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July 19, 2014

Abstract

This paper provides a large deviation principle for Non-Markovian, Brownian motion driven stochastic differential equations with random coefficients. Similar to Gao & Liu [19], this extends the corresponding results collected in Freidlin & Wentzell [18]. However, we use a different line of argument, adapting the PDE method of Fleming [14] and Evans & Ishii [10] to the path-dependent case, by using backward stochastic differential techniques. Similar to the Markovian case, we obtain a characterization of the action function as the unique bounded solution of a path-dependent version of the Eikonal equation. Finally, we provide an application to the short maturity asymptotics of the implied volatility surface in financial mathematics.

Key words: Large deviations, backward stochastic differential equations, viscosity solutions of path dependent PDEs.

AMS 2000 subject classifications: 35D40, 35K10, 60H10, 60H30.

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*University of Southern California, Department of Mathematics, jinma@usc.edu. Research supported in part by NSF grant DMS 1106853.

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¶We are grateful for Mike Tehranchi who introduced us to the application of large deviations to the problem of implied volatility asymptotics.

1 Introduction

The theory of large deviations is concerned with the rate of convergence of a vanishing sequence of probabilities $\big(\mathbb{P}[A_n]\big)_{n\geq 1}$, where $(A_n)_{n\geq 1}$ is a sequence of *rare* events. after convenient scaling and normalization, the limit is called *rate function*, and is typically represented in terms of a control problem.

The pioneering work of Freidlin and Wentzell [18] considers rare events induced by Markov diffusions. The techniques are based on the Girsanov theorem for equivalent change of measure, and classical convex duality. An important contribution by Fleming [14] is to use the powerful stability property of viscosity solutions in order to obtain a significant simplified approach. We refer to Feng and Kurtz [13] for a systematic application of this methodology with relevant extensions.

The main objective of this paper is to extend the viscosity solutions approach to some problems of large deviations with rare events induced by non-Markov diffusions

$$X_t \;=\; X_0 + \int_0^t b_s(W,X)ds + \int_0^t \sigma_s(W,X)dW_s, \quad t\geq 0, \tag{1.1}$$

where W is a Brownian motion, and b,σ are non-anticipative functions of the paths of (W,X) satisfying convenient conditions for existence and uniqueness of the solution of the last stochastic differential equation (SDE).

We should note that the Large Deviation Principle (LDP) for non-Markovian diffusions of type (1.1) is not new. For example, Gao & Liu [19] studied such a problem via the sample path LDP method by Fredlin-Wentzell, using various norms in infinite dimensional spaces. While the techniques there are quite deep and sophisticated, the methodology is more or less "classical." Our main focus in this work is to extend the PDE approach of Fleming [14] in the present path-dependent framework, with a different set of tools. These include the theories of backward SDEs, stochastic control, and the viscosity solution for path-dependent PDEs (PPDEs), among them the last one has been developed only very recently. Specifically, the theory of backward SDEs, pioneered by Pardoux & Peng [23], can be effectively used as a substitute to the partial differential equations in the Markovian setting. Indeed, the log-transformation of the vanishing probability solves a semilinear PDE in the Markovian case. However, due to the "functional" nature of the coefficients in (1.1), both backward SDE and PDE involved will become non-Markovian and/or path-dependent.

Several technical points are worth mentioning. First, since the PDE involved in our problem naturally has the nonlinearity in the gradient term (quadratic to be specific), we therefore need the extension by Kobylanski [21] on backward SDEs to this context. Second, in order to obtain the rate function, we exploit the stochastic control representation of the log-transformation, and proceed to the asymptotic analysis with crucial use of the BMO properties of the solution of the BSDE. Finally, we use the notion of viscosity solutions of path-dependent Hamilton-Jacobi equations introduced by Lukoyanov [22] in order to characterize the rate function as unique viscosity solution of a path dependent Eikonal equation.

Another main purpose, in fact the original motivation, of this work is an application in financial mathematics. It has been known that an important problem in the valuation and hedging of exotic options is to characterize the short time asymptotics of the implied volatility surface, given the prices of European options for all maturities and strikes. The need to resort to asymptotics is due to the fact that only a discrete set of maturities and strikes are available. This difficulty is bypassed by practitioners by using the asymptotics in order to extend the volatility surface to the un-observed regimes. We refer to Henry-Labordère [7]. The results available in this literature have been restricted to the Markovian case, and our results in a sense opens the door to a general non-Markovian, path-dependent paradigm.

We finally observe that the sequence of vanishing probabilities induced by non-Markov diffusions can be re-formulated in the Markov case by using the Gyöngy's [20] result which produces a Markov diffusion with the same marginals. However, the regularity of the coefficients of the resulting Markov diffusion $\sigma^X(t,x) := \mathbb{E}[\sigma_t | X_t = x]$ are in general not suitable for the application of the classical large deviation results.

The paper is organized as follows. Section 2 contains the general setting, and provides our main results. First, we solve the small noise large deviation problem for the Laplace transform induced by a non-Markov diffusion. Next, we state the small noise large deviation result for the probability of exiting from some bounded open domain before some given maturity. We then state the characterization of the rate function as a unique viscosity solution of the corresponding path-dependent Eikonal equation. Section 3 is devoted to the application to the short maturity asymptotics of the implied volatility surface. Finally, Sections 4, 5 and 6 contain the proofs of our large deviation results, and the viscosity characterization.

2 Problem formulation and main results

Let $\Omega_d := \{\omega \in C^0([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous paths starting from the origin, B the canonical process defined by $B_t := \omega_t$, $t \in [0,1]$, and $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$ the corresponding filtration. We shall use the following notation for the supremum norm:

$$\|\omega\|_t := \sup_{s\in[0,t]} |\omega_s| \quad \text{and} \quad \|\omega\| := \|\omega\|_T \quad \text{for all} \quad t \in [0,T],\ \omega \in \Omega_d.$$

Let $\mathbb{P}_0$ be the Wiener measure on Ω_d. For all $\varepsilon \geq 0$, we denote by $\mathbb{P}^\varepsilon := \mathbb{P}_0 \circ (\sqrt{\varepsilon}B)^{-1}$ the probability measure such that

$$\left\{W_t^\varepsilon := \tfrac{1}{\sqrt{\varepsilon}} B_t, 0 \leq t \leq T\right\} \quad \text{is a } \mathbb{P}^\varepsilon - \text{Brownian motion.}$$

Our main interest in this paper is on the solution of the path-dependent stochastic differential equation:

$$dX_t = b_t(B,X)dt + \sigma_t(B,X)dB_t, \quad X_0 = x_0, \qquad \mathbb{P}^\varepsilon\text{-a.s.} \tag{2.1}$$

where the process X takes values in $\mathbb{R}^n$ for some integer $n > 1$, and its paths are in $\Omega_n := C^0([0,T],\mathbb{R}^n)$.

The supremum norm on Ω_n is also denoted $\|.\|_t$, without reference to the dimension of the underlying space. The coefficients $b : [0,T] \times \Omega_d \times \Omega_n \longrightarrow \mathbb{R}^n$ and $\sigma : [0,T] \times \Omega_d \times \Omega_n \longrightarrow \mathbb{R}^{n\times d}$ are assumed to satisfy the following conditions which guarantee existence and uniqueness of a strong solution for all $\varepsilon > 0$.

Assumption 2.1 *The coefficients $f \in \{b,\sigma\}$ are:*
- *non-anticipative, i.e. $f_t(\omega,x) = f_t\big((\omega_s)_{s\leq t},(x_s)_{s\leq t}\big)$,*
- *$L-$Lipschitz-continuous in (ω,x), uniformly in t, for some $L > 0$:*

$$\big|f_t(\omega,x) - f_t(\omega',x')\big| \leq L(\|\omega-\omega'\|_t + \|x-x'\|_t); \ t \in [0,T], (\omega,x), (\omega',x') \in \Omega_d \times \Omega_n,$$

Under $\mathbb{P}^\varepsilon$, the stochastic differential equation (2.1) is driven by a small noise, and our objective is to provide some large deviation asymptotics in the present path-dependent case, which extend the corresponding results of Freidlin & Wentzell [18] in the Markovian case. Our objective is to adapt to our path-dependent case the PDE approach to large deviations of stochastic differential equation as initiated by Fleming [14] and Evans & Ishii [10], see also Fleming & Soner [15], Chapter VII.

2.1 Laplace transform near infinity

As a first example, we consider the Laplace transform of some path-dependent random variable $\xi\big((\omega_s)_{s\le T},(x_s)_{s\le T}\big)$ for some final horizon $T>0$:

$$L_0^\varepsilon \;:=\; -\varepsilon\ln\mathbb{E}^{\mathbb{P}^\varepsilon}\Big[e^{-\frac{1}{\varepsilon}\xi(B,X)}\Big]. \tag{2.2}$$

In the following statement $\mathbb{L}_d^2$ denotes the collection of measurable functions $\alpha:[0,T]\longrightarrow\mathbb{R}^d$ such that $\int_0^T[\alpha_t|^2dt<\infty$. Our first main result is:

Theorem 2.2 *Let ξ be a bounded uniformly continuous $\mathcal{F}_T-$measurable r.v. Then, under Assumption 2.1, we have:*

$$L_0^\varepsilon\longrightarrow L_0:=\inf_{\alpha\in\mathbb{L}_d^2}\ell_0^\alpha\quad\textit{as }\varepsilon\to 0,\ \textit{ where }\ \ell_0^\alpha:=\xi(\omega^\alpha,x^\alpha)+\frac{1}{2}\int_0^T|\alpha_t|^2dt,$$

and (ω^α,x^α) are defined by the controlled ordinary differential equations:

$$\omega_t^\alpha=\int_0^t\alpha_s ds,\quad x_t^\alpha=X_0+\int_0^t b_s(\omega^\alpha,x^\alpha)ds+\int_0^t\sigma_s(\omega^\alpha,x^\alpha)d\omega_s^\alpha,\quad t\in[0,T].$$

The proof of this result is reported in Section 4.

Remark 2.3 Theorem 2.2 is still valid in the context where the coefficient b depends also on the parameter ε, so that the process X is replaced by X^ε defined by:

$$dX_t^\varepsilon=b_t^\varepsilon(B,X^\varepsilon)dt+\sigma_t(B,X^\varepsilon)dB_t,\quad X_0^\varepsilon=x_0,\qquad \mathbb{P}^\varepsilon\text{-a.s.}$$

Since this extension will be needed for our application in Section 3, we provide a precise formulation. Let Assumption 2.1 hold uniformly in $\varepsilon\in[0,1)$, and assume further that $\varepsilon\longmapsto b^\varepsilon$ is uniformly Lipschitz on $[0,1)$. Then the statement of Theorem 2.2 holds with x^α defined by:

$$x_t^\alpha=X_0+\int_0^t b_s^0(\omega^\alpha,x^\alpha)ds+\int_0^t\sigma_s(\omega^\alpha,x^\alpha)d\omega_s^\alpha,\quad t\in[0,T].$$

This slight extension does not induce any additional technical difficulty in the proof. We shall therefore provide the proof in the context of Theorem 2.2.

2.2 Exiting from a given domain before some maturity

As a second example, we consider the asymptotic behavior of the probability of exiting from some given subset of $\mathbb{R}^n$ before the maturity T:

$$Q_0^\varepsilon := -\varepsilon \ln \mathbb{P}^\varepsilon[H < T], \quad \text{where} \quad H := \inf\{t > 0 : X_t \notin O\}, \tag{2.3}$$

and O is a bounded open set in $\mathbb{R}^n$. We also introduce the corresponding subset of paths in Ω_n:

$$\mathcal{O} \;\; := \;\; \big\{\omega \in \Omega : \omega_t \in O \text{ for all } t \le T\big\}. \tag{2.4}$$

The analysis of this problem requires additional conditions.

Assumption 2.4 *The coefficients b and σ are uniformly bounded, and σ is uniformly elliptic, i.e. $a := \sigma\sigma^{\mathrm{T}}$ is invertible with bounded inverse a^{-1}.*

The present example exhibits a singularity on the boundary ∂O because Q_0^ε vanishes whenever the path ω is started on the boundary ∂O. Our second main result is the following.

Theorem 2.5 *Let O be a bounded open set in $\mathbb{R}^n$ with C^3 boundary. Then, under Assumptions 2.1 and 2.4, we have:*

$$Q_0^\varepsilon \longrightarrow Q_0 := \inf\big\{q_0^\alpha : \alpha \in \mathbb{L}_d^2,\ x_{T\wedge\cdot}^\alpha \notin \mathcal{O}\big\}, \quad \textit{where} \quad q_0^\alpha := \frac{1}{2}\int_0^T |\alpha_s|^2 ds,$$

and x^α is defined as in Theorem 2.2.

The proof of this result is reported in Section 5.

Remark 2.6 (i) A similar result of Theorem 2.5 can be found in Gao-Liu [19]. However, our proof has a completely different flavor and, given the preparation of the PPDE theory, seems to be more direct, whence shorter.

(ii) The condition on the boundary ∂O can be slightly weakened. Examining the proof of Lemma 5.1, where this condition is used, we see that it is sufficient to assume that O can be approximated from outside by open bounded sets with C^3 boundary.

Remark 2.7 The result of Theorem 2.5 is still valid in the context of Remark 2.3. This can be immediately verified by examining the proof of Theorem 2.5.

2.3 Path-dependent Eikonal equation

We next provide a characterization of our asymptotics in terms of partial differential equations. We refer to Evans & Ishii [10], Fleming & Souganidis [16], Evans-Souganidis [11], Evans, Souganidis, Fournier & Willem [12], Fleming & Soner [15], for the corresponding PDE literature with a derivation by means of the powerful theory of viscosity solutions.

Due to the path dependence in the dynamics of our state process X, and the corresponding limiting system x^α, our framework is clearly not covered by any of these existing works. Therefore, we shall adapt the notion of viscosity solutions introduced in Lukoyanov [22].

Consider the truncated Eikonal equation:

$$\big\{ -\partial_t u - F_{K_0}\big(.,\partial_\omega u, \partial_x u\big)\big\}(t,\omega,x) = 0 \quad \text{for} \quad (t,\omega,x) \in \Theta^0, \tag{2.5}$$

where K_0 is a fixed parameter, and the nonlinearity F_{K_0} is given by:

$$F_{K_0}(\theta, p_\omega, p_x) := b(\theta)\cdot p_x + \inf_{|a|\le K_0} \Big\{\frac{1}{2}a^2 + a\big(p_\omega + \sigma(\theta)^{\mathrm{T}} p_x\big)\Big\}, \tag{2.6}$$

for all $\theta \in \Theta$, $p_\omega \in \mathbb{R}^d$ and $p_x \in \mathbb{R}^n$. Notice that

$$F_{K_0}(\theta, p_\omega, p_x) \longrightarrow b(\theta)\cdot p_x - \frac{1}{2}\big|p_\omega + \sigma^{\mathrm{T}} p_x\big|^2 \quad \text{as} \quad K_0 \to \infty,$$

the equation (2.5) thus leads to a path-dependent Eikonal equation. We note that the truncated feature of the equation (2.5) is induced by the fact that the corresponding solution will be shown to be Lipschitz under our assumptions.

2.3.1 Classical derivatives

Denote $\hat\Omega := \Omega_d \times \Omega_n$ and $\hat\omega = (\omega, x)$ a generic element of $\hat\Omega$, $\Theta := [0,T]\times\hat\Omega$, and $\Theta^0 := [0,T)\times\hat\Omega$. The set Θ is endowed with the pseudo-distance

$$d(\theta,\theta') := |t-t'| + \big\|\hat\omega_{t\wedge} - \hat\omega'_{t'\wedge}\big\| \quad \text{for all} \quad \theta = (t,\hat\omega), \theta' = (t',\hat\omega') \in \Theta.$$

For any integer $k > 0$, we denote by $C^0(\Theta, \mathbb{R}^k)$ the collection of all continuous function $u : \Theta \longrightarrow \mathbb{R}^k$. Notice, in particular, that any $u \in C^0(\Theta,\mathbb{R}^k)$ is non-anticipative, i.e. $u(t,\hat\omega) = u(t,(\hat\omega_s)_{s\le t})$ for all $(t,\hat\omega) \in \Theta$.

We denote $\hat{\Omega}_K$ as the set of all K-Lipschitz paths. For $\theta = (t, \hat{\omega}) \in \Theta^0$, we denote $\Theta(\theta) := \cup_{K \geq 0} \Theta_K(\theta)$, where:

$$\Theta_K(\theta) \;:=\; \big\{(t', \hat{\omega}') \in \Theta : t' \geq t,\ \hat{\omega}'_{t\wedge} = \hat{\omega}_{t\wedge},\ \text{and } \hat{\omega}'|_{[t,T]} \text{ is } K\text{−Lipschitz}\big\}.$$

Definition 2.8 *A function $\varphi : \Theta \longrightarrow \mathbb{R}$ is said to be $C^{1,1}(\Theta)$ if $\varphi \in C^0(\Theta, \mathbb{R})$, and we may find $\partial_t \varphi \in C^0(\Theta, \mathbb{R})$, $\partial_{\hat{\omega}} \varphi \in C^0(\Theta, \mathbb{R}^{d+n})$, such that for all $\theta = (t, \hat{\omega}) \in \Theta$:*

$$\varphi(\theta') = \varphi(\theta) + \partial_t \varphi(\theta)(t' - t) + \partial_{\hat{\omega}} \varphi(\theta)(\hat{\omega}'_{t'} - \hat{\omega}_t) + \circ_{\hat{\omega}'}(t' - t) \quad \textit{for all} \quad \theta' \in \Theta(\theta),$$

where $\circ_{\hat{\omega}'}(h)/h \longrightarrow 0$ as $h \searrow 0$. The derivatives ∂_ω and ∂_x are defined by the natural decomposition $\partial_{\hat{\omega}} \varphi = (\partial_\omega \varphi, \partial_x \varphi)^{\mathrm{T}}$.

The last collection of smooth functions will be used for our subsequent definition of viscosity solutions.

2.3.2 Viscosity solutions of the path-dependent Eikonal equation

Let $\Theta^0_K := [0, T) \times \hat{\Omega}_K$. The set of test functions is defined for all $K > 0$ and $\theta \in \Theta^0_K$ by:

$$\underline{\mathcal{A}}^K u(\theta) \;:=\; \big\{\varphi \in C^{1,1}(\Theta) : (\varphi - u)(\theta) = \min_{\theta' \in \Theta_K} (\varphi - u)(\theta')\big\}, \tag{2.7}$$

$$\overline{\mathcal{A}}^K u(\theta) \;:=\; \big\{\varphi \in C^{1,1}(\Theta) : (\varphi - u)(\theta) = \max_{\theta' \in \Theta_K} (\varphi - u)(\theta')\big\}. \tag{2.8}$$

Definition 2.9 *Let $u : \Theta \longrightarrow \mathbb{R}$ be a continuous function.*
(i) *u is a K-viscosity subsolution of* (2.5)*, if for all $\theta \in \Theta^0_K$, we have*

$$\big\{ - \partial_t \varphi - F_{K_0}(., \partial_{\hat{\omega}} \varphi)\big\}(\theta) \leq 0 \quad \textit{for all} \quad \varphi \in \underline{\mathcal{A}}^K u(\theta).$$

(ii) *u is a K-viscosity supersolution of* (2.5)*, if for all $\theta \in \Theta^0_K$, we have*

$$\big\{ - \partial_t \varphi - F_{K_0}(., \partial_{\hat{\omega}} \varphi)\big\}(\theta) \geq 0 \quad \textit{for all} \quad \varphi \in \overline{\mathcal{A}}^K u(\theta).$$

(iii) *u is a K-viscosity solution of* (2.5) *if it is both K-viscosity subsolution and supersolution.*

2.3.3 Wellposedness of the path-dependent Eikonal equation

We only focus on the asymptotics of Laplace transform. For simplicity, we adopt the following strengthened version of Assumption 2.1.

Assumption 2.10 *The coefficients b and σ are bounded and satisfy Assumption 2.1.*

A natural candidate solution of equation (2.5) is the dynamic version of the limit L^0 introduced in Theorem 2.2:

$$u(t,\hat{\omega}) \ := \ \inf_{\alpha\in\mathbb{L}^2_d([t,T])} \Big\{\xi^{t,\hat{\omega}}(\hat{\omega}^{\alpha,t,\hat{\omega}}) + \frac{1}{2}\int_t^T |\alpha_s|^2 ds\Big\}, \quad (t,\hat{\omega})\in\Theta, \tag{2.9}$$

where $\hat{\omega}^{\alpha,t,\hat{\omega}} := (\omega^{\alpha,t,\hat{\omega}}, x^{\alpha,t,\hat{\omega}})$ is defined by:

$$\omega^{\alpha,t,\hat{\omega}}_s = \int_0^s \alpha_{t+r}dr, \quad x^{\alpha,t,\hat{\omega}}_s = \int_0^s b_{t+r}(\hat{\omega}\otimes_t \hat{\omega}^{\alpha,t,\hat{\omega}})dr + \int_0^s \sigma_{t+r}(\hat{\omega}\otimes_t \hat{\omega}^{\alpha,t,\hat{\omega}})d\omega^{\alpha,t,\hat{\omega}}_r,$$

with the notation $(\hat{\omega}\otimes_t \hat{\omega}')_s := \mathbf{1}_{\{s\le t\}}\hat{\omega}_s + \mathbf{1}_{\{s>t\}}\big(\hat{\omega}_t + \hat{\omega}'_{s-t}\big)$, and

$$\xi^{t,\hat{\omega}}(\hat{\omega}') := \xi\big((\hat{\omega}\otimes_t \hat{\omega}')_{T\wedge\cdot}\big) \quad \text{for all} \quad \hat{\omega},\hat{\omega}'\in\hat{\Omega}.$$

Theorem 2.11 *Let Assumption 2.10 hold true, and let ξ be a bounded Lipschitz function on $\hat{\Omega}$. Then, for K and K_0 sufficiently large, the function u defined in* (2.9) *is the unique bounded K-viscosity solution of the path-dependent PDE* (2.5).

The proof of this result is reported in Section 6.

3 Application to implied volatility asymptotics

3.1 Implied volatility surface

The Black-Scholes formula $\mathrm{BS}(K,\sigma^2T)$ expresses the price of a European call option with time to maturity T and strike K in the context of a geometric Brownian motion model for the underlying stock, with volatility parameter $\sigma\ge 0$:

$$\widehat{\mathrm{BS}}(k,v) \ := \ \frac{\mathrm{BS}(K,v)}{S_0} \ := \ \begin{cases} (1-e^k)^+ & \text{for} \ \ v=0, \\ \mathbf{N}\big(d_+(k,v)\big) - e^k\mathbf{N}\big(d_-(k,v)\big), & \text{for} \ \ v>0, \end{cases}$$

where S_0 denotes the spot price of the underlying asset, $v := \sigma^2 T$ is the total variance, $k := \ln(K/S_0)$ is the log-moneyness of the call option, $\mathbf{N}(x) := (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2} dy$,

$$d_{\pm}(k, v) := \frac{-k}{\sqrt{v}} \pm \frac{\sqrt{v}}{2},$$

and the interest rate is reduced to zero.

We assume that the underlying asset price process is defined by the following dynamics under the risk-neutral measure $\mathbb{P}_0$:

$$dS_t = S_t \sigma_t(B, S) dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

so that the price of the $T-$maturity European call option with strike K is given by $\mathbb{E}^{\mathbb{P}_0}\big[(S_T - K)^+\big]$. The implied volatility surface $(T, k) \longmapsto \Sigma(T, k)$ is then defined as the unique non-negative solution of the equation

$$\mathbf{N}\big(d_+(k, \Sigma^2 T)\big) - e^k \mathbf{N}\big(d_-(k, \Sigma^2 T)\big) \;=\; \hat{C}(T, k) \;:=\; \mathbb{E}^{\mathbb{P}_0}\big[\big(e^{X_T} - e^k\big)^+\big],$$

where $X_t := \ln(S_t/S_0)$, $t \geq 0$.

Our interest in this section is on the short maturity asymptotics $T \searrow 0$ of the implied volatility surface $\Sigma(T, k)$ for $k > 0$. This is a relevant practical problem which is widely used by derivatives traders, and has induced an extensive literature initiated by Berestycki, Busca & Florent [1, 2]. See e.g. Henry-Labordère [7], Hagan, Lesniewski, & Woodward [8], Ford and Jacquier [17], Gatheral, Hsu, Laurence, Ouyang & Wang [9], Deuschel, Friz, Jacquier & Violante [5, 6], and Demarco & Friz [4].

Our starting point is the following limiting result which follows from standard calculus:

$$\lim_{v \to 0} v \ln \widehat{\mathrm{BS}}(k, v) \;=\; -\frac{k^2}{2}, \quad \text{for all} \quad k > 0.$$

We also compute directly that, for $k > 0$, we have $\hat{C}(T, k) \longrightarrow 0$ as $T \searrow 0$. Then $T\Sigma(T, k)^2 \longrightarrow 0$ as $T \searrow 0$, and it follows from the previous limiting result that

$$\lim_{T \to 0} T\Sigma(T, k)^2 \ln \hat{C}(T, k) \;=\; -\frac{k^2}{2}, \quad \text{for all} \quad k > 0. \tag{3.10}$$

Consequently, in order to study the asymptotic behavior of the implied volatility surface $\Sigma(T, k)$ for small maturity T, we are reduced to the asymptotics of $T \ln \hat{C}(T, k)$

for small T, which will be shown in the next subsection to be closely related to the large deviation problem of Subsection 2.2. Hence, our path-dependent large deviation results enable us to obtain the short maturity asymptotics of the implied volatility surface in the context where the underlying asset is a non-Markovian martingale under the risk-neutral measure.

3.2 Short maturity asymptotics

Recall the process $X_t := \ln(S_t/S_0)$. By Itô's formula, we deduce the dynamic for process X:

$$dX_t = -\frac{1}{2}\sigma_t^X(B,X)^2 d\langle B\rangle_t + \sigma_t^X(B,X)dB_t, \tag{3.11}$$

where $\sigma^X(\omega,x) := \sigma\big(\omega, S_0 e^{x\cdot}\big)$. For the purpose of the application in this section, we need to convert the short maturity asymptotics into a small noise problem, so as to apply the main results from the previous section. In the present path-dependent case, this requires to impose a special structure on the coefficients of the stochastic differential equation (3.11).

For a random variable Y and a probability measure $\mathbb{P}$, we denote by $\mathcal{L}^{\mathbb{P}}(Y)$ the $\mathbb{P}-$distribution of Y.

Assumption 3.1 *The diffusion coefficient $\sigma^X : [0,T]\times\Omega_d\times\Omega_n \longrightarrow \mathbb{R}$ is non-anticipative, Lipschitz-continuous, takes values in $[\underline{\sigma},\overline{\sigma}]$ for some $\overline{\sigma}\geq\underline{\sigma}>0$, and satisfies the following small-maturity small-noise correspondence:*

$$\mathcal{L}^{\mathbb{P}_0}(X_\varepsilon) = \mathcal{L}^{\mathbb{P}^\varepsilon}(X_1) \quad \textit{for all} \quad \varepsilon\in[0,1).$$

Remark 3.2 Assume that σ is independent of ω and satisfies the following time-indifference property:

$$\sigma_{ct}^X(x) = \sigma_t^X(x^c) \ \text{ for all } \ c>0, \quad \text{where} \quad x_s^c := x_{cs}, \quad s\in[0,T].$$

Then, $\mathcal{L}^{\mathbb{P}_0}\big((X_s)_{s\leq\varepsilon}\big) = \mathcal{L}^{\mathbb{P}^\varepsilon}\big((X_s)_{s\leq 1}\big)$ for all $\varepsilon\in[0,1)$, which implies that the small-maturity small-noise correspondence holds true. In particular, the time-indifference property holds in the homogeneous Markovian case $\sigma_t(x)=\sigma(x_t)$.

In view of (3.10) and the small-maturity small-noise correspondence of Assumption 3.1, we are reduced to the asymptotics of

$$\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1} - e^k)^+] \quad \text{as} \quad \varepsilon \to 0.$$

Under $\mathbb{P}^\varepsilon$ the dynamics of X is given by the stochastic differential equation:

$$dX_t \;=\; -\frac{\varepsilon}{2}\,\sigma_t^X(B,X)^2 dt + \sigma_t^X(B,X) dB_t, \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

whose coefficients satisfy the conditions given in Remarks 2.3 and 2.7. Consider the stopping time

$$H_{a,b} \;:=\; \inf\{t : X_t \notin (a,b)\} \quad \text{for} \quad -\infty < a < b < +\infty.$$

Then, it follows from Theorem 2.5 and Remark 2.7 that

$$Q_0^\varepsilon := -\varepsilon \ln \mathbb{P}^\varepsilon\big[H_{a,b} \le 1\big] \longrightarrow Q_0(a,b) \quad \text{as} \quad \varepsilon \searrow 0,$$

where $Q_0(a,b)$ is defined as in Theorem 2.5 in terms of the controlled function x^α of Theorem 2.2:

$$Q_0(a,b) := \inf\Big\{\frac{1}{2}\int_0^1 |\alpha_s|^2 ds : \alpha \in \mathbb{L}_d^2,\ x_{1\wedge\cdot}^\alpha \notin \mathcal{O}_{a,b}\Big\},$$

where $\mathcal{O}_{a,b} := \big\{x : x_t \in (a,b) \text{ for all } t \in [0,1]\big\}$. The rest of this section is devoted to the following result.

Proposition 3.3 $\lim_{\varepsilon\to 0} -\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1} - e^k)^+] = Q_0(k) := \lim_{a\to-\infty} Q_0(a,k)$.

Proof **1.** We first show that

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1} - e^k)^+] \;\le\; -Q_0(k). \tag{3.12}$$

Fix some $p > 1$ and the corresponding conjugate $q > 1$ defined by $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder inequality, we estimate that

$$\mathbb{E}^{\mathbb{P}^\varepsilon}\left[(e^{X_1} - e^k)^+\right] \le \mathbb{E}^{\mathbb{P}^\varepsilon}\left[e^{X_1}\mathbf{1}_{\{X_1 \ge k\}}\right] \le \mathbb{E}^{\mathbb{P}^\varepsilon}\left[e^{qX_1}\right]^{1/q} \mathbb{P}^\varepsilon[H_{a,k} \le 1]^{1/p}, \text{ for all } a < k.$$

By standard estimates, we may find a constant C_p such that $\mathbb{E}^{\mathbb{P}^\varepsilon}\left[e^{qX_1}\right] \le C_p$ for all $\varepsilon \in (0,1)$. Then,

$$\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon}\left[(e^{X_1} - e^k)^+\right] \;\le\; \frac{\varepsilon}{q}\ln C_p + \frac{\varepsilon}{p}\ln \mathbb{P}^\varepsilon[H_{a,k} \le 1],$$

which provides (3.12) by sending $\varepsilon \to 0$ and then $p \to 1$.

2. We next prove the following inequality:

$$\underline{\lim}_{\varepsilon\to 0} \varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1}-e^k)^+] \;\; \geq \;\; -Q_0(k). \tag{3.13}$$

For $n \in \mathbb{N}$, denote $f_n(x) := (e^{-n}-x)^+ + (x-e^k)^+$ for $x \in \mathbb{R}$. Since f_n is convex and e^X is $\mathbb{P}^\varepsilon$-martingale, the process $f\big(e^X\big)$ is a non-negative $\mathbb{P}^\varepsilon$-submartingale. For a sufficiently small $\delta > 0$, set $a_{n,\delta} := \ln(e^{-n}-\delta)$ and $k_\delta := \ln(e^k+\delta)$. Then, it follows from the Doob inequality that

$$\mathbb{P}^\varepsilon[H_{a_{n,\delta},k_\delta} \leq 1] \;=\; \mathbb{P}^\varepsilon\Big[\max_{t\leq 1} f_n\big(e^{X_t}\big) \geq \delta\Big] \;\leq\; \frac{1}{\delta}\,\mathbb{E}^{\mathbb{P}^\varepsilon}\big[f_n\big(e^{X_1}\big)\big]. \tag{3.14}$$

We shall prove in Step 3 below that

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{-n}-e^{X_1})^+]}{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1}-e^k)^+]} = 0 \quad \text{for large} \quad n. \tag{3.15}$$

Then, it follows from (3.14), by sending $\varepsilon \to 0$, that

$$-Q_0(a_{n,\delta},k_\delta) \;\;\leq\;\; \underline{\lim}_{\varepsilon\to 0} \varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1}-e^k)^+].$$

Finally, sending $\delta \to 0$ and then $n \to \infty$, we obtain (3.13).

3. It remains to prove (3.15). Since $\underline{\sigma} \leq \sigma \leq \overline{\sigma}$, by Assumption 3.1, it follows from the convexity of $s \longmapsto (e^{-n}-s)^+$ and $s \longmapsto (s-e^k)^+$ that

$$\frac{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{-n}-e^{X_1})^+]}{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1}-e^k)^+]} \;\;\leq\;\; \frac{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{-n}-e^{-\frac{1}{2}\varepsilon\overline{\sigma}^2+\overline{\sigma}B_1})^+]}{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{-\frac{1}{2}\varepsilon\underline{\sigma}^2+\underline{\sigma}B_1}-e^k)^+]}.$$

Further, we have

$$\mathbb{E}^{\mathbb{P}^\varepsilon}\Big[\big(e^{-n}-e^{-\frac{1}{2}\varepsilon\overline{\sigma}^2+\overline{\sigma}B_1}\big)^+\Big] \;\;\leq\;\; e^{-n}\mathbf{N}\Big(\frac{1}{2}\overline{\sigma}\sqrt{\varepsilon}-\frac{n}{\overline{\sigma}\sqrt{\varepsilon}}\Big),$$

and, by the Chebyshev inequality,

$$\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{-\frac{1}{2}\varepsilon\underline{\sigma}^2+\underline{\sigma}B_1}-e^k)^+] \geq \lambda\mathbb{P}^\varepsilon[e^{-\frac{1}{2}\varepsilon\underline{\sigma}^2+\underline{\sigma}B_1} \geq e^k+\lambda] = \lambda\mathbf{N}\Big(-\frac{1}{2}\underline{\sigma}\sqrt{\varepsilon}-\frac{\ln(e^k+\lambda)}{\underline{\sigma}\sqrt{\varepsilon}}\Big).$$

Using the estimate $\mathbf{N}(-x) \sim \frac{1}{\sqrt{2\pi}}x^{-1}e^{-\frac{x^2}{2}}$, we obtain that

$$\overline{\lim_{\varepsilon\to 0}} \frac{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{-n}-e^{X_1})^+]}{\mathbb{E}^{\mathbb{P}^\varepsilon}[(e^{X_1}-e^k)^+]} \;\;\leq\;\; C\exp\Big\{-\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\Big(\frac{n^2}{\overline{\sigma}^2}-\frac{(\ln(e^k+\lambda))^2}{\underline{\sigma}^2}\Big)\Big\} \;\;=\;\; 0,$$

whenever $n^2 > \frac{\overline{\sigma}^2}{\underline{\sigma}^2}(\ln(e^k+\lambda))^2$. ∎

4 Asymptotics of Laplace transforms

Our starting point is a characterization of Y_0^ε in terms of a quadratic backward stochastic differential equation. Let

$$Y_t^\varepsilon := -\varepsilon \ln \mathbb{E}_t^{\mathbb{P}^\varepsilon}\Big[e^{-\frac{1}{\varepsilon}\xi(B,X)}\Big], \quad t \in [0,T]. \tag{4.16}$$

where $\mathbb{E}_t^{\mathbb{P}^\varepsilon}$ denotes expectation operator under $\mathbb{P}^\varepsilon$, conditional to $\mathcal{F}_t$.

Proposition 4.1 *The processes Y^ε is bounded by $\|\xi\|_\infty$, and is uniquely defined as the bounded solution of the quadratic backward stochastic differential equation*

$$Y_t^\varepsilon \;=\; \xi - \frac{1}{2}\int_t^T \big|Z_s^\varepsilon\big|^2 ds + \int_t^T Z_s^\varepsilon \cdot dB_s, \quad \mathbb{P}^\varepsilon - a.s.$$

Moreover, the process Z^ε satisfies the BMO estimate

$$\|Z\|_{\mathbb{H}^2_{\rm BMO}(\mathbb{P}^\varepsilon)} := \sup_{t\in[0,T]} \Big\|\mathbb{E}_t^{\mathbb{P}^\varepsilon}\int_t^T \big|Z_s^\varepsilon\big|^2 ds\Big\|_{\mathbb{L}^\infty(\mathbb{P}^\varepsilon)} \;\le\; 4\|\xi\|_\infty. \tag{4.17}$$

Proof Since ξ is bounded, we see immediately that $Y_t^\varepsilon \le -\varepsilon \ln\big(e^{-\frac{1}{\varepsilon}\|\xi\|_\infty}\big) = \|\xi\|_\infty$ and, similarly $Y_t^\varepsilon \ge -\|\xi\|_\infty$. Consequently, the process

$$p^\varepsilon := e^{-\frac{1}{\varepsilon}Y^\varepsilon} = \mathbb{E}_t^{\mathbb{P}^\varepsilon}\big[e^{-\frac{1}{\varepsilon}\xi(B,X)}\big]$$

is a bounded martingale. By martingale representation, there exists a process q^ε, with $\mathbb{E}^{\mathbb{P}^\varepsilon}\big[\int_0^T |q_t^\varepsilon|^2 dt\big] < \infty$, such that $p_t^\varepsilon = p_0^\varepsilon + \int_0^t q_s^\varepsilon \cdot dB_s$, for all $t\in[0,T]$. Then, Y^ε solves the quadratic backward SDE by Itô's formula. The estimate $\|Z\|_{\mathbb{H}^2_{\rm BMO}(\mathbb{P}^\varepsilon)}$ follows immediately by taking expectations in the quadratic backward SDE, and using the boundedness of Y^ε by $\|\xi\|_\infty$. ■

We next provide a stochastic control representation for the process Y^ε. For all $\alpha \in \mathbb{H}^2_{\rm BMO}$, we introduce

$$M_T^{\varepsilon,\alpha} \;:=\; e^{\frac{1}{\varepsilon}\int_0^T \alpha_t\cdot dB_t - \frac{1}{2\varepsilon}\int_0^T |\alpha_t|^2 dt}.$$

Then $\mathbb{E}^{\mathbb{P}^\varepsilon}\big[M_T^{\varepsilon,\alpha}\big] = 1$, and we may introduce an equivalent probability measure $\mathbb{P}^{\varepsilon,\alpha}$ by the density $d\mathbb{P}^{\varepsilon,\alpha} := M_T^{\varepsilon,\alpha} d\mathbb{P}^\varepsilon$. Define:

$$Y_t^{\varepsilon,\alpha} \;=\; \mathbb{E}^{\mathbb{P}^{\varepsilon,\alpha}}\Big[\xi + \frac{1}{2}\int_t^T |\alpha_s|^2 ds\Big], \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

Lemma 4.2 *We have*

$$Y_0^\varepsilon \;=\; Y_0^{\varepsilon,Z^\varepsilon} \;=\; \inf_{\alpha\in\mathbb{H}^2_{\rm BMO}(\mathbb{P}^\varepsilon)} Y_0^{\varepsilon,\alpha}.$$

Proof Notice that $Y^{\varepsilon,\alpha}$ solves the linear backward SDE

$$dY_t^{\varepsilon,\alpha} \;=\; -Z_t^{\varepsilon,\alpha}\cdot dB_t - \big(Z_t^{\varepsilon,\alpha}\cdot\alpha_t - \frac{1}{2}|\alpha_t|^2\big)dt, \;\; \mathbb{P}^\varepsilon - \text{a.s.}$$

Since $-\frac{1}{2}z^2 = \inf_{a\in\mathbb{R}^d}\big\{-a\cdot z + \frac{1}{2}a^2\big\}$, it follows from the comparison of BSDEs that $Y^{\varepsilon,\alpha}\ge Y^\varepsilon$. The required result follows from the observation that the last supremum is attained by $a^* = z$, and that $Y^{\varepsilon,Z^\varepsilon} = Y^\varepsilon$. ∎

Proof of Theorem 2.2. First, it is clear that $\mathbb{L}^2_d \subset \cap_{\varepsilon>0}\mathbb{H}^2_{\rm BMO}(\mathbb{P}^\varepsilon)$. Let $\alpha\in\mathbb{L}^2_d$ and any $\varepsilon>0$ be fixed. Since α is deterministic, it follows from the Girsanov Theorem that

$$Y_0^{\varepsilon,\alpha} = \mathbb{E}^{\mathbb{P}_0}\Big[\xi(W^{\varepsilon,\alpha},X^{\varepsilon,\alpha}) + \frac{1}{2}\int_0^T|\alpha_t|^2dt\Big],$$

where

$$\begin{array}{lcl} W_t^{\varepsilon,\alpha} &:=& \sqrt{\varepsilon}B_t + \int_0^t\alpha_s ds,\\ X_t^{\varepsilon,\alpha} &=& X_0 + \int_0^t b_s(W^{\varepsilon,\alpha},X_s^{\varepsilon,\alpha})ds + \int_0^t\sigma_s(W^{\varepsilon,\alpha},X_s^{\varepsilon,\alpha})dW_s^{\varepsilon,\alpha}, \end{array} \quad \mathbb{P}_0\text{-a.s.}$$

By the given regularities, it is clear that $\lim_{\varepsilon\to 0}Y_0^{\varepsilon,\alpha} = l_0^\alpha$. Then it follows from Lemma 4.2 that

$$\overline{\lim_{\varepsilon\to0}}\,Y_0^\varepsilon \;\le\; \overline{\lim_{\varepsilon\to0}}\,Y_0^{\varepsilon,\alpha} \;=\; \ell_0^\alpha.$$

By the arbitrariness of $\alpha\in\mathbb{L}^2_d$, this shows that $\overline{\lim}_{\varepsilon\to0}Y_0^\varepsilon \le L_0$.

To prove the reverse inequality, we use the minimizer from Lemma 4.2. Note that $\mathbb{P}^\varepsilon$ is equivalent to $\mathbb{P}^{\varepsilon,Z^\varepsilon}$ and for $\mathbb{P}^\varepsilon$-a.e. ω, $\alpha^{\varepsilon,\omega} := Z^\varepsilon_\cdot(\omega)\in\mathbb{L}^2_d$. Then we compute that

$$\begin{array}{rcl} Y_0^\varepsilon \;=\; Y_0^{\varepsilon,Z^\varepsilon} &=& \mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\xi(B,X) + \frac{1}{2}\int_0^T\big|Z_t^\varepsilon\big|^2dt\Big]\\ &\ge& L_0 + \mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\xi(B,X) - \xi\big(\omega^{Z^\varepsilon(\omega)},x^{Z^\varepsilon(\omega)}(\omega)\big)\Big]\\ &\ge& L_0 - \mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\rho\big(\big\|B-\omega^{Z^\varepsilon(\omega)}\big\|_T + \big\|X - x^{Z^\varepsilon(\omega)}(\omega)\big\|_T\big)\Big]. \end{array}$$

By definition of ω^α, notice that $\omega \longmapsto W^\varepsilon(\omega) := \varepsilon^{-1/2}\big(B(\omega) - \omega^{Z^\varepsilon(\omega)}\big)$ defines a Brownian motion under $\mathbb{P}^{\varepsilon,Z^\varepsilon}$. Then it is clear that

$$\overline{\lim_{\varepsilon\to 0}}\,\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\big\|B-\omega^{Z^\varepsilon(\omega)}\big\|_T\Big] = \overline{\lim_{\varepsilon\to 0}}\,\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\sqrt{\varepsilon}\|W^\varepsilon\|_T\Big] = 0.$$

Furthermore, recall that σ and b are Lipschitz-continuous, it follows from the comparison of SDEs that $\underline{\delta}_t \le X - x^{Z^\varepsilon} \le \overline{\delta}_t$, where $\underline{\delta}_0 = \overline{\delta}_0 = 0$, and

$$\begin{aligned} d\underline{\delta}_t &= \sigma_t(B,X)\sqrt{\varepsilon}dW^\varepsilon_t - L\big(\sqrt{\varepsilon}\|W^\varepsilon\|_t + \|\underline{\delta}\|_t\big)\left(|Z^\varepsilon_t|+1\right)dt,\\ d\overline{\delta}_t &= \sigma_t(B,X)\sqrt{\varepsilon}dW^\varepsilon_t + L\big(\sqrt{\varepsilon}\|W^\varepsilon\|_t + \|\overline{\delta}\|_t\big)\left(|Z^\varepsilon_t|+1)\right)dt. \end{aligned}$$

We now estimate $\overline{\delta}$. The estimation of $\underline{\delta}$ follows the same line of argument. Denote $K_t := \int_0^t \sigma_s(B,X)dW^\varepsilon_s$. By Gronwall's inequality, we obtain

$$\begin{aligned} \varepsilon^{-1/2}\|\overline{\delta}_T\| &= L\|W^\varepsilon\|_T\int_0^T e^{L\int_t^T(|Z^\varepsilon_s|+1)ds}\left(|Z^\varepsilon_t|+1\right)dt + \int_0^T e^{L\int_t^T(|Z^\varepsilon_s|+1)ds}d\|K\|_t\\ &\le e^{L\int_0^T(|Z^\varepsilon_s|+1)ds}\left(\|W^\varepsilon\|_T + \|K\|_T\right). \end{aligned}$$

Then,

$$\begin{aligned} \varepsilon^{-1/2}e^{-LT}\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}[\|\overline{\delta}_T\|] &\le \mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[e^{L\int_0^T|Z^\varepsilon_s|ds}[\|W^\varepsilon\|_T + \|K\|_T\Big]\\ &\le \Big(\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[e^{2L\int_0^T|Z^\varepsilon_s|ds}\Big]\Big)^{\frac12}\Big(\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\|W^\varepsilon\|^2_T + \|K\|^2_T\Big]\Big)^{\frac12}. \end{aligned}$$

Recall that $\sigma_t(0,x)$ is bounded. One may easily check that, for some constant C independent of ε,

$$\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\|W^\varepsilon\|^2_T + \|K\|^2_T\Big] \le C.$$

Moreover, note that

$$Y^\varepsilon_t = \xi + \frac12\int_t^T|Z^\varepsilon_s|^2ds - \sqrt{\varepsilon}\int_t^T Z^\varepsilon_t dW^\varepsilon_t.$$

Then, it follows that $\|Z\|_{\mathbb{H}^2_{\rm BMO}(\mathbb{P}^{\varepsilon,Z^\varepsilon})} \le 4\|\xi\|_\infty$, and $\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[e^{\eta\int_0^T|Z^\varepsilon_s|^2ds}\Big] \le C$ for all $\varepsilon > 0$, for some $\eta > 0$ and $C > 0$ independent of ε, see e.g. [3]. This implies $\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[e^{2L\int_0^T|Z^\varepsilon_s|ds}\Big] \le C$ and thus

$$\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}[\|\overline{\delta}\|_T] \le C\sqrt{\varepsilon}, \quad \forall\varepsilon.$$

Similarly, $\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}[\|\underline{\delta}\|_T] \le C\sqrt{\varepsilon}$, and we may conclude that

$$\mathbb{E}^{\mathbb{P}^{\varepsilon,Z^\varepsilon}}\Big[\rho\big(\big\|B-\omega^{Z^\varepsilon}\big\|_T + \big\|X - x^{Z^\varepsilon}\big\|_T\big)\Big] \;\longrightarrow\; 0, \;\text{ as }\; \varepsilon\searrow 0,$$

completing the proof. ∎

5 Asymptotics of the exiting probability

This section is dedicated to the proof of Theorem 2.5. As before, we introduce the processes:

$$Y_t^\varepsilon := -\varepsilon \ln p_t^\varepsilon, \quad p_t^\varepsilon := \mathbb{P}_t^\varepsilon[H < T] \quad \text{for all} \quad t \le T.$$

Unlike the previous problem, the present example features an additional difficulty due to the singularity of the terminal condition:

$$\lim_{t \to T} Y_t^\varepsilon = \infty \quad \text{on} \quad \{H \ge T\}.$$

We shall first show that $\overline{\lim}_{\varepsilon \downarrow 0} Y_0^\varepsilon \le Q_0$. Adapting the argument of Fleming & Soner [15], this will follow from the following estimate.

Lemma 5.1 *There exists a constant K such that for any $\varepsilon > 0$ we have*

$$Y_t^\varepsilon \;\le\; \frac{Kd(X_t, \partial O)}{T-t} \quad \textit{for all} \quad t < T \quad \textit{and} \quad t \le H, \quad \mathbb{P}^\varepsilon\textit{-a.e.}$$

Proof First, fix $T_1 < T$. For $x \in \mathbb{R}^d$, we denote by x^1 its first component. Since O is bounded, there exists constant m such that $x^1 + \mu > 0$ for all $x \in O$. Define a function:

$$g^\varepsilon(t, x) \;:=\; \exp\left(-\frac{\lambda(x^1 + \mu)}{\varepsilon(T_1 - t)} \right), \quad \text{for} \quad t < T_1, \; x \in \text{cl}(O),$$

where λ is some constant to be chosen later. By Itô's formula, we have $\mathbb{P}^\varepsilon$-a.s.,

$$dg^\varepsilon(t, X_t) = \frac{g^\varepsilon(t, X_t)}{\varepsilon(T_1 - t)^2} \left[\frac{1}{2} a_t^{1,1}(B, X)\lambda^2 - \lambda(X_t^1 + \mu) - (T_1 - t)\lambda b_t^1(B, X) \right] dt + dM_t,$$

for some $\mathbb{P}^\varepsilon-$martingale M. Since $a^{1,1}$ is uniformly bounded away from zero and b^1 is uniformly bounded, the dt-term of the above expression is positive for a sufficiently large $\lambda = \lambda^*$. Hence, $g^\varepsilon(t, X_t)$ is a submartingale on $[0, T_1 \wedge H]$. Also, note that $g^\varepsilon(T_1, X_{T_1}) = 0 \le p_{T_1}^\varepsilon$ and $g^\varepsilon(H, X_H) \le 1 = p_H^\varepsilon$. Since p^ε is a martingale, we conclude that

$$g^\varepsilon(t, X_t) \;\le\; p_t^\varepsilon \quad \text{for all} \quad t \le T_1 \wedge H, \; \mathbb{P}^\varepsilon\text{-a.s.}$$

Denote $d(x) := d(x, \partial O)$. Since ∂O is C^3, there exists a constant η such that on $\{x \in O : d(x) < \eta\}$, the function d is C^2. Now, define

$$\tilde{g}^\varepsilon(t,x) \;:=\; \exp\left(-\frac{Kd(x)}{\varepsilon(T_1-t)}\right), \quad \text{for} \quad t < T_1,\; x \in \mathrm{cl}(O),$$

for some $K \geq \frac{\lambda^*(C+\mu)}{\eta}$. Clearly, for $t \leq T_1 \wedge H$ and $d(X_t) \geq \eta$, we have

$$\tilde{g}^\varepsilon(t, X_t) \;\leq\; g^\varepsilon(t, X_t) \;\leq\; p_t^\varepsilon, \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

In the remaining case $t \leq T_1 \wedge H$ and $d(X_t) < \eta$, we will now verify that

$$\left\{\tilde{g}^\varepsilon(s, X_s)1_{\{d(X_t)<\eta\}}, s \in [t, H_\eta \wedge H \wedge T]\right\} \quad \text{is a} \quad \mathbb{P}^\varepsilon - \text{submartingale},$$

where $H_\eta := \inf\{s : d(X_s) \geq \eta\}$. By Itô's formula, together with the fact that $|Dd(x)| = 1$,

$$\begin{aligned}
d\tilde{g}^\varepsilon(s, X_s) \;&=\; \frac{K\tilde{g}^\varepsilon(s,X_s)}{\varepsilon(T_1-s)^2}\Big[\frac{K}{2}a_s Dd(X_s)\cdot Dd(X_s) - \varepsilon\frac{T_1-s}{2}\mathrm{tr}\big(a_s D^2 d(X_s)\big) \\
&\qquad\qquad -(T_1-s)b_s\cdot Dd(X_s) - d(X_s)\Big]ds + dM_s \\
&\geq\; \frac{K\tilde{g}^\varepsilon(s,X_s)}{\varepsilon(T_1-s)^2}\Big(\frac{K}{2}\delta - \varepsilon\frac{T_1-s}{2}|a_s|\big|D^2 d(X_s)\big| - (T_1-s)\|b_s\|\Big)ds + dM_s.
\end{aligned}$$

Hence, for sufficiently large $K = K^*$, the dt-term is positive, and $\tilde{g}^\varepsilon(s, X_s)1_{\{d(X_t)<\eta\}}$ is a submartingale for $s \in [t, H_\eta \wedge H \wedge T]$. We also verify directly that

$$\tilde{g}^\varepsilon(H_\eta \wedge H \wedge T, X_{H_\eta\wedge H\wedge T})1_{\{d(X_t)<\eta\}} \;\leq\; p^\varepsilon_{H_\eta\wedge H\wedge T}, \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

Since p^ε is a $\mathbb{P}^\varepsilon-$martingale, we deduce that $\tilde{g}^\varepsilon(t, X_t) \leq p_t^\varepsilon$ for $t \leq T_1 \wedge H$ and $d(X_t) < \eta$. Thus, we may conclude that

$$\tilde{g}^\varepsilon(t, X_t) \leq p_t^\varepsilon \text{ for all } t \leq T_1 \wedge H,\; \mathbb{P}^\varepsilon\text{-a.s.}$$

Let $T_1 \to T$, we finally get

$$Y_t^\varepsilon \;\leq\; \frac{Kd(X_t)}{T-t} \quad \text{for all} \quad t < T \quad \text{and} \quad t \leq H, \quad \mathbb{P}^\varepsilon\text{-a.s.}$$

■

Proposition 5.2 $\overline{\lim}_{\varepsilon\downarrow 0} Y_0^\varepsilon \leq Q_0$.

Proof As in Proposition 4.1, we may show that there exists a process Z^ε such that for any $T_1 < T$:

$$Y_t^\varepsilon = Y_{T_1}^\varepsilon - \frac{1}{2}\int_t^{T_1} |Z_s^\varepsilon|^2 ds + \int_t^{T_1} Z_s^\varepsilon \cdot dB_s, \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

Define a sequence of BSDEs:

$$\overline{Y}_t^{\varepsilon,T_1} = \frac{Kd(X_{T_1}, O^c)}{T-T_1} - \frac{1}{2}\int_t^{T_1} |Z_s^{\varepsilon,T_1}|^2 ds + \int_t^{T_1} Z_t^{\varepsilon,T_1} \cdot dB_s, \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

Note that $Y_{T_1\wedge H}^\varepsilon \le \frac{Kd(X_{T_1\wedge H}, O^c)}{T-T_1\wedge H} \le \frac{Kd(X_{T_1}, O^c)}{T-T_1}$. By Lemma 5.1 and the comparison principle of BSDE, we deduce that

$$Y_0^\varepsilon \;\le\; \overline{Y}_0^{\varepsilon,T_1} \;\text{ for all }\; T_1 < T.$$

Since $\xi(x) := \frac{Kd(x_{T_1}, O^c)}{T-T_1}$ is bounded and uniformly continuous, it follows from Theorem 2.2 that

$$\lim_{\varepsilon\to 0} \overline{Y}_0^{\varepsilon,T_1} = y_0^{T_1} := \inf_{\alpha\in\mathbb{L}^2}\Big\{\frac{1}{2}\int_0^{T_1} \alpha_t^2 dt + \frac{Kd(x_{T_1}^\alpha, O^c)}{T-T_1}\Big\}.$$

Thus, we have

$$\overline{\lim_{\varepsilon\downarrow 0}}\, Y_0^\varepsilon \;\le\; \inf_{\alpha\in\mathbb{L}^2}\Big\{\frac{1}{2}\int_0^{T_1} \alpha_t^2 dt + \frac{Kd(x_{T_1}^\alpha, O^c)}{T-T_1}\Big\} \;\le\; \inf_{\alpha\in\mathbb{L}^2, x_{T_1}^\alpha\notin O}\Big\{\frac{1}{2}\int_0^{T} \alpha_t^2 dt\Big\}.$$

Finally, observe that

$$\inf_{\alpha\in\mathbb{L}^2, x_{T_1}^\alpha\notin O}\Big\{\frac{1}{2}\int_0^{T} \alpha_t^2 dt\Big\} = \inf_{\alpha\in\mathbb{L}^2, x_{T_1\wedge\cdot}^\alpha\notin \mathcal{O}}\Big\{\frac{1}{2}\int_0^{T} \alpha_t^2 dt\Big\} \longrightarrow Q_0, \quad \text{as} \;\; T_1 \to T.$$

■

To complete the proof of Theorem 2.5, we next complement the result of Proposition 5.2 by the opposite inequality.

Proposition 5.3 $\underline{\lim}_{\varepsilon\downarrow 0} Y_0^\varepsilon \ge Q_0$.

Proof We organize the proof in three steps.

1. Define another sequence of BSDEs:

$$\underline{Y}_t^{\varepsilon,T_1,m} = md(X_{T_1}, O^c)\wedge Y_{T_1}^\varepsilon - \frac{1}{2}\int_t^{T_1} |Z_s^{\varepsilon,T_1,m}|^2 ds + \int_t^{T_1} Z_t^{\varepsilon,T_1,m} \cdot dB_s, \; \mathbb{P}^\varepsilon\text{-a.s.}$$

By comparison of BSDEs, we have that $\underline{Y}_t^{\varepsilon,T_1,m} \le Y_t^\varepsilon$ for all $t \le T_1$. Then, by the stability of BSDEs, we know that $Y^{\varepsilon,T_1,m}$ converge to the solution of the following BSDE as $T_1 \to T$:

$$\underline{Y}_t^{\varepsilon,m} = md(X_T, O^c) - \frac{1}{2}\int_t^T |Z_s^{\varepsilon,m}|^2 ds + \int_t^T Z_t^{\varepsilon,m} \cdot dB_s, \quad \mathbb{P}^\varepsilon\text{-a.s.}$$

Again, we may apply Theorem 2.2 and get that

$$\varliminf_{\varepsilon\downarrow 0} Y_0^\varepsilon \ge \lim_{\varepsilon\downarrow 0} \underline{Y}_0^{\varepsilon,m} = y_0^m := \inf_{\alpha\in\mathbb{L}^2}\Big\{\frac{1}{2}\int_0^T \alpha_s^2 ds + md(x_T^\alpha, O^c)\Big\}. \tag{5.18}$$

2. We now prove that the sequence $\big(y_0^m\big)_m$ is bounded. Take $\alpha_t \equiv C \cdot 1$. Then

$$x_T^\alpha = x_0 + \int_0^T (b_t + C\sigma_t \cdot 1) dt.$$

Since b is bounded and σ is positive, when $C = C_0$ is sufficiently large, we will have $x_T^\alpha \notin O$. Hence, $y_0^m \le \frac{1}{2}C_0^2 T d$.

3. In view of (5.18), we now conclude the proof of the proposition by verifying that $y_0^m \longrightarrow Q_0$, as $m \to \infty$. Let $\rho > 0$. By the definition of y_0^m, there is a ρ-optimal α^ρ:

$$y_0^m + \rho > \frac{1}{2}\int_0^T |\alpha_t^\rho|^2 dt + md(x_T^\rho, O^c),$$

where we denoted $x^\rho := x^{\alpha^\rho}$. By the boundedness of $(y_0^m)_m$ in Step 2, we have $d(x_T^\rho, O^c) \le \frac{C}{m}$. So, there exists a point $x_0 \in \partial O$ such that $|x_T^\rho - x_0| \le \frac{C}{m}$. Define:

$$\tilde{\alpha}_t := \alpha_t^\rho + \sigma_t^{-1}\frac{x_0 - x_T^\rho}{T}.$$

Then, $x_T^{\tilde{\alpha}} = x_0 \notin O$. Also, note that $\sigma_t^{-1}\frac{x_0 - x_T^\rho}{T} = o(\frac{1}{m})$ when $m \to \infty$. Hence,

$$\frac{1}{2}\int_0^T |\alpha_t^\rho|^2 dt = \frac{1}{2}\int_0^T |\tilde{\alpha}_t - \sigma_t^{-1}\frac{x_0 - x_T^\rho}{T}|^2 dt \ge \inf_{\alpha\in\mathbb{L}^2, x_T^\alpha\notin O}\Big\{\frac{1}{2}\int_0^T |\alpha_t|^2 dt\Big\} + o(\frac{1}{m}).$$

Finally, sending $m \to \infty$, we see that $\lim_{m\to\infty} y_0^m + \rho \ge Q_0$. Since ρ is arbitrary, the proof is complete. ■

6 Viscosity property of the candidate solution

This section is devoted to prove Theorem (2.11).

Lemma 6.1 *Fix $K \geq 0$. There exists a constant C such that for any $t \in [0,T]$ and $\hat{\omega}^1, \hat{\omega}^2 \in \hat{\Omega}$,*

$$\sup_{\alpha: \int_t^T |\alpha|_s^2 ds \leq K} \|\hat{\omega}^{\alpha,t,\hat{\omega}^1} - \hat{\omega}^{\alpha,t,\hat{\omega}^2}\| \leq C\|\hat{\omega}^1 - \hat{\omega}^2\|_t$$

Proof By the definition of $\hat{\omega}^{\alpha,t,\hat{\omega}^i}$ ($i = 1,2$), we know that the components $\omega^{\alpha,t,\hat{\omega}^i}$ are equal. The difference comes from the component $x^{\alpha,t,\hat{\omega}^i}$. Denote $\delta x_t := \|x^{\alpha,t,\hat{\omega}^1} - x^{\alpha,t,\hat{\omega}^2}\|_t^2$. Then, by the definition of $x^{\alpha,t,\hat{\omega}^i}$ and the Lipschitz continuity of b and σ, we obtain that

$$\begin{aligned}
\delta x_s &\leq \int_0^s C(\|\hat{\omega}^1 - \hat{\omega}^2\|_t^2 + \delta x_r)dr + C\Big(\int_0^s (\|\hat{\omega}^1 - \hat{\omega}^2\|_t + \delta x_r)|\alpha_r|dr\Big)^2 \\
&\leq \int_0^s C(\|\hat{\omega}^1 - \hat{\omega}^2\|_t^2 + \delta x_r)dr + 2KC(\int_0^s (\|\hat{\omega}^1 - \hat{\omega}^2\|_t^2 + \delta x_r)dr)
\end{aligned}$$

Finally, the claim results from the Gronwall's inequality. ∎

By standard argument, one may easily show the following dynamic programming for the optimal control problem (2.9).

Lemma 6.2 (Dynamic programming) *Let u be the value function defined in* (2.9). *Then, for all $0 \leq t \leq s \leq T$ and $\hat{\omega} \in \hat{\Omega}$, we have*

$$u(t,\hat{\omega}) = \inf_{\alpha \in \mathbb{L}_d^2} \Big\{ \frac{1}{2}\int_t^s |\alpha_s|^2 ds + u^{t,\hat{\omega}}(s-t, \hat{\omega}^{\alpha,t,\hat{\omega}}) \Big\},$$

where $u^{t,\hat{\omega}}(t', \hat{\omega}') := u(t+t', \hat{\omega} \otimes_t \hat{\omega}')$.

Lemma 6.3 *The function u defined in (2.9) is bounded and Lipschitz-continuous.*

Proof Clearly, u inherits the bound of ξ. For $t \in [0,T]$, $\hat{\omega}^1, \hat{\omega}^2 \in \hat{\Omega}$, since ξ is bounded, there exists constant K such that

$$\begin{aligned}
u_t(\hat{\omega}^i) &= \inf_{\alpha \in \mathbb{L}_d^2} \Big\{ \frac{1}{2}\int_t^T |\alpha_s|^2 ds + \xi^{t,\hat{\omega}^i}(\hat{\omega}^{\alpha,t,\hat{\omega}^i}) \Big\} \\
&= \inf_{\alpha: \int_t^T |\alpha|_s^2 ds \leq K} \Big\{ \frac{1}{2}\int_t^T |\alpha_s|^2 ds + \xi^{t,\hat{\omega}^i}(\hat{\omega}^{\alpha,t,\hat{\omega}^i}) \Big\}.
\end{aligned}$$

It follows from Lemma 6.1 that:

$$\big|u(t,\hat{\omega}^1) - u(t,\hat{\omega}^2)\big| \leq \sup_{\alpha: \int_t^T |\alpha|_s^2 ds \leq K} \big\{ \big|\xi^{t,\hat{\omega}^1}(\hat{\omega}^\alpha) - \xi^{t,\hat{\omega}^2}(\hat{\omega}^\alpha)\big| \big\} \leq C\big\|\hat{\omega}^1_{t\wedge\cdot} - \hat{\omega}^2_{t\wedge\cdot}\big\|. \quad (6.19)$$

On the other hand, fixing $\hat{\omega}$, it follows from the dynamic programming principle that

$$u(t+h,\hat{\omega}_{t\wedge\cdot})-u(t,\hat{\omega})=\sup_{\alpha\in\mathbb{L}^2}\Big\{-\frac{1}{2}\int_t^{t+h}\alpha_s^2ds-u^{t,\hat{\omega}}(h,\hat{\omega}^{\alpha,t,\hat{\omega}})+u(t+h,\hat{\omega}_{t\wedge\cdot})\Big\}\geq 0, \tag{6.20}$$

where the last inequality is induced by the constant control $\alpha=0$. Moreover, since b and σ are bounded, note that $\|(\hat{\omega}\otimes_t\hat{\omega}^{\alpha,t,\hat{\omega}})_{(t+h)\wedge\cdot}-\hat{\omega}_{t\wedge\cdot}\|\leq C\int_t^{t+h}(1+|\alpha_s|)ds$. Then, using again the dynamic programming principle together with (6.19), we obtain

$$u(t+h,\hat{\omega}_{t\wedge\cdot})-u(t,\hat{\omega})\leq\sup_{\alpha\in\mathbb{L}^2}\Big\{\int_t^{t+h}\Big(-\frac{1}{2}\alpha_s^2+C|\alpha_s|+C\Big)ds\Big\}\leq\Big(\frac{C^2}{2}+C\Big)h. \tag{6.21}$$

Combining this with (6.19), we see that

$$\begin{aligned}\Big|u(t+h,\hat{\omega}^1)-u(t,\hat{\omega}^2)\Big| &\leq \Big|u(t+h,\hat{\omega}^1)-u(t+h,\hat{\omega}^1_{t\wedge\cdot})\Big|\\ &\quad+\Big|u(t+h,\hat{\omega}^1_{t\wedge\cdot})-u(t,\hat{\omega}^1)\Big|+\Big|u(t,\hat{\omega}^1)-u(t,\hat{\omega}^2)\Big|\\ &\leq C'(\|\hat{\omega}^1\|_t^{t+h}+h+\|\hat{\omega}^1_{t\wedge\cdot}-\hat{\omega}^2_{t\wedge\cdot}\|)\\ &\leq 3C'(h+\|\hat{\omega}^1_{(t+h)\wedge\cdot}-\hat{\omega}^2_{t\wedge\cdot}\|).\end{aligned}$$

■

Now, consider a functional u_K:

$$u_K(t,\hat{\omega}):=\inf_{\|\alpha\|_\infty\leq K}\Big[\xi(\hat{\omega}\otimes_t\hat{\omega}^{\alpha,t,\hat{\omega}})+\frac{1}{2}\int_t^T|\alpha_s|^2ds\Big];$$

Notice that $u_K\geq u_{K-1}\geq u$.

Proposition 6.4 *For K sufficiently large, we have $u=u_K$.*

Proof Similar to Lemma 6.3, for each K, one may easily see that $u_K(t,\cdot)$ is uniformly Lipschitz in ω with the same Lipschitz constant denoted as L. We first claim that there exists α^K such that

$$u_K(0,0)=\xi(\hat{\omega}^{\alpha^K})+\frac{1}{2}\int_0^T|\alpha_t^K|^2dt. \tag{6.22}$$

Then for any t and h, one can easily show that

$$u_K(t,\hat{\omega}^{\alpha^K})=u_K(t+h,\hat{\omega}^{\alpha^K})+\frac{1}{2}\int_t^{t+h}|\alpha_s^K|^2ds.$$

On the other hand, by the dynamic programming,

$$u_K(t, \hat{\omega}^{\alpha^K}) \le u_K(t+h, \hat{\omega}^{\alpha^K}_{t\wedge\cdot}).$$

Then

$$\begin{aligned}\frac{1}{2}\int_t^{t+h} |\alpha^K_s|^2 ds &\le u_K(t+h, \hat{\omega}^{\alpha^K}_{t\wedge\cdot}) - u_K(t+h, \hat{\omega}^{\alpha^K}) \\ &\le L\|\hat{\omega}^{\alpha^K} - \hat{\omega}^{\alpha^K}_{t\wedge\cdot}\|_{t+h} \le CL\int_t^{t+h}(1+|\alpha^K_s|)ds,\end{aligned}$$

where C is a common bound for the coefficients b and σ. Since t and h are arbitrary, we get $\|\alpha^K\|_\infty \le C'$ for some constant C' independent of K. Then $u_K = u_{C'}$ for any $K \ge C'$, and thus $u = u_{C'}$.

We now prove the existence claim (6.22). Let $\alpha^{K,n}$ be a minimum sequence of controls for $u_K(0,0)$, namely

$$u_K(0,0) = \lim_{n\to\infty}\Big[\xi(\hat{\omega}^{\alpha^{K,n}}) + \frac{1}{2}\int_0^T |\alpha^{K,n}_t|^2 dt\Big]. \tag{6.23}$$

By compactness of Ω_K, the sequence $\{\omega^{\alpha^{K,n}}, n \ge 1\}$ has a limit $\omega^K \in \Omega_K$, after possibly passing to a subsequence:

$$\lim_{n\to\infty}\|\omega^{\alpha^{K,n}} - \omega^K\|_T = 0. \tag{6.24}$$

By (6.23) and since ξ is bounded, it is clear that $\sup_n \int_0^T |\alpha^{K,n}_t|^2 dt < \infty$. Then without loss of generality we may assume $\{\alpha^{K,n}, n \ge 1\}$ converges to certain α^K weakly in $\mathbb{L}^2([0,T])$. Then for any t and h,

$$\omega^K_{t+h} - \omega^K_t = \lim_{n\to\infty}[\omega^{\alpha^{K,n}}_{t+h} - \omega^{\alpha^{K,n}}_t] = \lim_{n\to\infty}\int_t^{t+h}\alpha^{K,n}_s ds = \int_t^{t+h}\alpha^K_s ds.$$

This implies that $\omega^K = \omega^{\alpha^K}$. Further, by Gronwall's inequality, we obtain that

$$\lim_{n\to\infty}\|x^{\alpha^{K,n}} - x^{\alpha^K}\|_T = 0. \tag{6.25}$$

Now by Mazur's lemma, there exist convex combinations $\tilde{\alpha}^{K,n} = \sum_i c^n_i \alpha^{K,m^n_i}$, where $m^n_i \ge n$, such that $\{\tilde{\alpha}^{K,n}, n \ge 1\}$ converges to α^K strongly in $\mathbb{L}^2([0,T])$. Then by Jensen's inequality we see that

$$\int_0^T |\alpha^K_t|^2 dt = \lim_{n\to\infty}\int_0^T |\tilde{\alpha}^{K,n}_t|^2 dt \le \lim_{n\to\infty}\sum_i c^n_i \int_0^T |\alpha^{K,m^n_i}_t|^2 dt$$

On the other hand, by (6.24), (6.25) and since ξ is continuous, we have

$$\xi(\hat{\omega}^{\alpha^K}) = \lim_{n\to\infty} \sum_i c_i^n \xi(\hat{\omega}^{\alpha^{K,m_i^n}}).$$

Then

$$\xi(\hat{\omega}^{\alpha^K}) + \frac{1}{2}\int_0^T |\alpha_t^K|^2 dt \le \lim_{n\to\infty} \sum_i c_i^n \Big[\xi(\hat{\omega}^{\alpha^{K,m_i^n}}) + \frac{1}{2}\int_0^T |\alpha_t^{K,m_i^n}|^2 dt\Big] = u_K(0,0),$$

where the last equality follows from (6.23). This proves the claim. ∎

Clearly our equation (2.5) satisfies the conditions of Lukoyanov [22], so that a comparison result for bounded viscosity super and subsolutions holds true. Conseuently, uniqueness holds for (2.5) within the class of bounded functions and, in order to prove Theorem 2.11 it remains to verify that u satisfies the viscosity properties.

Proof of Theorem 2.11 Fix K_0 such that $u = u_{K_0}$. Recall that b and σ are bounded by C. Then, define $K := C(1+K_0)$, so that for all $\|\alpha\|_\infty \le K_0$ and $\hat{\omega} \in \hat{\Omega}_K$, we have $\hat{\omega}^{\alpha,t,\hat{\omega}} \in \hat{\Omega}_K$.

We first prove the viscosity subsolution property. Let $(t,\hat{\omega}) \in \Theta_K$, and $\varphi \in \underline{\mathcal{A}}^K u(t,\hat{\omega})$. By the dynamic programming principle, we have:

$$u(t,\hat{\omega}) = \inf_{\alpha\in\mathbb{L}^2} \Big\{\frac{1}{2}\int_t^{t+h} \alpha_r^2 dr + u^{t,\hat{\omega}}(h,\hat{\omega}^{\alpha,t,\hat{\omega}})\Big\} \quad \text{for} \quad h \ge 0. \tag{6.26}$$

Since $\varphi \in \underline{\mathcal{A}}^K u(t,\hat{\omega})$, we have for all $\|\alpha\|_\infty \le K_0$:

$$0 \le \frac{1}{2}\int_t^{t+h} |\alpha|_r^2 dr + u^{t,\hat{\omega}}(h,\hat{\omega}^{\alpha,t,\hat{\omega}}) - u(t,\hat{\omega}) \le \frac{1}{2}\int_t^{t+h} |\alpha|_r^2 dr + \varphi^{t,\hat{\omega}}(h,\hat{\omega}^{\alpha,t,\hat{\omega}}) - \varphi(t,\hat{\omega}).$$

By the smoothness of φ, this provides:

$$0 \le \frac{1}{h}\int_0^h \big(\partial_t\varphi + b\partial_x\varphi + \frac{1}{2}|\alpha|^2 + \alpha\cdot(\partial_\omega\varphi + \sigma^{\mathrm{T}}\partial_x\varphi)\big)^{t,\hat{\omega}}(r,\hat{\omega}^{\alpha,t,\hat{\omega}})dr. \tag{6.27}$$

By sending $h \to 0$, we obtain

$$-\Big(\partial_t\varphi + b.\partial_x\varphi + \inf_{|\alpha|\le K_0}\Big(\frac{1}{2}|\alpha|^2 + \alpha\cdot(\partial_\omega\varphi + \sigma^{\mathrm{T}}\partial_x\varphi)\Big)\Big)(t,\hat{\omega}) \le 0.$$

We next prove the viscosity supersubsolution property. Assume not, then there exists $\varphi \in \overline{\mathcal{A}}^K u(t,\hat{\omega})$ such that

$$c := -\Big(\partial_t\varphi + b.\partial_x\varphi + \inf_{|\alpha|\le K_0}\Big(\frac{1}{2}|\alpha|^2 + \alpha\cdot(\partial_\omega\varphi + \sigma^{\mathrm{T}}\partial_x\varphi)\Big)\Big)(t,\hat{\omega}) > 0.$$

Without loss of generality, we may assume that $\varphi(t,\hat{\omega}) = u(t,\hat{\omega})$. Recall that $u = u_{K_0}$. Now for any $h > 0$, by the dynamic programming,

$$\begin{aligned}\varphi(t,\hat{\omega}) = u(t,\hat{\omega}) &= \inf_{\|\alpha\|_\infty \le K_0} \Big[u_h^{t,\hat{\omega}}(\hat{\omega}^{\alpha,t\hat{\omega}}) + \frac{1}{2}\int_t^{t+h} |\alpha_s|^2 ds \Big] \\ &\ge \inf_{\|\alpha\|_\infty \le K_0} \Big[\varphi_h^{t,\hat{\omega}}(\hat{\omega}^{\alpha,t,\hat{\omega}}) + \frac{1}{2}\int_t^{t+h} |\alpha_s|^2 ds \Big].\end{aligned}$$

Then,

$$\begin{aligned}0 &\ge \inf_{\|\alpha\|_\infty \le K_0} \Big[\varphi_h^{t,\hat{\omega}}(\hat{\omega}^{\alpha,t,\hat{\omega}}) - \varphi_t(\hat{\omega}) + \frac{1}{2}\int_t^{t+h} |\alpha_s|^2 ds \Big] \\ &= \inf_{\|\alpha\|_\infty \le K_0} \int_0^h \Big[\partial_t\varphi + b.\partial_x\varphi + \frac{1}{2}|\alpha|^2 + \alpha\cdot(\partial_\omega\varphi + \sigma^{\mathrm{T}}\partial_x\varphi) \Big]^{t,\hat{\omega}}(s,\hat{\omega}^{\alpha,t,\hat{\omega}}) ds \\ &\ge \inf_{\|\alpha\|_\infty \le K_0} \int_0^h \Big[c - C\Big(|\partial_t\varphi^{t,\hat{\omega}}(s,\hat{\omega}^{\alpha,t,\hat{\omega}}) - \partial_t\varphi(t,\hat{\omega})| + |\partial_{\hat{\omega}}\varphi^{t,\hat{\omega}}(s,\hat{\omega}^{\alpha,t,\hat{\omega}}) - \partial_{\hat{\omega}}\varphi(t,\hat{\omega})| \Big) \Big] ds \\ &\ge \Big[c - \rho\big(d_\infty((1+K)h)\big) \Big] h,\end{aligned}$$

which leads to a contradiction by choosing h sufficiently small. ■

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