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SEGRE NUMBERS, A GENERALIZED KING FORMULA, AND
LOCAL INTERSECTIONS

MATS ANDERSSON & HÅKAN SAMUELSSON KALM
& ELIZABETH WULCAN & ALAIN YGER

Abstract. Let $\mathcal{J}$ be an ideal sheaf on a reduced analytic space $X$ with zero set $Z$. We show that the Lelong numbers of the restrictions to $Z$ of certain generalized Monge-Ampère products $(dd^c \log |f|^2)^k$, where $f$ is a tuple of generators of $\mathcal{J}$, coincide with the so-called Segre numbers of $\mathcal{J}$, introduced independently by Tworzewski and Gaffney-Gassler. More generally we show that these currents satisfy a generalization of the classical King formula that takes into account fixed and moving components of Vogel cycles associated with $\mathcal{J}$. A basic tool is a new calculus for products of positive currents of Bochner-Martinelli type. We also discuss connections to intersection theory.

1. Introduction

Let $X$ be a reduced analytic space of pure dimension and let $\mathcal{J}$ be a coherent ideal sheaf on $X$. Given a point $x \in X$, Tworzewski, [25], Achilles–Manaresi, [1], and Gaffney–Gassler, [10], independently introduced numbers $e_0(\mathcal{J}, X, x), \ldots, e_n(\mathcal{J}, X, x)$ that in a certain sense measure the singularities of $\mathcal{J}$ at $x$ and that generalize the classical Hilbert-Samuel multiplicity. Following [10] we will call them Segre numbers; Tworzewski used the term extended index of intersection whereas Achilles-Manaresi used the term multiplicity sequence. The definition in [25] goes via a local variant of the Stückrad-Vogel construction, [24], introduced in [25, 18], and a closely related, also geometric, procedure is used in [10]. In [1] the definition is purely algebraic; it is based on Hilbert functions of bigraded rings. It is proved in [2] that the definitions in [25, 1, 10] yield the same numbers; see also [19]. In this paper we give a (semi-)global representation of these numbers as the Lelong numbers of certain positive closed currents, constructed from a tuple of generators of $\mathcal{J}$. This is part of our main result Theorem 1.1, which is a generalization of King’s formula for these currents.

Let us first recall the definition in [25]. In that paper $X$ is a subvariety of a smooth manifold $Y$ and $\mathcal{J}$ is the pullback to $X$ of the sheaf associated with a smooth submanifold $A \subset Y$. However, we find it advantageous to avoid any ambient space and instead consider an arbitrary coherent ideal sheaf $\mathcal{J} \to X$. A sequence $h = (h_1, h_2, \ldots, h_n)$ in the local ideal $\mathcal{J}_x$ is called a Vogel sequence of $\mathcal{J}$ at $x$ if there is a neighborhood $U \subset X$ of $x$ where the $h_j$ are defined, such that

\begin{equation}
\text{codim } [(U \setminus Z) \cap (|H_1| \cap \ldots \cap |H_k|)] = k \text{ or } \infty, \quad k = 1, \ldots, n;
\end{equation}

here $Z$ is the (reduced) zero set of $\mathcal{J}$ and $|H_\ell|$ are the supports of the divisors $H_\ell$ defined by the $h_\ell$. Notice that, possibly after shrinking $U$, the common zero set $\{h = 0\}$ in $U$ equals $Z \cap U$ and that if $f_0, \ldots, f_m$ generate $\mathcal{J}_x$, then any generic
sequence of $n$ linear combinations of the $f_j$ is a Vogel sequence at $x$. Let $X_0 = X$ and let $X_0^Z$ denote the irreducible components of $X_0$ that are contained in $Z$ and let $X_0^{X\setminus Z}$ be the remaining components\(^1\), so that

$$X_0 = X_0^Z + X_0^{X\setminus Z}.$$  

By the Vogel condition (1.1), $H_1$ intersects $X_0^{X\setminus Z}$ properly. Set

$$X_1 = H_1 \cdot X_0^{X\setminus Z} \tag{1.2}$$

and decompose analogously $X_1$ into the components $X_1^Z$ contained in $Z$ and the remaining components $X_1^{X\setminus Z}$, so that $X_1 = X_1^Z + X_1^{X\setminus Z}$. Define inductively $X_{k+1} = H_{k+1} \cdot X_k^{X\setminus Z} \cdot X_1^{X\setminus Z}$, and $X_{k+1}^{X\setminus Z}$. Then

$$V_h := X_0^Z + X_1^Z + \cdots + X_n^Z$$

is the Vogel cycle\(^2\) associated with the Vogel sequence $h$. Let $V_k^h$ denote the components of $V^h$ of codimension $k$, i.e., $V_k^h = X_k^Z$. Tworzewski defines the extended index of intersection as $\min_{\text{lex}}(\mult_x V_0^h, \ldots, \mult_x V_n^h)$, where the $\min_{\text{lex}}$ is taken over all Vogel sequences $h$ of $\mathcal{J}_x$.

Let us next recall the definition of Segre numbers in [10], where also so-called polar multiplicities are introduced. Let $f$ be a tuple of generators $f_0, \ldots, f_m$ of $\mathcal{J}_x$ and let $h$ be a Vogel sequence of linear combinations $h_j = \alpha_j \cdot f = \alpha_j^0 f_0 + \cdots + \alpha_j^m f_m$; notice that any Vogel sequence is on this form for some choice of generators and $\alpha_j$. It is proved in [10, Section 2], see also Section 6 below, that the multiplicities $\mult_x V_k^h$ and $\mult_x X_k^{X\setminus Z}$ are independent of $\alpha_j$ for generic choices of $\alpha_j$ and also independent of $f$, and these numbers are the Segre numbers, $e_k(x) = e_k(\mathcal{J}, X, x)$, and polar multiplicities, $m_k(x) = m_k(\mathcal{J}, X, x)$, respectively. Throughout we will use this definition of the list $e(x) = e(\mathcal{J}, X, x) = (e_0(x), \ldots, e_n(x))$ of Segre numbers. In Section 8 below we prove that the definitions in [25] and [10] coincide, i.e.,

$$e(x) = \min_{\text{lex}}(\mult_x V_0^h, \ldots, \mult_x V_n^h). \tag{1.3}$$

It is not clear to us whether this coincidence has been noticed in the literature before. In [2], both notions are discussed, and (1.3) follows for the restrictive class of sheaves considered in [25], but it is not explicitly stated. The coincidence also follows from [22, Theorem 3.3] in combination with [10, Lemma 2.2].

We remark that both definitions above are local. Indeed, the Vogel condition (1.1) as well as the genericity of $\alpha_j$ depends on $x$, cf., Remark 1.4. Also the algebraic definition in [1] is local.

Let $f$ be a tuple of generators of the ideal sheaf $\mathcal{J}$. For $k = 0, 1, 2, \ldots, n$ we consider the closed positive currents $M_k^f$ introduced in [4]. The current $M_k^f$ coincides with $1_Z(dd^c \log |f|^2)^k$, where $1_Z$ is the characteristic function for $Z$ and

$$dd^c \log |f|^2)^k := \lim_{\epsilon \to 0} (dd^c \log(|f|^2 + \epsilon))^k; \tag{1.4}$$

\(^1\)In [10], $X_0^Z$ is empty by assumption, but for us it is convenient not to exclude the possibility that $\mathcal{J}$ vanishes identically on some irreducible component of $X$.

\(^2\)The notion Vogel cycle was introduced by Massey [17, 18]. For a generic choice of Vogel sequence the associated Vogel cycle coincides with the Segre cycle introduced by Gaffney and Gassler, [10], see Lemma 2.2 in [10].
for \( k \leq \text{codim}\{f = 0\} \) it is well-known that the definition (1.4) coincides with the standard one. Notice that \((dd^c \log |f|^2)^0 = 1\) and hence, \(M_0^f = 1_Z\) is the current of integration over the components of \(X\) that are contained in \(Z\); in particular, it vanishes unless \(f \equiv 0\) on some irreducible component of \(X\). See Section 4 for other expressions for \(M_k^f\). Our main result is the following.

**Theorem 1.1** (Generalized King’s formula). Let \(X\) be a reduced analytic space of pure dimension \(n\) and let \(\mathcal{J}\) be a coherent ideal sheaf on \(X\) generated by a tuple \(f\) of holomorphic functions. Let \(Z\) be the zero set of \(\mathcal{J}\) and let \(Z_j^k\) be the distinguished varieties of \(\mathcal{J}\) of codimension \(k\). Then

\[
(1.5) \quad M_k^f = 1_Z (dd^c \log |f|^2)^k = \sum_j \beta_j^k [Z_j^k] + N_k^f =: S_k^f + N_k^f, \quad k = 0, \ldots, n,
\]

where the \(\beta_j^k\) are positive integers, the \(N_k^f\) are positive closed currents, the Lelong numbers \(\ell_x(N_k^f)\) are nonnegative integers that only depend on the integral closure class of \(\mathcal{J}\) at \(x\), and the set where \(\ell_x(N_k^f) \geq 1\) has codimension at least \(k + 1\). The Lelong number of \(M_k^f\) at \(x\) is the Segre number \(e_k(\mathcal{J}, X, x)\). The polar multiplicity \(m_k(\mathcal{J}, X, x)\) coincides with the Lelong number at \(x\) of the current \(1_{X \setminus Z}(dd^c \log |f|^2)^k\).

Here \([Z_j^k]\) denotes the Lelong current\(^3\), i.e., the current of integration, associated with the variety \(Z_j^k\). Recall that the integral closure of \(\mathcal{J}_x\) consists of all holomorphic germs \(\phi\) such that \(|\phi| \leq C|f|\) for some \(C > 0\) at \(x\). For the definition of the (Fulton-MacPherson) distinguished varieties \(Z_j^k\) at \(x\) associated to \(\mathcal{J}_x\), see Section 7 below. It turns out that \(S_k^f = \sum_k S_k^f = \sum_{jk} \beta_j^k [Z_j^k]\) is precisely the cycle that appears in all Vogel cycles obtained from generic (enough) Vogel sequences. It is called the fixed part in [10]. The remaining parts of the Vogel cycles vary with the Vogel sequence and are called the moving parts. Notice that (1.5) is the Siu decomposition, [21], of \(M_k^f\).

Note that, contrary to the previous local definitions of Segre numbers, Theorem 1.1, gives a (semi-)global representation of the Segre numbers

\[
(1.6) \quad e_k(\mathcal{J}, X, x) = \ell_x(M_k^f).
\]

Moreover, the \(M_k^f\) are obtained as limits of explicit expressions in generators of \(\mathcal{J}\).

It follows from Lemma 2.2 that \(M_k^f = 0\) if \(k < \text{codim}\ Z\) and that \(N_{\text{codim}\ Z}^f = 0\). The case \(k = \text{codim}\ Z\) of (1.5) is precisely the classical King formula.

**Remark 1.2.** If \(\mathcal{J}_x\) is generated by \(p < n\) functions \(f_0, \ldots, f_{p-1}\), we will see that \(M_k^f = 0\) for \(k > p\) and hence \(e_k = 0\) for \(k > p\). However, \(M_k^f\) may be non-vanishing if \(g\) is another, larger, set of generators. If, in addition, \(\text{codim}\ Z_x = p\), i.e., \(\mathcal{J}_x\) is a complete intersection, then \(e_p(\mathcal{J}, X, x)\) is the only non-zero entry in \(e(\mathcal{J}, X, x)\). This number is the classical intersection number of the proper intersection of the divisors of the \(p\) generators \(f_j\), see, e.g., [7]. \(\square\)

**Corollary 1.3.** If \(\mathcal{J}\) is the radical ideal of a variety \(Z\) of pure codimension \(p\), then \(M_p^f = [Z]\).

\(^3\)We will often identify a cycle with its corresponding Lelong current.
Remark 1.4. Assume that \( x \) is a point where \( n_k(\mathcal{J}, X, x) \geq 1 \) for some \( k \) and let \( V^h \) be a generic Vogel cycle so that \( \text{mult}_x V^h_k = e_k(\mathcal{J}, X, x) \). Then \( V^h_k = S^I_k + W \) where the moving part \( W \) is a positive cycle of codimension \( k \), such that \( \text{mult}_x W = n_k(\mathcal{J}, X, x) \). Since \( n_k(\mathcal{J}, X, y) \geq 1 \) only on a set of codimension \( \geq k + 1 \), at most points \( y \) on \( V^h_k \) we have that \( e_k(\mathcal{J}, X, y) = \text{mult}_y (S^I_k) \) and hence \( \text{mult}_y V^h_k > e_k(\mathcal{J}, X, y) \). As soon as there is a moving part at \( x \) it is thus impossible to find a Vogel cycle that represents the Segre numbers in a whole neighborhood of \( x \). □

A fundamental ingredient in this paper is a current calculus described in Sections 3 to 5. It gives an expedient analytic approach to Vogel cycles; for instance, it becomes a straightforward matter to form mean values of (the Lelong currents of) such cycles. The currents \( M^f_k \) are in fact such mean values, see Section 6; this is the intuitive idea behind Theorem 1.1. The current calculus is fundamental for the proof of Theorem 1.1, which is given in Section 7, and it makes it possible to provide a proof of (1.3) in our slightly more general setting of a general sheaf \( \mathcal{J} \) than what was considered in [25], see Section 8. Our current calculus is also useful for concrete computations of Segre numbers, see Section 11. In Section 9 we prove a certain invariance property of Segre numbers. The motivation in [25] for introducing these numbers was to develop a new intersection theory. In Section 10 we discuss some local aspects of connections to intersection theory. Our technique to form new currents by averaging Vogel cycles will be the starting point in a forthcoming paper where we will study a kind of global intersection products.

Remark 1.5. This paper is a shortened and slightly elaborated version of [5]; in that paper can be found, additionally, a discussion of global intersections in the sense of Tworzewski and various examples. □

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2. Preliminaries

Let us fix some notation. Throughout this paper \( X \) is a reduced analytic space of pure dimension \( n \) and \( \mathcal{J} \) is a coherent ideal sheaf on \( X \). Given a tuple \( f = (f_0, \ldots, f_m) \) of holomorphic functions on an analytic space we will use \( \mathcal{J}(f) \) to denote the sheaf it generates. Similarly if \( W \subset X \) is an analytic subset we will use \( \mathcal{J}_W \) to denote the radical sheaf. We will denote the local ring of germs of holomorphic functions at \( x \) in \( X \) by \( \mathcal{O}_{X,x} \). We say that a sequence \( g_1, \ldots, g_m \) of functions on an analytic space \( X \) is a geometrically regular sequence if \( \text{codim} \{ g_1 = \ldots = g_k = 0 \} = k \) for \( 1 \leq k \leq m \). If \( X \) is smooth (or Cohen-Macaulay) a sequence is geometrically regular if and only if it is regular.

Though less natural at first sight it is often computationally more convenient to use regularizations based on analytic continuation rather than smooth regularizations as in (1.4). For instance, if \( h \) is a holomorphic function on \( X \) then

\[
\lambda \mapsto \bar{\partial} |h|^{2\lambda} \wedge \frac{\partial \log |h|^2}{2\pi i},
\]

a priori defined for \( \text{Re} \lambda \gg 0 \), has a current-valued analytic continuation to a neighborhood of 0 and the value at 0 is the integration current associated to the divisor
defined by $h$. In general, if $\alpha(\lambda)$ is a current-valued function, defined in a neighborhood of 0, we let $\alpha(\lambda)|_{\lambda=0}$ denote the value at $\lambda = 0$.

2.1. **Positive currents.** Let $d^c = (4\pi i)^{-1}(\partial - \bar{\partial})$ so that $dd^c = (2\pi i)^{-1}\bar{\partial}\partial$. We briefly recall some basic facts about positive currents, referring to [7, 8] for details. Let $\mu$ be a positive current of bidegree $(k,k)$ defined in some open set $\Omega \subset \mathbb{C}^N$. Then $\mu$ has order zero, so that the restriction $1_S\mu$ is well-defined for any Borel set $S \subset \Omega$. If in addition $\mu$ is closed and $S$ is analytic, then the Skoda-El Mir theorem asserts that $1_S\mu$ is closed as well. If $\mu$ is closed then one can define inductively

$$
(dd^c\log |z-x|^2)^{j+1}\wedge \mu = dd^c\left(\log |z-x|^2dd^c((\log |z-x|^2)^j\wedge \mu)\right),
$$

If in addition $\mu$ is closed and $S$ is analytic, then the right hand side of (2.2) shall be interpreted as

$$
\int (dd^c\log |z-x|^2)^{N-k}\wedge \mu \wedge \xi = \lim_{\lambda \to 0^+} \int \frac{|z|^{2\lambda} - 1}{\lambda} (dd^c\log |z|^2)^{N-k-1}\wedge \mu \wedge dd^c\xi.
$$

After an integration by parts, the right-hand side of (2.3) may be rewritten as

$$
\lim_{\lambda \to 0^+} \int \frac{\partial |z|^{2\lambda}}{2\pi i |z|^2} \wedge (dd^c\log |z|^2)^{N-k-1}\wedge \mu \wedge \xi
$$

The second term is precisely the action of $1_{\mathbb{C}^N\setminus \{0\}}(dd^c\log |z|^2)^{N-k}\wedge \mu$ on $\xi$, and consequently the point mass at 0 of $(dd^c\log |z|^2)^{N-k}\wedge \mu$ is the same as the point mass at 0 of the first term, which proves (2.2). \hfill \Box

---

4By slight abuse of notation we will write $[0]$, instead of the formally more correct $\{0\}$, to denote point evaluation at 0.
2.2. Currents on an analytic space.\footnote{For a more detailed exposition of currents on an analytic space we refer to \cite[Section 4.2]{13}.} Let $X$ be a reduced analytic space of pure dimension $n$. Given a local embedding $i: X \to \mathbb{C}^N$, we let $E_X$ be the sheaf of smooth forms on $X$, obtained from the sheaf of smooth forms in the ambient space, where two forms are identified if their pullbacks to $X_{\text{reg}}$ coincide; it is well-known that this definition does not depend on the particular embedding. We say that $\mu$ is a \textit{current} on $X$ of \textit{bidegree} $(p, q)$ if it acts on test forms on $X$ of bidegree $(n - p, n - q)$. Such currents $\mu$ are naturally identified with currents $\tau = i_*\mu$ of bidegree $(N - n + p, N - n + q)$ in the ambient space such that $\tau$ vanish on the kernel of $i^*$. Observe that the $d$-operator is well-defined on currents on $X$. If $W$ is a subvariety of $X$ of pure codimension $p \geq 0$, then

$$\phi \mapsto [W]\phi = \int_{W_{\text{reg}}} \phi$$

is a closed $(p, p)$-current on $X$; this is the \textit{current of integration} over $W$.

Recall that a current $\nu$ is \textit{normal} if both $\nu$ and $d\nu$ have order zero. The following lemma follows immediately from the corresponding one in $\mathbb{C}^N$.

\textbf{Lemma 2.2.} Suppose that $\mu$ is a normal current of bidegree $(p, p)$ on $X$ that has support on a subvariety $W$ of codimension $k$. If $k > p$ then $\mu = 0$. If $k = p$ and $\mu$ is closed, then $\mu = \sum_j \alpha_j[W_j]$ for some numbers $\alpha_j$, where $W_j$ are the irreducible components of $W$ of codimension $p$.

It is readily checked that if we have a proper holomorphic mapping $\nu: X' \to X$ between analytic spaces, then the push-forward $\nu_*$ is well-defined on currents on $X'$. Assume that $\mu$ is a positive closed current on the analytic space $X$. Fix $x \in X$ and let $i: X \to \mathbb{C}^N$ be a local embedding. We define the Lelong number $\ell_x(\mu)$ as $\ell_x(i_*\mu)$. After a suitable change of coordinates $i$ can be factorized as $i = j \circ i'$, where $i': X \to \mathbb{C}^M$ is a minimal embedding and $j$ is the natural embedding $\mathbb{C}^M \to \mathbb{C}^M \times \mathbb{C}^{N-M}$. Since the Lelong number is invariant under holomorphic changes of coordinates, all minimal embeddings are equal up to a holomorphic change of variables, and $\ell_x(\tau) = \ell_x(j_*\tau)$, it follows that $\ell_x(\mu)$ is well-defined. Thus if $Z$ is a subvariety of an analytic space $X$ and we have an embedding $X \to \mathbb{C}^N$, then the number $\ell_x([Z])$ is independent of whether we consider $[Z]$ as the Lelong current of $Z$ on $X$ or on $\mathbb{C}^N$.

Recall that if $Z$ is a variety in $\mathbb{C}^N$, then the \textit{multiplicity} $\text{mult}_x Z$ of $Z$ at $x$ coincides with the Lelong number $\ell_x([Z])$, see \cite[Prop. 3.15.1.2]{7}; here $\text{mult}_x Z$ is defined as in \cite[Ch. 2.11.1]{7}. In particular, the Lelong number of the function 1, considered as a current on an analytic space $X$, at $x$ is precisely $\text{mult}_x X$.

The classical \textit{Siu decomposition}, \cite{21}, of positive closed currents extends immediately to currents on our analytic space $X$. Let $\mu$ be a positive closed $(p, p)$-current on $X$; then there is a unique decomposition

$$\mu = \sum_i \beta_i[W_i] + N,$$

where $W_i$ are irreducible analytic varieties of codimension $p$, $\beta_i \geq 0$, and, for each $\delta > 0$, the set where $\ell_x(N) \geq \delta$ is analytic and has codimension strictly larger than $p$. 

2.3. Cycles and Lelong currents. Given an analytic cycle $Z = \sum \alpha_j W_j$, where $W_j$ are varieties, we let $[Z] = \sum \alpha_j [W_j]$ be the associated Lelong current. We will often identify analytic cycles with their Lelong currents. We let $|Z|$ denote the support of $Z$, and we let $1_Z$ mean $1_{[Z]}$. If $H$ is a Cartier divisor defined by (a germ of) a holomorphic function $h$, we will (sometimes) use the notation $[h]$ for $[H]$ and $1_h$ for $1_{[H]}$. Given an analytic cycle $Z = \sum \alpha_i W_i$ of pure dimension, the multiplicity of $Z$ at $x$ is defined as $\sum \alpha_i \operatorname{mult}_x W_i$ (this definition follows [10, p. 704]). It follows that
\[
\operatorname{mult}_x Z = \ell_x([Z]).
\]
If $Z = \sum_{k=0}^n Z_k$, where $Z_k$ is an analytic cycle of codimension $k$ we define
\[
(2.4) \quad \operatorname{mult}_x Z := (\operatorname{mult}_x Z_0, \ldots, \operatorname{mult}_x Z_n)
\]
Throughout this paper all analytic cycles are effective, unless otherwise stated.

2.4. Proper intersections. Let $Y$ be a complex manifold and let $Z_1, \ldots, Z_r$ be (effective) analytic cycles in $Y$ of pure codimensions $p_j$, $j = 1, \ldots, r$, that intersect properly, i.e., the intersection $V$ of their supports has codimension $p_1 + \cdots + p_r$. There is a well-defined cycle, called the (proper) intersection of the $Z_j$,
\[
(2.5) \quad Z_r \cdots Z_1 = \sum m_j V_j,
\]
where $V_j$ are the irreducible components of $V$ and $m_j$ are certain positive integers. One can obtain these numbers $m_j$ by defining the intersection number $i(x)$, algebraically or geometrically, at each fixed point $x$ of $V$, and prove that $i(x)$ is generically constant on each $V_j$, see, e.g., [7]. However, by means of currents, (2.5) can be obtained in a more direct way: By an appropriate regularization one can define the wedge product $[Z_r] \wedge \cdots \wedge [Z_1]$, see, e.g., [7, 8], and this current indeed coincides with the Lelong current of $Z_r \cdots Z_1$. In particular, if the $Z_j$ are (effective) divisors defined by holomorphic functions $h_j$, then the Lelong current of the intersection can be obtained explicitly as
\[
(2.6) \quad [Z_r \cdots Z_1] = \lim_{\epsilon \to 0} \int \frac{dd^c}{\epsilon} \log(|h_j|^2 + \epsilon).
\]
At each point $x$ there is a well-defined intersection number
\[
\epsilon(x) := \sum_j m_j \operatorname{mult}_x (V_j);
\]
here $\operatorname{mult}_x (V_j)$ is the multiplicity of the variety $V_j$ at $x$. The number $\epsilon(x)$ is precisely equal to the Lelong number $\ell_x([Z_r \cdots Z_1])$ of the positive closed current $[Z_r \cdots Z_1]$.

3. Multiplying a Lelong current by a Cartier divisor

In this section we will describe how the inductive construction of a Vogel cycle $V^h$ can be expediently expressed as certain products of Lelong currents. Notice that the map $[W] \mapsto 1_Z W$ is linear. Notice also that if $Z, Z'$ are analytic cycles in $X$, then
\[
(3.1) \quad 1_{Z'}[Z] = [Z Z'];
\]
recall that $Z Z'$ denotes the irreducible components of $Z$ that are contained in $Z'$. To see (3.1) we may assume that $Z$ is irreducible. If $|Z|$ is contained in $|Z'|$, then $1_{Z'}[Z] = [Z]$. Otherwise, $|Z| \cap |Z'|$ has higher codimension than $|Z|$, and thus $1_{Z'}[Z]$ vanishes by Lemma 2.2. Notice that $1_Z$ is 1 on the components of $X$ that are contained in $Z$ and 0 otherwise, i.e., it is the Lelong current of $X^Z$. 
If $h$ is a non-vanishing holomorphic function on (each irreducible component of) the analytic space $Z$, then $\log |h|^2$ is a well-defined $(0,0)$-current on $Z$. This is clear if $Z$ is smooth and follows in general, e.g., by means of a smooth resolution $\tilde{Z} \to Z$, cf., the proof below.

**Lemma 3.1.** Let $Z$ be an analytic cycle in $X$, $h$ be a holomorphic function, and let $u$ be a non-vanishing smooth function on $X$. Then

\[(3.2) \quad \lambda \mapsto \partial|uh|^{2\lambda} - \lambda \log |uh|^2 \omega(Z), \]

a priori defined when $Re \lambda$ is large, has an analytic continuation to a half-plane $Re \lambda > -\epsilon$, where $\epsilon > 0$. The value at $\lambda = 0$ is independent of $u$.

If $h$ does not vanish identically on any irreducible component of (the support of) $Z$, then this value is equal to $dd^c (\log |h|^2 [Z])$.

Notice that $v^\lambda := \partial|uh|^{2\lambda} - \lambda \log |uh|^2 / (2\pi i)$ has continuous coefficients when $Re \lambda > 1$, so the product in (3.2) is then well-defined.

**Proof.** First assume that $Z = X = \mathbb{C}^N$ and $h$ is a monomial $h = z_1^{a_1} \cdots z_N^{a_N}$. Then (3.2) is equal to

$$v^\lambda = \partial|uz_1^{a_1} \cdots z_N^{a_N}|^{2\lambda} - \lambda \sum_1^N a_j \frac{dz_j}{z_j} + \lambda \log |u|^2.$$ 

One can check that the desired analytic continuation exists, and that the value at $\lambda = 0$ is the current $\sum_1^N a_j [z_j] = d\bar{d} \log |h|^2$; in particular, it is independent of $u$.

Consider now the general case. By linearity, we may assume that $Z$ is irreducible. If $h$ vanishes identically on $Z$ and $Re \lambda$ is large, then $v^\lambda \wedge [Z] = 0$, and thus it trivially extends to $\lambda \in \mathbb{C}$. Assume that $h$ does not vanish identically on $Z$. Let $i: Z \hookrightarrow X$ be an embedding and let $\pi: \tilde{Z} \to Z$ be a smooth modification of $Z$ such that $\pi*i^*h$ is locally a monomial; such a modification exists due to Hironaka’s theorem on resolution of singularities. After a partition of unity we are back to the case above. It follows that $\pi*i^*v^\lambda$ has an analytic continuation to $Re \lambda > -\epsilon$ for some $\epsilon > 0$ and thus $v^\lambda \wedge [Z] = i_*\pi_*(\pi*i^*v^\lambda)$ has the desired analytic continuation. The value at $\lambda = 0$ is equal to

$$i_*\pi_*(d\bar{d} \log |\pi*i^*h|^2)$$

which proves the second statement, since $\log |h|^2 [Z] = i_*\pi_*(\log |\pi*i^*h|^2).$ \hfill \Box

Let $H$ denote the Cartier divisor defined by $h$. We define $[H] \wedge [Z]$ as the value of (3.2) at $\lambda = 0$. According to the lemma it does not depend on the particular choice of $h$ defining $H$. It follows from the definition that

\[(3.3) \quad [H] \wedge ([Z_1] + [Z_2]) = [H] \wedge [Z_1] + [H] \wedge [Z_2] \]

and thus $[Z] \mapsto [H] \wedge [Z]$ is a linear operator on Lelong currents, cf., (3.1). However, in general it is not true that $([H_1] + [H_2]) \wedge [Z] = [H_1] \wedge [Z] + [H_2] \wedge [Z]$ or $[H_1] \wedge [H_2] = [H_2] \wedge [H_1]$.

Since (3.2) is analytic by Lemma 3.1 it follows that $[H] \wedge 1_H[Z] = 0$, and so, using (3.3), we get

$$[H] \wedge [Z] = [H] \wedge (1_H[Z] + 1_{X \setminus H}[Z]) = [H] \wedge 1_{X \setminus H}[Z].$$
Proposition 3.4. Let \( 1_{X \setminus H}[Z] \) be the current of integration over \( Z^{X \setminus H} \), i.e., the components of \( Z \) that are not contained in \( |H| \). Since \( H \) and \( Z^{X \setminus H} \) intersect properly it follows that \( [H] \wedge 1_{X \setminus H}[Z] = [H \cdot Z^{X \setminus H}] \), cf. Section 2.4. Summarizing, we get the computation rules
\[
[H] \wedge [Z] = [H] \wedge 1_{X \setminus H}[Z] = [H \cdot Z^{X \setminus H}].
\]

Remark 3.2. It is important to emphasize that \( [H] \wedge [Z] \) is not the same as (the Lelong current associated with) the intersection \( H \cdot Z \) in [9]. In fact, if \( Z \) is irreducible and contained in \( H \), then \( [H] \wedge [Z] = 0 \), whereas in [9] the product is a cycle in \( Z \) of codimension 1 that is well-defined up to rational equivalence.

Example 3.3. Let \( H_1 \) and \( H_2 \) be Cartier divisors and let \( H = H_1 + H_2 \). Then \( [H_1] \wedge [H] = [H_1] \wedge [H_2] \) but \( [H] \wedge [H_1] = 0 \). Moreover \( [H_1] \wedge 1_{H_1}[H] = [H_1] \wedge [H_1] = 0 \) but \( 1_{H_1}[H_1] \wedge [H] = 1_{H_1}[H_1] \wedge [H_2] = [H_1] \wedge [H_2] \).

We can construct Vogel cycles, cf., Section 1, by inductively applying operators \( 1_Z [H] \wedge \).

**Proposition 3.4.** Let \( X \) be an analytic space of dimension \( n \) and let \( h = (h_1, \ldots, h_n) \) be a Vogel sequence of an ideal \( J \) with variety \( Z \) at \( x \in X \), with corresponding divisors \( H_1, \ldots, H_n \). Then on \( X \),
\[
[X_0] = 1, \quad [X_\ell] = [H_\ell] \wedge \cdots \wedge [H_1] \wedge [X], \quad \ell = 1, \ldots, n
\]
and
\[
[X_0^Z] = 1_Z, \quad [X_\ell^Z] = 1_Z [H_\ell] \wedge \cdots \wedge [H_1] \wedge [X], \quad \ell = 1, \ldots, n.
\]
In particular,
\[
[V^h] = 1_Z + 1_Z [H_1] + 1_Z [H_2] \wedge [H_1] + \cdots + 1_Z [H_n] \wedge \cdots \wedge [H_1].
\]
If we consider \( X \) as embedded in some larger analytic space \( Y \), then we have instead
\[
[X_0] = [X], \quad [X_\ell] = [H_\ell] \wedge \cdots \wedge [H_1] \wedge [X], \quad \ell = 1, \ldots, n
\]
and
\[
[X_0^Z] = 1_Z [X], \quad [X_\ell^Z] = 1_Z [H_\ell] \wedge \cdots \wedge [H_1] \wedge [X], \quad \ell = 1, \ldots, n.
\]

**Proof.** In view of (3.1), (3.6) follows from (3.5). Using (3.3), we have, in view of (1.2), that
\[
[X_1] = [H_1] \wedge [X_0^Z] = [H_1] \wedge ([X_0] - [X_0^Z]) = [H_1]
\]
since \( [H_1] \wedge [X_0^Z] = [H_1] 1_Z = 0 \). One obtains (3.5) by induction.

4. Bochner-Martinelli currents

Let \( f = (f_0, \ldots, f_m) \) be a tuple of holomorphic functions on \( X \) that generates \( J \) and let \( Z \) be the zero set of \( J \). For \( \text{Re} \lambda > 0 \), let
\[
M_0^f,\lambda := 1 - |f|^2 \lambda
\]
\[
M_k^f,\lambda := \partial |f|^2 \lambda \wedge \frac{1}{2\pi i} \partial \log |f|^2 \wedge (dd^c \log |f|^2)^{k-1} \text{ if } k \geq 1,
\]
and
\[
M^f,\lambda := \sum_{k=0}^{\infty} M_k^f,\lambda,
\]
where $|f|^2 = \sum_{j=0}^{m}|f_j|^2$. The sum in (4.1) is finite for degree reasons, and when $\Re \lambda > 0$, $M^{f,\lambda}$ is locally integrable. We will show that $\lambda \mapsto M^{f,\lambda}_k$ has a current-valued analytic continuation to $\Re \lambda > -\epsilon$, for some $\epsilon > 0$. We denote the value of $M^{f,\lambda}_k$ at $\lambda = 0$ by $M^{f}_k$ and we write $M^{f} := \sum_{k} M^{f}_k$. The current $M^{f}$ and its components $M^{f}_k$ will be referred to as Bochner-Martinelli currents, cf. Remark 4.2 below.

A computation yields that

$$M^{f,\lambda}_k = \frac{i}{2\pi} \frac{\partial |f|^2 \wedge \bar{\partial}|f|^2}{|f|^{4-2\lambda}} \wedge (dd^c \log |f|^2)^{k-1}$$

which is positive when $\lambda > 0$, and thus $M^{f}_k$ is a positive current. Note that $M^{f}_0$ is the current of integration over the components of $X$, on which $f \equiv 0$. In particular, if $f$ does not vanish identically on any component of $X$, then $M^{f}_0 = 0$.

Let $\pi: \tilde{X} \to X$ be a normal modification such that the pull-back ideal sheaf $\mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$ is principal; for instance one can take the normalization of the blow-up of $X$ along $\mathcal{J}$. Then $\pi^* f = f^0 f'$ where $f^0$ is a section of the holomorphic line bundle $L \to \tilde{X}$ corresponding to the exceptional divisor $D_f$ of $\pi$: $\tilde{X} \to X$, i.e., the divisor defined by $\mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$, and $f'$ is a nonvanishing tuple of sections of $L^{-1}$. Let $L$ be equipped with the metric defined by $|f_0|^2 = |\pi^* f| = |f^0 f'|$, and let

$$(4.2) \quad \omega_f := dd^c \log |f'|^2;$$

here the right hand side is computed locally for any local trivialization of $L^{-1}$. Then $-\omega_f$ is the first Chern form of $(L, |\cdot|_L)$, and clearly $\omega_f \geq 0$.

Since $\log |\pi^* f|^2 = \log |f^0|^2 + \log |f'|^2$ it follows from the Poincare-Lelong formula that

$$(4.3) \quad dd^c \log |\pi^* f|^2 = [D_f] + \omega_f.$$ 

In particular, $\pi^*(dd^c \log |f|^2) = \omega_f$ outside $\pi^{-1}\{f = 0\}$. Therefore, for $\Re \lambda > 0$,

$$(4.4) \quad \pi^* M^{f,\lambda}_0 = 1 - |f^0 f'|^{2\lambda}$$

$$(4.5) \quad \pi^* M^{f,\lambda}_k = (2\pi i)^{-1} \bar{\partial}|f^0 f'|^{2\lambda} \wedge \partial \log |f^0 f'|^{2\lambda} \wedge \omega_f^{k-1}, \quad k \geq 1.$$ 

Now Lemma 3.1 asserts that $\lambda \mapsto \pi^* M^{f,\lambda}_k$ has an analytic continuation to $\Re \lambda > -\epsilon$ and since $M^{f,\lambda}_k = \pi_* \pi^* M^{f,\lambda}_k$ for $\Re \lambda \gg 0$, it follows that $\lambda \mapsto M^{f,\lambda}_k$ has the desired analytic continuation. Moreover

$$(4.6) \quad M^f_0 = M^{f,\lambda}_0 |_{\lambda=0} = \pi_* (\pi^* M^{f,\lambda}_0 |_{\lambda=0}) = \pi_* (1_{D_f}) = 1_{\{f=0\}}.$$ 

$$(4.7) \quad M^f_k = M^{f,\lambda}_k |_{\lambda=0} = \pi_* (\pi^* M^{f,\lambda}_k |_{\lambda=0}) = \pi_* ([D_f] \wedge \omega_f^{k-1}), \quad k \geq 1.$$ 

Following for example [4] one can check that for $k \geq 1$,

$$(4.8) \quad M^f_k = 1_Z (dd^c \log |f|^2)^k$$

and

$$1_X \setminus Z (dd^c \log |f|^2)^k = \pi_* (\omega_f^k).$$

It is not hard to see that $M^{f,\lambda}_k$ is locally integrable for $\Re \lambda > 0$ and that $M^{f,\lambda}_k \to M^f_k$ as measures when $\lambda \to 0^+$. 
Remark 4.1. For future reference, let $g$ be a tuple of holomorphic functions such that $|g| \sim |f|$, i.e., there exists $C \in \mathbb{R}$ such that $|f|/C \leq |g| \leq C|f|$, and let $\pi: \tilde{X} \to X$ be a normal modification such that both $\mathcal{J}(f) \cdot \mathcal{O}_X$ and $\mathcal{J}(g) \cdot \mathcal{O}_X$ are principal. Then $|f^0| \sim |g^0|$ and since $f'$ and $g'$ are non-vanishing it follows that $f^0$ and $g^0$ define the same divisor on $\tilde{X}$. Therefore the corresponding negative Chern forms $\omega_f$ and $\omega_g$ are $dd^c$-cohomologous, i.e., there is a global smooth function $\gamma$ such that $dd^c \gamma = \omega_f - \omega_g$. \hfill $\square$

By combining [4, Proposition 3.2] and [23, Corollary 4] it follows that

\begin{equation}
(4.9) \quad M^f_k = \lim_{\epsilon \to 0} \frac{\epsilon(dd^c|f|^2)^k}{(|f|^2 + \epsilon)^{k+1}}.
\end{equation}

Remark 4.2. Given a tuple $f$ as above, associated residue currents of Bochner-Martinelli type were introduced in [20]. Let $E$ be a trivial vector bundle with basis elements $e_0, \ldots, e_m$ and consider $f = f_0e_0 + \cdots + f_me_m$ as a section of the dual bundle $E^*$ with basis elements $e^*_j$. Following [3] one can define a residue current $R^f = R^f_0 + \cdots + R^f_n$, where $R^f_0$ is a current of bidegree $(0,k)$ with values in the exterior product $\Lambda^kE$, such that the coefficients in $R^f$ are precisely the currents in [20]. It is proved in [4] that

\begin{equation}
M^f_k = R^f \cdot (df)/(2\pi i)^k k!,
\end{equation}

where $\cdot$ denotes the natural contraction. For more details, see, e.g., [4]. \hfill $\square$

5. Products of Bochner-Martinelli currents

Given tuples $f_1, \ldots, f_r$ of holomorphic functions in $X$, we will give meaning to the product

\begin{equation}
(5.1) \quad M^{f_r} \wedge \cdots \wedge M^{f_1}
\end{equation}

of Bochner-Martinelli currents. The construction is recursive. Assume that $M^{f_r} \wedge \cdots \wedge M^{f_1}$ is defined; it follows from the proof of Proposition 5.2 that

\begin{equation}
(5.2) \quad \lambda \mapsto M^{f_{r+1}} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1}
\end{equation}

is holomorphic for $\text{Re} \lambda > -\epsilon$, where $\epsilon > 0$. Set

\begin{equation}
(5.3) \quad M^{f_{r+1}} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1} := M^{f_{r+1}} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1}|_{\lambda=0}.
\end{equation}

We define the products $M^{f_r}_{k_r} \wedge \cdots \wedge M^{f_1}_{k_1}$ in the analogous way so that

\begin{equation}
(5.4) \quad M^{f_r} \wedge \cdots \wedge M^{f_1} = \sum_{k_r, \ldots, k_1 \geq 0} M^{f_r}_{k_r} \wedge \cdots \wedge M^{f_1}_{k_1}.
\end{equation}

Notice that if the $f_j$ are single functions, then $M^{f_j} = 1_{f_j} + [f_j]$ and

\begin{equation}
(5.5) \quad M^{f_r} \wedge \cdots \wedge M^{f_1} = (1_{f_r} + [f_r]) \wedge \cdots \wedge (1_{f_1} + [f_1]);
\end{equation}

cf. Section 3.

Proposition 5.1. If $h = (h_1, \ldots, h_n)$ is a Vogel sequence of some ideal with zero set $Z$ at $x$, then

\begin{equation}
M^{h_n} \wedge \cdots \wedge M^{h_1} = [V^h].
\end{equation}
Proof. In light of Lemma 2.2,
\[ 1_{h_\ell} \cdot \cdots \cdot 1_{h_{k+1}} [h_k] \wedge \cdots \wedge [h_1] = 1_Z [h_k] \wedge \cdots \wedge [h_1]. \]
Thus, by (3.4), \([h_{k+1}] \wedge 1_{h_\ell} \cdot \cdots \cdot 1_{h_{k+1}} [h_k] \wedge \cdots \wedge [h_1] = 0.\) Hence, in view of (5.5),
\[ M^{h_n} \wedge \cdots \wedge M^{h_1} = \sum_{k=0}^{n} 1_{h_\ell} \cdot \cdots \cdot 1_{h_{k+1}} [h_k] \wedge \cdots \wedge [h_1] = \sum_{k=0}^{n} 1_Z [h_k] \wedge \cdots \wedge [h_1]: \]
where we have used that \([h_n] \wedge \cdots \wedge [h_1] \) has support on \(Z\). Now, Proposition 3.4 asserts that the right hand side of (5.6) is equal to \([V^h]\). \(\square\)

In contrast to the definition of products of residue currents of Bochner-Martinelli type introduced in [26], the recursively defined products (5.1) are not commutative in general, even if the tuples just consist of one single function. For instance, in \(\mathbb{C}_x \times y\), we have that \(M^{x,y} \wedge M^y = 0\), whereas \(M^y \wedge M^{x,y} = [0]\), cf., Example 3.3. Various approaches to recursively defined products of residue currents are investigated in [15].

Proposition 5.2. Let \(f_1, \ldots, f_r\) be tuples of holomorphic functions in \(X\), with common zero set \(Z = \{f_1 = \ldots = f_r = 0\}\). Then the current \(M^{f_r} \wedge \cdots \wedge M^{f_1}\), defined by (5.3), is positive and has support on \(Z\).

Let \(\pi: X \to X\) be a normal modification such that the sheaves \(\mathcal{J}(f_\ell) \cdot \mathcal{O}_X\) are principal for \(\ell = 1, \ldots, r\). As in Section 4, let \(D_{f_\ell}\) and \(\omega_{f_\ell}\) be the corresponding divisors and negative Chern forms, respectively. Then
\[ M^{f_r} \wedge \cdots \wedge M^{f_1} = \pi_* ([D_{f_\ell}] \wedge \cdots \wedge [D_{f_1}] \wedge \omega_{f_r}^{-1} \wedge \cdots \wedge \omega_{f_1}^{-1}), \]
where, if \(k_\ell = 0\), the factor \([D_{f_\ell}]\) shall be replaced by \(1_{D_{f_\ell}}\) and the factor \(\omega_{f_\ell}^{-1}\) shall be removed.

Assume that \(g_1, \ldots, g_r\) are tuples of holomorphic functions in \(X\) such that \(|g_\ell| \sim |f_\ell|\) for \(\ell = 1, \ldots, r\). Then there is a normal current \(T\) with support on \(Z\) such that
\[ dd^c T = M^{f_r} \wedge \cdots \wedge M^{f_1} - M^{g_r} \wedge \cdots \wedge M^{g_1}. \]

Proof. Iteratively using Lemma 3.1, the computation rules (3.4), and (4.4)–(4.7) we see that the desired analytic continuation of (5.2) exists and that (5.7) holds. It follows that \(M^{f_r} \wedge \cdots \wedge M^{f_1}\) has its support contained in \(\pi([D_{f_\ell}] \cap \cdots \cap [D_{f_1}]) = Z\). Moreover \(M^{f_r} \wedge \cdots \wedge M^{f_1}\) is the push-forward of a product of positive \((1,1)\)-currents and positive forms, and hence it is positive.

To prove the last part, it suffices to change one of the \(f_\ell\) to \(g_\ell\) with \(|g_\ell| \sim |f_\ell|\). First notice that then \(M^{f_\ell}_{\ell} = 1_{f_\ell} = 1_{g_\ell} = M^{g_\ell}_{\ell}\). Let us then assume that \(k_\ell \geq 1\), and that the modification \(\pi\) is chosen so that also \(\mathcal{J}(g_\ell) \cdot \mathcal{O}_X\) is principal. By Remark 4.1, there is a smooth global function \(\gamma\) on \(\tilde{X}\) such that \(\omega_{f_\ell} - \omega_{g_\ell} = dd^c \gamma\) and thus we can find a smooth global form \(w\) such that \(dd^c w = \omega_{f_\ell}^{-1} - \omega_{g_\ell}^{-1}\). Let
\[ T := \pi_* \left(\tau_\ell \wedge \cdots \wedge \tau_{\ell+1} \wedge [D_{f_\ell}] \wedge w \wedge \tau_{\ell-1} \wedge \cdots \wedge \tau_1\right), \]
where \(\tau_j = 1_{D_{f_j}}\) if \(k_j = 0\) and \(\tau_j = [D_{f_j}] \wedge \omega_{f_j}^{k_j-1}\) otherwise. Then \(T\) satisfies (5.8). Note that \(\tau_\ell \wedge \cdots \wedge \tau_{\ell+1} \wedge [D_{f_\ell}] \wedge w \wedge \tau_{\ell-1} \wedge \cdots \wedge \tau_1\) is normal, and since normality is preserved under push-forward, so is \(T\). \(\square\)
We also define products of Bochner-Martinelli currents and Lelong currents. If $f_1, \ldots, f_r$ are tuples of holomorphic functions in $X$ and $Z$ is an analytic subset of $X$, we define recursively $M^f \wedge [Z] := M^{f \wedge} \wedge [Z] |_{\lambda = 0}$, and
\[
M^{f_{k+1}} \wedge \cdots \wedge M^{f_1} \wedge [Z] := M^{f_{k+1} \wedge} \wedge M^{f_k} \wedge \cdots \wedge M^{f_1} \wedge [Z] |_{\lambda = 0}.
\]
By arguments as in the proof of Proposition 5.2 one can prove that the desired analytic continuations exist, and thus $M^{f_r} \wedge \cdots \wedge M^{f_1} \wedge [Z]$ is well-defined. It is readily checked that if $i : Z \hookrightarrow X$, then, for any $k_1, \ldots, k_r \in \mathbb{N},$
\[
(5.9) \quad M^{f_r} \wedge \cdots \wedge M^{f_1} \wedge [Z] = i_*[M^{f_r} \wedge \cdots \wedge M^{f_1}] .
\]
Moreover, if $Z = Z' + Z''$, $Z'' \subset \{f_j = 0\}$, and $k_j > 0$, then one checks that
\[
(5.10) \quad M^{f_r} \wedge \cdots \wedge M^{f_1} \wedge [Z] = M^{f_r} \wedge \cdots \wedge M^{f_1} \wedge [Z'],
\]
cf. (the first equality of) (3.4).

For future reference, note that if $f$ is a tuple of holomorphic functions on the analytic space $X$ then
\[
(5.11) \quad M^f = M^f 1_X = \sum_j M^f 1_{X_j},
\]
where $X_j$ are the irreducible components of $X$.

**Proposition 5.3.** Let $f_1, \ldots, f_r$ be tuples of holomorphic functions in $X$ and let $\xi$ be a tuple of holomorphic functions such that $\{\xi = 0\} = \{x\}$, where $x \in X$. Then
\[
(5.12) \quad M^{\xi} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1} = M^{\xi} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1} = \alpha[x],
\]
where $k = k_1 + \cdots + k_r$ and $\alpha$ is a non-negative integer. If $\xi$ generates the maximal ideal at $x \in X$, then $\alpha = \ell_x(M^{f_r} \wedge \cdots \wedge M^{f_1})$.

**Proof.** By Proposition 5.2, $M^{\xi} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1}$ is positive and has support at $x$, and thus by Lemma 2.2 it is of the form $\alpha[x]$ for some non-negative $\alpha$. Let $\pi : \tilde{X} \to X$ be a normal modification such that $\mathcal{J}(f_j) \cdot \mathcal{O}_{\tilde{X}}$ and $\mathcal{J}(\xi) \cdot \mathcal{O}_{\tilde{X}}$ are principal. Let us use the notation from Section 4. Then, from (5.7), we see that $\alpha$ is an intersection number and hence an integer.

Now assume that $\xi$ generates the maximal ideal at $x$ and that $i : X \hookrightarrow \mathbb{C}^N$ is a local embedding such that $i(x) = 0$, so that $i_*[x] = [0]$. By the second part of Proposition 5.2 we may assume that $f_j = i^*F_j$ and $\xi = i^*z$ for some tuples $F_j$ and the standard coordinate system $z = (z_1, \ldots, z_N)$ in $\mathbb{C}^N$. Then
\[
(5.13) \quad i_*(M^{\xi} \wedge M^{f_r} \wedge \cdots \wedge M^{f_1}) = M^z \wedge M^{F_r} \wedge \cdots \wedge M^{F_1} \wedge [X],
\]
cf. (5.9). By Lemma 2.1, the right hand side of (5.13) is precisely the Lelong number of $M^{F_r} \wedge \cdots \wedge M^{F_1} \wedge [X]$ at 0 in $\mathbb{C}^N$ times $|0|$.

**Proposition 5.4.** The Lelong number at $x$ of $M^{f_r} \wedge \cdots \wedge M^{f_1}$ is unchanged if we replace $f_j$ by $g_j$ such that $|f_j| \sim |g_j|$.

**Proof.** It follows from the second part of Proposition 5.2, applied to $f_1, \ldots, f_r, \xi$ since then $T$, which has bidegree $(n-1, n-1)$, must vanish by Lemma 2.2.

One can replace all the evaluations in the definition of the product by one single evaluation in the following way; for the proof see [6].
Proposition 5.5. Assume that $\mu_j$ are strictly positive integers such that $\mu_1 > \mu_2 > \ldots > \mu_r$. Then $\lambda \mapsto M_{k_{r}}^{f_{r}, \lambda_{r}} \wedge \cdots \wedge M_{k_{1}}^{f_{1}, \lambda_{1}}$ is holomorphic in a neighborhood of the half-axis $[0, \infty)$ in $\mathbb{C}$ and

$$M_{k_{r}}^{f_{r}, \lambda_{r}} \wedge \cdots \wedge M_{k_{1}}^{f_{1}, \lambda_{1}} \big|_{\lambda=0}.$$  

In view of Proposition 5.1 we get

Corollary 5.6. If $h_1, \ldots, h_n$ is a Vogel sequence of some ideal at some point $x$ and $\mu_j$ are as in Proposition 5.5, then the Lelong current of the associated Vogel cycle is given as the value at $\lambda = 0$ of the function

$$\lambda \mapsto \bigwedge_{k=1}^{n} M^{h_k, \lambda_k} = \bigwedge_{k=1}^{n} \left(1 - |h_k|^2 \lambda_k^2 + \partial \bar{\partial} |h_k|^2 \lambda_k \wedge \partial \log |h_k|^2 / 2\pi i\right).$$

6. Bochner-Martinelli currents and Vogel cycles

For a tuple $f = (f_0, \ldots, f_m)$ of holomorphic functions and $\beta = [\beta_0 : \ldots : \beta_m] \in \mathbb{P}^m$ we write $\beta \cdot f := \beta_0 f_0 + \cdots + \beta_m f_m$. Note that $M^{\beta \cdot f}$ only depends on $\beta \in \mathbb{P}^m$ and not on the choice of homogeneous coordinates. Our first result in this section relates Lelong numbers of Bochner-Martinelli currents to multiplicities of Vogel cycles.

Theorem 6.1. Let $f = (f_0, \ldots, f_m)$ be a tuple of holomorphic functions in $X$, pick $x \in X$, and let $Z = \{f = 0\}$. Then for $k \geq 0$, and a generic choice of $\alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{P}^m)^k$,

$$\ell_x \left(1_Z [\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f] \right) = \ell_x (M_k^f).$$

Here the current on the left hand side of (6.1) should be interpreted as $1_Z$ if $k = 0$.

Assume that $f = (f_0, \ldots, f_m)$ generates the ideal sheaf $\mathcal{J}$. Then $\alpha_1 \cdot f, \ldots, \alpha_n \cdot f$ is a Vogel sequence for a generic choice of $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{P}^m)^n$. By Theorem 6.1 and Proposition 3.4,

$$\text{mult}_x V^\alpha_{k \cdot f} = \ell_x (M_k^f)$$

and by Proposition 5.4 the right hand side only depends on $\mathcal{J}$ and not on the particular choice of generators $f$; in fact, it only depends on the integral closure of $\mathcal{J}$. Thus this gives an independent proof of Gaffney-Gassler’s result, [10, Section 2], that the multiplicities of Vogel cycles $V^h$ are independent of $h$ for generic $h$, which guarantees that the Segre numbers are well-defined. Also, (1.6) immediately follows from (6.2).

Proof. Choose a normal modification $\pi : \widetilde{X} \to X$ such that $\mathcal{J}(f) \cdot \mathcal{O}_{\widetilde{X}}$ is principal; we will use the notation from Section 4. Assume moreover that the pullback of the maximal ideal at $x$ is principal, and let $D_{\xi}$ and $\omega_{\xi}$ be the corresponding divisor and Chern form, obtained from a tuple $\xi$ that defines the maximal ideal at $x$.

Let $W$ be any irreducible subvariety of $\widetilde{X}$. Since $f'$ is nonvanishing on $W$ it follows that for $\beta$ outside a hypersurface in $\mathbb{P}^m$, the section $\beta \cdot f'$ is not vanishing identically on $W$. By induction it follows that there is a Zariski-open dense subset $A \subset (\mathbb{P}^m)^n$ such that for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$, the sequence $\alpha_1 \cdot f', \ldots, \alpha_n \cdot f'$ is a geometrically regular sequence on each component of $\widetilde{X}$, $|D_{f'}|$, $|D_{\xi}|$, and on the support of $|D_{\xi}/D_{f'}|$.
Since \( \pi^*(\alpha \cdot f) = f^0 \alpha \cdot f' \), we have that \([\alpha \cdot f] = \pi_*([D_f] + [\alpha \cdot f'])\) and if \( \alpha \in A \), in light of (3.4), thus
\[
[\alpha_2 \cdot f] \wedge [\alpha_1 \cdot f] = \pi_*([D_f] \wedge [\alpha_1 \cdot f'] + [\alpha_2 \cdot f'] \wedge [\alpha_1 \cdot f']).
\]
By induction,
\[
(6.3) \quad [\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f] = \pi_*([D_f] \wedge [\alpha_{k-1} \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f'] + [\alpha_k \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f']),
\]
and so
\[
(6.4) \quad 1_Z[\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f] = \pi_*([D_f] \wedge [\alpha_{k-1} \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f']).
\]
Here we have used that \( 1_{D_f} [\alpha_k \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f'] \) vanishes by Lemma 2.2, and that
\[
(6.5) \quad 1_Z (\pi_* \tau) = \pi_* (1_{D_f} \tau).
\]

For \( k = 0, 1, (6.1) \) follows from (4.6), (4.7) and (6.4); in fact, the currents in (6.1) coincide in these cases. Let us now assume that \( k \geq 2 \). We claim that there is a normal current \( A_k \) such that
\[
(6.6) \quad d\dd^c A_k = [D_f] \wedge \omega^{k-1}_f \wedge [D_f] \wedge \omega^{k-1}_f = [\alpha_{k-1} \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f'].
\]
For \( \ell = 1, \ldots, k \), \( \log [\alpha \cdot f']^2 \) defines a singular metric on \( L^{-1} \) with first Chern form \([\alpha \cdot f']\), cf., (4.2), and thus \([\alpha \cdot f']\) is \( d\dd^c\)-cohomologous to \( \omega_f \). More precisely, \( c_\ell := \log ([f']^2 / [\alpha \cdot f']^2) \) is a global current on \( \bar{X} \) and \( \omega_f - [\alpha \cdot f'] = d\dd^c c_\ell \). Now, let
\[
A_k := [D_f] \wedge \omega^{k-1}_f \wedge [D_f] \wedge \sum_{\ell=1}^{k-1} \omega^{k-\ell}_f \wedge c_\ell \wedge [\alpha_{\ell-1} \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f'].
\]
Then \( A_k \) is normal. Since \( \alpha \cdot f' \) does not vanish identically on any irreducible component of \( M^f \), \( d\dd^c A_k \) does not vanish identically on \( M^f \). Hence \( d\dd^c A_k \) follows from Lemma 3.1 and the discussion after the proof of that lemma that (6.6) holds. From (the proof of) Proposition 5.3 and (6.4) we get
\[
(6.7) \quad d\dd^c \pi_* (A_k) = (\ell_x(M^f_k) - \ell_x(1_Z [\alpha \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f]))[x].
\]
On the other hand, \( \pi_* A_k \) is a normal \((n-1, n-1)\)-current, and so since it has support at \( x \), it vanishes according to Lemma 2.2.

Our next result concerns mean values of Bochner-Martinelli currents. In particular, it says that \( M^f \) can be represented a mean value of Volgo cycles.

**Theorem 6.2.** Assume that \( f = (f_0, \ldots, f_m) \) is a tuple of holomorphic functions on \( X \). Then
\[
(6.8) \quad M^f_k = \int_{\alpha \in \{P^m\}^k} 1_Z [\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f]
\]
where \( Z = \{ f = 0 \} \). Moreover, if \( \nu \geq \min(m + 1, n + 1) \), then
\[
(6.9) \quad M^f = \int_{\alpha = (\alpha_1, \ldots, \alpha_n) \in \{P^m\}^\nu} M^{\alpha \cdot f} \wedge \cdots \wedge M^{\alpha_1 \cdot f}.
\]
For the proof we will use the following lemma which is a simple variant of Croft’s formula that should be well-known so we omit the proof, see also [5].
Lemma 6.3. If $\phi$ is a non-vanishing holomorphic $(m+1)$-tuple on $X$, then, in the sense of currents,

$$
\int_{\beta \in \mathbb{P}^m} [\beta \cdot \phi]d\sigma(\beta) = dd^c \log |\phi|^2,
$$

where $d\sigma$ is the normalized Fubini-Study metric.

Proof of Theorem 6.2. We use the same notation as in the proof of Theorem 6.1. In view of Lemma 6.3 and (4.2) we have that

$$
(6.10) \quad \int_{\beta \in \mathbb{P}^m} [\beta \cdot f^*]d\sigma(\beta) = \omega_f.
$$

Since all currents are positive we can apply Fubini’s theorem and get (6.8) from (6.4) by repeated use of (6.10), cf., (4.7).

We now prove (6.9). By (4.6) and (4.7),

$$
M^{\alpha \cdot f} = M_{0}^{\alpha \cdot f} + M_{1}^{\alpha \cdot f} = 1_{\alpha \cdot f} + [\alpha \cdot f].
$$

As in the proof of Proposition 5.1 we get, for generic $(\alpha_1, \ldots, \alpha_\nu) \in (\mathbb{P}^m)^\nu$, that

$$
M^{\alpha_{\nu} \cdot f} \wedge \cdots \wedge M^{\alpha_1 \cdot f} = \sum_{\nu} 1_Z [\alpha_j \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f] + 1_{X \setminus Z} [\alpha_{\nu} \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f].
$$

Moreover, it follows from (6.3) and (6.4) that

$$
1_{X \setminus Z} [\alpha_{\nu} \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f] = \pi_*(\omega_{\nu} \cdot f') \wedge \cdots \wedge (\omega_1 \cdot f').
$$

Hence, using (6.8) and Lemma 6.3, we conclude that

$$
\int_{(\alpha_1, \ldots, \alpha_\nu) \in (\mathbb{P}^m)^\nu} M^{\alpha_{\nu} \cdot f} \wedge \cdots \wedge M^{\alpha_1 \cdot f} = M^f + \pi_*(dd^c \log |f'|^2)^\nu = M^f;
$$

indeed, $(dd^c \log |f'|^2)^\nu = 0$ since $\nu \geq \min(m+1, n+1)$.

By arguments as in the proof of Theorem 6.2 one can check that

$$
(6.11) \quad \int_{\alpha \in (\mathbb{P}^m)^k} 1_{X \setminus Z} [\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f] = 1_{X \setminus Z} (dd^c \log |f'|^2)^k.
$$

7. Proof of the generalized King formula (Theorem 1.1)

Let $X$ and $\mathcal{J}$ be as in Theorem 1.1 and let $Z$ be the variety of $\mathcal{J}$. The (Fulton-MacPherson) distinguished varieties associated with $\mathcal{J}$ are defined in the following way, cf., [9]: Let $\nu: X^+ \to X$ be the normalization of the blow-up of $X$ along $\mathcal{J}$ and let $E$ be the exceptional divisor of $\nu$. Then $Z_\mathcal{J} \subset X$ is a distinguished variety if it is the image under $\nu$ of an irreducible component of $E$. Let $Z^k_\mathcal{J}$ be the distinguished varieties of codimension $k$. Also, we define the irreducible components of $X$ contained in $Z$ to be distinguished varieties (of codimension 0).

Let us first consider the case $k = 0$. By (5.11) we may assume that $X$ is irreducible. Then either $\mathcal{J} = (0)$ or $Z$ is a proper subvariety of $X$. In the first case $M_0^f = M_0^0 = 1_X$ and if $h$ is a Vogel sequence of $\mathcal{J}$, then necessarily $h = (0, \ldots, 0)$ and so $V^h = V_0^h = X$. In the second case $M_0^f = 0$ and if $h$ is a Vogel sequence of $\mathcal{J}$, then $V^h = X_0^h = 0$, since $X \not\subset Z$. It follows that Theorem 1.1 holds for $k = 0$.

Next, consider the case $k \geq 1$. Let $\pi: \tilde{X} \to X$ be a normal modification such that $\mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$ principal. We use the notation from Section 4, so that $M^f_k = \pi_*([D] \wedge \omega_{\mathcal{J}}^{k-1})$,
where $D = D_j$. Moreover, we let $D^k$ denote the components of $D$ that are mapped to sets of codimension $k$ in $X$. Note that $D = D^0 + \ldots + D^n$, if $p = \text{codim } Z$.

If $\ell > k$, then $\pi_*([D^\ell] \wedge \omega_j^{k-1})$ is a positive closed $(k, k)$-current with support on a variety of codimension $\ell > k$, and hence it must vanish in view of Lemma 2.2. Thus

\begin{equation}
M_f^\ell = S_k^f + N_k^f,
\end{equation}

where

\begin{equation}
S_k^f = \pi_*([D^k] \wedge \omega_j^{k-1}), \quad N_k^f = \pi_*\left( \sum_{\ell < k} [D^\ell] \wedge \omega_j^{k-1} \right).
\end{equation}

Note that $M_k^f = 0$ for $k < p$ and $N_k^f = 0$. We claim that (7.1) is the Siu decomposition of $M_k^f$, cf., Section 2.2. By Lemma 2.2, $S_k^f$ is the Lelong current of a cycle of codimension $k$, so it is enough to show that $N_k^f$ does not carry any mass on varieties of codimension $k$. Let $W \subset X$ be such a variety. By (6.5),

\begin{equation}
1_W \pi_*([D^\ell] \wedge \omega_j^{k-1}) = \sum_j \pi_* (1_{\pi^{-1}W} [D_j^f] \wedge \omega_j^{k-1}),
\end{equation}

where $D_j^f$ are the irreducible components of $D^\ell$. Then, since $\ell < k$, $\pi^{-1}(W)$ does not contain any component $D_j^f$, thus each term in the right hand side of (7.3) vanishes, and thus the claim follows.

Since (7.1) is the Siu decomposition of $M_k^f$, it follows that $S_k^f$ is independent of $\pi : \tilde{X} \to X$. If we take $\pi$ to be the normalization of the blow-up of $J$, we see that the $Z_j^k$ in (1.5) has to be among the distinguished varieties of $J$. By Proposition 5.3 (for $r = 1$), the Lelong number of $M_k^f$ is an integer at each point, and since the Lelong number of $N_k^f$ generically vanishes on each $Z_j^k$, we conclude that the $\beta_j^k$ and $\ell_s(N_k^f)$ are integers. That $\ell_s(N_k^f)$ is an integer can also be seen directly by copying the proof of Proposition 5.3. Moreover, cf., Proposition 5.4, $\beta_j^k$ and $\ell_s(N_k^f)$ only depend on the integral closure of $J$ at $x$.

We shall now see that the coefficients $\beta_j^k$ of the distinguished varieties are, in fact, $\geq 1$, following the proof of Corollary 5.4.19, in [16]. The blow-up $\pi_{\mathcal{J}} : \text{Bl}_{\mathcal{J}}X \to X$ of $X$ along $\mathcal{J}$ can be seen as the subvariety of $X \times \mathbb{P}^m$ defined by the equations $t_j f_k - t_k f_j = 0$, where $0 \leq j < k \leq m$. Moreover, the line bundle associated with the exceptional divisor is the pullback of $\mathcal{O}_{\mathbb{P}^m}(-1)$ from $\mathbb{P}^m$ to $\text{Bl}_{\mathcal{J}}X$, so $\omega_t = \alpha^d \log |t|^2$ represents minus its first Chern class. This form is strictly positive on the fibers of $\pi_{\mathcal{J}}$, and since the normalization $X^{+} \to \text{Bl}_{\mathcal{J}}X$ is a finite map, the pullback $\omega_t$ of $\omega_t$ to $X^{+}$ remains strictly positive on the fibers of $\nu : X^{+} \to X$ as well. Let $E_j$ be one of the irreducible component of the exceptional divisor of $\nu$. We conclude that $\nu_*([E_j] \wedge \omega_j^{k-1})$ is a positive integer times $[Z_j^k]$, where $Z_j^k := \nu(E_j)$. On the other hand, this current is unaffected if we replace $\omega$ by $\omega_j$ since these two forms are first Chern forms of the same line bundle. It follows that $\beta_j^k \geq 1$.

We saw in the discussion after Theorem 6.1 that $\ell_s(M_k^f)$ is equal to the $k$-th Segre number of $J$ at $x$. Next, we show that the fixed Vorgel components of $J$ are precisely the $S_k^f$. Fix a point $x \in X$. As in proof of Theorem 6.1 we can construct, for $k \geq 1$ and a generic $\alpha \in (\mathbb{P}^m)^n$, a normal current $A_k$ with support on $|D^k|$ such that

\begin{equation}
\text{dd}^c A_k = |D^k| \wedge ([\alpha_{k-1} \cdot f] \wedge \ldots \wedge [\alpha_1 \cdot f] - \omega_j^{k-1}).
\end{equation}
Now $\pi_\ast A_k$ is a normal $(k - 1, k - 1)$-current with support on $\bigcup_j Z_j^k$, and thus it vanishes by Lemma 2.2. It follows that $\pi_\ast ([D_k^j] \wedge [\alpha_{k-1} \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f']) = S^f_k$ and hence $S^f_k$ occurs in a generic Vogel cycle at $x$, meaning that $S^f_k$ is a fixed Vogel cycle. On the other hand, the cycles

\begin{equation}
\pi_\ast (\sum_{\ell < k} [D^\ell] \wedge [\alpha_{k-1} \cdot f'] \wedge \cdots \wedge [\alpha_1 \cdot f'])
\end{equation}

must be moving. Indeed, by (the proof of) Theorem 6.2, taking mean values of (7.4) over all $\alpha \in (\mathbb{P}^n)^k$, we get the current $N^f_k$, which carries no mass on any variety of codimension $k$, as seen above.

By arguments as in the proof of Theorem 6.1 one shows that for a generic choice of $\alpha \in (\mathbb{P}^n)^k$,

\begin{equation}
\ell_x (1_{X \setminus Z} [\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f]) = \ell_x (1_{X \setminus Z} (dd^c \log |f|^2)^k)
\end{equation}

cf. (6.11). However, it follows from Proposition 3.4 that the left hand side of (7.5) is equal to $m_k(x)$. This concludes the proof of Theorem 1.1.

**Remark 7.1.** One can see more directly that only the distinguished varieties occur in $S^f_k$ if $S^f_k$ is defined by (7.2) from an arbitrary normal modification $\pi: \tilde{X} \to X$. To begin with, $\pi$ factors over $\nu$, i.e., there exists a modification $\tilde{\nu}: \tilde{X} \to X^+$ such that $\pi = \nu \circ \tilde{\nu}$. If $\omega_+ \subseteq \tilde{X}$ is the form associated with $\mathcal{J} \cdot \mathcal{O}_{X^+}$ in $X^+$, then $\tilde{\nu}^* \omega_+ = \omega_f$.

Let $D^f_j$ be an irreducible component of the divisor $D_k$. Since $|D^f_j| \subseteq \pi^{-1}(Z)$, it follows that $\tilde{\nu}(|D^f_j|)$ is contained in one of the components $E_j$ of $E$ in $X^+$. If $\tilde{\nu}(|D^f_j|)$ has codimension $\leq 1$ in $E_j$, then $\tilde{\nu}_* [D^f_j] = (\tilde{\nu}_*[D^f_j]) \wedge \omega_+^{k-1}$ vanishes by Lemma 2.2. Hence $\pi_* [D^f_j] \wedge \omega_+^{k-1}$ vanishes unless $\tilde{\nu}(|D^f_j|) = E_j$, in which case $\pi(|D^f_j|)$ is a distinguished variety.

Assume that $f_0, \ldots, f_{p-1}$ is a regular sequence. From the theory for proper intersections we know that

$$
[f_{p-1}] \wedge \cdots \wedge [f_0] = \sum \beta_j [Z_j]
$$

where $Z_j$ are the irreducible components of $Z = \{f = 0\}$, and that the intersection only depends on the ideal generated by the $f_j$, cf., Remark 1.2. In particular, the right hand side is unaffected if we replace $f_j$ by $\alpha_j \cdot f$ for generic $\alpha_j$. From (6.8) we conclude that

\begin{equation}
M^f_p = [f_{p-1}] \wedge \cdots \wedge [f_0].
\end{equation}

**Proof of Corollary 1.3.** If $Z$ is smooth, then locally there are coordinates $(z, w)$ so that $Z = \{w_1 = \cdots = w_p = 0\}$. In view of (7.6) we have that

$$
M^w_p = [w_p] \wedge \cdots \wedge [w_1] = [Z],
$$

and hence $\ell_x (M^w_p) = 1$ for $x \in Z$. If $Z$ is reduced and $f$ generates the ideal $\mathcal{J}_Z$, therefore $\ell_x (M^f_p) = 1$ for all smooth points $x \in Z$ in view of Proposition 5.4. From Theorem 1.1 we know that $M^f_p = \sum \beta_j [Z_j]$. Since the smooth points are dense, we conclude that $\beta_j = 1$ for each $j$. □
8. The minimality property

Recall that the lexicographical order on $\mathbb{R}^N$ is a total order, defined by $(x_1, \ldots, x_N) \leq_{\text{lex}} (y_1, \ldots, y_N)$ if there is an $1 \leq \ell \leq N$ such that $x_i = y_i$ for $i \leq \ell$ and $x_\ell < y_\ell$. We let $\min_{\text{lex}}$ denote the minimum with respect to the lexicographical order. We will now give a proof that Tworzewski’s extended index of intersection coincides with the list of Segre numbers. More precisely we will prove:

Theorem 8.1. Let $\mathcal{J}$ be a coherent ideal sheaf on $X$ and let $e(x)$ be the list of associated Segre numbers at $x$. Then (1.3) holds, where the $\min_{\text{lex}}$ is taken over all Vogel sequences $h$ of ideals with the same integral closure as $\mathcal{J}_x$.

Moreover, if $f$ is a tuple of generators of $\mathcal{J}$ (or any ideal with the same integral closure as $\mathcal{J}$) then it follows that $\lim_j e_j = (\alpha_1 \cdot f, \ldots, \alpha_n \cdot f)$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{P}^n)^n$.

As mentioned in the introduction, Theorem 8.1 is known in the case when $\mathcal{J}$ is the pullback to $X$ of the radical sheaf of a smooth manifold $A$ in some ambient space. For the proof of Theorem 8.1 we will need the following result; if $Z$ is smooth this is precisely Theorem 3.4 in [25].

Proposition 8.2. Assume that $(W_j)_{j \in \mathbb{N}}$ and $W$ are subvarieties of $X$ of pure dimension such that $\lim_{j \to \infty} W_j = W$ as currents on $X$. Let $Z$ be a fixed subvariety of $X$, let $x$ be a fixed point in $Z$, and assume that

\[
\ell_x(1_Z[W]) \leq \ell_x(1_Z[W_j]).
\]

for all $j$. Then there is a neighborhood $U$ of $x$ in $X$, in which $\lim_{j \to \infty} (1_Z[W_j]) = 1_Z[W]$ and $\lim_{j \to \infty} (1_{X \setminus Z}[W_j]) = 1_{X \setminus Z}[W]$.

Proof. Since the currents $[W_j]$ are positive and locally uniformly bounded, so are the currents $1_Z[W_j]$. Thus, there is a subsequence $(1_Z[W_{j_k}])_{k \in \mathbb{N}}$ converging to a positive closed current with support on $W \cap Z$. By Lemma 2.2 this current is the integration current $[V]$ for some effective cycle $V$ (with possibly real coefficients). Since $[W_j] - 1_Z[W_j]$ is positive, so is $[W] - [V] = \lim_k ([W_{j_k}] - 1_Z[W_{j_k}])$, and since $[V] \subset [Z]$, it follows that

\[
[V] = 1_Z[V] \leq 1_Z[W].
\]

By (8.1) and semicontinuity, (2.1), we have that

\[
\ell_x(1_Z[W]) \leq \limsup_k (\ell_x(1_Z[W_{j_k}])) \leq \ell_x([V]) \leq \ell_x(1_Z[W]).
\]

Thus $\ell_x(1_Z[W]) = \ell_x([V])$, and combined with (8.2) and the fact that $V$ and $W$ are effective cycles, it follows that $[V] = 1_Z[W]$ in some neighborhood of $x$.

Since each subsequence of $(1_Z[W_j])_{j \in \mathbb{N}}$ has a subsequence that tends to $1_Z[W]$, it follows that $\lim_{j \to \infty} (1_Z[W_j]) = 1_Z[W]$. The last statement follows by complementarity. \qed

Proof of Theorem 8.1. Since each Vogel sequence $h$ can be realized as $\alpha \cdot f$ for some choice of $f$ and $\alpha$, it is easy to check that the first statement follows from the second one. Let $f$ be a tuple of generators of $\mathcal{J}$. By definition, $e(x) = \mu_x V^{\alpha \cdot f}$ for almost all $\alpha$, and thus it is enough to prove that $e(x) \leq \min_{\text{lex}} \min_{\text{lex}} \mu_x V^{\alpha \cdot f}$ if $\alpha \cdot f$ is a Vogel sequence.

Suppose that $e(x) \nleq_{\text{lex}} \min_{\text{lex}} \mu_x V^{\alpha \cdot f}$. Then there is an $r$ and a Vogel sequence $\alpha \cdot f$ such that $e_k(x) = \mu_x V^{\ell_k \cdot f}$ for $k \leq r - 1$ but $\mu_x V^{\ell_k \cdot f} < e_r(x)$. Since $\alpha \cdot f$
is a Vogel sequence of $\mathcal{J}$ for a generic choice of $\alpha$, we can choose $(\alpha_j^i)_{j \in \mathbb{N}}$ in $(\mathbb{P}^m)^n$ such that $(\alpha_j^i)_{j \in \mathbb{N}} \to \alpha$ and such that $\alpha_j^i \cdot f$ is a Vogel sequence of $\mathcal{J}$ for each $j$, and moreover, by Theorem 6.1, such that $\text{mult}_x V^{\alpha_j^i \cdot f} = e(x)$. It then follows that, for $k \leq r$,

$$\ell_x(1_Z[\alpha_k \cdot f] \land \cdots \land [\alpha_1 \cdot f]) \leq e_k(x) = \ell_x(1_Z[\alpha_k^i \cdot f] \land \cdots \land [\alpha_1^i \cdot f]).$$

We claim that

$$\lim_{j \to \infty} [\alpha_k^j \cdot f] \land \cdots \land [\alpha_1^j \cdot f] = [\alpha_k \cdot f] \land \cdots \land [\alpha_1 \cdot f]$$

for $k \leq r$. For instance by [7, Chapter 2, Corollary 12.3.4], (8.4) holds for $k = 1$. Assume now that it holds for $k < r$. Then by (8.3) and Proposition 8.2,

$$\lim_{j \to \infty} (1_X \land Z)[\alpha_k^j \cdot f] \land \cdots \land [\alpha_1^j \cdot f]) = 1_X \land Z[\alpha_k \cdot f] \land \cdots \land [\alpha_1 \cdot f].$$

Since $\alpha_k \cdot f$ and $\alpha \cdot f$ are Vogel sequences, the currents in (8.5) intersect properly with $[\alpha_k^j \cdot f]$ and $[\alpha_k^j \cdot f]$, respectively. In light of [7, Chapter 2, Corollary 12.3.4] or [25, Theorem 3.6], (8.4) holds for $k + 1$, and the claim follows by induction.

Proposition 8.2 and (8.3) imply that

$$\lim_{j \to \infty} (1_Z[\alpha_k^j \cdot f] \land \cdots \land [\alpha_1^j \cdot f]) = 1_Z[\alpha_r \cdot f] \land \cdots \land [\alpha_1 \cdot f].$$

By semicontinuity, (2.1), the Lelong number of the right hand side of (8.6) is greater than or equal to the Lelong number of $1_Z[\alpha_r \cdot f] \land \cdots \land [\alpha_1 \cdot f]$. Thus, $\text{mult}_x V^{\alpha \cdot f} \geq e_r(x)$, which gives a contradiction. Hence $\text{min}_x \text{mult}_x V^{\alpha \cdot f} = e(x)$. □

Given a positive closed current $v$, we define $\ell_x(v) := (\ell_{x0}, \ldots, \ell_{xn})$, where $\ell_{xk}$ denotes the Lelong number at $x$ of the component of $v$ of bidegree $(k, k)$. If $v$ and $w$ are positive and closed, we let $v \leq x w$ mean that $\ell_x(v) \leq \ell_x(w)$, and $v = x w$ means that $\ell_x(v) = \ell_x(w)$. Observe that if $h$ is a Vogel sequence of an ideal $\mathcal{J}_x$, then the zero sets of $h$ and $\mathcal{J}_x$ coincide. If $f_1, \ldots, f_s$ is Vogel sequence of an ideal $\mathcal{J}_x$, then in view of Theorems 1.1 and 8.1 and Proposition 5.1, $M^f \leq x M^{f_1} \land \cdots \land M^{f_s}$. In fact we have:

**Proposition 8.3.** Let $f_1, \ldots, f_s$ be a sequence of elements in $\mathcal{O}_{X,x}$ and let $f = (f_1, \ldots, f_s)$. Then

$$M^f \leq x M^{f_1} \land \cdots \land M^{f_s}.$$

**Proof.** Let $Z := \{f = 0\}$. In order to prove (8.7), we proceed by induction on the number $s$ of functions. Clearly (8.7) holds for $s = 1$, so assume that it holds for $s - 1$ instead of $s$. Let $f := (f_2, \ldots, f_s)$. By (5.11) we may assume that $X$ is irreducible and that $f_1$ does not vanish identically on $X$, so that $M^{f_1} = M_1^{f_1} = [f_1]$; otherwise $M^{f_1} = M_0^{f_1} = 1_X$ and $M^f = M^f$ and we are back in the case $s - 1$.

Let $[W] := [f_1]$, and let $i_W : W \hookrightarrow X$ be the irreducible components of $W^{X \setminus Z}$. Theorem 6.1 asserts that for a generic choice of $\alpha \in (\mathbb{P}^{s-2})^{n-1}$, $\alpha \cdot \tilde{f}$ is a Vogel sequence of $\tilde{\mathcal{J}}(i_W^* \tilde{f})$ and $M^{\alpha \cdot \tilde{f}} \land \cdots \land M^{\alpha \cdot \tilde{f}} = x M^f$ on each $W_j$, so that

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6 If $s = 2$, then $\tilde{f} = f_2$ and $\mathbb{P}^0$ should be interpreted as $\{1\}$. 

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Proposition 9.1. Let \( \mathcal{J} \) be the pullback to \( X \) of the radical sheaf of a smooth manifold in some ambient space this was proved in [22, Section 5]. By the induction hypothesis
\[
M^\alpha \cap [W^X Z] \leq_x M^\alpha \cap [W^X Z].
\]
In view of (5.9) and (5.10), since \([f_1] = [W^Z] + [W^X Z] \) and \( f \) vanishes on \( Z \), we get
\[
M^{\alpha_1} \cap [W^X Z] \leq_x M^\alpha \cap [W^X Z].
\]
For a generic choice of \( \alpha \), the sequence \( f_1, \alpha_1 \cdot f, \ldots, \alpha_{n-1} \cdot f \) is a Vogel sequence of \( \mathcal{J}(f) \). Thus, by Theorem 8.1,
\[
M^f \leq_x M^{\alpha_1} \cap [W^X Z].
\]
Combining (8.8) and (8.9), we get (8.7).

\[\square\]

9. An invariance property

We have the following invariance property of Segre numbers. In the setting when \( \mathcal{J} \) is the pullback to \( X \) of the radical sheaf of a smooth manifold in some ambient space this was proved in [22, Section 5].

Proposition 9.1. Let \( \mathcal{J} \) be an ideal sheaf on an analytic space \( X \), let \( \mathcal{J}' \) be the pullback of \( \mathcal{J} \) to \( X \times \mathbb{C} w \) under the projection \( (x, w) \mapsto x \), and let \( i : X \hookrightarrow X \times \mathbb{C} w \) be the embedding \( x \mapsto (x, 0) \). Then
\[
e_k(\mathcal{J}', X \times \{0\}, i(x)) = e_k(\mathcal{J}, X, x)
\]
and
\[
e_{k+1}(\mathcal{J}' + (w), X \times \mathbb{C} w, i(x)) = e_k(\mathcal{J}, X, x).
\]

For the proof we will use the following invariance of Bochner-Martinelli currents.

Lemma 9.2. Let \( f \) be a tuple of holomorphic functions on \( X \) and let \( i : X \hookrightarrow X \times \mathbb{C} w \) be the embedding \( x \mapsto (x, 0) \). Then \( M_k(f, w) = 0 \) and
\[
M_k(f, w) = i_* M_k f, \quad k \geq 0.
\]
Moreover, if \( W \subset X \) is an analytic variety,
\[
M_k(f, w) \cap [W \times \{0\}] = i_* (M_k f \cap [W]).
\]
If we consider \( X \) as embedded in some larger analytic space \( X' \) and \( i : X' \hookrightarrow X' \times \mathbb{C} w, x \mapsto (x, 0) \), then (9.3) reads
\[
M_k(f, w) \cap [X \times \mathbb{C} w] = i_* (M_k f \cap [X]).
\]
In particular, if \( f = 0 \),
\[
M_{k+1}^f \cap [X \times \mathbb{C} w] = i_* [X] = [X \times \{0\}].
\]

Proof. Let \( z \) be local coordinates on \( X \). Since \( (z, w) \mapsto (f(z), w) \) does not vanish identically on \( X \times \mathbb{C} w \), it follows that \( M_k^f(0, w) = 0 \).

Let us now prove (9.3). First consider the case when \( k = 0 \). By (5.11) we may assume that \( X \) is irreducible. Then either \( f \equiv 0 \) on \( X \) or the zero set of \( f \) has at least codimension 1 in \( X \). In the first case
\[
M_1^f(0, w) = M_1^w = [w] = i_* 1 = i_* M_0^f = i_* M_0^f.
\]
In the latter case the zero set of \((f, w)\) has at least codimension 2 on \(X \times C_w\), and and so both sides of (9.3) vanish by Lemma 2.2. Thus (9.3) holds for \(k = 0\).

Next let \(\pi: \tilde{X} \to X\) be a smooth modification such that \(\mathcal{J} \cdot \mathcal{O}_\tilde{X}\) is principal and moreover \(f^0\) is locally a monomial; use the notation from Section 4. Observe that then \(\pi \otimes \text{id}_w: \tilde{X} \times C_w \to X \times C_w\) is a smooth modification with the same properties. It follows that it is enough to prove (9.3) in case \(X\) is smooth, \(\mathcal{J} = \langle f^0 \rangle\) is principal and \(f^0\) is (in local coordinates) a monomial.

In light of Section 4 we thus have to show that

\[
(2\pi i)^{-1} \partial((|f|^2 + |w|^2)\lambda \partial \log(|f|^2 + |w|^2) \wedge (dd^c \log(|f|^2 + |w|^2))^k)
\]

is equal to \([f^0]^\wedge (dd^c \log |f'|^{k-1} \wedge |w|) = i_* M^f_k\) when \(\lambda = 0\). Indeed, at \(\lambda = 0\), (9.6) is equal to \(M^f_{k+1, w}\). Note that (9.6) is locally integrable for \(\text{Re} \lambda > 0\). Moreover, if \(\text{Re} \lambda < 1\), it is integrable in the \(w\)-direction and thus acts on forms that are just bounded in the \(w\)-direction. Since \(M^f_{k+1, w}\) is of order zero and \(\text{supp} M^f_{k+1, w} \subset \{w = 0\}\), it follows that to check the action of \(M^f_{k+1, w}\) on test forms, it is enough to consider forms \(\xi(z, w) = \tilde{\xi}(z)\), where \(\tilde{\xi}(z)\) is any test form in \(X\). However, after the (generically \(1 - 1\)) change of variables \(f^0 \omega = w\), so that \(|f|^2 + |w|^2 = |f|^2(|f|^2 + |w|^2)|\), the action of (9.6) on \(\xi\) is equal to

\[
(2\pi i)^{-1} \int_{z, \omega} \partial(|f|^2 \lambda \partial \log |f|^2 \wedge (dd^c \log(|f'|^2 + |w|^2))^k) \wedge \tilde{\xi}(z).
\]

Taking \(\lambda = 0\), we get

\[
\int_z |f|^2 \wedge \tilde{\xi}(z) \wedge \int_\omega (dd^c \log(|f'|^2 + |w|^2))^k.
\]

One can check that the inner integral in (9.7) is equal to \((dd^c \log |f'|^2)^{k-1}\), which proves (9.3). Finally we prove (9.4). Let \(j: W \rightarrow X\). Then, using (5.9),

\[
M^{\otimes 1} \wedge [W \times \{0\}] = i_{\cdot j_*} M_j^\otimes \wedge f^{\otimes 1} = i_{\cdot j_*} M_j^\otimes f = i_* M_{f} \wedge [W].
\]

\[\Box\]

**Proof of Proposition 9.1.** Since the pullback of \(\mathcal{J}'\) to \(X \simeq X \times \{0\}\) is just \(\mathcal{J}\), (9.1) should be clear. More formally: Let \(f\) be a tuple that defines the ideal sheaf \(\mathcal{J}\) in \(X\). Then \(f \otimes 1\) defines \(\mathcal{J}' \subset X \times C_w\) and

\[
e_k(\mathcal{J}', X \times \{0\}, x) = \ell_x(M_{k}^{\otimes 1} \wedge [X \times \{0\}]) = \ell_x(M_{k}^{f} \wedge [X]) = e_k(\mathcal{J}, X, x),
\]

where we have used (9.4) for the second equality. This proves (9.1).

To see (9.2) notice that \(f, w\) defines \(\mathcal{J} + (w)\) in \(X \times C_w\). Thus, using (5.9) and (9.3) we have

\[
e_{k+1}(\mathcal{J}' + (w), X \times C_w, (x, 0)) = \ell_x(M_{k+1}^{f, w} \wedge [X \times C_w]) = \ell_x(M_{k}^{f} \wedge [X]) = e_k(\mathcal{J}, X, x).
\]

\[\Box\]

10. **Local intersection numbers**

Tworzezowski’s original motivation for introducing the extended index of intersection was to understand intersection theory in the nonproper case. Let \(Z_1, \ldots, Z_r\) be subvarieties of a smooth manifold \(Y\) that do not necessarily intersect properly.
A standard procedure to define an intersection product $Z_1 \cdots Z_r$ is to give some reasonable meaning to the intersection

\[(10.1) \Delta \cdot Z_1 \times \cdots \times Z_r,\]

where $i : Y \cong \Delta \to Y \times \cdots \times Y$ is the diagonal in $Y \times \cdots \times Y$. In this way one is reduced to the case of two varieties one of which is smooth.

Now assume that $A, Z$ are subvarieties of $Y$, that $A$ is smooth, and (initially) that $Z$ has pure dimension. Let $\mathcal{J}_A$ denote the radical sheaf of $A$, and also, for simplicity, the pullback of $\mathcal{J}_A$ to $Z$. Following Tworzewski we define \textit{local intersection numbers}

\[(10.2) g_\ell(A, Z, x) := e_{\dim Z - \ell}(\mathcal{J}_A, Z, x)\]

at $x$. If $Z = \sum_j \alpha_j Z_j$, where the $Z_j$ are pure dimensional, we set $g_\ell(A, Z, x) := \sum_j \alpha_j g_\ell(A, Z_j, x)$. The change of indices is made so that $\ell$ corresponds to the generic multiplicity of components of \textit{dimension} $\ell$ of Vogel cycles. We will use the notation $A \circ Z$ for these lists of local intersection numbers, i.e.,

\[(10.3) A \circ Z(x) = (g_{\dim Z}(A, Z, x), \ldots, g_1(A, Z, x), g_0(A, Z, x))\]

The local \textit{multiplicities of intersection} of general varieties $Z_1, \ldots, Z_r$ are then

\[(10.4) \epsilon_\ell(Z_1, \ldots, Z_r; x) = g_\ell(\Delta, Z_1 \times \cdots \times Z_r, (x, \ldots, x)),\]

cf. [25, Section 6]. We will write

\[(10.5) Z_1 \circ \cdots \circ Z_r(x) = (\epsilon_\nu(Z_1, \ldots, Z_r; x), \ldots, \epsilon_1(Z_1, \ldots, Z_r; x), \epsilon_0(Z_1, \ldots, Z_r; x)),\]

where $\nu$ is the dimension of the set-theoretical intersection $Z_1 \cap \cdots \cap Z_r$.

Note that the product $Z_1 \circ \cdots \circ Z_r$ by definition is commutative. It is also independent of the manifold $Y$ in the following sense: If $i : Y \to \tilde{Y}$ is an embedding of $Y$ in a larger manifold $\tilde{Y}$, then

\[(10.6) Z_1 \circ \cdots \circ Z_r(x) = i(Z_1) \circ \cdots \circ i(Z_r)(i(x)).\]

This follows from Proposition 9.1, see also [22, Section 5].

The next result, which relates the two local intersections (10.3) and (10.5), we have not found in the literature.

\textbf{Proposition 10.1.} Assume that $A, Z$ are subvarieties of a manifold $Y$, and that $A$ is smooth. Then

\[A \circ Z = A \circ Z.\]

In particular, if $A$ and $B$ are smooth submanifolds of $Y$, since $A \circ B$ is commutative, it follows that $A \circ B = B \circ A$.

\textbf{Proof.} Fix $x$ in $Y$. We may assume, without loss of generality, that $Y = \mathbb{C}^N$. Choose local coordinates $z = (z', z'')$ on $\mathbb{C}^N$ so that $A = \{z' = 0\}$, and local coordinates $(z, w)$ on $\mathbb{C}^N \times \mathbb{C}^N$. We will show that

\[(10.7) \epsilon_j(A, Z; x) = e_{\dim A + \dim Z - j}(\mathcal{J}_A, Z \times A, x) = \ell_x(M_{\dim A + \dim Z - j}^z[Z \times A]),\]

cf. (5.9), coincides with

\[(10.8) g_j(A, Z, x) = e_{\dim Z - j}(\mathcal{J}_A, Z, x) = \ell_x(M_{\dim Z - j}^z[Z]).\]

Note that $M_{k}^{z''}[Z \times A] = M_k^{z',z'' \setminus \eta''}[Z \times \{w' = 0\}]$. Let $(z', z'', w', \eta'')$, where $\eta'' = z'' - w''$, be new coordinates on $\mathbb{C}^N \times \mathbb{C}^N$. Then (9.3) implies that $M_{k+\dim A}(Z \times \{w' = 0\}) = i_* M_{k}^{z',z'',w'} \hookrightarrow \mathbb{C}^{2N}_{z',z'',w'}$.  

Moreover, by (9.4), \( M^w_k \wedge [Z \times \{ w' = 0 \}] = j_* M^{z'}_k \wedge [Z] \), where \( j : \mathbb{C}^N_{z', z''} \rightarrow \mathbb{C}^{2N - \dim A} \). Hence
\[
M^{z'-w}_{\dim A + \dim Z - j} \wedge [Z \times A] = i_! j_* M^{z'}_{\dim Z - j} \wedge [Z]
\]
and thus (10.7) is equal to (10.8).

**Example 10.2.** If \( Z_j \) intersect properly, then also (10.1) is a proper intersection and it is well-known that \( Z_1 \cdots Z_r \) coincides with the intersection (10.1) (after identifying \( Y \simeq \Delta \)). It follows that
\[
\epsilon_\ell(Z_1, \ldots, Z_r; x) = \mult_x(Z_1 \cdots Z_r)
\]
for \( x \in |Z_1 \cdots Z_r| \) and \( \ell = \dim(Z_1 \cdots Z_r) \), and 0 otherwise, cf. [25, Theorem 6.5]. In particular, if \( Z \) is a subvariety of the smooth manifold \( A \), in view of (10.6),
\[
\sum_k \mult_x(Z_1 \cdots Z_r) = \sum_\ell \epsilon_\ell(Z_1, \ldots, Z_r; x)
\]
for each point \( x \), where the index \( k \) denotes the component of dimension \( k \). By definition this product respects the (sum of the) local intersection multiplicities, but it does not respect Bezout’s formula in general. For instance, the self-intersection in \( \mathbb{P}^2 \) of any smooth curve \( C \) is just \( C \) itself, and thus \( \deg(C \cdot C) = (\deg C)^2 \) unless \( C \) is a line.

In a forthcoming paper we will introduce, in the case \( Y = \mathbb{P}^n \), a global current that represents, at each point, the local intersection multiplicities, and respects Bezout’s formula, in a reasonable sense. It is obtained as the mean value of various Vogel sequences, based on global variants of the ideas in Section 6 above.

11. Examples

Let us start by some computations of Bochner-Martinelli currents and Segre numbers. Our first example illustrates that the currents \( M^f \) in general depend on the set of generators \( f \) although the Lelong numbers only depend on (the integral closure of) the ideal generated by \( f \), cf. Remark 1.2.

**Example 11.1.** Let us consider the primary ideal \((x)\) in \( \mathbb{C}^2_{x,y} \). We know from Corollary 1.3 and Remark 1.2, respectively, that \( M^1_x = [x] \) and that \( M^2_y = 0 \). Let us now consider the pair \((x, xy)\) of generators for the same ideal. We first consider the Vogel cycles obtained from generic linear combinations of these generators. Since \( Z = \{x = 0\} \) and \([\alpha_0 x + \alpha_1 xy] = [x(\alpha_0 + \alpha_1 y)] = [x] + [\alpha_0 + \alpha_1 y]\) we have that
\[
1_Z[\alpha_0 x + \alpha_1 xy] = [x]
\]
as expected, since \([x]\) must be a fixed component in any Vogel cycle. A simple computation yields that

\[
1_{Z}[\beta_0 x + \beta_1 xy] \cdot [\alpha_0 x + \alpha_1 xy] = [x] \cdot [\alpha_0 + \alpha_1 y]
\]

for generic choices of \(\alpha\) and \(\beta\), cf. Section 3. Thus the component of the Vogel cycle of codimension 2 is non-vanishing for generic \(\alpha, \beta\). Taking mean values over \(\mathbb{P}^1\) we get, cf., Theorem 6.2,

\[
M^x,xy = [x] \cdot \frac{dy \wedge d\bar{y}}{\pi(1 + |y|^2)^2}.
\]

Here we have used that with the generic parametrization \(\mathbb{C} \ni t \mapsto [-t, 1] \in \mathbb{P}^1\), we have

\[
\int_{[\alpha] \in \mathbb{P}^1} [\alpha_0 + \alpha_1 y] d\sigma(\alpha) = \int_{t \in \mathbb{C}} [y-t] \cdot \frac{dt \wedge dt}{\pi(1 + |t|^2)^2} = \frac{dy \wedge d\bar{y}}{\pi(1 + |y|^2)^2}.
\]

\[\square\]

Next we will discuss a simple example where a moving component occurs.

**Example 11.2.** Consider the tuple \(f = t_3(t_1, t_2, t_3) = t_3t\) in \(X = \mathbb{C}^3\), with zero set \(Z = \{t_3 = 0\}\). We will compute the Segre numbers \(\epsilon_k(0) = \epsilon_k(\mathcal{J}(f), C^3, 0), k = 0, 1, 2, 3\). Let \(\alpha \cdot f\) be a Vogel sequence of \(\mathcal{J}(f)\) at \(0\) of the form \(\alpha_1 \cdot f, \ldots, \alpha_3 \cdot f\). Let us compute the corresponding Vogel cycle \(V^\alpha f\). First note that \(X_0 = X_0 = X\). Thus, by Proposition 3.4,

\[
[X_1] = M^t_{t_3(\alpha_1 \cdot t)} = [t_3] + [\alpha_1 \cdot t] = [X^t_1] + [X_1^X \setminus Z].
\]

Furthermore, using (3.3) and (3.4), we get

\[
[X_2] = M^t_{t_3(\alpha_2 \cdot t)} \cap M^t_{t_3(\alpha_1 \cdot t)} = [t_3] \cdot [\alpha_1 \cdot t] + [\alpha_2 \cdot t] \cdot [\alpha_1 \cdot t] = [X^t_2] + [X_2^X \setminus Z]
\]

and

\[
[X_3] = M^t_{t_3(\alpha_3 \cdot t)} \cap M^t_{t_3(\alpha_2 \cdot t)} \cap M^t_{t_3(\alpha_1 \cdot t)} = ([t_3] + [\alpha_3 \cdot t]) \cdot [\alpha_2 \cdot t] \cdot [\alpha_1 \cdot t] = 2[0] = [X^t_3],
\]

for a generic \(\alpha\). Hence

\[
[V^h] = [V^h_1] + [V^h_2] + [V^h_3] = [t_3] + [t_3] \cdot [\alpha_1 \cdot t] + 2[0]
\]

and, in particular,

\[
e_0(0) = 0, \ e_1(0) = 1, \ e_2(0) = 1, \ e_3(0) = 2.
\]

Observe that \(V^h_1\) and \(V^h_3\) are fixed, whereas \(V^h_2\) is moving. A computation, using Theorem 6.2 and Lemma 6.3, yields

\[
M^t_0 = 0, \ M^t_f = [t_3], \ M^t_2 = [t_3] \cdot d\bar{c} \log(|t_1|^2 + |t_2|^2), \ M^t_3 = 2[0] .
\]

\[\square\]

The following simple lemma is useful for computations.

**Lemma 11.3.** Let \(X\) and \(X'\) be two analytic spaces of dimension \(n\), let \(\tau : X' \to X\) be a holomorphic map, and let \(f\) be a tuple of holomorphic functions on \(X\). Assume that \(\tau\) is proper, surjective, and generically \(r\) to 1. Then

\[
(11.1) \quad r M^f_k = \tau_* M^{r f}_k.
\]
Moreover, if \( \xi \) is a tuple that defines the maximal ideal at \( x \in X \), then the Segre numbers at \( x \) associated with \( \mathcal{J} = \mathcal{J}(f) \) on \( X \) are given by

\[
(11.2) \quad e_k(x) = \frac{1}{r} \int_{X'} M_n^\tau \xi \wedge M_k^\tau f.
\]

**Proof.** Since \( \tau^* M_k^{f,\lambda} = M_k^\tau f,\lambda \) if \( \text{Re } \lambda \gg 0 \), we have that then

\[
\int_{X'} M_k^{f,\lambda} \wedge \psi = \frac{1}{r} \int_{X'} M_k^\tau f,\lambda \wedge \tau^* \psi
\]

for test forms \( \psi \). Taking analytic continuations to \( \lambda = 0 \), we get (11.1). In view of Proposition 5.5 we have

\[
e_k(x) = \ell_x M_k^f = \int_X M_n^\xi \wedge M_k^\tau f \big|_{\lambda=0} = \frac{1}{r} \int_{X'} M_n^\tau \xi \wedge M_k^\tau f \big|_{\lambda=0} = \frac{1}{r} \int_{X'} M_n^\tau \xi \wedge M_k^\tau f.
\]

\[\square\]

In particular, it follows from Lemma 11.3 that

\[
(11.3) \quad \text{mult}_x X = \int_X M_n^\xi = \frac{1}{r} \int_{X'} M_n^\tau \xi.
\]

**Example 11.4.** Let \( r, s \) be relatively prime integers and consider the cusp \( X = \{z_1^r - z_2^s = 0\} \) in \( \mathbb{C}_x^2 \). Since we have the parametrization \( \tau: t \mapsto (t^r, t^s) \) of \( X \), using (11.3) we get

\[
\text{mult}_0 X = \int_X M_1^{(z_1, z_2)} = \int_{\mathbb{C}_1} M_1^{(t^r, t^s)} = \int_{\mathbb{C}_1} M_1^{\min(s, r)} = \min(s, r).
\]

This multiplicity is of course well-known, and can be computed in various other ways. \[\square\]

We will now proceed with some computations of local intersection numbers.

**Example 11.5.** Let \( X = \{x_2 x_1^m - x_3^2 = 0\} \subset \mathbb{C}_x^3 \), where \( m \geq 1 \), and let \( A = \{x_2 = x_3 = 0\} \). Since \( A \) is smooth and contained in \( X \), and \( X \) is smooth outside the origin in \( \mathbb{C}_x^3 \), we must have that \( A \circ X(x) = (\text{mult}_x A, 0, \ldots, 0) \) for \( x \neq 0 \), cf., (10.9).

We shall now compute the local intersection numbers at 0. To this end we consider a generic Vogel sequence of \( \mathcal{J}_A \) on \( X \) at the origin and compute the corresponding Vogel cycle. Let \( H_1 \) be a generic hyperplane that contains \( A \), defined by \( h_1 = \alpha x_2 - x_3 \). Then \( X_1 = H_1 \cdot X \) is the curve \( \{x_2 x_1^m - (\alpha x_2)^2 = 0, \alpha x_2 - x_3 = 0\} \). It follows that \( X_1^A \) is equal to \( A \), whereas \( X_1^{Z/A} \) is the curve \( \{x_1^m - x_2^2 x_3^2 = 0, \alpha x_2 - x_3 = 0\} \). Next, let \( h_2 = \beta x_2 - x_3 \). Then \( X_2 = H_2 \cdot X_1^{X/A} \) is the cycle \( \{x_3 = x_2 = 0, x_1^m = 0\} \).

Since its support is contained in \( A \), it is equal to \( X_2^A \) and it has order \( m \) at the origin. We conclude that \( V^h = A + m[0] \). Thus \( e_k(A, X, 0) \) is equal to 1 when \( k = \dim A = 1 \) and \( m \) when \( k = 0 \).

As an illustration, let us also compute \( e_0(A, X, 0) = e_2(\mathcal{J}_A, X, 0) \) as the Le-long number of a certain Bochner-Martinelli current. Notice that \( \tau: (t_1, t_2) \mapsto (t_1^2, t_2^2, t_1^m t_2) \) is a surjective, generically 2 − 1, mapping \( \mathbb{C}_2^2 \rightarrow X \). If \( i: X \hookrightarrow \mathbb{C}_3 \) is the
inclusion map we have by Lemma 11.3 that
\[
e_2(J_A, X, 0) = t_x(M_2^{(x_2, x_3)} \cap [X]) = \int_{C^3} M_0^{(z_1, x_2, x_3)} \wedge M_2^{(x_2, x_3)} \wedge [x_2 x_1^m - x_3^2] = \\
\int_X M_0^{(t^* z_1, t x_2, i^* x_3)} \wedge M_2^{(t x_2, i^* x_3)} = \frac{1}{2} \int_{C_{31}^{t^*}} M_0^{(t^* z_1, t x_2, i^* x_3)} \wedge M_2^{(t x_2, i^* x_3)}.
\]
According to Theorem 6.2, \(M_2^{(t x_2, i^* x_3)}\) is the mean value of
\[
[(\beta t_2 - t_1^n) t_2] \wedge [(|\alpha t_2 - t_1^n|) t_2]
\]
for generic choices of \(\alpha, \beta \in \mathbb{C}\). For generic \(\alpha, \beta\), using the new variables \(v_1 = t_1, v_2 = \alpha t_2 - t_1^n\), we get
\[
[\beta t_2 - t_1^n] \wedge [\alpha t_2 - t_1^n] = [\beta' v_2 - \alpha' v_1^n] \wedge [v_2] = [v_1^n] \wedge [v_2] = m[0]
\]
for some \(\alpha', \beta' \in \mathbb{C}\). Since \([|\alpha t_2 - t_1^n|) t_2] = [t_2] + [\alpha t_2 - t_1^n]\), by (3.3) and (3.4), we thus have that
\[
[(\beta t_2 - t_1^n) t_2] \wedge [(|\alpha t_2 - t_1^n|) t_2] = ([\beta t_2 - t_1^n] + [t_2]) \wedge [\alpha t_2 - t_1^n] = 2m[0].
\]
Now, \(M_0^{(t x_2, i^* x_3)} = 1_{(0,0)}\), so \(e_0(A, X, 0) = m\) as expected. \(\square\)

The following example is related to Example 11.2 above.

**Example 11.6.** The mapping \(\gamma: \mathbb{C}^3 \to \mathbb{C}^6\) defined by
\[
(t_1, t_2, t_3) \mapsto \gamma(t) = (t_1, t_2, t_3 t_1, t_3 t_2, t_3^2, t_3^3)
\]
is proper and injective, so that \(X := \gamma(\mathbb{C}^3)\) is a subvariety of \(\mathbb{C}^6\). Let \(A = \{z_3 = z_4 = z_5 = z_6 = 0\}\). Then \(A\) is smooth and contained in \(X\) and, since \(X\) is smooth outside 0, it follows from (10.9) that \(A \cap X(x) = (\text{mult}_x A, 0, \ldots, 0)\) for \(x \neq 0\). We want to determine the local intersection numbers \(e_{3-k}(A, X, 0) = e_k(J_A, X, 0)\) at 0. Since \(J_A\) has codimension 1 in \(X, e_0(0) = 0\). Moreover, by King’s formula, \(M_{3^3: z_3, z_4: z_5: z_6}\) is a Lelong current on \(X\), and at a point \(x \neq 0\) we know that the Lelong number is 1 if \(x \in A\) and 0 otherwise. We conclude that \(M_{3^3: z_3, z_4: z_5: z_6} = [A]\), and hence \(e_1(1) = \text{mult}_0 A = 1\). By Lemma 11.3,
\[
e_k(A, X, 0) = \int_{\mathbb{C}^3} \gamma^* z_1 \wedge \gamma^* z_2 \wedge \gamma^* z_3 \wedge \gamma^* z_4 \wedge \gamma^* z_5 \wedge \gamma^* z_6 = \int_{\mathbb{C}^3} M_{3-k}^{(t x_2, i^* x_3)} \wedge M_{k}^{x_1(t_1, t_2, t_3)},
\]
where we have used that the ideal \(\gamma^* z\) is generated by \(t_1, t_2, t_3\), that the ideal \(\gamma^* J_A\) is generated by \(t_3(t_1, t_2, t_3)\), and that \(e_k(A, X, 0)\) only depends on the ideals. In light of Example 11.2 thus,
\[
e_2(A, X, 0) = \int_{\mathbb{C}^3} M_{1}^{(t x_2, i^* x_3)} \wedge [t_3] \wedge dd^c \log |t|^2 = \int_{\mathbb{C}^2} M_{1}^{(t x_2, i^* x_3)} \wedge [t_3] \wedge dd^c \log |t|^2 = \\
\int_{\mathbb{C}^2_{(t_1, t_2)}} M_{1}^{(t x_2, i^* x_3)} \wedge dd^c \log |t'|^2 = \ell_0(dd^c) \log |t'|^2 = 1,
\]
where \(t' = (t_1, t_2)\). To see the last equality, in view of Theorem 6.1, one can replace \(dd^c \log |t|^2\) by a generic hyperplane \([\alpha \cdot t]\). In a similar way one concludes that \(e_3(A, X, 0) = 2\). \(\square\)
References


Department of Mathematics, Chalmers University of Technology and the University of Gothenburg, S-412 96 Gothenburg, SWEDEN
E-mail address: matsa@chalmers.se & hasam@chalmers.se & wulcan@chalmers.se