_{L^2(0,t_m,H^{-1/2}(\Gamma))}^2 \).

**Theorem 5.4.** For all \( m = 1, ..., N \), we have the bound:
\[
\|\frac{\partial(u - u_h)}{\partial t}\|_{L^2(0,t_m,H^{-1/2}(\Gamma))} \leq C \left[ \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_h} (\eta^h_{n,\kappa})^2 + \tau_n (\eta^h_{n,\kappa})^2 \right] + \|u - u^0_h\|_{0,\Gamma}^2,
\]
where \( C \) is a constant independent of \( h_n \) and \( \tau_n \).

**Proof.** Let \( r(t) \in H^{1/2}(\Gamma) \) and consider the problem:
\[
\begin{cases}
\Delta w(t, x) = 0 & \text{in } ]0, T[ \times \Omega, \\
w(t, x) = r(t, x) & \text{on } ]0, T[ \times \Gamma.
\end{cases}
\]

(5.19)
It admits a unique solution \( w(t) \in H^1(\Omega) \) which verify
\[
\| \nabla w(t) \|_{0,\Omega} \leq C_1 \| r \|_{1/2,\Gamma},
\] (5.20)
where \( C_1 \) is a constant.

We consider the equation (5.5), use the relation (5.12) and (5.15), and use the Cauchy Schwartz inequality to get
\[
\frac{1}{|\nabla v(t)|_{1,\Omega}} \int_{\Gamma} \frac{\partial (u - u_h)}{\partial t} (t, s) v(t, s) \, ds \leq \beta \| \nabla (u - u_h)(t) \|_{0,\Omega} + \beta \left[ \frac{t_n - t}{\tau_n} \right] \left( \sum_{\kappa \in \mathcal{T}_{h,n}} \| \nabla (u^n_h - u^{n-1}_h) \|_{0,\kappa}^2 \right)^{1/2} + c \left( \sum_{\kappa \in \mathcal{T}_{h,n}} (\eta^h_{n,\kappa})^2 \right)^{1/2}.
\] (5.21)

For every \( v(t) \in H^{1/2}(\Gamma) \), we consider it lifting in \( v \in H^1(\Omega) \) verifying the system (5.19). Using (5.20), we deduce following bound
\[
\frac{1}{|\nabla v(t)|_{1/2,\Gamma}} \int_{\Gamma} \frac{\partial (u - u_h)}{\partial t} (t, s) v(t, s) \, ds \leq \frac{1}{|\nabla v(t)|_{1,\Omega}} \int_{\Gamma} \frac{\partial (u - u_h)}{\partial t} (t, s) v(t, s) \, ds \leq \beta \| \nabla (u - u_h)(t) \|_{0,\Omega} + \beta \left[ \frac{t_n - t}{\tau_n} \right] \left( \sum_{\kappa \in \mathcal{T}_{h,n}} \| \nabla (u^n_h - u^{n-1}_h) \|_{0,\kappa}^2 \right)^{1/2} + c \left( \sum_{\kappa \in \mathcal{T}_{h,n}} (\eta^h_{n,\kappa})^2 \right)^{1/2}.
\] (5.22)

Then we get
\[
\| \frac{\partial (u - u_h)}{\partial t} \|_{-1/2,\Gamma} = \sup_{v \in H^{1/2}(\Gamma)} \frac{1}{|\nabla v(t)|_{1/2,\Gamma}} \int_{\Gamma} \frac{\partial (u - u_h)}{\partial t} (t, s) v(t, s) \, ds \leq \beta \| \nabla (u - u_h)(t) \|_{0,\Omega} + \beta \left[ \frac{t_n - t}{\tau_n} \right] \left( \sum_{\kappa \in \mathcal{T}_{h,n}} \| \nabla (u^n_h - u^{n-1}_h) \|_{0,\kappa}^2 \right)^{1/2} + c \left( \sum_{\kappa \in \mathcal{T}_{h,n}} (\eta^h_{n,\kappa})^2 \right)^{1/2}.
\] (5.23)

We deduce the desired result after integrating over \( [t_{n-1}, t_n] \), summing on \( n \) from 1 to \( m \) for a \( m \in \{1, ..., N\} \), and using the theorem 5.3.

To conclude the upper bound of our \textit{a posteriori} error, we bound the term \( \| \nabla (u - \pi \tau u_h) \|_{L^2(0,t_m,L^2(\Omega))}^2 \).

**Theorem 5.5.** For all \( m = 1, ..., N \), we have the bound
\[
\| \nabla (u - \pi \tau u_h) \|_{L^2(0,t_m,L^2(\Omega))}^2 \leq C [ \sum_{n=1}^{m} \sum_{\kappa \in \mathcal{T}_{h,n}} (\eta^h_{n,\kappa})^2 + \tau_n (\eta^h_{n,\kappa})^2 ] + \| u_0 - u^n_h \|_{0,1}^2.
\] (5.24)

where \( C \) as a constant independent of \( h_n \) and \( \tau_n \).

**Proof.** First, we have
\[
\| \nabla (u - \pi \tau u_h) \|_{L^2(0,t_m,L^2(\Omega))} \leq \| \nabla (u - u_h) \|_{L^2(0,t_m,L^2(\Omega))} + \| \nabla (u_h - \pi \tau u_h) \|_{L^2(0,t_m,L^2(\Omega))}.
\] (5.25)

The first term of right hand of the last relation can be bounded, using theorem 5.3, as
\[
\| \nabla (u - u_h) \|_{L^2(0,t_m,L^2(\Omega))} \leq C \left[ \sum_{n=1}^{m} \sum_{\kappa \in \mathcal{T}_{h,n}} (\eta^h_{n,\kappa})^2 + \sum_{n=1}^{m} \sum_{\kappa \in \mathcal{T}_{h,n}} \tau_n (\eta^h_{n,\kappa})^2 + \| u_0 - u^n_h \|_{0,1}^2 \right]^{1/2}.
\] (5.26)

Now, we will bound the second term of the right hand side of (5.25). For \( t \in [t_{n-1}, t_n] \), we have \( \pi \tau u_h(t) = u^n_h \) and then
\[
u n_h(t) - \pi \tau u_h(t) = \frac{t - t_n}{\tau_n} (u^n_h - u^{n-1}_h).
\] (5.27)

We obtain the relation
\[
\| \nabla (u_h - \pi \tau u_h(t)) \|_{0,\kappa}^2 \leq \left( \frac{t - t_n}{\tau_n} \right)^2 \left( \sum_{\kappa \in \mathcal{T}_{h,n}} \| \nabla (u^n_h - u^{n-1}_h) \|_{0,\kappa}^2 \right).
\] (5.28)
In the following, we will bound the indicators $\eta_{n,\kappa}$ and get
\[
\int_{t_{n-1}}^{t_n} \|\nabla (u_h - \pi_h u_h)(t)\|_{0,\Omega}^2 \leq \frac{1}{3} \sum_{\kappa \in \Gamma_h,n} (\eta_{n,\kappa}^\tau)^2.
\]
(5.29)

Finally, we conclude the relation
\[
\|\nabla (u - \pi u u_h)\|_{L^2(0,t_n;L^2(\Omega))} \leq C' \left[ \sum_{n=1}^{m} \sum_{\kappa \in \Gamma_h,n} (\eta_{n,\kappa}^\tau)^2 + \sum_{n=1}^{m} \sum_{\kappa \in \Gamma_h,n} \tau_n (\eta_{n,\kappa}^h)^2 + \|u_0 - u_h\|_{0,\Gamma}^2 \right]^{1/2},
\]
(5.30)

where $C'$ is a constant independent of $h_n$ and $\tau_n$.
\[\square\]

**Corollary 5.6.** For all $m = 1,\ldots,N$, we have the following upper bound:
\[
\|\nabla (u - \pi u u_h)\|_{L^2(0,t_n;L^2(\Omega))}^2 + \beta \|\nabla (u - u_h)\|_{L^2(0,t_n;L^2(\Omega))}^2 + \|u(t_m) - u_h\|_{0,\Gamma}^2
\]
\[
+ \|\frac{\partial (u - u_h)}{\partial t}\|_{L^2(0,t_n;H^{-1/2}(\Gamma))}^2 \leq C \left[ \sum_{n=1}^{m} \sum_{\kappa \in \Gamma_h,n} (\eta_{n,\kappa}^\tau)^2 + \sum_{n=1}^{m} \sum_{\kappa \in \Gamma_h,n} \tau_n (\eta_{n,\kappa}^h)^2 + \|u_0 - u_h\|_{0,\Gamma}^2 \right],
\]
(5.31)

where is a constant independent of $h_n$ and $\tau_n$.

**Remark:** Estimates (5.31) constitutes our a posteriori error estimate.

### 5.3. Upper bounds of the indicators.

In this section, we bound the indicators $\eta_{n,\kappa}^\tau$ and $\eta_{n,\kappa}^h$ in order to satisfy the optimality on the a posteriori error. We begin with the time indicator $\eta_{n,\kappa}^\tau$.

**Theorem 5.7.** For all $m = 1,\ldots,N$, the following estimate holds
\[
(\eta_{n,\kappa}^\tau)^2 \leq C \left( \|\nabla (u - \pi u u_h)\|_{L^2(t_{n-1},t_n;L^2(\kappa))}^2 + \|\nabla (u - u_h)\|_{L^2(t_{n-1},t_n;L^2(\kappa))}^2 \right),
\]
(5.32)

where $C$ is a constant independent of $h_n$ and $\tau_n$.

**Demonstration** For $t \in [t_{n-1},t_n]$, (5.27) allows us to have
\[
\left| \frac{t - t_n}{\tau_n} \right| \|\nabla (u_h^n - u_h^{n-1})(x)\| \leq 2(\|\nabla (u - u_h)(t,x)\|^2 + \|\nabla (u - \pi u u_h)(t,x)\|^2).
\]
(5.33)

We integrate the last relation on $\kappa$ and on $[t_{n-1},t_n]$ and we get the following result.
\[
(\eta_{n,\kappa})^2 \leq 6(\|\nabla (u - u_h)\|_{L^2(t_{n-1},t_n;L^2(\kappa))}^2 + \|\nabla (u - \pi u u_h)\|_{L^2(t_{n-1},t_n;L^2(\kappa))}^2),
\]
(5.34)

\[\square\]

In the following, we will bound the indicators $\eta_{n,\kappa}^h$. For $t \in [t_{n-1},t_n]$, we have
\[
\beta \int_{\Omega} \nabla (u(t) - u_h^n)(x) \nabla v(t,x) \, dx + \int_{\Gamma} \frac{\partial (u - u_h)}{\partial t}(t,s) v(t,s) \, ds
\]
\[
= -\beta \sum_{\kappa \in \Gamma_h,n} \int_{\kappa} \nabla u_h^n(t,x) \nabla v(t,x) \, dx - \int_{\Gamma} \frac{u_h^n - u_h^{n-1}}{\tau_n}(s) v(t,s) \, ds
\]
(5.35)
\[
= -\beta \sum_{\kappa \in \Gamma_h,n} \sum_{\kappa \in \partial \kappa} \int_{\Gamma} \phi_h^n(x) v(t,x) \, dx.
\]
Theorem 5.8. For all \(m = 1, \ldots, N\), the following bound holds
\[
\tau_n (\eta_{m,n}^h)^2 \leq C \left( \|u(t) - u(t_h)\|_{L^2(t_{n-1},t_n;L^2(\Delta \kappa))}^2 + \sum_{e \in \partial \kappa} \delta_e \| \frac{\partial (u - u_h)}{\partial t} (t) \|_{L^2(t_{n-1},t_n;H^{-1/2}(e))}^2 \right),
\] (5.36)
where
\[
\delta_e = \begin{cases} 
1 & \text{if } e \in \varepsilon_n^m \cap \partial \kappa \\
0 & \text{elsewhere} \end{cases},
\]
and \(C\) is a constant independent of \(h_n\) and \(\tau_n\).

Proof. We consider the equation (5.35), an element \(\kappa \in T_{nh}\) and an edge \(e\) of \(\kappa\). We distinguish two cases

1. \(e \in \varepsilon_n\) is an interior edge. We set \(v(t, x) = L_e(\phi_{h,n}^e \psi_e)(x)(5.35)\) and we get
\[
\int_e (\phi_{h,n}^e)^2 (x) \psi_e (x) \, dx = \int_{\Delta_e} \nabla (u - \pi_r u_h) (t, x) \nabla L_e(\phi_{h,n}^e \psi_e)(x) \, dx.
\] (5.37)
By using the Hölder inequality and the property 5.2, we get
\[
\int_e (\phi_{h,n}^e)^2 (x) \, dx \leq C \| \nabla (u - \pi_r u_h) (t) \|_{0, \Delta_e} |L_e(\phi_{h,n}^e \psi_e)|_{1, \Delta_e}
\]
\[
\leq C' \| \nabla (u - \pi_r u_h) (t) \|_{0, \Delta_e} h_e^{-\frac{1}{2}} \| \phi_{h,n}^e \|_{0, e},
\] (5.38)
where \(C, C'\) are constants independent of \(h_n\) and \(\tau_n\). Then for all interior edge \(e\) we have
\[
h_e \| \phi_{h,n}^e \|_{0,e} \leq C' \| \nabla (u - \pi_r u_h)(t) \|_{0, \Delta_e}^2
\] (5.39)

2. \(e \in \varepsilon_n^m\) is an edge on \(\partial \Omega\). We set \(v(t, x) = L_e(\phi_{h,n}^e \psi_e)(x)(5.35)\) and we get
\[
\int_e (\phi_{h,n}^e)^2 (x) \psi_e (x) \, dx = \int_{\kappa} \nabla (u - \pi_r u_h)(t, x) \nabla L_e(\phi_{h,n}^e \psi_e)(x) \, dx
\]
\[
+ \frac{1}{\beta} \int_e \frac{\partial (u - u_h)}{\partial t} (t, x)(\phi_{h,n}^e \psi_e)(x) \, dx,
\] (5.40)
By using the Hölder inequality and the property 5.2, we get
\[
\| \phi_{h,n}^e \|_{0,e}^2 \leq C \| \nabla (u - \pi_r u_h) (t) \|_{0, \kappa} \|L_e(\phi_{h,n}^e \psi_e)\|_{1, \kappa}
\]
\[
+ \frac{1}{\beta} \| \frac{\partial (u - u_h)}{\partial t} (t) \|_{-1/2, e} \| \phi_{h,n}^e \psi_e \|_{1/2, e},
\] (5.41)
where \(C\) is a constant independent of \(h_n\) and \(\tau_n\). The trace theorem and the property 5.2 allow us to get
\[
h_e^{\frac{1}{2}} \| \phi_{h,n}^e \|_{0,e} \leq C' (\| \nabla (u - \pi_r u_h)(t) \|_{0, \kappa} + \| \frac{\partial (u - u_h)}{\partial t} (t) \|_{-1/2, e}),
\] (5.42)
and then
\[
h_e \| \phi_{h,n}^e \|_{0,e}^2 \leq 2 C' (\| \nabla (u - \pi_r u_h)(t) \|_{0, \kappa}^2 + \sum_{e \in \partial \kappa} \delta_e \| \frac{\partial (u - u_h)}{\partial t} (t) \|_{-1/2, e}^2).
\] (5.43)
We deduce, by using (5.39) and (5.43), the following bound
\[
(\eta_{m,n}^h)^2 \leq C' (\| \nabla (u - \pi_r u_h)(t) \|_{0, \Delta \kappa}^2 + \sum_{e \in \partial \kappa} \delta_e \| \frac{\partial (u - u_h)}{\partial t} (t) \|_{-1/2, e}^2).
\] (5.44)
Finally, by integrating on \([t_{n-1}, t_n]\), we deduce (5.36).  

6. Numerical results

To validate the theoretical results, we perform several numerical simulations using the FreeFem++ software (see [11]). We consider the two-dimensional unit circle and we choose \(\beta = 1\) and \(T = 1\).
6.1. **a priori error validations.** We begin with the numerical validation of the *a priori* error estimates. The Figure 1 represents the mesh with \( m = 50 \) vertices on \( \Gamma \) and a mesh step size \( h = \frac{2\pi}{m} \). We consider the following exact solution
\[
 u(t, x, y) = \frac{(e^{-tx})^2 - (e^{-ty})^2}{2} + e^{-ty} + \frac{1}{2},
\]
which verify the system (1.1). We choose \( k = h \) and we consider the following numerical scheme
\[
(\nabla u_h^{n+1}, \nabla v_h) + \frac{1}{k} (u_h^{n+1}, v_h) = \frac{1}{k} (u_h^n, v_h).
\]
We introduce the error
\[
 err_N = \frac{\sum_{n=1}^{N} k \| u_h^n - u(t_n) \|_{1,\Omega}}{\sum_{n=1}^{N} k \| u(t_n) \|_{1,\Omega}},
\]
where \( N = \left\lceil \frac{T}{dt} \right\rceil = \left\lceil \frac{M}{2\Pi} \right\rceil \) (\([.]\) is the integer part).

The figure 2 shows in logarithmic scale, the error curve between the exact and the numerical solution for different values of the mesh step where \( M \) takes the values 80, 90, 100, 110, 120. As \( k = h \), the error must be of order of \( h \) and the slope of the straight line must be of the order one. The figure 2 gives a straight line with a slope of 0.9284.
7. A POSTERIORI ERROR VALIDATIONS

For the numerical validation of the a posteriori error estimates, we consider the following initial data on $\Gamma$ of the problem (1.1)

$$u_0(x, y) = u_0(\cos \theta, \sin \theta) = \begin{cases} \cos(5\theta) & \text{if } \theta \in [0, \frac{\pi}{4}], \\ 0 & \text{sinon}, \end{cases}$$  \hspace{1cm} (7.1)

and the numerical scheme

$$\forall v_h \in X_{nh}, \quad \beta \int_\Omega \nabla u^n_h \nabla v_h(t)dx + \int_\Gamma \frac{1}{\tau_n} u^n_h v_h(t)d\sigma = \int_\Gamma \frac{1}{\tau_n} u^{n-1}_h v_h(t)d\sigma. \hspace{1cm} (7.2)$$

We introduce the following time and space indicators

$$\eta^2 = \sum_{n=1}^{N} \sum_{\kappa \in \tau_n, \phi} \tau_n \| \nabla (u^n_h - u^{n-1}_h) \|^2_{0,d} \quad \text{and} \quad \eta^{2}_h = \sum_{n=1}^{N} \sum_{\kappa \in \tau_n, e \in \theta} \tau_n h_e \| \phi^e_{n,h} \|^2_{0,e}. \hspace{1cm} (7.3)$$

We begin the iterations with an initial time step $\tau_1 = \frac{T}{50}$ and the initial mesh represented by the Figure 1. The numerical investigations are done between $t = 0$ and $t = 1$. Our goal is to validate the a posteriori error estimates.

We present here an adaptive algorithm based on our a posteriori error estimates which ensures that the relative energy error between the exact and the approximate solutions is below a prescribed tolerance $\varepsilon$. At the same time, it intends to equilibrate the space and time estimators $\eta_{sp}$ and $\eta_{tm}$. We aim at achieving

$$\frac{\eta^2 + \eta^{2}_h}{\|u_h\|_{L^2(0,T;H^1(\Omega))}} \leq \varepsilon^2. \hspace{1cm} (7.4)$$

For practical implementation purposes and because of computer limitations, we introduce maximal space refinement level parameters $N_{sp}$. For the adapt mesh (refinement and coarsening), we use routines in FreeFem++. We set $\varepsilon_1 = \frac{\varepsilon}{\sqrt{2}}$ and we introduce the time and space error

$$e_1 = \frac{\eta}{\|u_h\|_{L^2(0,T;H^1(\Omega))}} \quad \text{and} \quad e_2 = \frac{\eta_h}{\|u_h\|_{L^2(0,T;H^1(\Omega))}}.$$

The actual algorithm is as follows:

Choose an initial mesh $\mathcal{T}_{0h}$, an initial time step $\tau_1$, and set $t_0 = 0$ Set $n = 1$

Loop in time: While $t_n \leq T$

1. $t_n = t_{n-1} + \tau_n$
2. Solve $u^n_{h*} = \text{Sol}(u^{n-1}_h, \tau_n, \mathcal{T}_{nh})$
3. calculate $e_1(u^n_{h*})$ and $e_2(u^n_{h*})$
4. if $((e_1 > \varepsilon_1) \text{ or } (e_2 \geq \varepsilon_1))$
5. if $e_1 > e_2$
6. set $t_n = t_{n-1} - \tau_n$ and $\tau_n = \tau_n/2$
7. else if (the level of space refinement at $t_n$ is less than $N_{sp}$)
8. set $t_n = t_{n-1} - \tau_n$
9. refine and coarse the mesh using the routine "ReMeshIndicator" in FreeFem++, and create a new mesh called again $\mathcal{T}_{nh}$
10. end if
11. else if($e_1$ is very smaller than $\varepsilon_1)$
12. set $\tau_n = 2\tau_n$, $u^n_h = u^n_{h*}$ and $n = n + 1$
13. set $\mathcal{T}_{nh} = \mathcal{T}_{n-1h}$
14. else
15. set $u^n_h = u^n_{h*}$ and $n = n + 1$
16. set $\mathcal{T}_{nh} = \mathcal{T}_{n-1h}$

In this algorithm, if the error does not satisfying the criterium (7.4), it tests if the time error is bigger than the space error, it decreases the time step to the half, elsewhere, it adapt the space mesh using the indicators and the routine ”ReMeshIndicator” in FreeFem++. If the error satisfies the criterium (7.4), it run away in time iterations either by increasing the time step if the error is very smaller the $\varepsilon_1$, or not keeping the same time step.

The Figures 3 and 4 show the evolution of the solution and the mesh during the time. It is clear that the mesh is concentrated around the part of the boundary $\Gamma$ where we impose the initial data.

\begin{figure}[ht]
\centering
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{initial_mesh.png}
\caption{Initial mesh}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{t=0.01.png}
\caption{Mesh at t=0.01}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{t=0.02.png}
\caption{Mesh at t=0.02}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{t=1.04.png}
\caption{Mesh at t=1.04}
\end{subfigure}
\caption{Evolution of the mesh during the time iterations.}
\end{figure}
Figure 4. Numerical solutions during the time iterations

References


