



# Analysis of a fractal boundary: the graph of the Knopp function

Mourad Ben Slimane, Clothilde Melot

## ► To cite this version:

Mourad Ben Slimane, Clothilde Melot. Analysis of a fractal boundary: the graph of the Knopp function. *Abstract and Applied Analysis*, 2015, <http://www.hindawi.com/journals/aaa/2015/587347/.hal-01037859>

**HAL Id: hal-01037859**

**<https://hal.science/hal-01037859>**

Submitted on 23 Jul 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Analysis of a fractal boundary: the graph of the Knopp function

Mourad Ben Slimane

Department of Mathematics, College of Science, King Saud University,  
P. O. Box 2455, Riyadh 11451, Saudi Arabia,  
Email: mbenslimane@ksu.edu.sa

Clothilde Mélot

Aix-Marseille Univ, CNRS, LATP, UMR 6632,  
F-13453 Marseille, France,  
Email: melot@cmi.univ-mrs.fr

## Abstract

A usual classification tool to study a fractal interface is the computation of its fractal dimension. But a recent method developed by Y. Heurteaux and S. Jaffard proposes to compute either weak and strong accessibility exponents or local  $L^p$  regularity exponents (the so-called  $p$ -exponent). These exponents describe locally the behavior of the interface. We apply this method to the graph of the Knopp function which is defined for  $x \in [0, 1]$  as  $F(x) = \sum_{j=0}^{\infty} 2^{-\alpha j} \phi(2^j x)$  where  $0 < \alpha < 1$  and  $\phi(x) = \text{dist}(x, \mathbb{Z})$ . The Knopp function itself has everywhere the same  $p$ -exponent  $\alpha$ . Nevertheless, using the characterization of the maxima and minima done by B. Dubuc and S. Dubuc, we will compute the  $p$ -exponent of the characteristic function of domain under the graph of  $F$  at each point  $(x, F(x))$  and show that  $p$ -exponents, weak and strong accessibility exponents change from point to point. Furthermore we will derive a characterization of the local extrema of the function according to the values of these exponents.

---

2010 *Mathematics Subject Classification*: Primary 26A16, 26A30. Secondary: 26A27, 28A80.

*Key words and phrases*: Fractal interface, Knopp function, Hölder and  $L^p$  regularities, Weak and strong accessibility exponents, Dyadic expansion, Extrema.

# 1 Introduction

At the beginning of the century several examples of non differentiable functions were studied, such as the Weierstrass function or the example we will focus on in the following, i.e the Takagi-Knopp or so called Knopp function (see [A] and references therein for a review). The issue was the study of the regularity.

Indeed in 1918, Knopp [K] introduced a new family of non differentiable functions defined on the interval  $[0, 1]$ . Going beyond the construction of Weierstrass of a continuous non differentiable function, his goal was to build examples of continuous functions for which one sided limits of the difference quotient at all points don't exist. He considered the function  $F_{a,b}$  given by the series, for  $x \in [0, 1]$

$$(1.1) \quad F_{a,b}(x) = \sum_{j=0}^{\infty} a^j \phi(b^j x)$$

where  $\phi(x) = \text{dist}(x, \mathbb{Z})$ ,  $0 < a < 1$ ,  $b$  is an integer such that  $ab > 4$ .

For  $b = 2$  and  $a = 2^{-\alpha}$ , this function can be seen as a series expanded in the Faber-Schauder basis  $\Lambda_{j,k} : x \mapsto 2^{\frac{j}{2}} \Lambda(2^j x - k)$ ,  $j \in \mathbb{N}$ ,  $k = 0, \dots, 2^j - 1$ , where  $\Lambda$  is the Schauder function defined by  $\Lambda(x) = \inf(x, 1 - x)$  if  $x \in [0, 1]$  and 0 elsewhere. In fact

$$(1.2) \quad F_{2^{-\alpha}, 2}(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} 2^{-\alpha j} \Lambda(2^j x - k) .$$

*We will write  $F$  for  $F_{2^{-\alpha}, 2}$  in the following.*

Thus, for example, using the characterization of Lipschitz spaces with the help of coefficients in the Schauder-basis [C1], one gets immediately the fact that  $F$  belongs to  $C^\alpha([0, 1])$ .

A further step to study the regularity of this function can be to follow the ideas developed in multifractal analysis. The goal in multifractal analysis is to study the sets of points where the function has a given pointwise regularity, and doing so checking if the regularity changes from point to point and quantify these changes. Recall the definition of Hölder pointwise regularity and local  $L^p$  regularity.

**Definition 1.1.** Let  $x_0 \in \mathbf{R}^d$  and  $\alpha \geq 0$ . A locally bounded function  $f : \mathbf{R}^d \rightarrow \mathbf{R}$  belongs to  $C^\alpha(x_0)$  if there exists  $C > 0$  and a polynomial  $P = P_{x_0}$  with  $\deg(P) \leq [\alpha]$ , such that on a neighborhood of  $x_0$ ,

$$(1.3) \quad |f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha.$$

The pointwise Hölder exponent of  $f$  at  $x_0$  is  $h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$ .

**Definition 1.2.** [CZ] Let  $x_0 \in \mathbf{R}^d$ . Let  $p \in [1, \infty]$  and  $u$  such that  $u \geq -\frac{d}{p}$ . Let  $f$  be a function in  $L^p_{loc}$ . The function  $f$  belongs to  $T^p_u(x_0)$  if there exists  $R > 0$ , a polynomial  $P$  with  $\deg(P) \leq u$ , and  $C > 0$  such that

$$(1.4) \quad \forall \rho \leq R : \left( \frac{1}{\rho^d} \int_{|x-x_0| \leq \rho} |f(x) - P(x)|^p dx \right)^{\frac{1}{p}} \leq C\rho^u.$$

The  $p$ -exponent of  $f$  at  $x_0$  is  $u^p_f(x_0) = \sup\{u : f \in T^p_u(x_0)\}$ .

Then again with the help of the Faber-Schauder basis one can prove that for all  $x_0 \in [0, 1]$ ,  $F$  is in  $C^\alpha(x_0)$  ( details for this technique can be found in [JMa]). It is then easy to check that actually  $u^p_f(x_0) = h_f(x_0) = \alpha$  at all  $x_0 \in [0, 1]$ . Thus from the point of view of various notions of regularity, even if it is not differentiable, the function  $F$  is rather ‘regular’ since one can compute at each point  $x_0$  the same regularity exponent. This remark was actually the starting point of this work.

Indeed obviously the graph of the function has a very irregular behavior, and it has also some selfsimilarity properties. What can we say on the domain  $\Omega = \{X = (x, y) : y \leq F(x)\}$  under the graph of  $F$  ?

*Denote in the following by  $\mathbb{I}_\Omega$  the characteristic function of  $\Omega$ , which takes the value 1 on  $\Omega$  and 0 outside  $\Omega$ .*

A first reflex is to compute fractal dimensions of the boundary  $\partial\Omega$ . The box dimension of the graph can be derived by standard methods (see Tri-cot [T1]) and is exactly  $\dim_B(\partial\Omega) = 2 - \alpha$ . Let us mention that Ciesielski [C2, C3] proved results of this type for Schauder and Haar bases expansions in the case of more general families of functions. Jaffard [J], Kamont and Wolnik [KW] obtained then general formulas that allow to derive the box dimensions of the graphs of arbitrary functions from their wavelet expansions.

For what concerns the Hausdorff dimension of the graph of  $F$ , as far as we know, the question is not solved yet in its all generality. It was proved

by Ledrappier [L] in 1992 to be  $2 - \alpha$  in the special case where  $a = 2^{\alpha-1}$  is an Erdős number. By the results of Solomyak [S] on Erdős numbers this amounts to have the computation for almost every  $\alpha$  in  $[0,1]$ .

Beside the computation of the box and Hausdorff dimension, which provide global quantities to describe the graph of the function, several methods were recently developed to classify fractal boundaries with the help of pointwise exponents. The idea was to be able to give a finer description of the geometry of the boundary, since the pointwise behavior was studied. In [JMe], Jaffard and Mélot focused on the computation of the dimension of the set of points where  $\mathbb{I}_\Omega$  has a given  $p$ -exponent in the sense of Definition 1.2. In [JH], Jaffard and Heurteaux studied pointwise exponents more related to the geometry. These are the exponents we are actually interested in.

Indeed denote by *meas* the Lebesgue measure in  $\mathbf{R}^d$  and  $B(X, r)$  the  $d$  dimensional open ball of center  $X$  and radius  $r > 0$ . Jaffard and Heurteaux [JH] gave the following definitions.

**Definition 1.3.** Let  $\Omega$  be a domain of  $\mathbf{R}^d$  and let  $X_0 \in \partial\Omega$ . The point  $X_0$  is weak  $\alpha$ -accessible in  $\Omega$  if there exists  $C > 0$  and  $r_0 > 0$  such that

$$(1.5) \quad \forall r \leq r_0 \quad \text{meas}(\Omega \cap B(X_0, r)) \leq Cr^{\alpha+d}.$$

The supremum of all the values of  $\alpha$  such that (1.5) holds is called the weak accessibility exponent in  $\Omega$  at  $X_0$ . We denote it by  $E_\Omega^w(X_0)$ .

**Example:** Let  $0 < \beta < 1$  and  $\Omega = \{X = (x, y) \in \mathbf{R}^2 : |y| \leq |x|^\beta\}$ . Denote  $\Omega^c$  the complement of  $\Omega$ . Then one can easily check that at each point  $X_1 \neq (0, 0)$  of the boundary  $\partial\Omega$  we have  $E_\Omega^w(X_1) = 0 = E_{\Omega^c}^w(X_1)$  and at  $X_0 = (0, 0)$  we have  $E_{\Omega^c}^w(X_0) = \frac{1}{\beta} - 1$  and  $E_\Omega^w(X_0) = 0$ .

**Definition 1.4.** Let  $\Omega$  be a domain of  $\mathbf{R}^d$  and let  $X_0 \in \partial\Omega$ . The point  $X_0$  is strong  $\alpha$ -accessible in  $\Omega$  if there exists  $C > 0$  and  $r_0 > 0$  such that

$$(1.6) \quad \forall r \leq r_0 \quad \text{meas}(\Omega \cap B(X_0, r)) \geq Cr^{\alpha+d}.$$

The infimum of all the values of  $\alpha$  such that (1.6) holds is called the strong accessibility exponent in  $\Omega$  at  $X_0$ . We denote it by  $E_\Omega^s(X_0)$ .

The following proposition is given in [JH].

**Proposition 1.5.** *Let  $X_0 \in \partial\Omega$ . Then*

$$(1.7) \quad \begin{aligned} d + E_\Omega^w(X_0) &= \liminf_{r \rightarrow 0} \frac{\log(\text{meas}(\Omega \cap B(X_0, r)))}{\log r}, \\ d + E_\Omega^s(X_0) &= \limsup_{r \rightarrow 0} \frac{\log(\text{meas}(\Omega \cap B(X_0, r)))}{\log r}. \end{aligned}$$

Obviously  $E_\Omega^s(X_0) \geq E_\Omega^w(X_0)$ . We will see that thanks to our result one can prove that these two exponents can be different.

C. Tricot [T2] proved that these exponents are related to local dimension computation. Let us mention, without entering too much the details, the relationship of this work [T2] with these exponents. Indeed the author focus on the formula

$$(1.8) \quad H^\phi(E) = \liminf_{\varepsilon \rightarrow 0} \left\{ \sum_{i \geq 0} \phi(E_i) : E \subset \bigcup_{i \geq 0} E_i, \text{diam}(E_i) \leq \varepsilon \right\}$$

with  $\phi : \mathbb{B}_E \rightarrow (0, \infty)$  some "set function" and  $\mathbb{B}_E$  the set of closed balls centered on  $E$ .

Given an open set  $V$  such that  $E \subset \partial V$  the special choice of

$$\phi_\alpha(B) = \frac{\text{Vol}(B \cap V)}{\text{Vol}(B)} \text{diam}(B)^\alpha$$

lead to definitions of Hausdorff, exterior and interior dimensions, Packing, exterior and interior dimensions.

The following characterization, written for the setting we are interested in, holds

**Theorem 1.6.** *[T2] Let  $\Omega$  be a bounded open set in  $\mathbf{R}^d$  with boundary  $\partial\Omega$  such that  $\text{meas}(\partial\Omega) = 0$ . Let  $X_0 \in \Omega$ . Let  $r > 0$  and*

$$\alpha(B(X_0, r)) = d - \frac{\log(\text{meas}(\Omega \cap B(X_0, r)))}{\log(r)}.$$

*Then*

$$\liminf_{r \rightarrow 0} \alpha(B(X_0, r)) = \dim_{int}(\{X_0\}) \quad \text{and} \quad \limsup_{r \rightarrow 0} \alpha(B(X_0, r)) = \text{Dim}_{int}(\{X_0\})$$

*with  $\dim_{int}$  the Hausdorff interior dimension and  $\text{Dim}_{int}$  the Packing Hausdorff dimension.*

We clearly have  $\dim_{int}(\{X_0\}) = -E_{\Omega}^w(X_0)$  and  $\dim_{int}(\{X_0\}) = -E_{\Omega}^s(X_0)$ . Let us stress that in the setting of Tricot  $\dim_{ext}(\{X_0\}) = -E_{\Omega^c}^w(X_0)$  and  $\dim_{ext}(\{X_0\}) = -E_{\Omega^c}^s(X_0)$  with  $\Omega^c$  the complementary of  $\Omega$  in  $\mathbf{R}^d$ . We rather refer to [T2] for more details on local dimensions in their all generality.

We will compute these quantities at the points of the boundary  $\partial\Omega$  of  $\Omega = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq F(x)\}$ , where  $F$  is the function defined by (1.2). For that we will use the characterization of the maxima and minima done in [DD]. This will yield the  $p$ -exponent at each point of  $\mathbb{I}_{\Omega}$ . We will actually derive the fact that the set of local extrema of the function is fully characterized by the set of points where this  $p$ -exponent has a given value.

We will also prove that the weak and strong accessibility exponents in  $\Omega$  and  $\Omega^c$  change from point to point on the graph  $\partial\Omega$  of  $F$ . They also help to provide exact characterization of the sets of local maxima and local minima. Finally we will prove that there is a set of non trivial Hausdorff dimension such that the strong accessibility exponents in  $\Omega$  and  $\Omega^c$  are the same and the weak and strong accessibility exponents different.

Let us emphasize that this is to our knowledge the first time that the computation of these exponents was done in a nearly exhaustive study on a given example. The characterization we get for the set of extremas raise several questions: is it a general property ? Do other functions share it ? Could it lead to a finer classification of functions in Hölder classes ? We would like to adress them in future works.

Let us come back now to our work. The outline of the paper is the following. In Section 2 we set our main result. In Section 3 some notations, preliminary remarks and technical lemmas, help us to prepare Section 4 where are the main proofs.

## 2 Main results

### 2.1 Statement of our main result

Our goal is to prove the following Theorem.

**Main Theorem 2.1.** *Let  $a = 2^{-\alpha}$  with  $0 < \alpha < 1$  and  $b = 2$ . Let  $F$  be the function defined by (1.2).*

*Let  $\Omega = \{X_0 = (x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq F(x)\}$  and let  $f = \mathbb{I}_\Omega$ .*

*Then at each point  $X_0$  of  $\partial\Omega$ , the graph of  $F$ , we have*

1.  $u_f^p(X_0) = \frac{1}{p} \left( \frac{1}{\alpha} - 1 \right)$  *if and only if  $F(X_0)$  is a local extremum of  $F$ . Furthermore*

(a)  $E_\Omega^w(X_0) = E_\Omega^s(X_0) = \frac{1}{\alpha} - 1$  *if and only if  $F(X_0)$  is a local maximum of  $F$ . And in this case  $E_{\Omega^c}^w(X_0) = E_{\Omega^c}^s(X_0) = 0$ .*

(b)  $E_{\Omega^c}^w(X_0) = E_{\Omega^c}^s(X_0) = \frac{1}{\alpha} - 1$  *if and only if  $F(X_0)$  is a local minimum of  $F$ . And in this case  $E_\Omega^w(X_0) = E_\Omega^s(X_0) = 0$ .*

2. *In the other cases where  $F(X_0)$  is not a local extremum of  $F$ , we have  $E_\Omega^w(X_0) = E_{\Omega^c}^w(X_0) = 0$ .*

3. *Furthermore one can find a subset  $D_\alpha \subset \partial\Omega$  such that for each  $X_0 \in D_\alpha$   $E_\Omega^s(X_0) = E_{\Omega^c}^s(X_0) = \frac{1}{\alpha} - 1$  and  $E_\Omega^w(X_0) = E_{\Omega^c}^w(X_0) = 0$ . The orthogonal projection of  $D_\alpha$  on  $[0, 1]$  has the Hausdorff dimension  $\alpha$ .*

### 3 Useful notations and results

#### 3.1 Lemmas for practical computation of the exponents

From the computation of the weak accessibility exponent in  $\Omega$  and  $\Omega^c$  it is easy to derive the  $p$ -exponent. In [JMe], Jaffard and Mélot proved that  $\mathbb{I}_\Omega \in T_{\alpha/p}^p(X_0)$  if and only if either  $X_0$  is weak  $\alpha$ -accessible in  $\Omega$  or  $X_0$  is weak  $\alpha$ -accessible in  $\Omega^c$ . As a consequence we have

$$(3.1) \quad p \ u_f^p \mathbb{I}_\Omega(X_0) = \max(E_\Omega^w(X_0), E_{\Omega^c}^w(X_0)) .$$

We will also need the following lemma.

**Lemma 3.1.** *Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be in  $C^\alpha(x_0)$  with  $0 < \alpha < 1$  and  $\Omega$  be the domain below (resp. above) the graph of  $f$ . Consider  $X_0 = (x_0, f(x_0))$ . Then  $X_0$  is strong  $\frac{1}{\alpha} - 1$  accessible in both  $\Omega$  and  $\Omega^c$ .*

**Proof.** Suppose that  $\Omega$  is the domain below the graph of  $f$ . Without any loss of generality, we can assume that  $X_0 = (0, 0)$ . Let  $r > 0$ . Since  $f$

is in  $C^\alpha(0)$  and  $0 < \alpha < 1$  then there exists a constant  $C \geq 0$  such that in neighborhood of 0

$$(3.2) \quad |f(x)| \leq C|x|^\alpha .$$

Thus

$$(3.3) \quad -C|x|^\alpha \leq f(x) \leq C|x|^\alpha .$$

Obviously  $meas(\Omega^c \cap B(X_0, r))$  (resp.  $meas(\Omega \cap B(X_0, r))$ ) is greater than the area  $\mathcal{A} = C' \int_0^r y^{1/\alpha} dy = C'' r^{1+1/\alpha}$  above (resp. below) the graph of  $x \mapsto C|x|^\alpha$  and below (resp. above) the square of side  $r$  and center  $X_0$ .

The same results hold if  $\Omega$  is the domain above the graph of  $f$  (we have just to replace  $\Omega$  by  $\Omega^c$ ).

One of our goals for the points which are not extrema of  $F$  will be to find sequences of local maxima or minima such that the following key-lemma proved in [H] holds.

**Lemma 3.2.** *Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be in  $C^\alpha(\mathbf{R})$  and  $\Omega$  be the domain below the graph of  $f$ . Consider  $X_0 = (x_0, f(x_0))$ . Suppose that there exist  $c_\alpha > 0$ , a sequence  $r_n$  of positive numbers, such that  $r_n \rightarrow 0$  as  $n \rightarrow +\infty$ ,  $x_n \in ]x_0 - r_n, x_0 + r_n[$ , and  $n_0 \in \mathbf{N}$ , such that*

$$(3.4) \quad \forall n \geq n_0 \quad f(x_n) = f(x_0) - c_\alpha r_n^\alpha .$$

Then  $E_{\Omega^c}^w(X_0) = 0$ .

**Proof.** We can suppose that  $x_n \in ]x_0 - r_n, x_0[$  (the case  $x_n \in ]x_0, x_0 + r_n[$  is similar). Then by the mean value theorem we can find  $b_n \in ]x_n, x_0[$  such that  $f(b_n) = f(x_0) - r_n$ . Let  $b'_n = \inf\{b_n \in ]x_n, x_0[ ; f(b_n) = f(x_0) - r_n\}$ . Since  $f$  is continuous we get  $f(b'_n) = f(x_0) - r_n$  and  $b'_n \in ]x_n, x_0[$ . It follows from the definition of  $b'_n$  and the mean value theorem that

$$\forall t \in ]x_n, b'_n[ \quad f(t) < f(x_0) - r_n .$$

Thus

$$]x_n, b'_n[ \times [f(x_0) - r_n, f(x_0) + r_n] \subset B(X_0, r_n) \cap \Omega^c .$$

Therefore

$$(3.5) \quad meas(B(X_0, r_n) \cap \Omega^c) \geq 2|b'_n - x_n|r_n .$$

Since

$$\begin{aligned} f(x_n) - f(b'_n) &= f(x_0) - c_\alpha r_n^\alpha - (f(x_0) - r_n) \\ &= r_n - c_\alpha r_n^\alpha \\ &\simeq -r_n^\alpha, \end{aligned}$$

in the sense that there exists a constant  $C \geq 1$  such that for every  $n$  we have  $\frac{1}{C}r_n^\alpha \leq c_\alpha r_n^\alpha - r_n \leq Cr_n^\alpha$ .

Since  $f$  belongs to  $C^\alpha(\mathbf{R})$  we get

$$(3.6) \quad C|b'_n - x_n|^\alpha \geq |f(b'_n) - f(x_n)| \simeq r_n^\alpha.$$

Thus

$$(3.7) \quad |b'_n - x_n| \geq C' r_n.$$

Following (3.5) and (3.7) we get

$$(3.8) \quad \text{meas}(B(X_0, r_n) \cap \Omega^c) \geq Cr_n^2.$$

Since  $\text{meas}(B(X_0, r) \cap \Omega^c) \leq \text{meas}(B(X_0, r)) \leq r^2$  for all  $r \geq 0$  we get

$$\lim_{n \rightarrow +\infty} \frac{\log(\text{meas}(B(X_0, r_n) \cap \Omega^c))}{\log(r_n)} = 2.$$

Thus thanks to Proposition 1.5 we have  $E_{\Omega^c}^w(X_0) \leq 0$  which yields  $E_{\Omega^c}^w(X_0) = 0$ .

By replacing  $f$  by  $-f$  we also have the following result.

**Lemma 3.3.** *Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be in  $C^\alpha(\mathbf{R})$  and  $\Omega$  be the domain below the graph of  $f$ . Consider  $X_0 = (x_0, f(x_0))$ . Suppose that there exist  $c_\alpha > 0$ ,  $r_n \rightarrow 0$  as  $n \rightarrow +\infty$ ,  $x_n \in ]x_0 - r_n, x_0 + r_n[$ , and  $n_0 \in \mathbf{N}$ , such that*

$$(3.9) \quad \forall n \geq n_0 \quad f(x_n) = f(x_0) + c_\alpha r_n^\alpha.$$

*Then  $E_\Omega^w(X_0) = 0$ .*

### 3.2 Dyadic expansions and approximation by dyadics

We give some properties of the approximation of a point by the dyadics. Such properties will be used later.

Let  $x \in [0, 1]$ . Set  $i_1(x), \dots, i_j(x), \dots$  the binary digits of  $x$ , i.e.

$$(3.10) \quad x = \sum_{l=1}^{\infty} \frac{i_l(x)}{2^l}$$

- Note that dyadic points, i.e points  $x = 2^{-N}K$  with  $K \in 2\mathbb{N} + 1$  are characterized by the fact that one can find  $N > 0$  such that  $i_N(x) = 1$  and  $i_n(x) = 0$  for  $n > N$ , or equivalently  $i_N(x) = 0$  and  $i_n(x) = 1$  for  $n > N$ .

Furthermore for  $n > N$  the number  $x - 2^{-n}$  is dyadic. Since  $2^{-n} = \sum_{j=n+1}^{\infty} 2^{-j}$ , then  $x - 2^{-n} = (\sum_{j=1}^{N-1} i_j 2^{-j}) + 2^{-(N+1)} + 2^{-(N+2)} + \dots + 2^{-n}$ .

On the other hand  $x + 2^{-n}$  has the simple expansion  $\sum_{j=1}^N \frac{i_j(x)}{2^j} + \frac{1}{2^n}$ .

We will denote  $\mathcal{D}$  the set of all dyadic points in  $[0, 1]$ .

- Let us come back to the general case with  $x$  any point in  $[0, 1]$ .

For each  $j \in \mathbb{N}$ , define  $K_j(= K_j(x))$  by

$$(3.11) \quad |K_j 2^{-j} - x| = \inf_{k \in \mathbb{N}} |k 2^{-j} - x| .$$

Set

$$r_j(x) = \frac{\log |K_j 2^{-j} - x|}{\log 2^{-j}} .$$

Define the rate of approximation of  $x$  by dyadics as

$$r(x) = \limsup_{j \rightarrow \infty} r_j(x) .$$

Since  $|K_j 2^{-j} - x| \leq 2^{-j}$ , then for every  $x$ , we have  $r(x) \geq 1$ . If  $x$  is dyadic then  $r(x) = \infty$  (by taking the convention  $\log 0 = -\infty$ ). If  $x$  is normal (i.e. the frequency of ones (or zeros) in the binary expansion of  $x$  is equal to  $1/2$ ) then  $r(x) = 1$ .

- If  $r(x) > 1$ , following the definition of  $r(x)$ , then for any  $\delta > 0$  such that  $r(x) - \delta > 1$  one can find a subsequence  $J_n \rightarrow +\infty$  for  $n \rightarrow +\infty$  such that

$$(3.12) \quad r_{J_n(x)} \leq 2^{-J_n(r(x)-\delta)} .$$

Let  $J'_n = [J_n(r(x) - \delta)]$ . We have then

$$(3.13) \quad K_{J_n} 2^{-J_n} - 2^{-J'_n} \leq x \leq K_{J_n} 2^{-J_n} + 2^{-J'_n}$$

Thus, either  $x$  belongs to the dyadic interval  $[K_{J_n} 2^{-J_n} - 2^{-J'_n}, K_{J_n} 2^{-J_n}]$ , and in this case it satisfies  $i_{J_n+1}(x) = \dots = i_{J'_n-1}(x) = 1$ , or it belongs to the other interval  $[K_{J_n} 2^{-J_n}, K_{J_n} 2^{-J_n} + 2^{-J'_n}]$  and in this case it satisfies  $i_{J_n+1}(x) = \dots = i_{J'_n-1}(x) = 0$ .

In both cases let us notice that the binary expansion of  $x$  contains chains of 0 or 1 whose length  $J'_n - J_n \sim J'_n$  increases when  $n \rightarrow +\infty$ .

### 3.3 Approximation by sequences of maxima of $F$

We will see in the following that points in  $[0, 1]$  of the set

$$(3.14) \quad \mathcal{S} = \left\{ k \in \mathbb{N}, N_0 \in \mathbb{N}, \frac{k}{2^{N_0}} + \frac{1}{3} \frac{1}{2^{N_0}}, \frac{k}{2^{N_0}} + \frac{2}{3} \frac{1}{2^{N_0}} \right\}$$

will play a big role in this work, since they actually are the locations of the local maxima of the function  $F$  (see below). Remark that they are characterized by the fact that for each  $x \in \mathcal{S}$ , one can find  $j_0 \in \mathbb{N}$  such that for  $j \geq j_0$  we have  $i_j(x) + i_{j+1}(x) = 1$ .

As in the case of dyadic approximation we can define a rate of approximation by this kind of points.

Indeed let for  $x \in [0, 1]$

$$(3.15) \quad |m_j - x| = \inf_{k \in \mathbb{N}} \left\{ \left| \frac{k}{2^j} + \frac{1}{3} \frac{1}{2^j} - x \right|, \left| \frac{k}{2^j} + \frac{2}{3} \frac{1}{2^j} - x \right| \right\}.$$

Define

$$s_j(x) = \frac{\log(|m_j - x|)}{\log 2^{-j}}.$$

Then the rate of approximation of  $x$  by elements of  $\mathcal{S}$  is given by

$$s(x) = \limsup_{j \rightarrow \infty} s_j(x).$$

Since  $|m_j - x| \leq 2^{-j}$ , then for every  $x$ , we have  $s(x) \geq 1$ .

In the case of dyadic numbers, we have  $s(x) = 1$ . But remark that in other non trivial cases there is no obvious relationship between  $s(x)$  and  $r(x)$ . Indeed one can check on the following examples that  $s$  and  $r$  can take independently any value.

- Let  $x = \sum_{j=1}^{+\infty} \frac{i_j(x)}{2^j}$  with  $i_{3k+1}(x) = 0 = i_{3k+2}(x)$  and  $i_{3k+3}(x) = 1$  for all  $k \in \mathbb{N}$ . Then we have  $r(x) = s(x) = 1$ .
- Let  $u > 1$ . Then  $x = \sum_{n=0}^{+\infty} 2^{-[u^n]}$  with  $[u^n]$  the integer part of  $u^n$ . We have  $r(x) = u$  whereas  $s(x) = 1$ .
- Let  $u > 1$ . Then  $x = \sum_{n=1}^{+\infty} 2^{-2n} - \sum_{n=0}^{+\infty} 2^{-2[u^n]}$ . We have  $s(x) = u$  whereas  $r(x) = 1$ .
- Let  $u > 1$  and  $s > 1$ . Let  $x = \sum_{n=1}^{+\infty} 2^{-2[s^{n-1}u^n]} + \sum_{k=[s^{n-1}u^n]+1}^{[s^n u^{n+1}]-1} 2^{-2k}$ . Then  $r(x) = u$  and  $s(x) = s$ .

### 3.4 The shift operator

Since  $0 < \alpha < 1$  it is easy to check that we obtain from (??) with  $a = 2^{-\alpha}$  and  $b = 2$

(3.16)

$$F(x) = \sum_{j=0}^{\infty} \sum_{i=(i_1, \dots, i_j) \in \{0,1\}^j} 2^{-\alpha j} \Lambda(2^j x - 2^{j-1} i_1 - \dots - 2i_{j-1} - i_j).$$

The term of (3.16) corresponding to  $j = 0$  is  $\Lambda(x)$ . But, the function  $\Lambda$  is supported in  $[0, 1]$ , therefore  $F$  vanishes outside  $[0, 1]$  and for  $x \in [0, 1]$

$$(3.17) \quad F(x) = \sum_{j=0}^{\infty} 2^{-\alpha j} \Lambda(2^j x - 2^{j-1} i_1(x) - \dots - 2i_{j-1}(x) - i_j(x)).$$

For dyadic rationals  $x$ ,  $x = 2^{-N}K$  with  $K \in 2\mathbb{N} + 1$ , as we already said it, there exist two binary expansions, one such that  $i_N(x) = 1$  and  $i_n(x) = 0$  for  $n > N$ , and another one such that  $i_N(x) = 0$  and  $i_n(x) = 1$  for  $n > N$ . The two right-hand sides of (3.17) corresponding to the two choices of  $i(x)$  give identical results.

Denote by  $\tau$  the shift operator

$$\tau x = \sum_{l=2}^{\infty} i_l(x) 2^{-l+1} = \sum_{l=1}^{\infty} i_{l+1}(x) 2^{-l}.$$

Observe that

$$\tau x = \begin{cases} 2x & \text{if } x \in [0, 1/2[ \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

Hence

$$(3.18) \quad F(x) = \sum_{j=0}^{\infty} 2^{-\alpha j} \Lambda(\tau^j x)$$

and

$$\tau^j x = \sum_{l=1}^{\infty} i_{l+j}(x) 2^{-l}.$$

Our selfsimilar function is of the form  $F(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} C_{j,k} \Lambda(2^j x - k)$

with

$$\begin{aligned} C_{j,k} &= 2^{-\alpha j} \text{ if } j \neq 0, k \neq 0 \\ C_{0,0} &= 1 \end{aligned}$$

For  $n \geq 1$ , denote

$$(3.19) \quad F_n(x) = \sum_{j=0}^n 2^{-\alpha j} \Lambda(\tau^j x).$$

Remark that  $F_n$  is affine on intervals of type  $I_{n+1} = [\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}]$ . Remark also that if  $t \in [0, 1]$ , then  $\Lambda(t) = (-1)^{i_1(t)} t + i_1(t)$ . So, if  $t' \in [0, 1]$  and  $i_1(t) = i_1(t')$ , then  $\Lambda(t) - \Lambda(t') = (-1)^{i_1(t)} (t - t')$ .

It follows that if  $\frac{k}{2^{n+1}} = \sum_{j=1}^{n+1} \frac{i_j}{2^j}$  then the slope of  $F_n$  at any point  $x$  of the interval  $] \frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}[$  is exactly

$$(3.20) \quad C_n = C_n(x) = \sum_{j=0}^n (-1)^{i_{j+1}(x)} 2^{(1-\alpha)j} = \sum_{j=0}^n (-1)^{i_{j+1}} 2^{(1-\alpha)j}.$$

### 3.5 Extrema of $F$

We will need the following characterization of the extremas of  $F$  proved in [DE] and [DD]. Let us start with the local and global minima.

**Proposition 3.4.** *Let  $0 < \alpha < 1$  and  $F$  the function defined by (3.18), then*

- *0 and 1 are the abscissas of the global minima of  $F$ .*
- *The dyadic points are the abscissas of the minima of  $F$  and furthermore*

$$(3.21) \quad \min_{x \in I_N} F(x) = \min \left[ F_{N-1} \left( \frac{k}{2^N} \right), F_{N-1} \left( \frac{k+1}{2^N} \right) \right]$$

*with  $I_N = \left[ \frac{k}{2^N}, \frac{k+1}{2^N} \right]$ .*

In the case of the maxima, the statement of the result is slightly more technical. We need the following proposition of [DE] using the same notations as previously.

**Proposition 3.5.** *Let  $0 < \alpha < 1$  and  $F$  the function defined by (3.18). Let  $t = 2^{1-\alpha}$  and  $X(p)$  the list of positions where  $F(x) + px$  attains its maximum on  $[0, 1]$ . Let  $\mathcal{M}(p)$  be the maximum on  $[0, 1]$  of  $F(x) + px$ . Then*

- $X \left( \frac{-(t^N - 1)}{t - 1} \right) = \left\{ \frac{1}{3} \frac{1}{2^N}, \frac{2}{3} \frac{1}{2^N} \right\}$  for  $N = 0, 1, \dots$
- $X(p) = \left\{ \frac{1}{3} \frac{1}{2^N} \right\}$  if  $\frac{-(t^{N+1} - 1)}{t - 1} < p < \frac{-(t^N - 1)}{t - 1}$ .
- $X(p) = 1 - X(-p)$  for all  $p \neq 0$ .
- $\max_{x \in I_N} F(x) = F_{N-1} \left( \frac{k}{2^N} \right) + 2^{-N\alpha} \mathcal{M} \left( \frac{C_{N-1} \left( \frac{2k+1}{2^{N+1}} \right)}{t^N} \right)$ .

The following proposition is a consequence of the previous one.

**Proposition 3.6.** *Let  $0 < \alpha < 1$  and  $F$  the function defined by (3.18). Then*

- *1/3 and 2/3 are the abscissas of the global maxima of  $F$ .*
- *The abscissas of the local maxima of  $F$  are the points of  $\mathcal{S}$ .*

### 3.6 Approximation of slopes of $F_n$

Suppose first we have some informations about the dyadic expansion of  $x$ . Then we have the following Lemma, which helps to control the behavior of the slopes of the affine function  $F_{n-1}$ .

**Lemma 3.7.** 1. Let  $x = \frac{K}{2^N}$  be a dyadic number. Then one can find  $N_0 > N$ ,  $A > 0$  and  $B > 0$  depending only on  $x$  such that if  $n \geq N_0$  then

$$(3.22) \quad \forall y \in ]x, x + 2^{-n}[ \quad A2^{(1-\alpha)n} \leq C_{n-1}(y) \leq B2^{(1-\alpha)n}$$

and

$$(3.23) \quad \forall y \in ]x - 2^{-n}, x[ \quad -A2^{(1-\alpha)n} \geq C_{n-1}(y) \geq -B2^{(1-\alpha)n}.$$

2. Let  $x$  be the abscissa of a local maximum of  $F$ . Then one can find  $J_0 > 0$ ,  $A > 0$  and  $B > 0$  such that for  $n \geq J_0$

$$(3.24) \quad \begin{aligned} i_{n-1}(x) + i_n(x) &= 1, \\ C_{n-1}(x)C_n(x) &< 0 \\ A2^{(1-\alpha)n} &\leq (-1)^{i_{n+1}(x)}C_{n-1}(x) \leq B2^{(1-\alpha)n}. \end{aligned}$$

3. Let  $x$  be a non dyadic point such that  $r(x) > 1$ . Then one can find two subsequences  $J_n$  and  $J'_n$  with  $\frac{J'_n}{J_n} > 1$  for all  $n$ , such that  $i_{J_n}(x) = i_{J'_n+1}(x)$  and  $i_j(x) + i_{J_n}(x) = 1$  for  $J_n < j < J'_n + 1$ . Furthermore one can find  $J_0 > 0$ ,  $A > 0$  and  $B > 0$  such that for  $n > J_0$

$$(3.25) \quad A2^{(1-\alpha)J'_n} \leq (-1)^{i_{J_n+1}(x)}C_{J'_n-1}(x) \leq B2^{(1-\alpha)J'_n}.$$

4. Let  $x$  be a non dyadic point such that  $s(x) > 1$ . Then one can find two subsequences  $J_n$  and  $J'_n$  with  $\frac{J'_n}{J_n} > 1$  for all  $n$ , such that  $i_j(x) + i_{j+1}(x) = 1$  for  $J_n < j < J'_n$  and  $i_{J_n}(x) = i_{J'_n+1}(x)$ . Furthermore one can find  $J_0 > 0$ ,  $A > 0$  and  $B > 0$  such that for  $n > J_0$

$$(3.26) \quad A2^{(1-\alpha)J'_n} \leq (-1)^{i_{J'_n}(x)}C_{J'_n-1}(x) \leq B2^{(1-\alpha)J'_n}.$$

**Proof.**

- Case 1: the idea is very simple since it is a direct computation.

Indeed following (3.20) we have for  $y \in ]x, x + 2^{-n}[$

$$\begin{aligned} C_{n-1}(y) &= \sum_{j=0}^{n-1} (-1)^{i_{j+1}(y)} 2^{-\alpha j} 2^j \\ &= \sum_{j=0}^{N-2} (-1)^{i_{j+1}(y)} 2^{j(1-\alpha)} - 2^{(N-1)(1-\alpha)} + \sum_{j=N}^{n-1} 2^{j(1-\alpha)} \\ &= \sum_{j=0}^{N-2} (-1)^{i_{j+1}(y)} 2^{j(1-\alpha)} + 2^{N(1-\alpha)}(1 - 2^{\alpha-1}) + 2^{(N+1)(1-\alpha)} \frac{2^{(n-N-1)(1-\alpha)} - 1}{2^{(1-\alpha)} - 1}. \end{aligned}$$

The second equation with  $y \in ]x - 2^{-n}, x[$  can be computed in the same way, up to a change of signs.

Thus one can find  $N_0 > N$ ,  $A > 0$  and  $B > 0$  such that (3.23) holds for  $n > N_0$ .

- Case 2: it is enough to remark that  $x$  has the following binary expansion

$$(3.27) \quad x = \frac{k_{N_0}}{2^{N_0}} + \sum_{l=0}^{\infty} \frac{1}{2^{2l+1+N_0}} .$$

As a consequence of Proposition 3.6, the same kind of computation yields Case 2.

- Case 3: since  $r(x) > 1$ , for any  $\delta > 0$  one can find two subsequences  $J_n$  and  $J'_n$  such that  $i_{J_n+1}(x) = \dots = i_{J'_n-1}(x)$  and  $|x - K_{J_n} 2^{-J_n}| \leq 2^{-J'_n}$  with  $J'_n = [(r(x) - \delta)J_n]$ .

Suppose first eventually up to a small change of definition of  $J'_n$  that  $i_{J_n}(x) = 1 = i_{J'_n+1}(x)$  and  $i_{J_n+1}(x) = \dots = i_{J'_n-1}(x) = i_{J'_n}(x) = 0$ . Then with the same kind of computation as in Case 1 one gets

$$(3.28) \quad A2^{(1-\alpha)J'_n} \leq C_{J'_n-1}(x) \leq B2^{(1-\alpha)J'_n} .$$

In the other case  $i_{J_n+1}(x) = \dots = i_{J'_n-1}(x) = i_{J'_n}(x) = 1$ , the sign of the slope will be changed.

- Case 4: this follows exactly the same ideas than previously. Since  $s(x) > 1$  for any  $\delta > 0$  one can find two subsequences  $J_n$  and  $J'_n$  such that for all  $J_n < j < J'_n$   $i_j(x) + i_{j+1}(x) = 1$  and  $i_{J_n}(x) = i_{J_n-1}(x)$ ,  $i_{J'_n}(x) = i_{J'_n+1}(x)$ . Then with the same kind of computation as in Case 1 one gets

$$(3.29) \quad A2^{(1-\alpha)J'_n} \leq (-1)^{i_{J'_n}(x)} C_{J'_n-1}(x) \leq B2^{(1-\alpha)J'_n} .$$

Hence the result.

If we don't have any further information on  $x$ , the following Lemma will be useful.

**Lemma 3.8.** *Let  $x \in [0, 1]$  be a non dyadic number.*

1. *[G] Then there exists  $\delta > 0$  and  $\delta' = \frac{1}{2^{1-\alpha}-1}$  such that for all  $n \in \mathbb{N}$  one can find  $J_n \geq n$  such that*

$$(3.30) \quad \delta' 2^{J_n(1-\alpha)} > |C_{J_n-1}(x)| > \delta 2^{J_n(1-\alpha)} .$$

2. *If  $x \notin \mathcal{S}$  then there exists  $\delta > 0$  such that for all  $n \geq 0$  there exists  $J_n \geq n$  such that (3.30) holds and*

- *either  $[C_{J_n-1}(x) > 0 \text{ and } i_{J_n+1}(x) = 0]$ ,*
- *or  $[C_{J_n-1}(x) < 0 \text{ and } i_{J_n+1}(x) = 1]$ .*

**Proof.**

1. The upper bound is a straightforward computation.

Suppose the contrary, i.e for all  $\delta > 0$  one can find  $N \in \mathbb{N}$  such that for all  $n \geq N$

$$(3.31) \quad -\delta 2^{(1-\alpha)n} \leq C_{n-1}(x) \leq \delta 2^{(1-\alpha)n} .$$

If we suppose without loss of generality that  $i_{n+1}(x) = 0$  then at step  $n$

$$(3.32) \quad \begin{aligned} & -\delta 2^{(1-\alpha)n} \leq C_{n-1}(x) \leq \delta 2^{(1-\alpha)n} \\ & -\delta 2^{(1-\alpha)n} + 2^{(1-\alpha)n} \leq C_n(x) \leq \delta 2^{(1-\alpha)n} + 2^{(1-\alpha)n} \\ & -\frac{\delta}{2^{1-\alpha}} + \frac{1}{2^{1-\alpha}} \leq \frac{C_n(x)}{2^{(1-\alpha)(n+1)}} \leq \frac{\delta}{2^{1-\alpha}} + \frac{1}{2^{1-\alpha}} . \end{aligned}$$

It is enough to choose  $\delta$  such that  $-\frac{\delta}{2^{1-\alpha}} + \frac{1}{2^{1-\alpha}} > \delta$  to have a contradiction.

2. Suppose the contrary, i.e there exists  $x \notin \mathcal{S}$  and that for all  $\beta > 0$ , there exists  $N \in \mathbb{N}$ , such that for all  $n \geq N$

- (a) either  $|C_{n-1}(x)| \leq \beta 2^{J_n(1-\alpha)}$ ,
- (b) or  $[C_{n-1}(x) < 0 \text{ and } i_{n+1}(x) = 0]$ ,
- (c) or  $[C_{n-1}(x) > 0 \text{ and } i_{n+1}(x) = 1]$ .

Remark first that the points of  $\mathcal{S}$  satisfy exactly (2b) and (2c). Indeed for  $x \in \mathcal{S}$ , and assuming that  $i_{n_0+1}(x) = 1$ ,  $x$  has a binary expansion (3.27).

Thus following (3.20) the slope  $C_{n-1}(x)$  satisfies

$$\begin{aligned} C_{n-1}(x) &= \sum_{j=0}^{N_0-1} (-1)^{i_{j+1}(x)} 2^{(1-\alpha)j} + \sum_{j=N_0}^{n-1} 2^{(1-\alpha)j} (-1)^{i_{j+1}(x)} \\ &= \sum_{j=0}^{N_0-1} (-1)^{i_{j+1}(x)} 2^{(1-\alpha)j} + 2^{N_0(1-\alpha)} \frac{(-2^{(1-\alpha)})^{n-N_0-1}}{1+2^{1-\alpha}}. \end{aligned}$$

Hence, for  $n$  large enough (2b) and (2c) are satisfied.

Our goal is thus to prove that if we choose  $\beta$  small enough then only (2b) and (2c) can be satisfied, which will lead to the fact that  $x \in \mathcal{S}$ , and thus to a contradiction.

Let start by the following special cases.

- We claim that if one can find  $k$  large enough such that  $C_{k-1}(x) = 0$  then  $x \in \mathcal{S}$ , which is a contradiction.

Let us prove this claim.

We will need the following sequence: let for  $n \in \mathbb{N}^*$

$$(3.33) \quad d_n = 2^{-(1-\alpha)} \sum_{j=0}^n (-1)^j 2^{-j(1-\alpha)} = \frac{2^{-(1-\alpha)}}{1 + 2^{-(1-\alpha)}} \left( 1 - (-1)^{n+1} 2^{-(n+1)(1-\alpha)} \right).$$

We have clearly  $d_n \geq d_1 > 0$  for all  $n \geq 1$ .

Choose  $\beta \leq \frac{d_1}{2}$  and  $N$  such that the hypothesis are satisfied.

Suppose that  $k \geq N + 1$  is such that  $C_{k-1}(x) = 0$ . Then

$$|C_k(x)| = 2^{k(1-\alpha)} = 2^{-(1-\alpha)} 2^{(k+1)(1-\alpha)}.$$

Remark that  $\beta < d_1 \leq 2^{-(1-\alpha)}$  thus  $|C_k(x)| > \beta 2^{(k+1)(1-\alpha)}$ .

Suppose without lost of generality that  $C_k(x) > 0$  (the case  $C_k(x) < 0$  is symmetrical and can be proved in exactly the same

way). Thus  $i_{k+2}(x) = 1$  and

$$(3.34) \quad \begin{aligned} C_{k+1}(x) &= -2^{(k+1)(1-\alpha)} + 2^{k(1-\alpha)} = -2^{(k+2)(1-\alpha)} (2^{-(1-\alpha)} - 2^{-2(1-\alpha)}) \\ &< -\beta 2^{(k+2)(1-\alpha)}. \end{aligned}$$

Let us prove by induction on  $n$  that for all  $n \in \mathbb{N}^*$

$$|C_{k+n}(x)| = d_n 2^{(k+n+1)(1-\alpha)}, (-1)^n C_{k+n}(x) > 0 \quad (\text{P}).$$

We just proved that (P) is true for  $n = 1$ .

Suppose that for  $n \in \mathbb{N}^*$  (P) is true. Suppose without loss of generality that  $C_{k+n}(x) > 0$  (the case  $C_{k+n}(x) < 0$  is symmetrical and can be proved in exactly the same way). Thus  $i_{k+n+2}(x) = 1$  and

$$(3.35) \quad \begin{aligned} C_{k+n+1}(x) &= C_{k+n}(x) - 2^{(n+k+1)(1-\alpha)} = d_n 2^{(k+n+1)(1-\alpha)} - 2^{(n+k+1)(1-\alpha)} \\ &= -(2^{-(1-\alpha)} d_n + 2^{-(1-\alpha)}) 2^{(k+n+2)(1-\alpha)}. \end{aligned}$$

Since  $d_n$  satisfies exactly  $d_{n+1} = -2^{-(1-\alpha)} d_n + 2^{-(1-\alpha)}$ , we have the result and (P) is satisfied at level  $n + 1$ .

Thus for all  $n \in \mathbb{N}^*$  (P) is true. Remind that since  $\beta < d_1 \leq d_n$  for all  $n \in \mathbb{N}^*$ , this implies that for all  $n \in \mathbb{N}^*$   $i_{k+n+2}(x) + i_{k+n+3}(x) = 1$ , which is exactly the characterization of the points in  $\mathcal{S}$ , and is in contradiction with the hypothesis  $x \notin \mathcal{S}$ .

In the following we will always keep the hypothesis  $0 < \beta \leq \frac{d_1}{2}$  so that for  $n$  large enough we have always  $C_n(x) \neq 0$ .

- We now consider the case where  $|C_{n-1}(x)|$  is close to the value of  $\beta 2^{n(1-\alpha)}$  and prove that this yields that  $x \in \mathcal{S}$ , and thus a contradiction.

Let  $0 < \beta \leq \frac{d_1}{2}$ , and  $\beta' > 0$  whose value will be precised later on.

Suppose  $n \geq N$  is such that  $\beta' 2^{n(1-\alpha)} > C_{n-1}(x) > \beta 2^{n(1-\alpha)}$  (the case  $C_{n-1}(x) < -\beta 2^{n(1-\alpha)}$  can be done exactly in the same way). Then  $i_{n+1}(x) = 1$  and  $C_n(x) = C_{n-1}(x) - 2^{n(1-\alpha)}$ , hence

(3.36)

$$\begin{aligned} \beta 2^{n(1-\alpha)} - 2^{n(1-\alpha)} &\leq C_n(x) \leq \beta' 2^{n(1-\alpha)} - 2^{n(1-\alpha)} \\ (\beta - 1) 2^{-(1-\alpha)} 2^{(n+1)(1-\alpha)} &\leq C_n(x) \leq (\beta' - 1) 2^{-(1-\alpha)} 2^{(n+1)(1-\alpha)} . \end{aligned}$$

Choose  $\beta'$  such that  $(\beta' - 1) 2^{-(1-\alpha)} < -\beta$ , hence  $\beta < \beta' < \frac{1-\beta}{2^{-(1-\alpha)}}$ , which is possible since  $\beta \leq \frac{d_1}{2} < \frac{2^{-(1-\alpha)}}{1+2^{-(1-\alpha)}}$ .

This yields  $i_{n+2}(x) = 0$ . Thus

(3.37)

$$\begin{aligned} C_{n+1}(x) &= C_{n-1}(x) + (-2^{n(1-\alpha)} + 2^{(n+1)(1-\alpha)}) \\ &> (2^{-(1-\alpha)} - 2^{-2(1-\alpha)}) 2^{(n+2)(1-\alpha)} > \beta 2^{(n+2)(1-\alpha)} . \end{aligned}$$

Let us prove by induction on  $k$  that for all  $k \in \mathbb{N}$ ,

$$(-1)^{k+1} C_{n+k}(x) > \beta 2^{(n+k+1)(1-\alpha)} \quad (\text{Q}) .$$

We just prove that the case  $k = 0$  is true.

Suppose one can find  $k \in \mathbb{N}$  such that (Q) is true for all  $0 \leq k' \leq k$ .

Let us prove that it is true at  $k + 1$ . Without lost of generality suppose  $C_{n+k}(x) > 0$ , thus  $i_{n+k+2}(x) = 1$ .

We have

(3.38)

$$\begin{aligned} C_{n+k+1}(x) &= C_{n+k-1}(x) + 2^{(n+k)(1-\alpha)} - 2^{(n+k+1)(1-\alpha)} \\ &< -(2^{-(1-\alpha)} - 2^{-2(1-\alpha)}) 2^{(n+k+2)(1-\alpha)} < -\beta 2^{(n+k+2)(1-\alpha)} . \end{aligned}$$

This proves that (Q) is true at  $k + 1$ .

Thus by induction (Q) is true for all  $k \in \mathbb{N}$ . This means that for all  $k \in \mathbb{N}$   $i_{n+k+2}(x) + i_{n+k+3}(x) = 1$ , and thus  $x \in \mathcal{S}$ . Hence the contradiction.

- We now study the case where  $|C_{n-1}(x)| \leq \beta 2^{n(1-\alpha)}$  and prove that if we choose  $\beta$  small enough then it will lead to  $x \in \mathcal{S}$ .

Indeed let  $\beta \leq \inf(\frac{d_1}{2}, \frac{d'_1}{2})$  with

$$(3.39) \quad d'_1 = \frac{2^{-(1-\alpha)} - 2^{-2(1-\alpha)}}{1 + 2^{-2(1-\alpha)}} .$$

And suppose  $n \geq N$  such that  $|C_{n-1}(x)| \leq \beta 2^{n(1-\alpha)}$ . Suppose  $i_{n+1}(x) = 0$  without loss of generality. Thus we have

$$\begin{aligned} C_n(x) &= C_{n-1}(x) + 2^{n(1-\alpha)} \\ (-\beta + 1)2^{-(1-\alpha)}2^{(n+1)(1-\alpha)} &\leq C_n(x) \leq (\beta + 1)2^{-(1-\alpha)}2^{(n+1)(1-\alpha)}. \end{aligned}$$

Remark that with the choice of  $\beta$  we made, we have on one hand  $\beta < (-\beta + 1)2^{-(1-\alpha)}$  and on the other hand  $\beta < (\beta + 1)2^{-(1-\alpha)} < \frac{1-\beta}{2^{-(1-\alpha)}}$ . Thus following the previous result using  $\beta' = (\beta + 1)2^{-(1-\alpha)}$ ,  $x \in \mathcal{S}$  and we have a contradiction.

- We consider the case where  $C_{n-1}(x) > 0$  and  $C_n(x) < 0$  for  $n$  large enough under the previous range of values of  $\beta$ .

Let  $\beta \leq \inf(\frac{d_1}{2}, \frac{d'_1}{2})$  (recall that  $d_1$  is defined by (3.33) and  $d'_1$  by (3.39)).

And suppose that for  $n \geq N$  we have  $C_{n-1}(x) > 0$  and  $C_n(x) < 0$ . Following the previous case we have  $C_{n-1}(x) > \beta 2^{n(1-\alpha)}$  and  $C_n(x) < -\beta 2^{(n+1)(1-\alpha)}$ . Thus  $i_{n+1}(x) = 1$  and  $i_{n+2}(x) = 0$ .

Then

$$\begin{aligned} (3.40) \quad C_{n+1}(x) &= C_{n-1}(x) - 2^{n(1-\alpha)} + 2^{(n+1)(1-\alpha)} \\ &> \beta 2^{(n+2)(1-\alpha)} \end{aligned}$$

since by definition of  $\beta$  and  $d_1$  we have  $2^{-(1-\alpha)} - 2^{-2(1-\alpha)} \geq d_1 > \beta$ . Thus  $i_{n+3}(x) = 1$ .

We have  $C_{n+2}(x) = C_n(x) + 2^{(n+1)(1-\alpha)} - 2^{(n+2)(1-\alpha)} < -\beta 2^{(n+3)(1-\alpha)}$ .

A proof by induction exactly in the same way as previously yields that for  $k \geq 0$  we have  $C_{n+2k+1}(x) > 0$  and  $C_{n+2k}(x) < 0$ , thus  $i_{n+2k+2}(x) + i_{n+2k+3}(x) = 1$  for all  $k \in \mathbb{N}$  and we have  $x \in \mathcal{S}$ , hence a contradiction.

We will now go the main proof, taking into account what we just proved.

In the following we will consider  $\delta > 0$  and  $J_n$  defined as in Point 1,  $\beta = \inf(\frac{d_1}{2}, \frac{\delta}{2}, \frac{d'_1}{2})$  and  $n$  such that  $J_n \geq N$ . Thus for all  $n \in \mathbb{N}$

$$\frac{|C_{J_n-1}(x)|}{2^{J_n(1-\alpha)}} > \delta > \beta.$$

Suppose  $C_{J_n-1}(x) > 0$ . This means that  $C_{J_n}(x) = C_{J_n-1}(x) - 2^{J_n(1-\alpha)} < C_{J_n-1}(x)$ . The only case we want to consider is  $C_{J_n}(x) > \beta 2^{(J_n+1)(1-\alpha)}$  since for all the other cases the previous points yield  $x \in \mathcal{S}$ .

Thus  $i_{J_n+2}(x) = 1$ . It is clear that one can find  $k \in \mathbb{N}$  such that for all  $0 \leq k' \leq k$   $C_{J_n+k'}(x) > \beta 2^{(J_n+k'+1)(1-\alpha)}$  and  $i_{J_n+k'+2}(x) = 1$  and  $C_{J_n+k+1}(x) < \beta 2^{(J_n+k+2)(1-\alpha)}$ .

Hence either  $|C_{J_n+k+1}(x)| \leq \beta 2^{(J_n+k+2)(1-\alpha)}$  and  $x \in \mathcal{S}$ , or  $C_{J_n+k+1}(x) < -\beta 2^{(J_n+k+2)(1-\alpha)}$  and since  $C_{J_n+k}(x) > \beta 2^{(J_n+k+1)(1-\alpha)}$  we have also the result.

In all cases we proved that Points (2a), (2b), (2c) lead to  $x \in \mathcal{S}$ , which is a contradiction. Hence the Lemma.

## 4 Computation of weak and strong accessible exponents

### 4.1 Case of dyadic points

We will prove the following proposition.

**Proposition 4.1.** *If  $x$  is a dyadic point, and  $X = (x, F(x))$  then*

$$(4.1) \quad E_{\Omega^c}^w(X) = \frac{1}{\alpha} - 1 \text{ and } E_{\Omega}^w(X) = 0$$

$$(4.2) \quad E_{\Omega^c}^s(X) = \frac{1}{\alpha} - 1 \text{ and } E_{\Omega}^s(X) = 0$$

$$(4.3) \quad u_f^p(X) = \frac{1}{p} \left( \frac{1}{\alpha} - 1 \right).$$

If  $x$  is a dyadic point, i.e.  $x = 2^{-N}K$  with  $K \in 2\mathbb{N} + 1$ , we consider its binary expansion in which  $i_N(x) = 1$  and  $i_n(x) = 0$  for  $n > N$ . For  $n > N$  the number  $x - 2^{-n}$  is dyadic. Since  $2^{-n} = \sum_{j=n+1}^{\infty} 2^{-j}$  then

$$x - 2^{-n} = \left( \sum_{j=1}^{N-1} i_j 2^{-j} \right) + 2^{-(N+1)} + 2^{-(N+2)} + \dots + 2^{-n}. \text{ On the other hand}$$

$$x + 2^{-n} \text{ has the simple expansion } \left( \sum_{j=1}^N \frac{i_j(x)}{2^j} \right) + \frac{1}{2^n}.$$

Remark that  $F_{N-1}(x) = F_{n-1}(x) = F(x)$  for  $n > N$  and  $F_{n-1}(x + 2^{-n}) = F(x + 2^{-n})$ .

Any point  $y$  in the interval  $]x, x + 2^{-n}[$  satisfies the expansion  $i_N(y) = i_{N+1}(y) = \dots = i_{n-1}(y) = i_n(y) = 0$ .

It follows that

$$F(x + 2^{-n}) - F(x) = F_{n-1}(x + 2^{-n}) - F_{n-1}(x) = 2^{-n}C_{n-1}(y)$$

with  $y$  any of the points of the interval  $]x, x + 2^{-n}[$ .

Following Lemma 3.7 and Case 1 there exist two constants  $A > 0$  and  $B > 0$  and  $J_0 \geq N$  (which depend only on the given dyadic point  $x$ ) such that

$$(4.4) \quad \forall n \geq J_0 \quad A2^{-\alpha n} \leq F(x + 2^{-n}) - F(x) \leq B2^{-\alpha n}.$$

Thus we have  $F(x + 2^{-n}) - F(x) \geq A.2^{-\alpha n}$ .

On the other hand, following remarks of Section ??, for any  $y \in ]x - 2^{-n}, x[$  we have  $i_N(y) = 1 = \dots = i_n(y)$ . Thus

$$(4.5) \quad F(x - 2^{-n}) - F(x) = F_{n-1}(x - 2^{-n}) - F_{n-1}(x) = -C_{n-1}(y)2^{-n}.$$

Whence, following Lemma 3.7 and Case 1 we have for  $n \geq J_0$

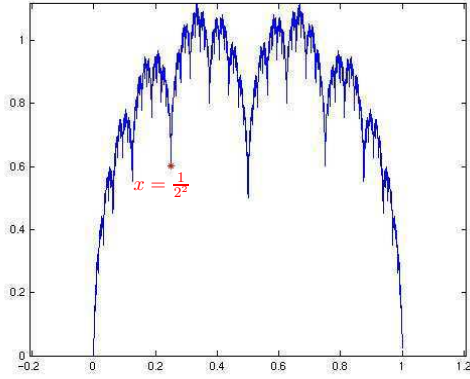
$$(4.6) \quad \forall n \geq J_0 \quad A2^{-\alpha n} \leq -F(x - 2^{-n}) + F(x) \leq B2^{-\alpha n}.$$

Let  $\rho > 0$  and  $J \geq J_0$  such that  $2^{-J-1} \leq \rho \leq 2^{-J}$ .

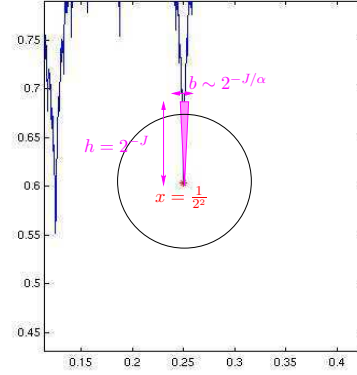
Since  $F \geq F_J$ , then  $\Omega_j \subset \Omega$  where  $\Omega_J$  is the domain below the graph of  $F_J$ . So

$$(4.7) \quad \text{meas}(B(X, \rho) \cap \Omega^c) \leq \text{meas}(B(X, \rho) \cap \Omega_J^c).$$

But  $\text{meas}(B(X, \rho) \cap \Omega_J^c)$  is smaller than the area  $hb/2$  of a triangle with altitude  $h$  issued from  $X$  and a corresponding hypotenuse  $b$  (see Figures below).



Overview of the function

Zoom at the point  $x = \frac{1}{2^2}$ 

Clearly, we can take  $h \sim 2^{-J}$ . On the other hand, if we write  $2^{-(j+1)} < b/2 < 2^{-j}$  with  $j \geq J$ , then using properties (4.4) and (4.6) (in which we replace  $n\alpha$  by  $J$ ), we get  $b/2 \sim 2^{-J/\alpha}$ . Since  $\alpha < 1$ , Equations (4.4) and (4.6) are valid with  $n = \frac{J}{\alpha} \geq J_0$ .

Whence

$$(4.8) \quad \text{meas}(B(X, \rho) \cap \Omega^c) \leq C\rho^{1+\frac{1}{\alpha}}.$$

We conclude that

$$(4.9) \quad E_{\Omega^c}^w(X) \geq \frac{1}{\alpha} - 1.$$

Since  $E_{\Omega^c}^w(X) \leq E_{\Omega^c}^s(X) \leq \frac{1}{\alpha} - 1$  this yields

$$E_{\Omega^c}^s(X) = \frac{1}{\alpha} - 1.$$

Since  $f = 1_\Omega$  then

$$(4.10) \quad u_f^p(X) \geq \frac{1}{p} \left( \frac{1}{\alpha} - 1 \right).$$

Since  $\text{meas}(B(X, r)) = \text{meas}(B(X, r) \cap \Omega) + \text{meas}(B(X, r) \cap \Omega^c)$

we get  $E_\Omega^s(X) = 0$ , hence  $E_\Omega^w(X) = 0$ .

Whence Proposition 4.1.

## 4.2 Case of a local maximum of $F$

We will prove the following proposition.

**Proposition 4.2.** *Let  $X = (x, F(x))$ .*

*If  $x$  is a local maximum,*

$$(4.11) \quad E_{\Omega}^w(X) = \frac{1}{\alpha} - 1 \text{ and } E_{\Omega^c}^w(X) = 0$$

$$(4.12) \quad E_{\Omega}^s(X) = \frac{1}{\alpha} - 1 \text{ and } E_{\Omega^c}^s(X) = 0$$

$$(4.13) \quad u_f^p(X) = \frac{1}{p} \left( \frac{1}{\alpha} - 1 \right).$$

Let  $x$  be a local maximum of  $F$ . There is an interval  $I$  containing  $x$  such that for all  $x' \in I$ ,  $F(x) \geq F(x')$ . Let  $N$  be such that the dyadic interval  $\left[\frac{k_N}{2^N}, \frac{k_N+1}{2^N}\right]$  which contains  $x$  is contained in  $I$ .

Following Lemma 3.6, we know that  $x$  has the binary expansion (3.27), i.e  $x = \frac{k_{N_0}}{2^{N_0}} + \sum_{l=0}^{\infty} \frac{1}{2^{2l+1+N_0}}$ .

As a consequence of Lemma 3.7, and following Case (2), one can find  $J_0$  and two constants  $A$  and  $B$  such that for  $n \geq J_0$  Equation (3.24) holds. Remark that it implies clearly that for  $n \geq J_0$   $i_n(x) = 1$  if  $n$  is odd, and  $i_n(x) = 0$  if  $n$  is even.

Our goal now is to evaluate  $F(x) - F(x')$  with  $x'$  in the interval  $\left[\frac{k_n}{2^n}, \frac{k_n+1}{2^n}\right] \subset \left[\frac{k_N}{2^N}, \frac{k_N+1}{2^N}\right]$  and  $x' \neq x$ . If  $x'$  is a dyadic then we take its expansion of type  $i_j(x') = 0$  for  $j$  large enough.

Let  $m \geq n$  be the smallest integer such that  $i_m(x) = i_m(x')$  and  $i_{m+1}(x) \neq i_{m+1}(x')$ . To fix the ideas, suppose that  $i_{m+1}(x) = 1$  and  $i_{m+1}(x') = 0$ . Thus

$$(4.14) \quad \frac{1}{32^{m-1}} \geq x - x' \geq \frac{1}{2^{m+1}} + \frac{1}{2^{m+3}} - \sum_{j=m+2}^{\infty} \frac{1}{2^j} \geq \frac{1}{2^{m+3}}.$$

Since  $i_m(x) = 0$  we have  $C_{m-1}(x) > 0$ .

Thus

$$(4.15) \quad F(x) - F(x') = \underbrace{C_{m-1}(x)(x - x')}_{(I)} + \underbrace{\sum_{k=m}^{+\infty} 2^{-k\alpha} (\Lambda(\tau^k x) - \Lambda(\tau^k x'))}_{(II)}.$$

We have

$$(4.16) \quad A2^{m(1-\alpha)}2^{-m-3} = C_12^{-m\alpha} \leq (I) \leq B2^{m(1-\alpha)}2^{-m-1}/3 = C_22^{-m\alpha}.$$

Since for  $k \geq m - 1$  we have  $\Lambda(\tau^k x) = 1/3$ , this yields

$$(4.17) \quad 0 \leq (II) \leq C_3 2^{-m\alpha} .$$

Thus we have

$$(4.18) \quad C_1 2^{-m\alpha} \leq F(x) - F(x') \leq (C_3 + C_2) 2^{-m\alpha} .$$

Let us compute the weak and strong exponents at  $x$ .

Let  $\rho$  and  $J \geq J_0$  such that  $2^{-J-1} \leq \rho \leq 2^{-J}$ . Thus obviously

$$meas(B(X, \rho) \cap \Omega) \leq meas(B(X, 2^{-J}) \cap \Omega) .$$

Remark first that if  $(x', y') \in B(X, 2^{-J}) \cap \Omega$  then  $|x - x'| < 2^{-J}$ ,  $|y' - F(x)| < 2^{-J}$  and  $y' \leq F(x')$ . Since  $x$  is a local maximum on the interval  $[\frac{k_I}{2^J}, \frac{k_I+1}{2^J}]$ , then  $y' \leq F(x') \leq F(x)$  and so  $0 \leq F(x) - F(x') < 2^{-J}$ . Hence  $(x', F(x')) \in B(X, 2^{-J}) \cap \Omega$ .

Furthermore since  $F(x')$  satisfies  $0 \leq F(x) - F(x') \leq 2^{-J}$ , and following Equation (4.18)  $x'$  belongs to  $[x - C2^{-J/\alpha}, x + C2^{-J/\alpha}]$  with  $C$  depending only on  $C_3 + C_2$ .

Thus  $B(X, 2^{-J}) \cap \Omega$  is contained in a rectangle of length  $2^{-J}$  and width  $C2^{-J/\alpha}$ .

This yields

$$(4.19) \quad meas(B(X, \rho) \cap \Omega) \leq C 2^{-J(1+\frac{1}{\alpha})} \leq C' \rho^{1+\frac{1}{\alpha}} .$$

We can conclude that

$$E_{\Omega}^w(X) \geq \frac{1}{\alpha} - 1 .$$

Since  $E_{\Omega}^w(X) \leq E_{\Omega}^s(X) \leq \frac{1}{\alpha} - 1$  this yields

$$(4.20) \quad E_{\Omega}^w(X) = E_{\Omega}^s(X) = \frac{1}{\alpha} - 1 .$$

Since  $meas(B(X, \rho)) = meas(B(X, \rho) \cap \Omega) + meas(B(X, \rho) \cap \Omega^c)$  we get

$$E_{\Omega^c}^s(X) = E_{\Omega^c}^w(X) = 0 .$$

And finally

$$(4.21) \quad u_f^p(X) = \frac{1}{p} \left( \frac{1}{\alpha} - 1 \right) .$$

### 4.3 Case of $x \notin \mathcal{D} \cup \mathcal{S}$

If  $x \notin \mathcal{D} \cup \mathcal{S}$  then we will compute separately the weak and strong exponents. We will first prove that for any point  $x$  in  $[0, 1]$  which is not a maximum or a minimum of  $F$  the two weak exponents vanish.

**Proposition 4.3.** *Let  $x \notin \mathcal{D} \cup \mathcal{S}$  and  $X = (x, f(x))$ .*

*Then  $E_{\Omega}^w(X) = E_{\Omega^c}^w(X) = 0$ .*

**Proof.** We will prove first that we have always  $E_{\Omega^c}^w(X) = 0$ , but will separate the proofs in cases  $r(x) > 1$  and  $r(x) = 1$ . Then we will prove that  $E_{\Omega}^w(X) = 0$  and prove it separately for  $s(x) > 1$ , and  $s(x) = 1$ .

- Case  $r(x) > 1$ . We follow the notations of Case 3 of Proposition 3.7, i.e one can find two subsequences  $J_n$  and  $J'_n$  such that  $\frac{J'_n}{J_n} > 1$  and  $i_{J_n}(x) = i_{J'_n+1}(x)$ ,  $i_j(x) + i_{J_n}(x) = 1$  for  $J_n < j < J'_n + 1$ . Suppose without loose of generality that  $i_{J_n+1}(x) = 0$ . Let  $\tilde{x}_n = \frac{K_{J_n}}{2^{J_n}} = \sum_{j=1}^{J_n} \frac{i_j(x)}{2^j}$ . Thus we have

$$(4.22) \quad 2^{-J'_n-1} \leq x - \tilde{x}_n \leq 2^{-J'_n}.$$

Since Case 3 of Proposition 3.7 holds, we get

$$(4.23) \quad \begin{aligned} A2^{(1-\alpha)J'_n}2^{-J'_n-1} &\leq F_{J'_n-1}(x) - F_{J'_n-1}(\tilde{x}_n) \leq B2^{(1-\alpha)J'_n}2^{-J'_n} \\ A'2^{-\alpha J'_n} &\leq F_{J'_n-1}(x) - F_{J'_n-1}(\tilde{x}_n) \leq B'2^{-\alpha J'_n}. \end{aligned}$$

We have  $F(x) = F_{J'_n-1}(x) + \underbrace{\sum_{k=J'_n}^{+\infty} 2^{-k\alpha} \Lambda(\tau^k x)}_{\geq 0}$  and  $F_{J'_n-1}(\tilde{x}_n) = F(\tilde{x}_n)$ .

Thus following (4.23) we have

$$(4.24) \quad \begin{aligned} A2^{(1-\alpha)J'_n}2^{-J'_n-1} &\leq F(x) - F(\tilde{x}_n) \leq B2^{(1-\alpha)J'_n}2^{-J'_n} + 2^{-J'_n\alpha} \sum_{k=0}^{+\infty} 2^{-k\alpha} \Lambda(\tau^{k+J'_n} x) \\ A'2^{-\alpha J'_n} &\leq F(x) - F(\tilde{x}_n) \leq B'2^{-J'_n} + 2^{-\alpha J'_n} F(\tau^{J'_n} x) \\ A'2^{-\alpha J'_n} &\leq F(x) - F(\tilde{x}_n) \leq B'2^{-J'_n} + 2^{-\alpha J'_n} F(\tau^{J'_n} 1/3) \\ A'2^{-\alpha J'_n} &\leq F(x) - F(\tilde{x}_n) \leq C2^{-\alpha J'_n} \end{aligned}$$

indeed the maximum of  $F$  is reached at abscissas  $1/3$  or  $2/3$ .

We can now apply the mean value theorem and get that for each  $n \geq J_0$  we can find  $y_n \in ]x - 2^{-J'_n}, x + 2^{-J'_n}[$  such that  $F(x) - F(y_n) = A' \cdot 2^{-\alpha n} / 2$ .

Thus using Lemma 3.2 we can conclude that  $E_{\Omega^c}^w(X) = 0$ .

- Case  $r(x) = 1$ .

Let  $J_n$  be defined just as in Lemma 3.8, i.e that one can find  $\delta > 0$ ,  $\delta' > 0$  and  $J_n$  such that Equation (3.30) is satisfied.

Following the definition of  $r(x)$ , for all  $\gamma > 0$  there exists  $n_0$  such that for all  $j \geq J_{n_0}$   $|K_j 2^{-j} - x| > 2^{-j(1+\gamma)}$ . Thus in particular for all  $n \geq n_0$  we have

$$(4.25) \quad 2^{-J_n} > |K_{J_n} 2^{-J_n} - x| > 2^{-J_n(1+\gamma)} .$$

Suppose on one hand  $C_{J_n-1}(x) \geq 0$ . Then choose  $\tilde{x}_n = K_{J_n} 2^{-J_n}$  if  $x \in ]K_{J_n} 2^{-J_n}, 2^{-J_n} + K_{J_n} 2^{-J_n}[$  (respectively  $\tilde{x}_n = K_{J_n} 2^{-J_n} - 2^{-J_n}$  if  $x \in ]K_{J_n} 2^{-J_n} - 2^{-J_n}, K_{J_n} 2^{-J_n}[$ ).

We have obviously

$$(4.26) \quad F_{J_n-1}(x) - F_{J_n-1}(\tilde{x}_n) = C_{J_n-1}(x)(x - \tilde{x}_n) \geq 0 .$$

If we suppose on the other hand  $C_{J_n-1}(x) \leq 0$ , then we can choose in the same way a dyadic number  $\tilde{x}_n = \frac{k}{2^{J_n}}$  such that

$$(4.27) \quad F_{J_n-1}(x) - F_{J_n-1}(\tilde{x}_n) = C_{J_n-1}(x)(x - \tilde{x}_n) \geq 0 .$$

Together with Equation (3.30) this yields in any of these cases that

$$(4.28) \quad \begin{aligned} \delta' 2^{-\alpha J_n} &\geq |F_{J_n-1}(x) - F_{J_n-1}(\tilde{x}_n)| \geq \delta 2^{-J_n(1+\gamma)} 2^{(1-\alpha)J_n} \\ \delta' 2^{-\alpha J_n} &\geq F_{J_n-1}(x) - F_{J_n-1}(\tilde{x}_n) \geq \delta 2^{-J_n(1+\gamma)} 2^{(1-\alpha)J_n} . \end{aligned}$$

Since  $\sum_{k=J'_n}^{+\infty} 2^{-k\alpha} \Lambda(\tau^k x) \geq 0$  and  $F_{J_n-1}(\tilde{x}_n) = F(\tilde{x}_n)$  we get

$$(4.29) \quad \begin{aligned} \delta' 2^{-\alpha J_n} + 2^{-\alpha J_n} F(1/3) &\geq F(x) - F(\tilde{x}_n) \geq F_{J_n-1}(x) - F_{J_n-1}(\tilde{x}_n) \\ &\geq \delta 2^{-J_n(1+\gamma)} 2^{(1-\alpha)J_n} \\ C 2^{-\alpha J_n} &\geq F(x) - F(\tilde{x}_n) \geq \delta 2^{-J_n(\gamma+\alpha)} . \end{aligned}$$

To get  $E_{\Omega^c}^w(X)$  we only have to adapt the proof of Lemma 3.2 to the case  $r_n = 2^{-J_n}$ . Suppose without lost of generality that  $x < \tilde{x}_n$  (the other case can be treated in a similar way) and let  $r_n = 2^{-J_n}$  for  $n \geq n_0$ .

Indeed, since for  $\gamma$  small enough and for  $n$  large enough  $2^{-J_n}$  is negligible in front of  $2^{-(\alpha+\gamma)J_n}$  (what we denote  $2^{-(\alpha+\gamma)J_n} \gg 2^{-J_n}$ ), following the mean value theorem we can find  $b_n \in ]\min(x, \tilde{x}_n), \max(x, \tilde{x}_n)[$  such that  $b_n = \sup\{u_n \in ]x, \tilde{x}_n[, f(u_n) = f(x) - r_n\}$ . For all  $t \in ]b_n, \tilde{x}_n[$ , we have  $f(t) < f(x) - r_n$ . Thus following the same method as in Lemma 3.2 we can find  $C > 0$  such that

$$(4.30) \quad C' r_n^2 \geq \text{meas} \left( B(X, r_n) \cap \Omega^c \right) \geq C r_n^{(1+\frac{\gamma}{\alpha})+1}.$$

This yields

$$(4.31) \quad 2 \leq \liminf_{n \rightarrow +\infty} \frac{\log(\text{meas}(B(X, r_n) \cap \Omega^c))}{\log(r_n)} \leq \limsup_{n \rightarrow +\infty} \frac{\log(\text{meas}(B(X, r_n) \cap \Omega^c))}{\log(r_n)} \leq 2 + \frac{\gamma}{\alpha}.$$

Since  $\gamma > 0$  is arbitrary and  $r_n$  is independent of  $\gamma$ , we have the result and  $E_{\Omega^c}^w(X) = 0$ .

- Case  $s(x) > 1$ .

Following Case 4, then one can find two subsequences  $J_n$  and  $J'_n$  with  $\frac{J'_n}{J_n} > 1$  for all  $n$ , such that  $i_j(x) + i_{j+1}(x) = 1$  for  $J_n < j < J'_n$  and  $i_{J'_n}(x) = i_{J'_n+1}(x)$ . Suppose without loosing generality that  $i_{J'_n}(x) = 0$ .

Let  $\tilde{X}_n$  such that  $\tilde{X}_n = \frac{k_{J_n}}{2^{J_n}} + \frac{2}{3(2^{J_n})} = \sum_{j=1}^{J_n} \frac{i_j(x)}{2^j} + \frac{2}{3(2^{J_n})}$ . We have clearly

$$(4.32) \quad 2^{-J'_n-1} \leq -x + \tilde{X}_n \leq 2^{-J'_n}.$$

Following the same sketch as in the proof with  $r(x) > 1$  we can say that, using Case 4 of Proposition 3.8

$$(4.33) \quad A' 2^{-\alpha J'_n} \leq F_{J'_n-1}(\tilde{X}_n) - F_{J'_n-1}(x) \leq B' 2^{-\alpha J'_n}$$

and since  $2^{-\alpha J'_n} F(\tau^{J'_n} 1/3) \geq \sum_{k=J'_n}^{+\infty} 2^{-k\alpha} \Lambda(\tau^k \tilde{X}_n) - \sum_{k=J'_n}^{+\infty} 2^{-k\alpha} \Lambda(\tau^k x) \geq 0$

we have indeed

$$(4.34) \quad A' 2^{-\alpha J'_n} \leq F(\tilde{X}_n) - F(x) \leq C 2^{-\alpha J'_n}.$$

Thus using the mean value theorem and Lemma 3.3 as in the previous case we conclude that  $E_\Omega^w(X) = 0$ .

- Case  $s(x) = 1$ .

Let  $J_n$  be defined just as in Lemma 3.8, i.e that one can find  $\delta > 0$ ,  $\delta' > 0$  and  $J_n$  such that Equation (3.30) is satisfied as well as Point 2 of Lemma 3.8.

Following the definition of  $s(x)$ , for all  $\gamma > 0$  there exists  $n_0$  such that for all  $j \geq J_{n_0}$   $|m_j - x| > 2^{-j(1+\gamma)}$ . Thus in particular for all  $n \geq n_0$  we have

$$(4.35) \quad 2^{-J_n} > |m_{J_n} - x| > 2^{-J_n(1+\gamma)} .$$

Suppose on one hand that  $C_{J_n-1}(x) > 0$ . Then take  $\tilde{X}_n = \frac{K_{J_n}}{2^{J_n}} + \frac{2}{3(2^{J_n})}$ . Since  $i_{J_n+1}(x) = 0$  we have  $F_{J_n-1}(\tilde{X}_n) - F_{J_n-1}(x) > 0$ .

Together with Equation (3.30) this yields that

$$(4.36) \quad \begin{aligned} \delta' 2^{-\alpha J_n} &\geq \left| F_{J_n-1}(x) - F_{J_n-1}(\tilde{X}_n) \right| \geq \delta 2^{-J_n(1+\gamma)} 2^{(1-\alpha)J_n} \\ \delta' 2^{-\alpha J_n} &\geq -F_{J_n-1}(x) + F_{J_n-1}(\tilde{X}_n) \geq \delta 2^{-J_n(1+\gamma)} 2^{(1-\alpha)J_n} . \end{aligned}$$

The same computation as previously yields

$$(4.37) \quad \begin{aligned} \delta' 2^{-\alpha J_n} + 2^{-\alpha J_n} F(1/3) &\geq -F(x) + F(\tilde{X}_n) \geq -F_{J_n-1}(x) + F_{J_n-1}(\tilde{X}_n) \\ &\geq \delta 2^{-J_n(1+\gamma)} 2^{(1-\alpha)J_n} \\ C 2^{-\alpha J_n} &\geq -F(x) + F(\tilde{X}_n) \geq \delta 2^{-J_n(\gamma+\alpha)} . \end{aligned}$$

To get  $E_\Omega^w(X)$  we only have to adapt the proof of Lemma 3.3 in the same way we adapt the one of Lemma 3.2 in the case  $r(x) = 1$ .

Thus taking  $r_n = 2^{-J_n}$  and following the same method as previously we can find  $C > 0$  such that

$$(4.38) \quad C' r_n^2 \geq \text{meas} \left( B(X, r_n) \cap \Omega \right) \geq C 2^{-J_n(1+\frac{\gamma}{\alpha})} 2^{-J_n} .$$

This yields

$$(4.39) \quad 2 \leq \liminf_{n \rightarrow +\infty} \frac{\log(\text{meas}(B(X, r_n) \cap \Omega))}{\log(r_n)} \leq \limsup_{n \rightarrow +\infty} \frac{\log(\text{meas}(B(X, r_n) \cap \Omega))}{\log(r_n)} \leq 2 + \frac{\gamma}{\alpha} .$$

Since  $\gamma > 0$  is arbitrary, we have the result and  $E_\Omega^w(X) = 0$ .

Hence the proof of Proposition 4.3.

For what concerns the strong accessibility exponent we have the following result.

**Proposition 4.4.** *Suppose  $x \notin \mathcal{D} \cup \mathcal{S}$  and let  $X_0 = (x, F(x))$ . Then*

1. *If  $r(x) > \frac{1}{\alpha}$  then  $E_{\Omega^c}^s(X_0) = \frac{1}{\alpha} - 1$ .*
2. *If  $s(x) > \frac{1}{\alpha}$  then  $E_\Omega^s(X_0) = \frac{1}{\alpha} - 1$ .*
3. *Let  $D_\alpha$  the set of  $x \notin \mathcal{D} \cup \mathcal{S}$  such that  $r(x) > \frac{1}{\alpha}$  and  $s(x) > \frac{1}{\alpha}$ . Then the Hausdorff dimension of  $D_\alpha$  is  $\alpha$ .*

**Proof.**

1. Let us prove Point 1. Since  $r(x) > \frac{1}{\alpha}$ , and following Point 3 of Lemma 3.7, for  $\delta > 0$  such that  $r(x) - \delta > \frac{1}{\alpha}$  we can find  $J'_n$  and  $J_n$  such that

- $x_n = K_{J_n} 2^{-J_n}$  and  $|x - x_n| \leq 2^{-J'_n}$ .
- $|F(x) - F(x_n)| \leq 2^{-\alpha J'_n}$ .

Since we can choose  $J'_n$  such that  $2^{-J'_n} \leq C 2^{(r(x)-\delta)J_n}$  (see the proof of Point 3 of Lemma 3.7), then  $|x - x_n|$  is negligible in front of  $2^{-J_n}$  (what we denote  $|x - x_n| \ll 2^{-J_n}$ ) and  $|F(x) - F(x_n)| \leq 2^{-\alpha J'_n} \ll 2^{-J_n}$ .

Thus we can choose a constant  $C$  such that with  $\rho_n = C 2^{-J_n}$  and  $B(X_0, \rho_n/2) \subset B((x_n, F(x_n)), \rho_n)$ . Following the proof of Proposition 4.1 and more precisely Equation (4.8) we have

$$\begin{aligned} \text{meas}((B(X, \rho_n/2) \cap \Omega^c)) &\leq \text{meas}(B((K_{J_n} 2^{-J_n}, F(K_{J_n} 2^{-J_n}), \rho_n) \cap \Omega^c)) \\ &\leq C \rho_n^{1+1/\alpha}. \end{aligned}$$

This yields  $E_{\Omega^c}^s(X) \geq \frac{1}{\alpha} - 1$ .

Since  $E_{\Omega^c}^s(X) \leq \frac{1}{\alpha} - 1$ , we get  $E_{\Omega^c}^s(X) = \frac{1}{\alpha} - 1$ .

2. We follow exactly the same proof as previously replacing  $x_n$  by  $\tilde{X}_n$  the sequence of local maxima defined in the proof of Point 4 of Lemma 3.7.

3. We follow here the results proved by [D] and summarized in [AB] for our special case. Indeed recall the definition given in [D] of an ubiquitous system in a real interval of  $\mathbf{R}$ .

**Definition 4.5.** Let  $U$  be a real open interval. Let  $(x_i)_{i \geq 1}$  be points in  $U$  and let  $(r_i)_{i \geq 1}$  be a sequence of positive real numbers such that  $\lim_{i \rightarrow \infty} r_i = 0$ . The family  $(x_i, r_i)_{i \geq 1}$  is a homogeneous ubiquitous system in  $U$  if the set  $\limsup_i B(x_i, r_i)$  is of full Lebesgue measure in  $U$ .

Theorem D of [AB] proved in [D] yields the following result.

**Theorem 4.6.** *Let  $\tau$  be a real number with  $\tau \geq 1$ . With the above notations if the families  $(x_i, r_i)_{i \geq 1}$  and  $(x'_i, r'_i)_{i \geq 1}$  are two homogeneous ubiquitous systems in  $U$ , then the Hausdorff dimension of the set  $\limsup B(x_i, r_i^\tau) \cap \limsup B(\tilde{x}_j, \tilde{r}_j^\tau)$  is at least equal to  $\frac{1}{\tau}$ .*

Let  $U = ]0, 1[$  and consider  $\mathcal{K}_1 = \{(\frac{k}{2^j}, 2^{-j}), k \in \mathbf{N}, 0 < k < 2^j, j \geq 1\}$ . It is a countable set and can be written as  $\mathcal{K}_1 = \{(x_i, r_i), i \geq 1\}$  with  $x_i$  a dyadic number for all  $i \geq 1$ . Let  $\mathcal{K}_2 = \{(x, r), x \in \mathcal{S}, r = \frac{2^{-j}}{3} \text{ for } j \geq 1\}$ . It is again a countable set and we can rewrite it as  $\mathcal{K}_2 = \{(\tilde{x}_i, \tilde{r}_i), i \geq 1\}$  with  $\tilde{x}_i \in \mathcal{S}$  for all  $i \geq 1$ .

It is clear that  $\limsup B(x_i, r_i)$  and  $\limsup B(\tilde{x}_i, \tilde{r}_i)$  are of full Lebesgue measure.

Remark then that  $D_\alpha = \limsup B(x_i, r_i^\tau) \cap \limsup B(\tilde{x}_i, \tilde{r}_i^\tau)$  with  $\tau = \frac{1}{\alpha}$ . Since  $D_\alpha \subset \bigcup_{j \geq J, 0 \leq k \leq 2^j} B(x_i, r_i^\tau)$  the Hausdorff dimension of  $D_\alpha$  is less or equal than  $\alpha$ . We apply Theorem 4.6 and we find it is exactly  $\alpha$ .

## 4.4 Proof of Theorem 2.1

Propositions 4.1, 4.2, 4.3 and 4.4 achieve the proof of Theorem 2.1.

## Acknowledgements

M. Ben Slimane was supported by the Research Center, College of Science, King Saud University.

## References

- [A] P. C. Allaart and K. Kawamura, *The Takagi function: a survey*, Real Anal. Exchange 37 (2011), no. 1, 1–54.
- [AB] M. Amou, Y. Bugeaud, *Exponents of Diophantine approximation and expansions in integer bases*, J. London Math. Soc. 81, no. 2 (2010), 297–316.
- [CZ] A. P. Calderón, A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. 20 (1961), 171–227.
- [C1] Z. Ciesielski, *On the isomorphisms of the spaces  $H$ , and  $M$* , Bull. Acad. Polon. Sci. Skr. Sci. Math. Astronom. Phys. 8 (1960), 217–222.
- [C2] Z. Ciesielski, *Fractal functions and Schauder bases*, Comput. Math. App. 30, no. 3-6 (1995), 283–291.
- [C3] Z. Ciesielski, *Spline orthogonal system and fractal functions*, Acta. Math. Hungar. 68, no. 4 (1995), 287–293.
- [DE] S. Dubuc, A. Elqortobi, *Le maximum de la fonction de Knopp*, INFOR 26, no. 4 (1990), 311–323.
- [DD] B. Dubuc, S. Dubuc, *Error bounds on the estimation of fractal dimension*, SIAM Journal on Numerical Analysis 33, no. 2 (1996), 602–626.
- [D] A. Durand, *Sets with large intersection and ubiquity*, Math. Proc. Cambridge Philos. Soc. 144 (2008), 119–144.
- [G] A. Gaudillièr, Personal communication.
- [H] Y. Heurteaux, Personal communication.
- [JH] Y. Heurteaux, S. Jaffard, *Multifractal analysis of images: new connexions between analysis and geometry*, Proceedings of the NATO-ASI Conference on Imaging for Detection and Identification, Springer, 2006.
- [JMe] S. Jaffard, C. Mélot, *Wavelet analysis of fractal boundaries. Part 1: local exponents*, Commun. Math. Phys. 258 (2005), 513–539.
- [JMa] S. Jaffard, B. Mandelbrot, *Local regularity of nonsmooth wavelet expansions and application to Polya’s function*, Adv. Math. 120, no. 2 (1996), 265–282.

- [J] S. Jaffard, *Oscillation spaces: Properties and applications to fractal and multifractal functions*, J. Math. Phys. 39 (1998), 4129–4141.
- [KW] A. Kamont, B. Wolnik, *Wavelet expansions and fractal dimensions*, Constr. Approx. 15 (1998), 97–108.
- [K] K. Knopp, *Ein einfaches verfahren zur bildng stetiger nirgends differenzierbarer funktionen*, Math. Z. 2 (1918), 1–26.
- [L] F. Ledrappier, *On the dimension of some graphs*, Contemp. Math. 135 (1992), 285–293.
- [S] B. Solomyak, *On the random series  $\sum \pm \lambda^n$  (an Erdős problem)*, Ann. Math. 142 (1995), 611–625.
- [T1] C. Tricot, *Curves and fractal dimension*, Springer, 1995.
- [T2] C. Tricot, *General Hausdorff functions, and the notion of one-sided measure and dimension*, Arkiv fr Matematik, 48 (2010), 149–176 .