Consistent Shared Data Types: Beyond Memory

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ABSTRACT

In large scale distributed systems, shared objects provide a valuable abstraction of communication. However, these objects can only be used reliably if they are specified precisely. Until now, a lot of work has been done on shared memory, to the detriment of other objects. This paper aims at extending this work to any update-query abstract data type, the types in which the operations are updates or queries. A shared object should be fully specified by two complementary aspects: a sequential specification that defines how the updates influence the queries, and a consistency criterion that discriminates which distributed histories are eligible according to the sequential specification. This paper formalizes the notions of sequential specification and consistency criterion. It then extends the definition of many consistency criteria, including causal consistency, to all update-query abstract data types. It also explores the notion of composability for consistency criteria and proves that no consistency criterion between pipelined consistency and sequential consistency is composable, which includes causal consistency.

Categories and Subject Descriptors

E.1 [**Data**]: Data structures—*Distributed data structures*

General Terms

Design, Reliability, Theory

Keywords

Abstract data types, causal consistency, composability, consistency criteria, eventual consistency, pipelined consistency, sequential consistency.

1. INTRODUCTION

Overview. Distributed systems are often viewed as more difficult to program than sequential systems because they require to solve many issues due to communication. Shared objects, that can be accessed concurrently by the processes of the system, can be used as a practical abstraction of communications to let processes enjoy a more general view of the system. A precise specification of these objects is essential to ensure their adoption as well as reliability of distributed systems.

Many models have been proposed to specify shared memory, and several inventories [17, 1] can be found in the literature. In [12], Lamport defines linearizable registers, that ensures that everything appears as if all the operation where executed instantaneously at a point situated between the moment when the operation is called and the moment when it returns. Sequential consistency [11] is a little weaker: it only guaranties that all the operations appear totally ordered, and that this order is compatible with the *program order*, the order in which each process performs its own operations.

These strong consistency criteria are very expensive to implement in message-passing systems. In terms of latency, it is necessary to wait for answers for the reads or the writes for sequential consistency [13] and for all kinds of operations in the case of linearizability [3]. In terms of fault tolerance, strong hypothesis must be respected by the system: it is impossible to resist to partitioning [5, 8]. Many weaker consistency criteria have been proposed to solve this problem. Among them, PRAM [13], causal memory [2] and eventual consistency [19] are best documented. PRAM is a local consistency criterion: each process only sees its own reads and all the writes, and all these operations appear to it as totally ordered by an order that respects the program order. Causal consistency strengthens PRAM by imposing to these total orders to be compatible with a partial order that is common to all processes, the *causal order*, that not only contains the process order, but also a *writes-into* relation, that encodes data dependencies between the write and the read operations. Eventual consistency expects that, if all the processes stop writing, they will eventually read the same values.

Motivation. Memory is a good abstraction in sequential programming models because all kinds of objects can be implemented using variables. Things are more complicated for distributed computing because of race conditions: complex concurrent editing can often lead to inconsistent states. Critical sections offer a generic solution to this problem but at a high cost: they reduce parallelism and they are unable to tolerate faulty behaviors. A better solution is to design the shared objects directly, without using shared memory.

We believe that an object should be totally specified by two facets: a sequential specification given by an abstract data type, and a consistency criterion, that defines a link between distributed histories and sequential specifications. Sequential specifications are very easy to design because they are based on the well studied and understood notions of languages and automata. This makes possible to apply all the tools developed for sequential systems, from their simple definition using structures and classes to the most advanced techniques like model checking and formal verification. Sequential objects cannot be used directly in distributed environments. A consistency criterion is necessary to adapt the sequential specification to the environment. Graphically, we can imagine a consistency criterion as a way to take a picture of the distributed histories so that they look sequential.

The definition of eventual consistency is independent from memory, but it fails to fully specify shared objects and requires additional non-intuitive and error-prone distributed specification techniques [**?**]. Other consistency criteria, such as linearizability and sequential consistency, have already been defined for many data types [10], including memory. The definition of causal memory is based on a semantic matching between the reads and the writes, that does not exist so clearly for other objects. For example, the value of a counter depends on all the increments and decrements made on it from the beginning. Because of the *ad hoc* substitute for the writes-into order it requires for each data type, this approach cannot be used for a generic definition of causal consistency. Moreover, Ahamad's definition has a serious limit even for memory: it does not handle multiple writings of the same value correctly. To our knowledge, the only solution to this issue is to forbid those editions [16], which is very damaging for the ease of use of the memory.

Contributions of the paper. This paper aims at introducing a clear distinction between the notions of sequential specification and consistency criteria that are both necessary to fully specify shared objects. It has four main contributions. It (1) formalizes the notions of abstract data types and consistency criteria and (2) extends the definitions of many famous consistency criteria to a larger class of abstract data types. In particular, it (3) explores the limits of causal memory and gives a new definition of causal consistency. It also (4) formalizes the notion of composability and proves several results on it.

The remainder of this paper is organized as follows. Section 2 formalizes the type of objects we target in this paper as well as the notion of consistency criteria and studies general properties on them. Section 3 illustrates the formalism with several examples of consistency criteria. Finally, Section 4 concludes the paper.

2. SPECIFYING SHARED OBJECTS

In distributed systems, sharing objects is a way to abstract communications between processes. The abstract type of these objects has a sequential specification, defined in this paper by a transition system, that characterizes the sequential histories allowed for this object. As shared objects are implemented in a distributed system, typically using replication, the events in a distributed history are partially ordered. A consistency criterion is necessary to make the link between the sequential specifications and the distributed executions that invoke them, by a characterization of the histories that are admissible for a program that uses the objects, depending on their type.

2.1 Update-query abstract data types

We use transition systems to specify sequential abstract data types. Our modeling of update-query abstract data type is intermediate between Mealy machines [15] and transition systems. We separate the input alphabet into two classes of operations: the updates and the queries[1]. On the one hand, the updates can have a side-effect that usually affects everyone, but do not return a value. They correspond to transitions between abstract states in the transition system. On the other hand, the queries can only read the state of the object but not modify it. Like in mealy machines, they produce an output that depends on the state on which they are executed.

Definition 1. An *update-query abstract data type* (UQ-ADT) is a 7-tuple $T = (U, Q_i, Q_o, S, s_0, \mu, \varphi)$ such that:

- U is a countable set of *update* operations;
- Q_i and Q_o are countable sets called *input* and *output* alphabets. $Q = Q_i \times Q_o$ is the set of *query* operations. A query operation (q_i, q_o) is denoted q_i/q_o;
- S is a countable set of *states*;
- $s_0 \in S$ is the *initial state*;
- $\mu : S \times U \to S$ is the *transition function* that specifies the updates;
- $\varphi : S \times Q_i \to Q_o$ is the *output function* that specifies the queries.

We consider the UQ-ADTs up to isomorphism as the names of the states are never used in our work. The set of all the UQ-ADTs is denoted by $\mathscr{T}$.

An infinite sequence of operations $(l_i)_{i\in\mathbb{N}} \in (U \cup Q)^\omega$ is recognized by T if there exists an infinite sequence of states $(s_i)_{i\geq 1} \in S^\omega$ such that for all $i \in \mathbb{N}$, $\mu(s_i, l_i) = s_{i+1}$ if $l_i \in U$ or $s_i = s_{i+1}$ and $\varphi(s_i, q_i) = q_o$ if $l_i = q_i/q_o \in Q$. The set of all infinite sequences recognized by T and their finite prefixes is denoted by $L(T)$. Informally, $L(T)$ contains the sequential histories that are admissible for T.

[1]Some objects, such as Compare&Swap and Test&Set for example, have operations that are both an update and a query. Such objects are out of the scope of this paper.

Suppose one wants to use several objects together in their program. We can model the composition of these objects as another object on which it is possible to apply the operations of both objects. The composition of two UQ-ADTs T_1 and T_2 is a parallel, asynchronous product of transitions systems. A word is recognized by the composition if it is the interleaving of two words recognized by T_1 and T_2 respectively. This composition is associative and commutative, and there is one neutral element T_1 that contains one state and no transition and an absorbing element T_0 that contains no state and no transition.

Definition 2. We define the composition of two UQ-ADTs $(U, Q_i, Q_o, S, s_0, \mu, \varphi) \times (U', Q'_i, Q'_o, S', s'_0, \mu', \varphi')$ as the UQ-ADT $(U \sqcup U', Q_i \sqcup Q'_i, Q_o \cup Q'_o, S \times S', s'_0, \mu'', \varphi'')$ with, for all $s \in S$, $s' \in S'$, $u \in U$, $u' \in U'$, $q \in Q_i$ and $q' \in Q'_i$, $\mu''((s, s'), u) = T(s, u)$, $\mu''((s, s'), u') = \mu'(s', u')$, $\varphi''((s, s'), q) = \varphi(s, q)$ and $\varphi''((s, s'), q') = \varphi'(s', q')$. The symbol $\sqcup$ stands for the disjoint union of sets.

We illustrate UQ-ADTs by two examples. We first give the full sequential specification of the register and the memory. Then, we define the graph as a class, like in sequential object-oriented programming languages.

The integer *register* $\mathcal{M}_x$ can be accessed by a *write* update operation $\mathrm{w}_x(n)$, where $n \in \mathbb{N}$ and a *read* query operation r_x that returns the last value written, if there is one, or the default value 0 otherwise. The integer *memory* $\mathcal{M}_X$ is the collection of the integer registers of X. More formally, they correspond to the UQ-ADTs given in Example 1.

Example 1. Let x be any symbolic *register name*. We define the integer *register* on x by the UQ-ADT:

$$\mathcal{M}_x = (\{\mathrm{w}_x(n) : n \in \mathbb{N}\}, \{\mathrm{r}_x\}, \mathbb{N}, \mathbb{N}, 0, \mu, \varphi)$$

with, for all $n, p \in \mathbb{N}$, $\mu(n, \mathrm{w}_x(p)) = p$ and $\varphi(n, \mathrm{r}_x) = n$.

Let X be a countable set of register names, we define the integer *memory* on X by the UQ-ADT:

$$\mathcal{M}_X = \prod_{x \in X} \mathcal{M}_x.$$

Figure 1 presents another more usual way to define sequential specifications. Here, the graph type $\mathcal{G}$ is specified by a class. A graph is constituted of a set of vertices, here represented by integers, and a set of edges, that are pairs of vertices. It provides four update operations, to insert and delete edges and vertices, and two query operation that allow to check if a vertex or an edge is present in the graph and return a boolean value. A graph must remain consistent: it cannot contain edges between vertices that do not exist in the graph. To ensure that, vertices are inserted or edges are deleted to avoid inconsistencies. An UQ-ADT can easily been deduced from this specification. The states and the operations are defined by the member variables and methods of the class, the output alphabet contains all the values that can be returned by the queries, here the booleans values true ($\top$) and false ($\bot$), the transition function is defined by the implementation of the update methods and the output function by the query methods. We consider the pruned transition system, in which all the states are reachable.

```
class G
    var vertices ⊂ ℕ ← ∅;
    var edges ⊂ ℕ × ℕ ← ∅;
    update I_v (v ∈ ℕ)                       /* insert vertex */
        vertices ← vertices ∪ {v};
    end
    update D_v (v ∈ ℕ)                       /* delete vertex */
        edges ← edges \ (V × {v} ∪ {v} × V);
        vertices ← vertices \ {v};
    end
    query Q_v (v ∈ ℕ) ∈ {⊥, ⊤}               /* get vertex */
        return v ∈ vertices;
    end
    update I_e (v1 ∈ ℕ, v2 ∈ ℕ)              /* insert edge */
        vertices ← vertices ∪ {v1, v2};
        edges ← edges ∪ {(v1, v2)};
    end
    update D_e (v1 ∈ ℕ, v2 ∈ ℕ)              /* delete edge */
        edges ← edges \ {(v1, v2)};
    end
    query Q_e (v1 ∈ ℕ, v2 ∈ ℕ) ∈ {⊥, ⊤}      /* get edge */
        return (v1, v2) ∈ edges;
    end
end
```

Figure 1: Sequential specification of the graph

2.2 Distributed histories

During an execution, the participants call the operations of an object, an instance of the abstract data type, which produces a set of events labelled by the operations of the abstract data type. In general, the computing entities are sequential, which imposes a strict ordering between their own operations, so the events are partially ordered. For example, in the case of communicating sequential processes, an event a precedes an event b in the *program order* if they are executed by the same process in that sequential order. In this case, the processes correspond to the maximal chains of the history. This representation of distributed histories is generic enough to model a large number of distributed systems, such as peer-to-peer systems where peers join and leave permanently, or more complex modern systems in which new threads are created and destroyed dynamically.

Definition 3. A distributed history (or simply history) is a 5-tuple $H = (U, Q, E, \Lambda, \mapsto)$ such that:

- U and Q are disjoint countable sets of *update* and *query* operations, and all queries are in the form q_i/q_o;
- E is a countable set of *events*;
- $\Lambda : E \to U \cup Q$ is a *labelling function*;
- $\mapsto \subset E \times E$ is a partial order called *program order*, such that for all $e \in E, \{e' \in E : e' \mapsto e\}$ is finite, i.e. all event has a finite past.

The set of all the distributed histories is denoted by $\mathscr{H}$.

Let $H = (U, Q, E, \Lambda, \mapsto)$ be a distributed history. We now introduce a few notations.

- The update and query events of H are denoted by $U_H = \{e \in E : \Lambda(e) \in U\}$, $Q_H = \{e \in E : \Lambda(e) \in Q\}$.
- Two events e and e' are concurrent (denoted $e||e'$) if they are not comparable for $\mapsto$, i.e. $e \not\mapsto e'$ and $e' \not\mapsto e$.
- $\mathscr{P}_H$ denotes the set of the maximal chains of H:

  $$\mathscr{P}_H = \left\{ p \subset E : \begin{array}{l} \forall e, e' \in p, (e \mapsto e' \vee e' \mapsto e) \\ \wedge \forall e'' \in E \setminus p, (e||e'' \vee e'||e'') \end{array} \right\}.$$

  In the case of sequential processes, each $p \in \mathscr{P}_H$ corresponds to the events produced by a process. In the remainder of this article, we use the term "process" to designate such a chain, even in models that are not based on a collection of communicating sequential processes.
- We also define some projections on the histories. The first one allows to withdraw some events: for $F \subset E$, $H_F = (U, Q, F, \Lambda|_F, \mapsto \cap (F \times F))$ is the history that contains only the events of F. The second one allows to replace the program order by another order $\to$: if $\to \cap (E \times E)$ respects the definition of a program order, $H^{\to} = (U, Q, E, \Lambda, \to \cap (E \times E))$ is the history in which the events are ordered by $\to$. Note that the projections commute, which allows the notation $H_F^{\to}$.
- A history in which a composition of two objects is used can also be seen as the composition of two sub-histories that only contain the events of one of the objects. Let $H = (U, Q, E, \Lambda, \mapsto)$ be a distributed history and $\{E_1, E_2\}$ be a partition of E. We say that H is a composition of H_{E_1} and H_{E_2}. There is more than one way to compose histories, so the composition of two histories is a set of histories. The set of all the compositions of H_1 and H_2 is denoted by $H_1 \times H_2$.
- Finally, a linearization of H corresponds to a sequential history that contains the events of H in an order consistent with the program order. More precisely, it is a word $\Lambda(e_0) \dots \Lambda(e_n) \dots$ such that $\{e_0, \dots, e_n, \dots\} = E$ and for all i and j, if $i < j$, then $e_j \not\mapsto e_i$. We denote by $\mathrm{lin}(H)$ the set of all linearizations of H.

2.3 Consistency criteria

A consistency criterion characterizes which histories are admissible for a given data type. More formally, it is a function C that associates a set of consistent histories $C(T)$ to all UQ-ADTs $T = (U, Q_i, Q_o, S, s_0, \mu, \varphi)$ such that, for all $H = (U', Q', E, \Lambda, \mapsto)$ in $C(T)$, $U' \subset U$ and $Q' \subset Q_i \times Q_o$. The set of all consistency criteria is denoted by $\mathscr{C}$. A shared object is C-consistent for a consistency criterion C and a UQ-ADT T if all the histories it admits are in $C(T)$.

We say that a criterion $C1$ is *stronger* than a criterion $C2$, denoted $C_2 \leq C_1$, if for all $T \in \mathcal{T}$, $C1(T) \subset C2(T)$. A strong consistency criterion guaranties stronger properties on the histories it admits, so a C_1-consistent implementation can always be used instead of a C_2-consistent implementation of the same abstract data type if $C_2 \leq C_1$.

Sometimes, one wants an object to respect several consistency criteria simultaneously (e.g. a causally consistent and eventually consistent memory). We define a join operator $C_1 + C_2 : T \mapsto C_1(T) \cap C_2(T)$. $(\mathscr{C}, \leq, +)$ is a join-semilattice. It has a minimal element, $C_\perp$, that accepts all the histories for all the objects and a maximal element, $C_\top$, that accepts none of them.

We can now define the composition of two consistency criteria. If C_1 and C_2 are two consistency criteria, $C_1 \times C_2$ denotes the set of the histories that are the composition of a C_1 consistent history and a C_2 consistent history.

Definition 4. Let C_1 and C_2 be two consistency criteria. We define $C_1 \times C_2$ as the strongest consistency criterion such that, for all $T_1, T_2 \in \mathscr{T}$ and for all histories $H_1 \in C_1(T_1)$ and $H_2 \in C_2(T_2)$, $H_1 \times H_2 \subset (C_1 \times C_2)(T_1 \times T_2)$. This strongest criterion exists since the property that we require on it is conserved by $+$.

$(\mathscr{C}, \times)$ is a commutative monoid (i.e. $\times$ is commutative and associative and $C^\top$ is neutral), $C_\perp$ is absorbing, $\times$ is distributive for $+$ (i.e. $C_1 \times (C_2 + C_3) = C_1 \times C_2 + C_1 \times C_3$), $\leq$ is compatible with $\times$ (i.e. $C_1 \leq C_2 \Rightarrow C_1 \times C_3 \leq C_2 \times C_3$) and $C_1 \times C_2 \leq C_1$.

We use the classical power notation C^n for the composition of n C-consistent objects, and the compositional closure $C^\star : T \mapsto \bigcup_{n \in \mathbb{N}} C^n(T)$ stands for the composition of any number of C-consistent objects. Intuitively, $C^\star$ is the limit of C^n when n grows to infinity.

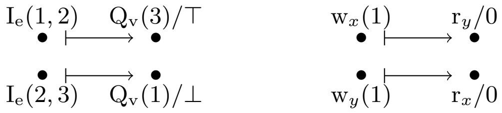
One could expect that a program that uses two C-consistent objects together will remain C-consistent with respect to the composition of the objects. This property, called composability, is an important property because it allows to program in a modular way, but only $C^2 \leq C$ is true in general.

Definition 5. A consistency criterion C is *composable* if it is idempotent for composition, i.e. $C^2 = C$.

As we will see in Section 3, composability is difficult to achieve. The reciprocal, however, is a natural request to all consistency criteria: if a history is globally consistent, it should also be consistent for all the objects that are involved in it. Decomposability means that, for all $T_1, T_2 \in \mathscr{T}$ and all history $H \in C(T_1 \times T_2)$, there exists histories $H_1 \in C(T_1)$, $H_2 \in C(T_2)$ such that $H \in H_1 \times H_2$. All the consistency criteria defined thereafter are decomposable.

3. COMMON CONSISTENCY CRITERIA

We now illustrate the concept of consistency criteria with common examples. We first illustrate the formalism with sequential consistency [11] and its derivatives, cache consistency [9] and linearizability [10]. We then formalize pipelined consistency [13], that is very close in its definition to local consistency. Causal consistency [2] is more complicated to extend because its definition is closely related to the semantics of the operations on the registers. We finish this presentation with eventual consistency [19] and strong eventual consistency [18]. All these criteria are illustrated on small examples on the memory and graph data types.



(a) $SC(\mathcal{G})$ (b) $SC^\star$, not SC

Figure 2: A sequential history for an instance of $\mathcal{G}$ and a cache consistent history for an instance of $\mathcal{M}_{\{x,y\}}$ that is not sequentially consistent

3.1 Sequential consistency

Sequential consistency was originally defined by Lamport in [11] as: "the result of any execution is the same as if the operations of all the processors were executed in some sequential order, and the operations of each individual processor appear in this sequence in the order specified by its program". In our formalism, such a sequence is a word of update and query operations that has two properties: it is correct with respect to the sequential specification of the object (i.e. it belongs to $L(T)$) and the total order is compatible with the program order (i.e. it belongs to $\mathrm{lin}(H)$).

Definition 6. A history H is *sequentially consistent* (SC) with an UQ-ADT T if $\mathrm{lin}(H) \cap L(T) \neq \emptyset$.

Figure 2a shows a sequentially consistent history. It can be viewed as two processes sharing a graph of $\mathcal{G}$. The first process first inserts the edge $(1,2)$ and then reads that the vertex 3 was inserted, while the second process inserts the edge $(2,3)$ and then reads that the vertex 1 was not inserted yet. The word $\mathrm{I_e}(2,3).\mathrm{Q_v}(1)/\bot.\mathrm{I_e}(1,2).\mathrm{Q_v}(3)/\top$ is in both $\mathrm{lin}(H)$ and $L(\mathcal{G})$, so this history is sequentially consistent.

The history of Figure 2b is very close, but the shared object is a memory, and each query returns the initial value of the register. This history is not sequentially consistent because the first event of any linearization must be a write which precedes the read of the same variable, that should return a 1. To be sequentially consistent, at least one read should return a 1.

Cache consistency. The compositional closure seems to be an easy way to define composable consistency criteria, as $C^\star$ is the strongest composable consistency criterion weaker than C. However, this criterion can be very weak. For example, $SC^\star$, known as *cache consistency* [9] or simply *coherence* [7] for memory in the literature, does not allow any guaranty of synchronization between the variables.

As $\mathrm{r}_x/0.\mathrm{w}_x(1)$ is a possible linearization for the events on $\mathcal{M}_x$ and $\mathrm{r}_y/0.\mathrm{w}_y(1)$ is a possible linearization for the events on $\mathcal{M}_y$, the history of Figure 2b is cache consistent for $\mathcal{M}_{\{x,y\}} = \mathcal{M}_x \times \mathcal{M}_y$.

Linearizability: the case of real time. Linearizability is very close to sequential consistency, as it also imposes the existence of a total order on all the events in the history. Real time must be respected by this total order in linearizability: if an event e_1 finishes before another event e_2 starts, then e_1 must precede e_2 in the total order.

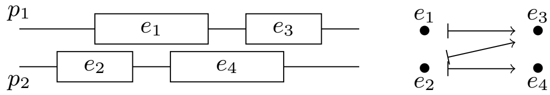


Figure 3: The program order is defined considering real time dependencies. We cannot have both $e_1||e_4$ and $e_2||e_3$

We did not introduce real time in our model yet because it is not relevant for most consistency criteria, and no global clock can be implemented in asynchronous distributed systems. However, it is possible to model real-time dependencies between events directly in the histories, by only considering interval orders [6] for the program orders. In this paragraph, we change the modeling of the executions: an event e_1 precedes an event e_2 in the program order if and only if e_1 returns before e_2 starts. Let us consider Figure 3. A process p_1 produces the events e_1 and e_3 while a process p_2 produces the events e_2 and e_4. It is impossible that e_3 starts before e_2 finishes and e_4 starts before e_1 finishes at the same time, which implies $e_1 \mapsto e_4$ or $e_2 \mapsto e_3$ if the program order models real time.

Definition 7. A history $(U, Q, E, \Lambda, \mapsto)$ is *real time consistent* (RT) if $(e_1 \mapsto e_3 \wedge e_2 \mapsto e_4) \Rightarrow (e_1 \mapsto e_4 \vee e_2 \mapsto e_3)$ for all $e_1, e_2, e_3, e_4 \in E$.

Real time consistency is not composable. However, if real time is respected during the composition of sequentially consistent histories, the composed histories are also sequentially consistent.

PROPOSITION 1. $SC^2 + RT = SC + RT$.

PROOF. It is clear that $SC^2 + RT \leq SC + RT$ and that $RT \leq SC^2 + RT$. We prove that $SC \leq SC^2 + RT$. Let $T_1, T_2 \in \mathscr{T}$, $H_1 = (U_1, Q_1, E_1, \Lambda_1, \mapsto_1) \in SC(T_1)$ and $H_2 = (U_2, Q_2, E_2, \Lambda_2, \mapsto_2) \in SC(T_2)$. We suppose that $H = (U, Q, E, \Lambda, \mapsto) \in H_1 \times H_2 \cap RT(T_1 \times T_2)$. We prove that $H \in SC(T_1 \times T_2)$.

By definition of sequential consistency, for $i \in \{1,2\}$ there is a total order $\leq_i$ on E_i and a unique word l_i such that $\mathrm{lin}\left(H_i^{\leq_i}\right) \cap L(T_i) = \{l_i\}$ and $\mapsto_i \subset \leq_i$. For sake of simplicity, we extend $\leq_i$ on E by taking its reflexive closure. We pose $\circ = \leq_1 \cup \leq_2 \cup \mapsto$ and $\bullet$ as the transitive closure of $\circ$. We first prove that $\bullet$ is antisymmetric.

Let $e_1 \circ e_2 \circ ... \circ e_{n-1} \circ e_n \in E$, with $e_1 \in E_1$. We prove by induction on n that there are $e, e' \in E$ such that $e_1 \leq_1 e \mapsto e' \leq_2 e_n$ (and by symmetry, $e_1 \leq_2 e \mapsto e' \leq_1 e_n$ if $e_1 \in E_2$). If $n = 1$, we have $e_1 \leq_1 e_1 \mapsto e_1 \leq_2 e_1$. Suppose the result is true for n and let us examine it for $n+1$. We have $e_1 \leq_1 e \mapsto e' \leq_2 e_n \circ e_{n+1}$. There are three cases for $\circ$.

- If $e_n \leq_2 e_{n+1}$, by transitivity, $e_1 \leq_1 e \mapsto e' \leq_2 e_{n+1}$.
- If $e_n \leq_1 e_{n+1}$ and $e_n \neq e_{n+1}$, then $e_n \in E_1$ so $e' = e_n$. Moreover, $e \in E_1$ because $e_1 \leq_1 e$, so $e \leq_1 e_n$. We have $e_1 \leq_1 e_{n+1} \mapsto e_{n+1} \leq_2 e_{n+1}$.
- If $e_n \mapsto e_{n+1}$, by real-time, either $e \mapsto e_{n+1}$ or $e_n \mapsto e'$ holds. In the first case, $e_1 \leq_1 e \mapsto e_{n+1} \leq_2 e_{n+1}$. In the second case, $e_n \leq_2 e' \leq_2 e_n$, so $e' = e_n$. By transitivity of $\mapsto$, $e_1 \leq_1 e \mapsto e_{n+1} \leq_2 e_{n+1}$.

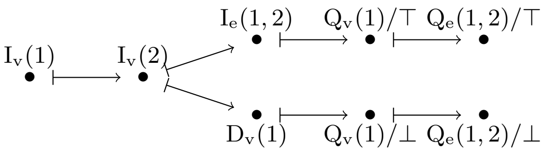


Figure 4: A locally consistent history on $\mathcal{G}$

(a) $SC^\star$ but not PC (b) PC but not $SC^\star$

Figure 5: These histories on $\mathcal{M}_{x,y}$ show that PC is not comparable with $SC^\star$

Let $e_1, e_2 \in E$ such that $e_1 \bullet e_2$ and $e_2 \bullet e_1$. Let us prove that $e_1 = e_2$.

- Suppose $e_1, e_2 \in E_1$ (the case $e_1, e_2 \in E_1$ is symmetric). We have $e_1 \leq_1 e \mapsto e' \leq_2 e_2 \leq_1 e'' \mapsto e''' \leq_2 e_1$, so $e_1 \leq_1 e \leq_1 e' = e_2 \leq_1 e'' \leq_1 e''' = e_1$ and $e_1 = e_2$.
- Suppose now that $e_1 \in E_1$ and $e_2 \in E_2$ (which proves the case $e_1 \in E_2$ and $e_2 \in E_1$ by symmetry). We have $e_1 \leq_1 e \mapsto e' \leq_2 e_2 \leq_2 e'' \mapsto e''' \leq_1 e_1$. By real-time, we have either $e \mapsto e'''$ or $e'' \mapsto e'$. Suppose that $e \mapsto e'''$ (the other case is symmetric). We have $e_1 \leq_1 e \mapsto_1 e''' \leq_1 e_1$, so $e_1 = e = e'''$. Moreover, $e'' \mapsto e''' = e \mapsto e'$, which implies $e_2 = e'' = e'$. Finally, $e_1 = e \mapsto e' = e_2 = e'' \mapsto e''' = e_1$, so $e_1 = e_2$.

As $\bullet$ is a partial order, we can extend it in a total order $\leq$. $\mathrm{lin}\left(H^{\leq}\right)$ contains exactly one word l, that is an interleaving of l_1 and l_2. As $l_1 \in L(T_1)$ and $l_2 \in L(T_2)$, $l \in L(T_1 \times T_2)$. Finally, $H \in SC(T_1 \times T_2)$. $\square$

3.2 Pipelined consistency

Pipelined consistency is an extension of pipelined random access memory (PRAM) [13] for other data types. It allows the processes to be aware of some, but not all, of the events. The definition of pipelined consistency is very close to those of local consistency, a very easily understood consistency criterion that we introduce first to illustrate the formalism needed for pipelined consistency.

Local consistency. Local consistency is the criterion respected by local variables. A local variable contains an object of type T that is not shared on the network. All the events on it are done by a sequential process, so they form a maximal chain p in the history. We recall that the maximal chains of the history H are contained into $\mathscr{P}_H$. This means that $\mathrm{lin}\left(H_p\right)$ is a singleton $\{l\}$ that only contains the sequential history seen by p. Local consistency requires that this history is correct with respect to the sequential specification of the object, i.e. $l \in L(T)$. More formally, Definition 8 requires that $\mathrm{lin}\left(H_p\right) \cap L(T)$ is not empty, as it must contain l.

Definition 8. A history H is *locally consistent* (LC) with an UQ-ADT T if $\forall p \in \mathscr{P}_H, \mathrm{lin}\left(H_p\right) \cap L(T) \neq \emptyset$.

The history on Figure 4 is locally consistent. It represents a graph edited by a thread. At one point, this thread forks and the graph is edited separately by the father thread and its son. There are two maximal chains in this history, and the first events are part of both. The operations made by one thread are ignored by the other thread. Local consistency does not make sense when threads are allowed to join, because one value must be discarded. If no join is allowed for the histories, for example because the computation model is based on parallel sequential processes, local consistency is composable, in the manner of sequential consistency with real time.

Pipelined consistency. Pipelined consistency is close in its definition to local consistency, but the processes are also aware of the updates made by the other processes, and of the order in which they are made. Each process must be able to explain the history individually by a linearization of their own knowledge. The consistency is local to each process, as different processes can see concurrent updates in a different order.

Definition 9. A history H is *pipelined consistent* (PC) with an UQ-ADT T if $\forall p \in \mathscr{P}_H, \mathrm{lin}\left(H_{U_H \cup p}\right) \cap L(T) \neq \emptyset$.

Pipelined consistency is weaker than sequential consistency, for which the linearizations seen by different processes must be identical (for a complete proof, see Proposition 3 in Section 3.3). It is not comparable with cache consistency, as illustrated on Figure 5. On the graph, that cannot be decomposed into simpler data types, cache consistency is equal to sequential consistency so we illustrate this on a shared memory.

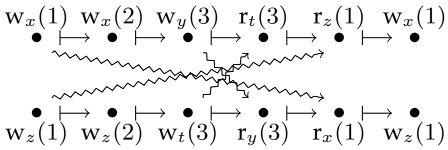
For memory, cache consistency is very weak as finite locally consistent histories are also cache consistent. The history of Figure 5a is also cache consistent since $r_x/0.w_x(1)$ and $w_y(1).r_y/1$ are correct linearizations of the sub-histories that only consider one register at a time. It is not pipelined consistent because there is no linearization of all the events, that is required for the second process.

As there is only one register in the history of Figure 5b, all the events must be considered in the same linearization for cache consistency, which is not possible. However, it is pipelined consistent: $w_x(1).w_x(2).r_x/2$ is a linearization for the first process and $w_x(2).w_x(1).r_x/1$ is a linearization for the second process.

We now prove a negative result on composability: there exists no composable consistency criterion between pipelined consistency and sequential consistency.

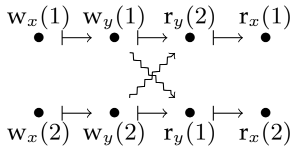
PROPOSITION 2. $\forall C, PC \leq C \leq SC \Rightarrow C^2 \neq C$.

PROOF. If there existed a C such that $PC \leq C \leq SC$ and $C = C^2$, we would have $PC \leq C = C^2 = C^\star \leq SC^\star$. The example of Figure 5a proves that $PC \not\leq SC^\star$, so such a C does not exist. $\square$



$l_1 = \mathrm{w}_x(1)\mathrm{w}_x(2)\mathrm{w}_y(3)\mathrm{w}_z(1)\mathrm{w}_z(2)\mathrm{w}_t(3)\mathrm{r}_t(3)\mathrm{w}_z(1)\mathrm{r}_z(1)\mathrm{w}_x(1)$
$l_2 = \mathrm{w}_x(1)\mathrm{w}_x(2)\mathrm{w}_y(3)\mathrm{w}_z(1)\mathrm{w}_z(2)\mathrm{w}_t(3)\mathrm{r}_y(3)\mathrm{w}_x(1)\mathrm{r}_x(1)\mathrm{w}_z(1)$

Figure 6: This writes-into order is not a correct explanation of the data dependencies in this history



$l_3 = \mathrm{w}_x(2)\mathrm{w}_x(1)\mathrm{w}_y(1)\mathrm{w}_y(2)\mathrm{r}_y(2)\mathrm{r}_x(1)$
$l_4 = \mathrm{w}_x(1)\mathrm{w}_x(2)\mathrm{w}_y(2)\mathrm{w}_y(1)\mathrm{r}_y(1)\mathrm{r}_x(2)$

Figure 7: In the case of concurrent writes, at least one process should eventually see the value of the other process

3.3 Causal Consistency

Causal memory. *Causal Consistency* [2] is an intermediate criterion between pipelined consistency and sequential consistency. Pipelined consistency is only local; each process has a consistent vision of the events it is aware of, but they can disagree on the order in which updates happen. On the other hand, sequential consistency imposes a total order on all the operations of the history. Causal consistency supposes the existence of a logical time, the causal order, composed of two kinds of dependences. On the one hand, the program order must be respected: like in pipelined consistency, if two operations happen on the same process, all the processes that see both operations must see them in the same order. On the other hand, two events related by data dependencies must be ordered in the causal order. More precisely, if a read returns the value written by a write, these events are related by a *writes-into* order that can affect the linearizations of all processes. We now recall the formal definition of causal memory.

Definition 10. Let $\mathcal{M}_X$ be a memory update-query abstract data type. A relation $\rightsquigarrow$ is a writes-into order if:

- for all $e, e' \in E$ such that $e \rightsquigarrow e'$, there are $x \in X$ and $n \in \mathbb{N}$ such that $\Lambda(e) = \mathrm{w}_x(n)$ and $\Lambda(e') = \mathrm{r}_x/n$,
- for all $e \in E$, $|\{e' \in E : e' \rightsquigarrow e\}| \leq 1$,
- for all $e \in E$ such that $\Lambda(e) = \mathrm{r}_x/n$ and there is no $e' \in E$ such that $e' \rightsquigarrow e$, then $n = 0$.

A history H is $\mathcal{M}_X$*-causal* if there exists a writes-into order $\rightsquigarrow$ such that:

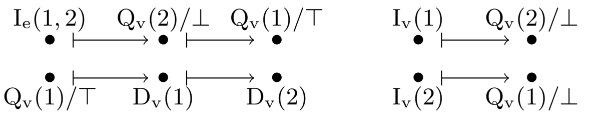
- the transitive closure $\rightarrow$ of $\rightsquigarrow \cup \mapsto$ is a partial order,
- $\forall p \in \mathscr{P}_H, \mathrm{lin}\left(H^{\rightarrow}_{U_H \cup p}\right) \cap L(\mathcal{M}_X) \neq \emptyset$.

Limits of causal memory. The first limit of causal memory is that it cannot be easily extended to other data types. Actually, the definition of writes-into orders is deeply related to the semantics of registers. For other abstract data types, e.g. graphs, counters or stacks, the value returned by a query does not depend on one particular update, but on all the updates that happened before. Moreover, in the case of the stack, these updates must be sorted to take into account the order of the elements in the stack. It might be possible to define a writes-into order for each of these objects individually, but this approach cannot be used in a data type-independent definition of causal consistency.

The second limit comes from the fact that the writes-into order is not unique. This weakens the role of the logical time, as the intuition that a read must be bound to its corresponding write is not always captured by the definition. Let us illustrate that point with the history on Figure 6. In this history, we consider the writes-into order in which the reads on x and z are related to the first write of the other process. This writes-into order is correct, as each read is related to exactly one write, and the variables and the values are the same. Moreover, l_1 and l_2 are correct linearizations, so this history is causally consistent. However, in these linearizations, the value read by the two last reads was not written by their predecessors in the writes-into relation, but if we change this relation to restore the real data dependencies, we obtain a cycle in the causal order. This example shows that the approach of Definition 10, that uses the semantics of the operations, is not well suited to define the consistency criteria. This issue is solved in [16] by the hypothesis that all written values are distinct. Even if this can be achieved by the addition of unique timestamps on the values stored by the memory, this solution is not acceptable because it changes the way the final object can be used. Who could accept to sacrifice the ease of use only for the need of specification?

Another example that cannot be solved by stamping is illustrated by Figure 7. The causal order from Definition 10 only considers write-read dependencies, while concurrent writes are not considered at all. In the example, there is no ambiguity on the writes-into order, as all the values written in each variable are different. The linearizations l_3 and l_4 show that this history is consistent for causal memory. Nevertheless, many people, thinking of causal reception, would reject it for causal consistency because it is impossible that the first process receives the notification message for $\mathrm{w}_x(2)$ before it sends the one for $\mathrm{w}_x(1)$, and the second process receives the notification message for $\mathrm{w}_x(1)$ before it sends the one for $\mathrm{w}_x(2)$: a message cannot be received before it is sent.

Causal Consistency. The example of Figure 7 leads to the consideration that, if an event e is concurrent to an event e' (i.e. incomparable in the causal order), e must precede e' in the linearization of the process that performed e. This is the principle of our definition of causal consistency: the past of an event e in the linearization of the process that performed e is exactly the causal past of e. As the causal order must be respected in all the linearizations, it results that if e' is in this past, e' must precede e in the linearizations of all the processes.



(a) PC but not CC (b) CC but not SC

Figure 8: Histories for an instance of $\mathcal{G}$, with different consistency criteria

Definition 11. A history H is *causally consistent* (CC) for a UQ-ADT $T \in \mathscr{T}$ if there exists a family $(\rightarrow_p)_{p\in\mathscr{P}_H}$ of total orders on E such that:

- Each $\rightarrow_p$ defines a correct linearization for pipelined consistency: $\forall p \in \mathscr{P}_H, \mathrm{lin}\left(H^{\rightarrow_p}_{U_H\cup p}\right) \cap L(T) \neq \emptyset$ and $\mapsto\subset\rightarrow_p$.
- If a process p has already seen an event e when it performs an event e' (according to $\rightarrow_p$), then every process sees those two events in the same order: $\forall p, p' \in \mathscr{P}_H, \forall e \in E, \forall e' \in p, (e \rightarrow_p e') \Rightarrow (e \rightarrow_{p'} e')$.

The relation $\rightarrow = \left(\bigcap_{p\in\mathscr{P}_H} \rightarrow_p\right)$ is a partial order called *causal order*.

The history of Figure 8a is pipelined consistent but not causally consistent. There is only one possible linearization for the first process, $\mathrm{D_v}(1).\mathrm{I_e}(1,2).\mathrm{D_v}(2).\mathrm{Q_v}(2)/\bot.\mathrm{Q_v}(1)/\top$, so $\mathrm{D_v}(1)$ must precede $\mathrm{I_e}(1,2)$ in the causal order, but there is no linearization for the second process in which the two events are in this order.

On the contrary, the history of Figure 8b is causally consistent because $\mathrm{I_v}(1).\mathrm{Q_v}(2)/\bot.\mathrm{I_v}(2)$ and $\mathrm{I_v}(2).\mathrm{Q_v}(1)/\bot.\mathrm{I_v}(1)$ are correct linearizations for both processes, and the causal order is equal to the program order. It is not sequentially consistent because at least one of the query should be aware of the update made by the other process.

PROPOSITION 3. *$PC \leq CC \leq SC$*

PROOF. Let $T \in \mathscr{T}$ and $H \in \mathscr{H}$.

Suppose H is sequentially consistent. There is a linearization $l \in \mathrm{lin}(H) \cap L(T)$. Let $\rightarrow$ be the total order on the events that define this linearization. Let us pose, for all $p \in \mathscr{P}_H$, $\rightarrow_p = \rightarrow$ and l_p be the word obtained by removing the queries that are not in p from l. We have $l_p \in \mathrm{lin}\left(H^{\rightarrow_p}_{U_H\cup p}\right)$ by construction and $l_p \in L(T)$ because $l \in L(T)$ and removing queries does not change the path in the UQ-ADT. Moreover, for all $p, p' \in \mathscr{P}_H$, $\rightarrow_p = \rightarrow_{p'}$, so H is causally consistent.

Suppose H is causally consistent. Let $p \in \mathscr{P}_H$. As $\mapsto\subset\rightarrow_p$, $\mathrm{lin}\left(H^{\rightarrow_p}_{U_H\cup p}\right) \subset \mathrm{lin}\left(H_{U_H\cup p}\right)$, and $\mathrm{lin}\left(H_{U_H\cup p}\right) \cap L(T) \neq \emptyset$. Thus, H is pipelined consistent. $\square$

COROLLARY 1. *Causal consistency is not composable.*

Consistency criteria are used as hypothesis to prove the correctness of the programs that use the objects. The strongest the consistency criterion, the least programs will be impossible to prove while they are actually correct. As an object is normally implemented only once and used many times in a program, it is profitable to prove a strongest property once if it simplifies all the other proofs later. Definition 11 is interesting because it characterizes exactly the histories that can be achieved by causal reception. However, it can be inconvenient to prove that all possible families of linearizations, for all possible processes, does not fit the definition. It is often easier to find data inconsistencies that impose loops in the causal order, like we are used to do with causal memory. Using Proposition 4, it is often possible to show there is no possible causal order for a given history such that all the queries can be explained by the causal past.

PROPOSITION 4. *If a history H is causally consistent for an UQ-ADT T, the causal order $\rightarrow$ on E is such that:*

- *H is pipelined consistent, and the linearizations respect the causal order:*
$$\forall p \in \mathscr{P}_H, \mathrm{lin}\left(H^{\rightarrow}_{U_H\cup p}\right) \cap L(T) \neq \emptyset;$$
- *all the queries can be explained by only considering the updates made in its causal past:* $\forall q \in Q_H, \mathrm{lin}\left(H^{\rightarrow}_{\{u\in U_H: u\rightarrow q\}\cup\{q\}}\right) \cap L(T) \neq \emptyset$.

PROOF. Let $T \in \mathscr{T}$ and $H \in CC(T)$. We consider the family of total orders on E $(\rightarrow_p)_{p\in\mathscr{P}}$ given by Definition 11 such that $\rightarrow = \left(\bigcap_{p\in\mathscr{P}_H} \rightarrow_p\right)$.

- Let $p \in \mathscr{P}_H$. As $\rightarrow\subset\rightarrow_p$, $\mathrm{lin}\left(H^{\rightarrow_p}_{U_H\cup p}\right) \subset \mathrm{lin}\left(H^{\rightarrow}_{U_H\cup p}\right)$ so $\mathrm{lin}\left(H^{\rightarrow}_{U_H\cup p}\right) \cap L(T) \neq \emptyset$.
- Let $p \in \mathscr{P}_H$ and $q \in p$. For all $u \in U_H$, $u \rightarrow_p q$ iff $u \rightarrow q$. Let us consider $l_q \in \mathrm{lin}\left(H^{\rightarrow_p}_{\{u\in U_H: u\rightarrow q\}\cup\{q\}}\right)$ and $l_p \in \mathrm{lin}\left(H^{\rightarrow_p}_{U_H\cup p}\right)$. $l_p \in L(T)$ and l_q can be obtained from l_p by taking a prefix and removing some queries. As $L(T)$ is closed under prefixing and queries do not change the state in T, $l_q \in L(T)$. Moreover, as $\rightarrow\subset\rightarrow_p$, $l_q \in \mathrm{lin}\left(H^{\rightarrow}_{\{u\in U_H: u\rightarrow q\}\cup\{q\}}\right)$. All q is contained in a $p \in \mathscr{P}_H$, so for all $q \in Q_H$, $\mathrm{lin}\left(H^{\rightarrow}_{\{u\in U_H: u\rightarrow q\}\cup\{q\}}\right) \cap L(T) \neq \emptyset$.

$\square$

Applied to memory, Proposition 4 claims that each query $\mathrm{r}_x(v)$, where $v \neq v_0$, contains an update $\mathrm{w}_x(v)$ that is not followed by a $\mathrm{w}_x(v')$ in its causal past. This defines a correct writes-into relation, which proves that causal consistency is stronger than causal memory. However, there is no equivalence between causal memory and the histories defined by Proposition 4 (Figure 6 is a counter-example) and no equivalence between the histories defined by Proposition 4 and causal consistency (Figure 7 is a counter-example).

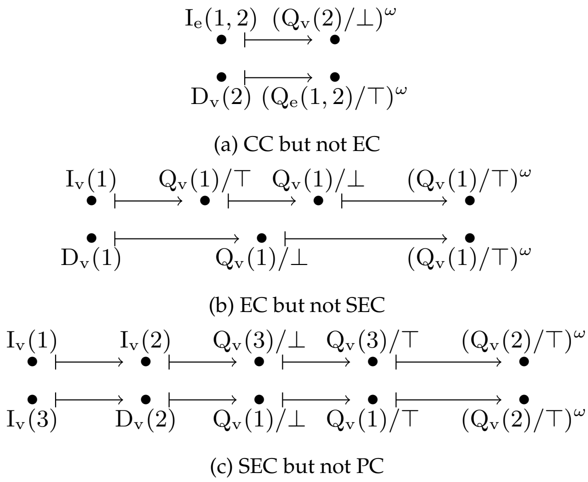


Figure 9: On these histories for an instance of $\mathcal{G}$, An event labelled ω is repeated an infinity of times. We can see that $SEC \neq EC$ and that PC and SC are not comparable with EC and SEC.

3.4 Eventual consistency

Eventual consistency [19] is one of the few consistency criteria that was not originally designed for memory. It requires that, if all the participants stop updating, all the replicas eventually converge to the same state. In other word, H is eventually consistent if it contains an infinite number of updates (i.e. the participants never stop to write) or if there exists a state compatible with all but a finite number of queries (the consistent state).

In our formalism, a consistency criterion must impose to all the events to be labelled by correct operations of the data type. It is not clear in the literature whether it is the case for eventual consistency. A lot of work has been done to fully specify CRDTs [**?**], a kind of objects especially designed to achieve eventual consistency. In [4], it is explicitly mentioned that if an insertion and a deletion of the same element are done concurrently on the set, then any state can be specified as consistent state, including, for example, an error state. Moreover, no assumption is made on the queries made before convergence, so we can imagine that data inconsistencies are acceptable for a short amount of time. For example, a query operation that returns a local copy of the graph could return an edge starting from a vertex that does not exists. These exotic behaviors may cause issues with data integrity, and our definition does not allow them. It is necessary to explicitly modify the sequential specification, for example by adding unreachable states, to take them into account.

Definition 12. A history H is *eventually consistent* (EC) if it contains an infinite number of updates or there exists a state $s \in S$ such that the set $\{q_i/q_o \in Q_H : \varphi(s, q_i) \neq q_o\}$ of queries that cannot be issued while in the state s is finite.

The history of Figure 9a is not eventually consistent since there are only two updates and no valid state can contain the edge $(1, 2)$ but not the vertex 2. The other histories of Figure 9 are eventually consistent since only one query is repeated infinitely often.

Strong eventual consistency. Strong eventual consistency [18] requires that two replicas of the same object converge as soon as they have received the same updates. The problem with that definition is that the notions of replica and message reception are inherent to the implementation, and are hidden to the programmer that uses the object, so they should not be used in a specification. To capture this, a visibility order is introduced to explain the history.

Definition 13. A history H is *strong eventually consistent* (SEC) if there exists an acyclic and reflexive relation $\rightarrow$ (called *visibility* relation) that contains $\mapsto$ and such that:

- Eventual delivery: when an update is viewed by a replica, it is eventually viewed by all replicas, so there can be at most a finite number of operations that do not view it: $\forall u \in U_H, \{e \in E, u \not\rightarrow e\}$ is finite;
- Growth: if an event has been viewed once by a process, it will remain visible forever: $\forall e, e', e'' \in E, (e \rightarrow e' \wedge e' \mapsto e'') \Rightarrow (e \rightarrow e'')$;
- Strong convergence: if two query operations view the same past of updates V, they are issued in the same state s: $\forall V \subset U_H, \exists s \in S, \forall q_i/q_o \in Q_H$, $\{u \in U_H : u \rightarrow q_i/q_o\} = V \Rightarrow \varphi(s, q_i) = q_o$.

Strong eventual consistency is strictly stronger than eventual consistency. Figure 9b shows a history that is strong eventually consistent but not eventually consistent. The update $I_e(1)$ must be visible by all the queries of the first process (by reflexivity and growth), so there are only two possible sets of visible updates ($\{I_e(1)\}$ and $\{I_e(1), D_e(1)\}$) for these events. By the growth property, the query event $Q_e(1)/\bot$ must have the same view as the previous event or the following event, which is impossible.

Eventual consistency and strong eventual consistency are not comparable with pipelined consistency and causal consistency. The history of Figure 9a is causally consistent but not eventually consistent. Conversely, the history of Figure 9c is strong eventually consistent but not pipelined consistent. To build the linearization of the first process, it is necessary to insert the $I_v(3)$ between the two $Q_v(3)$, but it is impossible to insert the $D_v(2)$ before any $Q_v(2)$. If these queries returned $\bot$, the history would be causally consistent but no more eventually consistent.

Eventual consistency can hardly be used directly to program reliable applications because it gives too few guaranties on the operations made before convergence. It can be used, however, for applications in which the object is controlled by humans. For example, it makes sense for a collaborative text editor like Logoot [20] to ensure that all the collaborators will eventually see the same document.

Another application is to strengthen other consistency criteria. For example, pipelined consistency and causal consistency do not ensure eventual consistency, since different processes can see the same updates in a different order. We can define causal convergence as $CC + EC$. In [14], it is proved that a variation of this criterion is the strongest that can be implemented for memory in asynchronous message-passing systems with omission-failures and unreliable networks.

$$\begin{tikzcd}
 & LC & & \underline{SC^\star} & (RT) \arrow[dr] & & \\
(\underline{C_\bot}) \arrow[r] \arrow[ur] \arrow[dr] & \underline{EC} \arrow[r] & \underline{SEC} \arrow[ur] \arrow[dr] & & SC \arrow[r] & SC + RT \arrow[r] & (\underline{C_\top}) \\
 & PC \arrow[r, Rightarrow] & CC \arrow[r, Rightarrow] & CC + EC \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

Figure 10: A summary of the criteria exposed in this article. The underlined criteria are composable. An arrow between C_1 and C_2 express the fact that $C_1 \leq C_2$. This arrow is thick if no composable consistency criteria exists along it

4. CONCLUSION

Sharing objects is essential to abstract communication complexity in large scale distributed systems. A lot of work has been done until now to specify many kinds of shared memory, but very few for other data types. In this paper, we propose a framework to easily specify shared objects. This framework is based on a clear separation between sequential specifications and consistency criteria. The interest of this approach is that sequential specifications are easy to understand as they are already widely used in sequential object-oriented programming.

This paper also extends the definition of many consistency criteria to all update-query abstract data types, the data types for which all the operations are whether updates or queries. Figure 10 sums up all the consistency criteria evoked above. Some, like $C_\bot$, $C_\top$ and RT only have a theoretical interest, but others have real practical applications.

Programming in a modular way is very important for reliability because it helps focussing on simpler pieces of codes. Composability is required to put the pieces together in the final program. However, this property seems very difficult to achieve, in particular for the strongest criteria. In this paper, we have shown that there exists no composable consistency criterion between pipelined consistency and sequential consistency. The example of linearizability shows that it might be possible to add constraints on the composition to make another consistency criterion composable. We could imagine an algorithm to compose C-consistent objects with respect to a criterion C' such that $C^2 + C' = C + C'$. Such a C' always exists, as $C^2 + C = C + C$ but a weaker criterion could lead to more efficient implementations. We leave all this as future work.

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