Extension from Precoloured Sets of Edges

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Abstract

We consider precolouring extension problems for proper edge-colourings of graphs and multigraphs, in an attempt to prove stronger versions of Vizing's and Shannon's bounds on the chromatic index of (multi)graphs in terms of their maximum degree Δ. We are especially interested in the following question: when is it possible to extend a precoloured matching to a colouring of all edges of a (multi)graph? This question turns out to be related to the notorious List Colouring Conjecture and other classic notions of choosability.

1 Introduction

Let $G = (V, E)$ be a (multi)graph and let $\mathcal{K} = [K] = \{1, \ldots, K\}$ be a palette of available colours. (In this paper, a *multigraph* can have multiple edges, but no loops; while a *graph* is always simple.) We consider the following question: given a subset $S \subseteq E$ of edges and a proper colouring of elements of S (i.e., adjacent edges must receive distinct colours) using only colours from $\mathcal{K}$, is there a proper colouring of all edges of G (again using only colours from $\mathcal{K}$) in concordance with the given colouring on S? We may consider the set S as a set of *precoloured* edges, while the full colouring, if it exists, may be considered as *extending* the precolouring. If the set S forms a matching in G, then the precolouring of S may be arbitrary from $\mathcal{K}$.

An early appearance of a problem regarding precolouring extension of edge-colourings can be found in Marcotte and Seymour [25]. Nevertheless, since then its counterpart for vertex-colourings has been more comprehensively studied. We hope to provoke interest in edge-precolouring extension and in the following question especially.

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Question 1.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$, let $\mathcal{K} = [\Delta(G)+\mu(G)]$, and let M be a matching of G precoloured from the palette $\mathcal{K}$. What conditions on G and M ensure that the precolouring of M extends to a proper $\mathcal{K}$-edge-colouring of all of G?

The obvious relationship between edge-precolouring and its vertex counterpart — in which we can see edge-precolouring extension of G as vertex-precolouring extension in its line graph $L(G)$ — yields immediate implications. For us, the distance between two edges in G is their corresponding distance in $L(G)$, i.e., the number of *vertices* contained in a shortest path in G between any of their end-vertices. A *distance-t matching* is a set of edges having pairwise distance greater than t. (This means that a matching is a distance-1 matching, while an induced matching is a distance-2 matching. Any set of edges is a distance-0 matching.) We point out the following consequence of a result of Albertson [1, Thm. 4] (see Subsection 1.2) and Vizing's theorem.

Proposition 2.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Using the palette $\mathcal{K} = [\Delta(G) + \mu(G) + 1]$, any precoloured distance-3 matching can be extended to a proper edge-colouring of all of G.

Albertson and Moore [3, Conj. 1] conjectured that when G is a simple graph, any precoloured distance-3 matching can be extended to a proper edge-colouring of G using the palette $\mathcal{K} = [\Delta(G) + 1]$. We propose a stronger conjecture.

Conjecture 3.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Using the palette $\mathcal{K} = [\Delta(G) + \mu(G)]$, any precoloured distance-2 matching can be extended to a proper edge-colouring of all of G.

Conjecture 3 strengthens Proposition 2 in two ways: we impose a weaker constraint on the distance between precoloured edges, and we use a smaller palette. Evidently, we believe that in edge-precolouring the distance requirement ought to be not as strong as it is for vertex-precolouring extension. In Section 2, however, we show that Conjecture 3 becomes false if we are allowed to precolour a distance-1 rather than a distance-2 matching. In the other direction, one might wonder if a strong enough distance requirement on the precoloured matching yields an improved palette bound, i.e., with palette $[\Delta(G)]$ or $[\chi'(G)]$, where $\chi'(G)$ is the edge-chromatic number of G. This fails however, even for bipartite graphs; see Figure 1. Note that even for trees, Conjecture 3 becomes false if we replace the palette $\mathcal{K}$ by $[\Delta(G)]$ or by $[\chi'(G)]$: consider a star with all leaves subdivided exactly once; see Figure 2.

If true, Conjecture 3 would extend Vizing's theorem [36], which is independently due to Gupta, cf. [18]. A variant of Conjecture 3 was proved by Berge and Fournier [7, Cor. 2] — they showed that extension is guaranteed, even from precoloured distance-1 matchings, provided that all edges of the matching have been precoloured with the same colour.

In this paper, we prove several special cases of Conjecture 3, in particular, for bipartite multigraphs, subcubic multigraphs, and planar graphs of large enough maximum degree. Indeed, for these classes we show that Conjecture 3 holds even when the precoloured set is

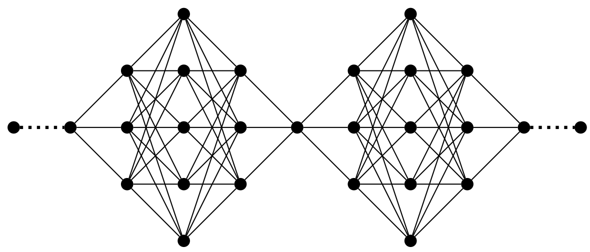

Figure 1: A representative of a class of bipartite graphs G, with a non-extendable matching consisting of two edges, using the palette $[\Delta(G)] = [\chi'(G)]$. The precoloured edges can be made arbitrarily distant. Dashed lines indicate edges precoloured with colour 1. (There is a general construction for $\Delta(G)$ even: within each diamond-shaped block replace each of the two independent sets of size three with one of size $\frac{1}{2}\Delta(G)$ and replace the central independent set of size five with one of size $\Delta(G) - 1$; next increase the distance between the precoloured edges by replacing the two blocks by a chain of arbitrarily many blocks.)

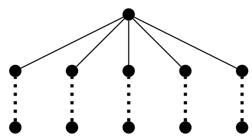

Figure 2: A representative of a class of bipartite planar graphs G, with a non-extendable precoloured (distance-2) matching, using the palette $[\Delta(G)] = [\chi'(G)]$. Dashed lines indicate edges precoloured with colour 1.

allowed to be distance-1 matching. Moreover, we prove a variant of Conjecture 3, where the extended edge-colouring *avoids* some prescribed colours on a (distance-1) matching. We discuss this further in Subsection 1.1. However, first allow us to place the conjecture in context by giving some preliminary observations.

By the following easy observation, Conjecture 3 is also related to list edge-colouring, and therefore to the *List Colouring Conjecture* (LCC), which states that $\mathrm{ch}'(G) = \chi'(G)$ for any multigraph G (where $\mathrm{ch}'(G)$ is, as usual, the list chromatic index of G). For a non-precoloured edge, we define its *precoloured degree* as the number of adjacent precoloured edges.

Proposition 4.
Let G be a multigraph with list chromatic index $\mathrm{ch}'(G)$. For a positive integer k, take the palette as $\mathcal{K} = [\mathrm{ch}'(G) + k]$. If G is properly precoloured so that the precoloured degree of any non-precoloured edge is at most k, then the precolouring can be extended to a proper edge-colouring of all of G.

So, if we assume that the LCC holds, then the following weak form of Conjecture 3 holds as well: using the palette $\mathcal{K} = [\Delta(G) + \mu(G) + 1]$, any precoloured distance-2 matching extends to all of G.

Due to the remarkable work of Kahn [22, 23, 24] on edge-colourings and list edge-colourings of (multi)graphs, not only does an asymptotic form of Conjecture 3 hold, but so does a precolouring extension of an asymptotic form of the Goldberg–Seymour Conjecture (which we review in Subsection 1.2). Kahn's theorem and Proposition 4 together imply the following.

Proposition 5.
For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds. For any multigraph G with $\chi'(G) \geq C_\varepsilon$, any precoloured matching using the palette $\mathcal{K} = [(1+\varepsilon)\chi'(G)]$ can be extended to a proper edge-colouring of all of G.

If we replace $\chi'(G)$ in the statement above by $\Delta(G) + \mu(G)$ or by the Goldberg–Seymour bound, then the statement remains valid, either due to Vizing's theorem or due to another theorem of Kahn.

One of our motivations for the formulation and study of Conjecture 3 comes from the close connections with vertex-precolouring and with the LCC.

1.1 Main Results

Although it appears that the LCC and our conjecture are independent statements, we have obtained several results corresponding to specific areas of success in list edge-colouring. In summary, we confirm Conjecture 3 for bipartite multigraphs, subcubic multigraphs, and planar graphs of large enough maximum degree. We also obtain a precolouring extension variant of Shannon's theorem, and we confirm a relaxed version of Conjecture 3, where the extended edge-colouring *avoids* some prescribed colours on a matching. Furthermore, all of these partial results hold in the more general context where the precoloured set is allowed to be a distance-1 matching, rather than the distance-2 matching required by Conjecture 3. In fact, in this section we mostly present our main results restricted to precoloured matchings, to aid clarity, even when yet more general statements hold.

Our first result is an edge-precolouring extension of Kőnig's theorem that any bipartite multigraph G is $\Delta(G)$-edge-colourable, whereas the subsequent result is an edge-precolouring analogue of Shannon's theorem that any multigraph G is $\lfloor \frac{3}{2}\Delta(G) \rfloor$-edge-colourable.

Theorem 6.
Let G be a bipartite multigraph with maximum degree $\Delta(G)$. With the palette $\mathcal{K} = [\Delta(G)+1]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

As we saw in Figures 1 and 2, the palette size in Theorem 6 is sharp.

Theorem 7.
Let G be a multigraph with maximum degree $\Delta(G)$. With the palette $\mathcal{K} = \left[\lfloor \frac{3}{2}\Delta(G) + \frac{1}{2} \rfloor \right]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

Due to the Shannon multigraphs, this last statement is sharp if $\Delta(G)$ is even, and within 1 of being sharp if $\Delta(G)$ is odd. Theorems 6 and 7 are proved in Section 3 using powerful list colouring tools developed by Borodin, Kostochka and Woodall [11].

The following theorem concerns multigraphs that are subcubic, i.e, of maximum degree at most 3. Note that Theorem 8 improves upon Theorem 7 for $\Delta(G) = 3$.

Theorem 8.
Let G be a subcubic multigraph. With the palette $\mathcal{K} = [4]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

The example we give in Section 2 shows that 3 is the largest value of $\Delta(G)$ for which we are guaranteed that the palette $[\Delta(G) + 1]$ is enough to extend every precoloured matching to a proper edge-colouring of the whole graph. In other words, Theorem 8 is best possible with respect to $\Delta(G)$. A form of Theorem 8, for subcubic simple graphs and with a distance condition on the precoloured matching, was observed by Albertson and Moore [3]. Although the LCC remains open for subcubic graphs, Juvan, Mohar and Škrekovski [20] have made a significant attempt. (They showed that for any subcubic graph G, if lists of 3 colours are given to the edges of a subgraph H with $\Delta(H) \leq 2$ and lists of 4 colours to the other edges, then G has a proper edge-colouring using colours from those lists.)

Theorem 8 is a direct corollary of the following theorem, which may be of interest in its own right. Its proof uses a degree-choosability condition and can be found in Section 4.

Theorem 9.
Let G be a connected multigraph with maximum degree $\Delta(G)$. Choose a non-negative integer k such that $\Delta(L(G)) \leq \Delta(G) + k$, and take the palette as $\mathcal{K} = [\Delta(G) + k]$. If G is properly precoloured so that the precoloured degree of any vertex is at most k, then the precolouring can be extended to a proper edge-colouring of all of G, except in the following cases:

(a) $k = 0$ *and G is a simple odd cycle;*

(b) *G is a triangle with edges of multiplicity m_1, m_2, m_3 and* $k = \min\{m_1, m_2, m_3\} - 1$.

Note that, when restricted to precoloured matchings, this theorem produces weak or limited bounds for larger maximum degree. On the other hand, if we replace every precoloured edge in the example of Figure 2 by a precoloured multi-edge of multiplicity k (or $k + 1$) and a precolouring from $[k]$ (or $[k + 1]$), we see that the palette bound (or precoloured degree condition) is best possible.

In the case where $k = \Delta(L(G)) - \Delta(G)$ in Theorem 9, the number of colours used is equal to the maximum degree of the line graph. In that sense the theorem can be considered as a precolouring extension of Brooks's theorem restricted to line graphs. It is relevant to mention that vertex-precolouring extension versions of Brooks's theorem [2, 6, 28] require, among other conditions, a large minimum distance between the precoloured vertices.

The class of planar graphs could be of particular interest. There is a prominent line of work on (list) edge-colouring for this class, which we discuss further in Subsection 1.2 and Section 5. Our main contributions to this area are the following results, the second one of which can be viewed as a strengthening of another old result of Vizing [37], provided the graph's maximum degree is large enough.

Theorem 10.
Let G be a planar graph with maximum degree $\Delta(G) \geq 19$. Using the palette $\mathcal{K} = [\Delta(G) + 1]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

Theorem 11.
Let G be a planar graph with maximum degree $\Delta(G) \geq 20$. Using the palette $\mathcal{K} = [\Delta(G)]$, any precoloured distance-3 matching can be extended to a proper edge-colouring of all of G.

Due to the trees exhibited in Figure 2, the palette size in Theorem 10 cannot be reduced, while the minimum distance condition in Theorem 11 cannot be weakened. In Section 5, we

	Palette $\mathcal{K}$	Distance t	Max. degree Δ	Reference
1.	$[\Delta+4]$	1	all Δ	Thm. 7 ($\Delta \leq 7$) and Prop. 4 with [8] ($\Delta \geq 8$)
2.	$[\Delta+3]$	1	$\Delta \leq 5$; $\Delta \geq 8$	Thm. 7; Prop. 4 with [8]
3.	$[\Delta+2]$	1	$\Delta \geq 12$	Prop. 4 with [11]
4.	$[\Delta+2]$	2	$\Delta \leq 4$; $\Delta \geq 8$	[21]; Prop. 4 with [8]
5.	$[\Delta+2]$	3	all Δ	Prop. 2
6.	$[\Delta+1]$	1	$\Delta \leq 3$; $\Delta \geq 19$	Thm. 8; Thm. 10
7.	$[\Delta+1]$	2	$\Delta \geq 12$	Prop. 4 with [11]
8.	$[\Delta]$	3	$\Delta \geq 20$	Thm. 11

Table 1: Summary of edge-precolouring extension results for planar graphs with maximum degree Δ, when a distance-t matching M is precoloured using the palette $\mathcal{K}$. See Section 5 for further details how these results can be obtained.

give some more results on when it is possible for a precoloured matching in a planar graph to be extended. A summary of the results is given in Table 1.

Suppose that we would go to any means to obtain an extension form of Vizing's theorem, say, by weakening the precolouring condition. We still let $\mathcal{K} = [K]$ be a palette of available colours. Given a subset $S \subseteq E$ of edges and an arbitrary (i.e., not necessarily proper) colouring of elements of S using only colours from $\mathcal{K}$, is there a proper colouring of all edges of G (using colours from $\mathcal{K}$) that *disagrees* with the given colouring on every edge of S? We may consider the coloured set S as a set of *forbidden* (coloured) edges, while the full colouring, if it can be produced, is called an *avoidance* of the forbidden edges. We can show the following result, which, while it is in one sense weaker than the statement in Conjecture 3, is directly implied by neither the LCC nor other existing precolouring results, implies Vizing's theorem, and provides further evidence in support of Conjecture 3. (This result was stated as a conjecture in an earlier version of this paper.)

Theorem 12.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Using the palette $\mathcal{K} = [\Delta(G) + \mu(G)]$, any forbidden matching can be avoided by a proper edge-colouring of all of G.

We use an aforementioned result of Berge and Fournier and a recolouring argument to prove this theorem in Section 6.

Some basic knowledge of edge-colouring is a prerequisite to the consideration of edge-precolouring extension problems — we provide some related background in the next subsection. To our frustration, many of the major methods for colouring edges (such as Kempe chains, Vizing fans, Kierstead paths, Tashkinov trees) seem to be rendered useless by precoloured edges. Though Conjecture 3 may at first seem as if it should be an "easy extension" of Vizing's theorem, it might well be difficult to confirm (if true). We are keen to learn of related edge-precolouring results independent of current list colouring methodology.

1.2 Further Background

Edge-colouring is a classic area of graph theory. We give a quick overview of some of the most relevant history for our study. The reader is referred to the recent book by Stiebitz, Scheide, Toft and Favrholdt [33] for detailed references and fuller insights. The lower bound $\chi'(G) \geq \Delta(G)$ is obviously true for any multigraph G. Close to a century ago, Kőnig proved that all bipartite multigraphs meet this lower bound with equality. Shannon [32] in 1949 proved that $\chi'(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$ for any multigraph G. Somewhat later, Gupta (as mentioned in [18]) and, independently, Vizing [36] proved that $\chi'(G) \leq \Delta(G) + \mu(G)$ for any multigraph G, so $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ if G is simple. Both the Shannon bound and the Gupta–Vizing bound are tight in general due to the Shannon multigraphs, which are triangles whose multi-edges have balanced multiplicities. (Note however that the latter bound can be improved for specific choices of $\Delta(G)$ and $\mu(G)$, as described in the work of Scheide and Stiebitz [30].)

A notable conjecture on edge-colouring arose in the 1970s, on both sides of the iron curtain. The Goldberg–Seymour Conjecture, due independently to Goldberg [17] and Seymour [31], asserts that $\chi'(G) \in \{\, \Delta(G), \Delta(G) + 1, \lceil \rho(G) \rceil \,\}$ for any multigraph G, where

$$\rho(G) = \max\left\{ \frac{2|E(G[T])|}{|T| - 1} \;:\; T \subseteq V,\ |T| \geq 3,\ |T| \text{ odd} \right\}.$$

The parameter $\rho(G)$ is a lower bound on $\chi'(G)$ based on the maximum ratio between the number of edges in H and the number of edges in a maximum matching of H, taken over induced subgraphs H of G. This conjecture remains open and is regarded as one of the most important problems in chromatic graph theory. Perhaps the most outstanding progress on this problem is due to Kahn [23], who established an asymptotic form.

The list variant of edge-colouring can be traced as far back as list colouring itself. The concept of list colouring was devised independently by Vizing [38] and Erdős, Rubin and Taylor [14], with the iron curtain playing its customary role here too. The List Colouring Conjecture (LCC) was already formulated by Vizing as early as 1975 and was independently reformulated several times, a brief historical account of which is given by, e.g., Häggkvist and Janssen [19]. For more on the LCC, particularly with respect to the probabilistic method, consult the monograph of Molloy and Reed [27]. The results on the LCC most relevant to our investigations also happen to be two of the most striking, both from the mid-1990s. First, Galvin [16] used a beautiful short argument to prove Dinitz's Conjecture (which concerned the extension of partially completed Latin squares), thereby confirming the LCC for bipartite multigraphs. Not long after Galvin's work, Kahn applied powerful probabilistic methods, with inspiration from extremal combinatorics and statistical physics, to asymptotically affirm the LCC [23, 24]. For more background on Kahn's proof, related methods, and improvements, consult [19, 26, 27].

Inspiration for this class of problems may also be taken from list vertex-colouring. For instance, we utilise a degree-choosability criterion due independently to Borodin [9] and Erdős, Rubin and Taylor [14]. See for example a survey of Alon [4] for an excellent (if older) survey on list colouring in somewhat more generality. We should mention that part of the motivation for studying list colouring was to use it to attack other, less constrained colouring problems. The connection has gone back in the other direction as well, as precolouring extension demonstrates.

Activity in the area of precolouring extension increased dramatically as a result of the startling proof by Thomassen of planar 5-choosability [34]; a key ingredient in that proof was a particular type of precolouring extension from some pair of adjacent vertices, according to a specific planar embedding. A little bit later, Thomassen asked about precolouring extension for planar graphs under a more general setup [35]. Eliding the planarity condition, Albertson [1] quickly answered Thomassen's question and proved more: in any k-colourable graph, for any set of vertices with pairwise minimum distance at least 4, any precolouring of that set from the palette $[k+1]$ can be extended to a proper colouring of the entire graph. (This implies Proposition 2 above.)

Since Albertson's seminal work, a large body of research has developed around precolouring extension. But this research has focused almost exclusively on extension of vertex-colourings. One of the few papers we are aware of that deals with edge-precolouring extension is by Marcotte and Seymour [25], in which a different type of necessary condition for extension is studied — curiously, this paper predates the above mentioned activity in vertex-precolouring.

For planar graphs, there has been significant interest in both edge-colouring and list edge-colouring. It is known that planar graphs G with $\Delta(G) \geq 7$ satisfy $\chi'(G) = \Delta(G)$. This was proved in 1965 by Vizing [37] in the case $\Delta(G) \geq 8$, and much later by Sanders and Zhao [29] for $\Delta(G) = 7$. We remark that Theorem 11 strengthens this for $\Delta(G)$ somewhat larger. Vizing conjectured that the same can be said for planar graphs G with $\Delta(G) = 6$, but this long-standing question remains open. Vizing also noted that not every planar graph G with $\Delta(G) \in \{4,5\}$ is $\Delta(G)$-edge-colourable. Regarding list edge-colouring, Borodin, Kostochka and Woodall [11] proved the LCC for planar graphs with maximum degree at least 12, i.e., they proved that such graphs have list chromatic index equal to their maximum degree. The LCC remains open for planar graphs with smaller maximum degree, though it is known that if $\Delta(G) \leq 4$, or $\Delta(G) \geq 8$, then $\mathrm{ch}'(G) \leq \Delta(G) + 1$. This is due to Juvan, Mohar and Škrekovski [21], and to Bonamy [8] and Borodin [10], respectively. As noted above, it is not true that planar graphs G with $\Delta(G) \in \{4,5\}$ are always $\Delta(G)$-edge-choosable.

2 Necessity of the Distance-2 Condition

In this section, we show that if we omit the distance-2 condition on the precoloured matching then Conjecture 3 becomes false whenever $\Delta(G) \geq 4$. For each $t \geq 3$, we construct a graph G_t of maximum degree $t+1$ with the property that, using the palette $\mathcal{K} = [t+2]$, there is a matching M and a precolouring of M that cannot be extended to a proper edge-colouring of all of G_t.

Our construction is based on an observation by Anstee and Griggs [5]. For $t \geq 3$, let H_t be the graph obtained from $K_{t,t}$ by subdividing one edge.

Lemma 13 (Anstee and Griggs [5]).
For every $t \geq 3$, the equality $\chi'(H_t) = \Delta(H_t) + 1 = t + 1$ holds.

Proof. Since H_t has $2t+1$ vertices, its largest matching has size t. Since H_t has t^2+1 edges, we cannot cover all the edges with t matchings. □

Let $A, B \subseteq V(H_t)$ be the original partite sets of $K_{t,t}$, so that A and B are independent

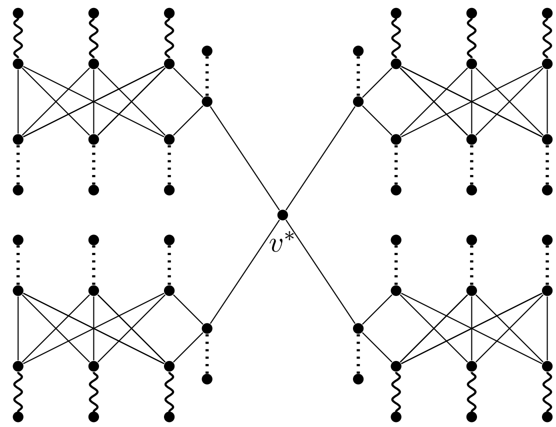

Figure 3: A graph G_3 with maximum degree 4 and a non-extendable precoloured matching using the palette [5]. Wavy edges are precoloured 1, while dotted edges are precoloured 2.

sets of size t in H_t, and the only vertex of H_t not contained in $A \cup B$ is the vertex of degree 2. Let H_t' be the graph obtained from H_t by attaching a pendant edge to each vertex of H_t, and for each $v \in V(H_t)$, let v' be the other endpoint of the pendant edge at v. Finally, set $M_0 = \{ vv' : v \in V(H_t) \}$. We precolour the matching M_0 by colouring vv' colour 1 if $v \in A$, and colouring vv' colour 2 otherwise. Now we define the full graph G_t by taking $t+1$ disjoint copies of H_t', and adding a new vertex v^* adjacent to the unique vertex of degree 3 in each copy of H_t. The precoloured matching M in G is just the union of each precoloured matching M_0 in each copy of H_t', with the same precolouring. Figure 3 shows G_3.

Theorem 14.
For every $t \geq 3$, using the palette $\mathcal{K} = [t+2] = [\Delta(G_t) + \mu(G_t)]$, the precolouring of the matching M as described above cannot be extended to a proper edge-colouring of all of G_t.

Proof. Suppose to the contrary that G_t has an edge-colouring from $\mathcal{K}$ that extends the precolouring of M. Since every neighbour of v^* has an incident edge precoloured 2, no edge incident to v^* can be coloured 2. Therefore, since $d(v^*) = t+1$, each of the $t+1$ colours excluding 2 is used exactly once on the edges incident to v^*. In particular, some edge e incident to v^* has colour 1. Let H be the copy of H_t containing the other endpoint of e. Observe that no edge of H can be coloured 1 or 2: every edge joining A and B has an edge precoloured 1 at one endpoint and an edge precoloured 2 at the other, while the vertex of degree 2 in H is incident to an edge precoloured 2 as well as the edge e coloured 1. Hence all edges of H use only the t remaining colours. Since $\chi'(H_t) = t+1$ by Lemma 13, this is impossible. □

3 Extensions of Kőnig's and Shannon's Theorems

Theorem 15 below implies Theorem 6, and hence verifies Conjecture 3 for bipartite multigraphs. Theorem 16 implies Theorem 7. Recall that the precoloured degree of a vertex is the number of incident precoloured edges.

Theorem 15.
Let G be a bipartite multigraph and $k \geq 1$. Take the palette as $\mathcal{K} = [\Delta(G)+k]$. If G is properly

precoloured so that the precoloured degree of any vertex is at most k, then this precolouring can be extended to a proper edge-colouring of all of G.

Theorem 16.
Let G be a multigraph and $k \geq 1$. Take the palette as $\mathcal{K} = \left[\, \lfloor \frac{3}{2}\Delta(G) + \frac{1}{2}k \rfloor \,\right]$. If G is properly precoloured so that the precoloured degree of any vertex is at most k, then this precolouring can be extended to a proper edge-colouring of all of G.

The two results are corollary to two theorems of Borodin, Kostochka and Woodall [11]. Let G be a multigraph and $f : E(G) \to \mathbf{Z}^+$. We say G is *f-edge-choosable* if for any assignment of lists where every edge e receives a list of size at least $f(e)$, there is a proper edge-colouring of G using colours from the lists.

Theorem 17 (Borodin, Kostochka & Woodall [11])**.**
Let G be a bipartite multigraph, and set $f(uv) = \max\{d(u), d(v)\}$ for each edge $uv \in E(G)$. Then G is f-edge-choosable.

Theorem 18 (Borodin, Kostochka & Woodall [11])**.**
Let G be a multigraph, and set $f(uv) = \max\{d(u), d(v)\} + \lfloor \frac{1}{2}\min\{d(u), d(v)\} \rfloor$ for each edge $uv \in E(G)$. Then G is f-edge-choosable.

Note that Theorem 17 is a strengthening of Galvin's theorem; while Theorem 18 is a list colouring version of Shannon's theorem (and in fact follows from Theorem 17).

Proof of Theorems 15 and 16. For a vertex v in a properly precoloured multigraph G, let $k(v)$ be the number of precoloured edges incident with v. Let G' be obtained from G by deleting all precoloured edges. To each edge $e \in E(G')$, assign a list $\ell(e)$ consisting of those colours in $\mathcal{K}$ that do not appear on precoloured edges adjacent to e in G.

Next consider any uncoloured edge $e = uv$, and assume that $d_G(u) - k(u) = d_{G'}(u) \leq d_{G'}(v) = d_G(v) - k(v)$.

In the bipartite case, since $\Delta(G) \geq d_G(v)$ and $k \geq k(u)$, we infer that

$$\begin{aligned} |\ell(e)| &\geq (\Delta(G) + k) - k(u) - k(v) = (\Delta(G) - k(v)) + (k - k(u)) \\ &\geq d_{G'}(v) = \max\{\, d_{G'}(u), d_{G'}(v) \,\}. \end{aligned}$$

As this holds for every edge $e = uv$ in G', Theorem 17 guarantees a colouring of the edges of G' from their lists. This colouring is an extension of the precolouring of G, completing the proof of Theorem 15.

In the general case, again because $\Delta(G) \geq \max\{\, d_G(u), d_G(v) \,\}$ and $k \geq k(u)$, we infer that

$$\begin{aligned} |\ell(e)| &\geq \left\lfloor \tfrac{3}{2}\Delta(G) + \tfrac{1}{2}k \right\rfloor - k(u) - k(v) = (\Delta(G) - k(v)) + \left\lfloor \tfrac{1}{2}\Delta(G) + \tfrac{1}{2}k - k(u) \right\rfloor \\ &\geq d_{G'}(v) + \left\lfloor \tfrac{1}{2}(\Delta(G) - k(u)) \right\rfloor \geq d_{G'}(v) + \left\lfloor \tfrac{1}{2} d_{G'}(u) \right\rfloor \\ &= \max\{\, d_{G'}(u), d_{G'}(v) \,\} + \left\lfloor \tfrac{1}{2}\min\{\, d_{G'}(u), d_{G'}(v) \,\} \right\rfloor. \end{aligned}$$

Analogously to the previous paragraph, we use Theorem 18 to complete the proof of Theorem 16. $\square$

4 An Approach using Gallai Trees

In this section, we use a slight refinement of a result due independently to Borodin [9] and Erdős, Rubin and Taylor [14]. This is a list version of an older result of Gallai [15] on colour-critical graphs, and it follows for instance from Theorem 4.2 in [35]. A connected graph all of whose blocks are either complete graphs or odd cycles is called a *Gallai tree.*

Theorem 19 (Borodin [9], Erdős, Rubin & Taylor [14]).
Given a connected graph $G = (V, E)$, let $\ell(v)$, for $v \in V$, be an assignment of lists where each vertex v receives at least $d(v)$ colours. Then there is a proper colouring of G using colours from the lists, unless G is a Gallai tree and $|\ell(v)| = d(v)$ for all v.

With this we prove Theorem 9, which implies Theorem 8.

Proof of Theorem 9. Assume to the contrary that the connected multigraph G and the non-negative integer k satisfy $\Delta(L(G)) \leq \Delta(G) + k$, but that, using the palette $\mathcal{K} = [\Delta(G) + k]$, there is a proper edge-precolouring of G of the required type that does not extend to a proper edge-colouring of G. For a vertex v, let $K(v) \subseteq \mathcal{K}$ be the set of colours appearing on the precoloured edges incident with v, and set $k(v) = |K(v)|$.

Let G' be obtained from G by deleting all precoloured edges. To each edge $e = uv$ in G', we assign a list $\ell(e)$ containing those colours in $\mathcal{K}$ not appearing on precoloured edges adjacent to e in G. For any edge $e = uv$ in G' we obtain, using that $\Delta(G) + k \geq \Delta(L(G)) \geq d_{L(G)}(e)$,

$$\begin{aligned} |\ell(e)| &= |\mathcal{K}| - |K(u) \cup K(v)| = (\Delta(G) + k) - |K(u) \cup K(v)| \\ &\geq d_{L(G)}(e) - |K(u) \cup K(v)| \geq d_{L(G')}(e). \end{aligned} \tag{1}$$

Since there is no extension of the precolouring of $L(G)$ to a full colouring of $L(G)$, it follows that $L(G')$ is not vertex-choosable with the lists $\ell(e)$, for $e \in E(G')$. In particular, there is a component C' of G' such that $L(C')$ is not vertex-choosable with the lists $\ell(e)$, for $e \in E(C')$. By Theorem 19, $L(C')$ must be a Gallai tree such that $|\ell(e)| = d_{L(C')}(e)$ for every e. This also means that we must have equality in all inequalities used to derive (1); in particular:

$$\text{for all } e \in E(C'): \qquad d_{L(G)}(e) = \Delta(L(G)) = \Delta(G) + k; \tag{2a}$$

$$\text{for all } e = uv \in E(C'): \qquad |\ell(e)| = d_{L(C')}(e) = d_{L(G)}(e) - |K(u) \cup K(v)|. \tag{2b}$$

Now note that $d_{C'}(v) + k(v) = d_G(v) \leq \Delta(G)$ for every vertex v. So, analogously to (1) above, we infer that for each edge $e = uv$ in C' the order of $\ell(e)$ is at least the degree in C' of each of its end-vertices:

$$\begin{aligned} |\ell(e)| &= (\Delta(G) + k) - |K(u) \cup K(v)| = (\Delta(G) + k) - k(u) - k(v) + |K(u) \cap K(v)| \\ &\geq d_{C'}(v) + (k - k(u)) + |K(u) \cap K(v)| \geq d_{C'}(v). \end{aligned} \tag{3}$$

We require the following statements.

Claim 1. *Every vertex in C' has at least two neighbours.*

Proof. Suppose to the contrary that the vertex u has the vertex v as its unique neighbour. Then for the edge $e = uv$ we have $d_{L(C')}(e) = d_{C'}(v) - 1$ (this holds even if uv is a multi-edge). But since $|\ell(e)| = d_{L(C')}(e)$, this gives $|\ell(e)| < d_{C'}(v)$, contradicting (3). □

Claim 2. *If C' is a simple odd cycle, then $k = 0$ and $G = C'$.*

Proof. Suppose that C' is a simple odd cycle. If $e = uv$ is an edge in C', then $|\ell(e)| = d_{L(C')}(e) = d_{C'}(v) = 2$. From this we can assume, by permuting the colours, that $\ell(e) = \{1, 2\}$ for every $e \in L(C')$. (Indeed, the only way to assign lists of length 2 to the edges of an odd cycle in such a way that there is no proper colouring of the cycle using colours from the lists is by making all lists identical.) There must also be equality everywhere in (3). Combining that with (2b) means in particular that for every edge $e = uv$ we have $K(u) \cup K(v) = \{\,3, 4, \ldots, \Delta(G) + k\,\}$ and $K(u) \cap K(v) = \varnothing$. By an easy parity argument, we can see that this is only possible if all the sets $K(u)$, for $u \in V(G)$, are empty. This means that $k = 0$ (and $\Delta(G) = 2$). Since G is connected, if there are no precoloured edges, then G can have only one component, which must be C'. □

We continue by considering the case that C' is not an odd cycle. Since line graphs are claw-free, it follows that odd cycle blocks of length at least five are impossible in $L(C')$. We deduce that all blocks of $L(C')$ are cliques. The only way that a leaf block B of $L(C')$ could be part of a nontrivial block structure is if it corresponds to a set of edges in C' that are all incident with a unique vertex, with one of the edges corresponding to the cut-vertex of B. This is ruled out by Claim 1. We conclude that $L(C')$ must itself be a clique. In turn, the only way that a line graph $L(C')$ of a multigraph is a clique is if C' is a star or a triangle, with possibly multiple edges. The first option is ruled out by Claim 1, so C' must be a triangle, possibly with multi-edges.

Let u, v, w be the vertices in C' and set $m = |E(C')|$. Then for all $e \in E(C')$ we have $|\ell(e)| = d_{L(C')} = m - 1$. It is easy to check that with lists of this size, the only way that C' is not edge-choosable is if all the lists are the same. This also means that the sets $K(u) \cup K(v)$, $K(u) \cup K(w)$ and $K(v) \cup K(w)$ are the same.

Let $A(u)$ be the set of colours that appear on precoloured edges incident with u, but not with v or w; define $A(v)$ and $A(w)$ analogously. (In other words, these are colours on the edges that connect C' to the rest of the graph G.) Let D be the set of colours that appear on precoloured edges with end-vertices contained in $\{u, v, w\}$. From (2a) and (2b) we deduce that $|\ell(e)| = d_{L(C')}(e)$ for every edge e in C', which, applied to an edge between u and v, implies that $A(u) \cap A(v) = \varnothing$, $A(u) \cap D = \varnothing$ and $A(v) \cap D = \varnothing$. By symmetry, $A(v) \cap A(w) = \varnothing$, $A(u) \cap A(w) = \varnothing$ and $A(w) \cap D = \varnothing$.

Now recall that all edges in C' must have the same list. Consequently, the disjointedness of the sets $A(u)$, $A(v)$ and $A(w)$ implies that these three sets are empty. Thus we find that there are no precoloured edges between any of u, v, w and the rest of the graph. Since G is connected, it follows that $V(G) = \{u, v, w\}$. Let $m(uv), m(uw), m(vw)$ be the multiplicities of the edges of G. Then $\Delta(L(G)) = m(uv) + m(uw) + m(vw) - 1$, while $\Delta(G) = m(uv) + m(uw) + m(vw) - \min\{\,m(uv), m(uw), m(vw)\,\}$. Since $\Delta(L(G)) = \Delta(G) + k$, we have shown that part (b) of the statement of the theorem holds, completing the proof. □

5 Planar Graphs

In this section, for brevity we usually write Δ for $\Delta(G)$.

In the next subsection we prove Conjecture 3 for planar graphs of large enough maximum degree (at least 19), which is the assertion of Theorem 10. As mentioned earlier, the LCC is known to hold for planar graphs with maximum degree at least 12. This is yet another result of Borodin, Kostochka and Woodall [11]: they indeed show that $\mathrm{ch}'(G) \leq \Delta$ for such graphs G. Combining this with Proposition 4 gives the bounds in lines 3 and 7 of Table 1. Since the former bound will be useful for us later on, let us state it formally.

Observation 20.
Let G be a planar graph with maximum degree $\Delta(G) \geq 12$. Using the palette $\mathcal{K} = [\Delta(G)+2]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

Borodin [10] showed that $\mathrm{ch}'(G) \leq \Delta + 1$ for planar graphs G of maximum degree $\Delta \geq 9$. Recently, Bonamy [8] extended this last statement to the case $\Delta = 8$. Combining this result with Proposition 4, implies that for planar graphs with maximum degree $\Delta \geq 8$ a precoloured matching can be extended to a proper colouring of the entire graph with the palette $[\Delta + 3]$, while a precoloured distance-2 matching can be extended with the palette $[\Delta + 2]$.

For smaller values of Δ, we can use Theorems 7 and 8, and the result of Juvan, Mohar and Škrekovski [21] that $\mathrm{ch}'(G) \leq \Delta(G) + 1$ for a planar graph G with $\Delta(G) \leq 4$, to achieve several of the bounds in Table 1. In particular, it follows that $\Delta + 4$ colours suffice for any planar graph with maximum degree Δ.

The final proof we present is of Theorem 11. As discussed in Subsection 1.2, Vizing conjectured [37] that any planar graph with maximum degree $\Delta \geq 6$ has a Δ-edge-colouring. The examples in Figure 2 show that this statement is false if we allow an adversarial precolouring of a distance-2 matching. But does it remains true with the adversarial precolouring of any distance-3 matching? We prove that this is indeed the case if $\Delta \geq 20$. We expect that this lower bound on Δ can be reduced, though, as noted before, certainly not below 6.

The proofs of Theorems 10 and 11 can be found in the next two subsections. They use a common framework, terminology and notation, which we outline now. Note that both adapt a nice trick of Cohen and Havet [12], which shortens the argument considerably.

Whenever considering a planar graph G, we fix a drawing of G in the plane. (So we really should talk about a *plane graph.*) Because of this fixed embedding we can talk about the *faces* of the graph. If G is connected, then the boundary of any face f forms a closed walk W_f.

We adopt the following notation to classify vertices of a graph G according to degree and incidence with vertices of degree 1. Let V_i be the set of vertices of degree i. Also, identify by $T_i \subseteq V_i$ those vertices of degree i that are adjacent to a vertex of degree 1, and set $U_i = V_i \setminus T_i$. Write $T = \cup_{i\geq 1} T_i$ and $U = V(G) \setminus T$. We also adopt the shorthand notation $V_{[i,j]}$, $U_{[i,j]}$ and $T_{[i,j]}$ to mean, respectively, the sets of vertices in V, U and T with degrees between i and j inclusively.

5.1 Proof of Theorem 10

The statement of Theorem 10 is true for graphs with maximum degree 19 and exactly 19 edges. We use induction on $E(G)$, and proceed with the induction step. We may easily assume that G is connected with at least 20 vertices, since $\Delta \geq 19$. Let M be a precoloured matching.

We first observe that

$$\text{if } uv \in E(G) \setminus M, \text{ then } d(u) + d(v) \geq \Delta + 3. \tag{4}$$

Indeed, suppose that the inequality does not hold for some edge $uv \notin M$. Then, by induction if $\Delta(G - uv) \geq 19$ and by Observation 20 if $\Delta(G - uv) = 18$, there exists an extension of M to a colouring of all $G - uv$ using the palette $\mathcal{K}$. Since at most Δ colours are used on the edges adjacent to uv, we can easily extend the colouring further to uv. It follows from this observation that G has no vertices of degree 2, that every vertex with degree 1 is incident with an edge in M and that any vertex has at most one neighbour of degree 1. We will use these facts often without reference in the remainder of the proof.

For a face f, let $V^-(f) = V(f) \setminus V_1$, and denote by W_f^- the sequence of vertices on the boundary walk W_f after removing vertices from V_1. For a vertex v, let $v_1, v_2, \ldots, v_{d(v)}$ be the neighbours of v, listed in clockwise order according to the drawing of G. Write f_i for the face incident with v lying between the edges vv_i and vv_{i+1} (taking addition modulo $d(v)$ in $\{1, \ldots, d(v)\}$).

If $v \in T$ has a (unique) neighbour in V_1, then we always choose v_1 to be this neighbour. In that case we have $f_{d(v)} = f_1$; we denote that face by f_1 again. Note that it is possible for other faces to be the same as well (if v is a cut-vertex), but we will not identify those multiple names of the same face. So, if $v \in U$, then the faces around v in consecutive order are $f_1, f_2, \ldots, f_{d(v)}$; while, if $v \in T$, then the faces around v are $f_1, f_2, \ldots, f_{d(v)-1}$.

Claim 3. $|V_\Delta| > |V_3|$.

Proof. Consider the set F of edges in $E(G) \setminus M$ with one end-vertex in V_3 and the other in V_Δ. The subgraph with vertex set $V_3 \cup V_\Delta$ and edge set F is bipartite; we assert it is acyclic. For suppose there exists an (even) cycle $C \in F$. By induction if $\Delta(G - C) \geq 19$ and by Observation 20 if $\Delta(G - C) \in \{17, 18\}$, we can extend the precolouring of $E(G) \setminus M$ to $G - C$ using the palette $\mathcal{K}$. But then we can further extend this colouring to the edges of C, since each edge of C is adjacent to only $\Delta - 1$ coloured edges, and even cycles are 2-edge-choosable.

Since each vertex in V_3 is incident with at least two edges in F, we have $|V_\Delta| + |V_3| > |F| \geq 2|V_3|$. The claim follows. $\square$

We use a discharging argument to continue the proof of the theorem. First, let us assign to each vertex v a charge

$\alpha 1$: $\alpha(v) = 3d(v) - 6$,

and to each face f a charge

$\alpha 2$: $\alpha(f) = -6$.

For each vertex v we define $\beta(v)$ as follows.

$\beta 1$: If $v \in V_\Delta$, then $\beta(v) = -2$.

$\beta 2$: If $v \in V_3$, then $\beta(v) = 2$.

$\beta 3$: In all other cases, $\beta(v) = 0$.

For each vertex v and edge $e = vu$, we define $\gamma_e(v)$ and $\gamma_e(u)$ as follows.

$\gamma 1$: If $v \in V_1$, then $\gamma_e(v) = -\gamma_e(u) = 3$.

$\gamma 2$: If $v, u \notin V_1$, then $\gamma_e(v) = \gamma_e(u) = 0$.

Finally, for each face f and vertex $v \in W_f^-$ we define $\delta_f(v)$ and $\delta_v(f)$ as follows.

$\delta 1$: If $v \in T_3$, then $\delta_v(f) = -\delta_f(v) = 1$.

$\delta 2$: If $v \in U_3$, then $\delta_v(f) = -\delta_f(v) = \frac{5}{3}$.

$\delta 3$: If $v \in T$ and $4 \leq d(v) \leq \Delta - 2$, then $\delta_v(f) = -\delta_f(v) = 3 - \dfrac{6}{d(v)-1}$.

$\delta 4$: If $v \in U$ and $4 \leq d(v) \leq \Delta - 2$, then $\delta_v(f) = -\delta_f(v) = 3 - \dfrac{6}{d(v)}$.

$\delta 5$: If $d(v) \geq \Delta - 1$, $|V^-(f)| = 3$, and both neighbours of v in $V^-(f)$ are vertices in $U_{[3,8]}$ that are joined by an edge in M, then $\delta_v(f) = -\delta_f(v) = 3$.

$\delta 6$: If $d(v) \geq \Delta - 1$, $|V^-(f)| = 3$, and v has a neighbour in $V^-(f) \cap T_{[3,6]}$, then $\delta_v(f) = -\delta_f(v) = \frac{5}{2}$.

$\delta 7$: If $d(v) \geq \Delta - 1$, $|V^-(f)| = 2$, none of $\delta 5$ and $\delta 6$ applies, and v has a neighbour in $V^-(f) \cap U_{[3,5]}$, then $\delta_v(f) = -\delta_f(v) = \frac{9}{4}$.

$\delta 8$: If $d(v) \geq \Delta - 1$, $|V^-(f)| = 3$, and none of $\delta 5$ and $\delta 6$ applies, then $\delta_v(f) = -\delta_f(v) = 2$.

$\delta 9$: If $d(v) \geq \Delta - 1$, $|V^-(f)| \geq 4$, and v has a neighbour in $V^-(f) \cap T_{[3,6]}$, then $\delta_v(f) = -\delta_f(v) = 2$.

$\delta 10$: If $d(v) \geq \Delta - 1$, $|V^-(f)| \geq 4$, and $\delta 9$ does not apply, then $\delta_v(f) = -\delta_f(v) = \frac{3}{2}$.

For a vertex v, write $\gamma(v)$ for the sum of $\gamma_e(v)$ over all edges e that have v as an end-vertex. For a vertex v of degree 1 we set $\delta(v) = 0$. For every other vertex v, write $\delta(v)$ for the sum over the faces f around v of $\delta_f(v)$. Similarly, for a face f, write $\delta(f)$ for the sum over the vertices v on the reduced walk W_f^- around f of the values of $\delta_v(f)$.

By the definitions of γ and δ,

$$\textstyle\sum_v \gamma(v) + \sum_v \delta(v) + \sum_f \delta(f) = 0.$$

It follows from Claim 3 that

$$\textstyle\sum_v \beta(v) < 0.$$

Finally, from Euler's Formula for simple plane graphs, we obtain

$$\textstyle\sum_v \alpha(v) + \sum_f \alpha(f) < 0.$$

Thus, in order to reach a contradiction, it is enough to show that for every vertex v:

$$\alpha(v) + \beta(v) + \delta(v) + \gamma(v) \geq 0, \tag{5}$$

and that for every face f:

$$\alpha(f) + \delta(f) \geq 0. \tag{6}$$

Let f be a face. As G is simple, $|V^-(f)| \geq 3$. Since $\alpha(f) = -6$, to establish (6) it is enough to show that $\delta(f) \geq 6$. Let v be a vertex in $V^-(f)$ for which $\delta_v(f)$ is minimum. If $\delta_v(f) \cdot |V^-(f)| \geq 6$, then (6) clearly holds, and so we only need to deal with cases $\delta 1$–$\delta 4$. Also, if $v \in T_{[7,\Delta-2]} \cup U_{[6,\Delta-2]}$, then $\delta 3$ and $\delta 4$ give $\delta_v(f) \geq 2$, and hence again (6) is verified.

If $v \in T_{[3,4]}$, then $\delta_v(f) \geq 1$ by $\delta 1$ or $\delta 3$ and, by (4), the neighbours u and w of v in $V^-(f)$ have degree at least $\Delta - 1$. If $|V^-(f)| = 3$, then $\delta 6$ applies to both u and w, so

$\delta_u(f) = \delta_w(f) = \frac{5}{2}$. If $|V^-(f)| \geq 4$, then $\delta 9$ applies to both u and w, so $\delta_u(f) = \delta_w(f) = 2$, while a fourth vertex z in $V^-(f)$ satisfies $\delta_z(f) \geq 1$ by the definition of v. So (6) always follows.

If $v \in T_{[5,6]}$, then $\delta_v(f) \geq \frac{3}{2}$, so we may assume that $|V^-(f)| = 3$ (as $\delta_v(f) \leq \delta_u(f)$ whenever $u \in V^-(f)$). Moreover, v has neighbours u, w in $V^-(f)$ with degree at least $\Delta - 3 \geq 9$ as $\Delta \geq 12$. If $d(u) \geq \Delta - 1$, then $\delta 6$ gives $\delta_u(f) = \frac{5}{2}$. If $d(u) \in \{\Delta - 3, \Delta - 2\}$, then $\delta 3$ and $\delta 4$ give $\delta_u(f) \geq \frac{9}{4}$. Since similar bounds hold for $\delta_w(f)$, we deduce that (6) holds.

We are left with the case where $v \in U_{[3,5]}$. By $\delta 2$ and $\delta 4$ we find that $\delta_v(f) \geq \frac{3}{2}$, and hence we again only have to consider the case where $|V^-(f)| = 3$. Rules $\delta 3 - \delta 7$ ensure that any other vertex u in $V^-(f)$ with $d(u) \geq 9$ satisfies $\delta_u(f) \geq \frac{9}{4}$. So we may suppose that there is a vertex $u \in V^-(f)$ with $d(u) \leq 8$. Since $|V^-(f)| = 3$, we must in fact have $uv \in E(G)$. Moreover, as $\Delta \geq 11$, we know by (4) that the edge uv belongs to the matching M. This means that $u \in U_{[3,8]}$. Let w be the third vertex in $V^-(f)$. If $v \in U_{[3,4]}$, then $d(w) \geq \Delta - 1$ by (4), and so $\delta_w(f) = 3$ by $\delta 5$, confirming (6). As the final case, assume that $v \in U_5$ and recall that $\delta_u(f) \geq \delta_v(f) = \frac{9}{5}$. Since also $d(w) \geq \Delta - 2$, one of $\delta 3 - \delta 5$ applies to w, yielding that $\delta_w(f) \geq \frac{12}{5}$. So again $\delta(f) \geq 6$, confirming (6) for all faces.

Now let v be a vertex. Recall the convention that if $v \in T$, then the two consecutive faces incident with v neighbouring the neighbour of degree 1 are counted as one face, while all other faces are counted separately.

If $d(v) = 1$, then $\alpha(v) = -3$ and $\gamma(v) = 3$. Since $\beta(v) = \delta(v) = 0$, we immediately obtain (5).

Recall that G has no vertices of degree 2. If $d(v) = 3$, then $\alpha(v) = 3$, while $\beta(v) = 2$ by $\beta 2$. If $v \in T_3$, then $\gamma(v) = -3$ and $\delta 1$ implies that $\delta(v) = -2$. If $v \in U_3$, then $\gamma(v) = 0$ and $\delta 2$ implies that $\delta(v) = -5$. This confirms (5) if $d(v) = 3$.

Next suppose that $4 \leq d(v) \leq \Delta - 2$. Recall that $\alpha(v) = 3d(v) - 6$, and observe that $\beta(v) = 0$. If $v \in T$, then $\gamma(v) = -3$ by $\gamma 1$. By $\delta 3$ we have $\delta(v) = (d(v)-1)\cdot\left(-3+\dfrac{6}{d(v)-1}\right) = 9 - 3d(v)$. Similarly, if $v \in U$, then $\gamma(v) = 0$, and $\delta 4$ implies that $\delta(v) = 6 - 3d(v)$. This proves (5) for those vertices v.

Now suppose that $d(v) \geq \Delta - 1$. As a next step towards proving (5), we consider the average value of $\delta_f(v)$ over the faces incident with v.

Claim 4. *For any five consecutive faces f_i, f_{i+1},…,f_{i+4} incident with v it holds that*

$$\delta_{f_i}(v) + \delta_{f_{i+1}}(v) + \delta_{f_{i+2}}(v) + \delta_{f_{i+3}}(v) + \delta_{f_{i+4}}(v) \geq -\tfrac{51}{4}$$

(where the addition of the indices is modulo $d(v)$ if $v \in U$, and modulo $d(v) - 1$ if $v \in T$).

Proof. For convenience, set $I = \{ f_i, f_{i+1}, f_{i+2}, f_{i+3}, f_{i+4} \}$. First, we observe that the stated inequality holds if $\delta 5$ applies to no face in I, since then $\delta_f(v) \geq -\frac{5}{2}$ for each $f \in I$ and therefore $\sum_{f\in I} \delta_f(v) \geq -\frac{25}{2} > -\frac{51}{4}$. So we assume that $\delta 5$ applies to at least one face in I. Let f_j be a face incident to v to which $\delta 5$ applies.

We assert that if f' is a face consecutive to f_j around v, (so $f' \in \{f_{j-1}, f_{j+1}\}$), then $\delta_f(v) + \delta_{f'}(v) \geq -\frac{21}{4}$. To prove this, we may assume by symmetry that $f' = f_{j+1}$. The definition of $\delta 5$ implies that v_j and v_{j+1} both belong to $U_{[3,8]}$ and are joined by an edge in M. Therefore, either $|V^-(f')| \geq 4$ or v_{j+2} has degree at least $\Delta - 5 > 6$ by (4). Consequently, none of $\delta 5$ and $\delta 6$ applies to f', which yields the inequality.

A consequence of the previous assertion is that the sought inequality holds if $\delta5$ applies to only one face f in I. Indeed, every other face $f' \in I \setminus \{f\}$ then satisfies $\delta_{f'}(v) \geq -\frac{5}{2}$ and at least one of them is consecutive to f, so $\sum_{k=0}^{4} \delta_{f_{i+k}}(v) \geq -3 - 3 \cdot \frac{5}{2} - \frac{9}{4} = -\frac{51}{4}$.

We now observe that if $\delta5$ applies to both f_j and f_{j+2}, then $\delta_{f_{j+1}}(v) = -\frac{3}{2}$. This follows as above from the definition of $\delta5$: the vertices v_j, v_{j+1}, v_{j+2} and v_{j+3} all belong to $U_{[3,8]}$ and M contains the edges $\{v_j, v_{j+1}\}$ and $\{v_{j+2}, v_{j+3}\}$. Consequently, $|V^-(f_{j+1})| \geq 4$ and $\delta9$ does not apply to f_{j+1} as none of v_{j+1} and v_{j+2} belongs to T. Therefore, $\delta10$ applies to f_{j+1} and the observation follows.

Let us consider several cases depending on which faces $\delta5$ applies to. As proved above, $\delta5$ can apply to at most three faces of I and, if it does, these faces must be f_i, f_{i+2} and f_{i+4}. Then, the previous observation implies that $\delta_{f_{i+1}}(v)$ and $\delta_{f_{i+3}}(v)$ are both $-\frac{3}{2}$, hence the inequality holds.

It remains to deal with the case where $\delta5$ applies to exactly two faces f and f' of I. Since $-3 \cdot 2 - \frac{9}{4} \cdot 3 = -\frac{51}{4}$, we can assume also that $\delta6$ applies to at least one face g in I. Furthermore, we know that no two faces among f, f' and g are consecutive. Therefore, by symmetry, there are only two cases to consider: either $\delta5$ applies to f_i and f_{i+4} and $\delta6$ applies to f_{i+2}, or $\delta5$ applies to f_i and f_{i+2} and $\delta6$ applies to f_{i+4}. In both cases, it suffices to prove that one of $\delta8 - \delta10$ applies to (at least) one face in I.

This follows from our observation above in the first case: $\delta10$ must apply to f_{i+1}. In the latter case, since $\delta6$ applies to f_{i+2} we know that one of v_{i+2} and v_{i+3} belongs to T: we assume by symmetry that $v_{i+2} \in T$. Then, since $\delta6$ does not apply to f_{i+1} by our first assertion, we deduce that either $\delta9$ or $\delta10$ applies to f_{i+1}, which concludes the proof. $\square$

If we add up the values of $\delta_{f_i}(v) + \delta_{f_{i+1}}(v) + \delta_{f_{i+2}}(v) + \delta_{f_{i+3}}(v) + \delta_{f_{i+4}}(v)$ for $i \in \{1, 2, \ldots, d(v)\}$ if $v \in U$, or for $i \in \{1, 2, \ldots, d(v) - 1\}$ if $v \in T$, then Claim 4 yields that

$$\delta(v) = \frac{1}{5} \sum_i \big(\delta_{f_i}(v) + \delta_{f_{i+1}}(v) + \delta_{f_{i+2}}(v) + \delta_{f_{i+3}}(v) + \delta_{f_{i+4}}(v)\big) \geq \begin{cases} -\frac{51}{20} d(v) + \frac{51}{20}, & \text{if } v \in T; \\ -\frac{51}{20} d(v), & \text{if } v \in U. \end{cases} \tag{7}$$

First suppose that $d(v) = \Delta - 1$. Then $\alpha(v) = 3\Delta - 9$ and $\beta(v) = 0$. If $v \in T$, then $\gamma(v) = -3$ and (7) gives $\delta(v) \geq -\frac{51}{20}(\Delta - 1) + \frac{51}{20}$. Since $\Delta \geq 16$, inequality (5) follows. If $v \in U$, then $\gamma(v) = 0$ and $\delta(v) \geq -\frac{51}{20}(\Delta - 1)$. The hypothesis that $\Delta \geq 15$ guarantees that (5) is valid again.

Finally, suppose that $d(v) = \Delta$. Now $\alpha(v) = 3\Delta - 6$ and $\beta(v) = -2$. If $v \in T$, then $\gamma(v) = -3$ and $\delta(v) \geq -\frac{51}{20}\Delta + \frac{51}{20}$. We see that (5) holds, as $\Delta \geq 19$. If $v \in U$, then $\gamma(v) = 0$ and $\delta(v) \geq -\frac{51}{20}\Delta$. So (5) is verified, provided that $\Delta \geq 18$.

This confirms (5) for all vertices and completes the proof of the theorem. $\square$

5.2 Proof of Theorem 11

Recall the notation and terminology introduced in the introduction of this section. The statement of Theorem 11 is true for graphs with maximum degree 20 and exactly 20 edges. We use induction on $E(G)$, and proceed with the induction step. We may easily assume that G is connected with at least 21 vertices, since $\Delta \geq 20$. Let M be a precoloured distance-3 matching.

We first observe that

$$\text{if } uv \in E(G) \setminus M, \text{ then } d(u) + d(v) \geq \Delta + 2. \tag{8}$$

Indeed, suppose that the inequality does not hold for some $uv \notin M$. Then by induction if $\Delta(G - uv) \geq 20$ and by Theorem 10 if $\Delta(G - uv) = 19$, there exists an extension of M to a colouring of $G - uv$ using the palette $\mathcal{K}$. Since at most $\Delta - 1$ colours are used on the edges adjacent to uv, we can easily extend the colouring further to uv. From (8) it follows that every vertex with degree 1 is incident with an edge in M and that if v has degree 2 and $uv \notin M$, then $d(u) = \Delta$. In particular, if a vertex v with degree greater than 1 has a neighbour in T_2, then $d(v) = \Delta$. Moreover, since edges in M are at distance at least 4 in G, a vertex can have at most one neighbour in $V_1 \cup T_2$.

Let V_2' be the set of vertices of degree 2 that are not incident with an edge of M. For a face f, let $V^-(f) = V(f) \setminus (V_1 \cup T_2)$, and let W_f^- be the sequence of vertices on the boundary walk W_f after removing vertices from $V_1 \cup T_2$. For a vertex v, let $v_1, v_2, \ldots, v_{d(v)}$ be the neighbours of v, listed in clockwise order according to the drawing of G. Write f_i for the face incident with v lying between the edges vv_i and vv_{i+1} (taking addition modulo $d(v)$ in $\{1, \ldots, d(v)\}$).

If a vertex v has a (unique) neighbour in $V_1 \cup T_2$, then we always choose v_1 to be this neighbour. In that case $f_{d(v)} = f_1$, and that face is called f_1 again. Note that it is possible for other faces to be the same as well (if v is a cut-vertex), but we will not identify those multiple names of the same face.

Claim 5. $|V_\Delta| > |V_2'|$.

Proof. Consider the set F of edges in $E(G) \setminus M$ with one end-vertex in V_2' and the other in V_Δ. The subgraph with vertex set $V_2' \cup V_\Delta$ and edge set F is bipartite; we assert it is acyclic. For suppose there exists an (even) cycle $C \in F$. By induction if $\Delta(G - C) \geq 20$, by Theorem 10 if $\Delta(G - C) = 19$ and by Observation 20 if $\Delta(G - C) = 18$, we can extend the precolouring of $E(G) \setminus M$ to $G - C$ using the palette $\mathcal{K}$. But then we can further extend this colouring to the edges of C, since each one sees only $\Delta - 2$ coloured edges, and even cycles are 2-edge-choosable.

Since each vertex in V_2' is incident with at least two edges in F, we have $|V_\Delta| + |V_2'| > |F| \geq 2|V_2'|$. The claim follows. $\square$

We use a discharging argument to complete the proof. First, let us assign to each vertex v a charge

$\alpha 1$: $\alpha(v) = 3d(v) - 6$,

and to each face f a charge

$\alpha 2$: $\alpha(f) = -6$.

For each vertex v we define $\beta(v)$ as follows.

$\beta 1$: If $v \in V_\Delta$, then $\beta(v) = -2$.

$\beta 2$: If $v \in V_2'$, then $\beta(v) = 2$.

$\beta 3$: In all other cases, $\beta(v) = 0$.

For each vertex v and edge $e = vu$, we define $\gamma_e(v)$ and $\gamma_e(u)$ as follows.

$\gamma 1$: If $v \in V_1$, then $\gamma_e(v) = -\gamma_e(u) = 3$.

$\gamma 2$: If $v \in T_2$ and $u \in V_\Delta$, then $\gamma_e(v) = -\gamma_e(u) = 3$.

$\gamma 3$: If $v \in U_2 \setminus V_2'$ and $u \in V_\Delta$, then $\gamma_e(v) = -\gamma_e(u) = 2$.

$\gamma 4$: In all other cases, $\gamma_e(v) = 0$.

Finally, for each face f and vertex $v \in W_f^-$ we define $\delta_f(v)$ and $\delta_v(f)$ as follows.

$\delta 1$: If $v \in U_2$, then $\delta_v(f) = -\delta_f(v) = 1$.

$\delta 2$: If $v \in T$ and $3 \le d(v) \le \Delta - 4$, then $\delta_v(f) = -\delta_f(v) = 3 - \dfrac{6}{d(v)-1}$.

$\delta 3$: If $v \in U$ and $3 \le d(v) \le \Delta - 4$, then $\delta_v(f) = -\delta_f(v) = 3 - \dfrac{6}{d(v)}$.

$\delta 4$: If $d(v) \ge \Delta - 3$, $|V^-(f)| = 3$, and both neighbours of v in $V^-(f)$ are joined by an edge in M, then $\delta_v(f) = -\delta_f(v) = 4$.

$\delta 5$: If $d(v) \ge \Delta - 3$ and v has a neighbour in $V^-(f) \cap T_3$, then $\delta_v(f) = -\delta_f(v) = 3$.

$\delta 6$: If $d(v) \ge \Delta - 3$ and none of $\delta 4$ and $\delta 5$ applies, then $\delta_v(f) = -\delta_f(v) = \frac{5}{2}$.

For a vertex v and face f, let $\gamma(v)$, $\delta(v)$ and $\delta(f)$ be defined as in the proof of Theorem 10. By definition we have $\sum_v \gamma(v) + \sum_v \delta(v) + \sum_f \delta(f) = 0$. It follows from Claim 5 that $\sum_v \beta(v) < 0$. From Euler's Formula we obtain $\sum_v \alpha(v) + \sum_f \alpha(f) < 0$.

Thus, in order to reach a contradiction, it is enough to show that for every vertex v:

$$\alpha(v) + \beta(v) + \delta(v) + \gamma(v) \ge 0, \tag{9}$$

and that for every face f:

$$\alpha(f) + \delta(f) \ge 0. \tag{10}$$

Let f be a face. As G is simple, $|V^-(f)| \ge 3$. Since $\alpha(f) = -6$, it follows that (10) is verified if we can show that $\delta(f) \ge 6$. Let v be a vertex in $V^-(f)$ for which $\delta_v(f)$ is minimum. If $\delta_v(f) \cdot |V^-(f)| \ge 6$, then (10) clearly holds. So, by checking $\delta 1 - \delta 6$, we see we only have to consider the case where $v \in T_{[3,6]} \cup U_{[2,5]}$. (Recall that vertices from $V_1 \cup T_2$ do not appear in W_f^-.)

If $v \in U_2$, then let u and w be the neighbours of v. Consider first the case where both u and w have degree Δ. Then they both belong to $V^-(f)$, so (10) follows, since $\delta_v(f) = 1$ and $\delta_u(f) \ge \frac{5}{2}$, $\delta_w(f) \ge \frac{5}{2}$ by $\delta 4 - \delta 6$. Suppose now that u has degree less than Δ, which implies by (8) that $uv \in M$ and, consequently, $vw \notin M$. In particular, $w \in V^-(f)$ and w has degree Δ. Note also that necessarily $u \in V^-(f)$. If $|V^-(f)| = 3$, then $\delta_w(f) = 4$ by $\delta 4$. As $\delta_u(f) \ge \delta_v(f) = 1$, it follows that (10) holds. If $|V^-(f)| \ge 4$, then u has a neighbour u' in $V^-(f) \setminus \{v, w\}$. We assert that $\delta_u(f) + \delta_{u'}(f) \ge \frac{5}{2}$. Indeed, this holds if either of u and u' has degree at least $\Delta - 3$, by $\delta 5$ and $\delta 6$. Otherwise, $\delta 2$ or $\delta 3$ applies to both u and u'. So $\delta_u(f) + \delta_{u'}(f) \ge 6 - \dfrac{6}{d(u)-1} - \dfrac{6}{d(u')-1}$. Since $d(u) \ge 3$, $d(u') \ge 3$ and $d(u') \ge \Delta + 2 - d(u)$, one readily sees that the expression $6 - \dfrac{6}{d(u)-1} - \dfrac{6}{d(u')-1}$ is minimised if one of $d(u)$ and $d(u')$ is 3 and the other one is $\Delta - 1$. We infer that $\delta_u(f) + \delta_{u'}(f) \ge 3 - \dfrac{6}{\Delta - 2} \ge \frac{5}{2}$, because $\Delta \ge 14$. The assertion follows. As a result, we deduce that (10) holds, since $\delta_w(f) \ge \frac{5}{2}$ by $\delta 5$ and $\delta 6$.

If $v \in T_3$, then $\delta_v(f) = 0$, but v has two neighbours in $V^-(f)$ that have degree at least $\Delta - 1$ each. Equation (10) then follows from $\delta 5$.

For the remaining cases we always have $\delta_v(f) \geq 1$. Rules $\delta 2 - \delta 6$ ensure that any vertex $u \in V^-(f)$ with $d(u) \geq 13$ satisfies $\delta_u(f) \geq \frac{5}{2}$; hence there can be at most one such vertex and, in particular, a neighbour u of v in $V^-(f)$ must have degree at most 12. As v itself has degree at most 6, by (8) we have $uv \in M$, which also implies that $u, v \in U$. Hence in particular $v \in U_{[3,5]}$. Let w be the neighbour of v in $V^-(f) \setminus \{u\}$. Since $v \in U_{[3,5]}$ and $vw \notin M$, it necessarily holds that $d(w) \geq \Delta - 3$. If $|V^-(f)| = 3$, then (10) holds by $\delta 4$ since $\delta_u(f) \geq \delta_v(f) \geq 1$. If $|V^-(f)| \geq 4$, then u has a neighbour u' in $V^-(f) \setminus \{v, w\}$, which has degree at least $\Delta + 2 - d(u) \geq 10$. Consequently, $\delta_{u'}(f) \geq \frac{12}{5}$ by $\delta 3$, $\delta 5$ or $\delta 6$. We deduce that (10) holds, as $\delta_w(f) \geq \frac{5}{2}$ by $\delta 5$ or $\delta 6$.

This confirms (10) for all faces.

Now let v be a vertex. Recall that $\alpha(v) = 3d(v) - 6$. Furthermore, if v has a neighbour in $V_1 \cup T_2$, then the two consecutive faces incident with that neighbour are counted as one face; all other faces are counted separately. Finally, as noted earlier, a vertex can have at most one neighbour in $V_1 \cup T_2$

If $d(v) = 1$, then $\alpha(v) = -3$ and $\gamma(v) = 3$. Since $\beta(v) = \delta(v) = 0$, we immediately obtain (9).

If $d(v) = 2$, then $\alpha(v) = 0$. If $v \in T_2$, then both $\gamma 1$ and $\gamma 2$ apply; hence $\gamma(v) = 0$. Again one can check that $\beta(v) = \delta(v) = 0$, confirming (9). Otherwise $v \in U_2$, and $\delta 1$ implies that $\delta(v) \geq -2$, as v is incident with at most two faces. If $v \in V_2'$ as well, then $\beta 2$ yields that $\beta(v) = 2$ and $\gamma(v) = 0$. If $v \notin V_2'$, then $\gamma(v) = 2$ while $\beta(v) = 0$. In either case (9) follows.

Next suppose that $3 \leq d(v) \leq \Delta - 4$. Observe that $\beta(v) = 0$. If $v \in T$, then $\gamma(v) = -3$ by $\gamma 1$. Since v has a neighbour with degree one, we know that v is incident with $d(v) - 1$ regions, and so $\delta 2$ yields that $\delta(v) = (d(v) - 1) \cdot \left(-3 + \dfrac{6}{d(v) - 1}\right) = 9 - 3d(v)$. Similarly, if $v \in U$, then $\gamma(v) = 0$, and $\delta 4$ yields that $\delta(v) = 6 - 3d(v)$. This proves (9) for those vertices v.

Suppose now that $d(v) \in \{\Delta - 3, \Delta - 2, \Delta - 1\}$. Then $\beta(v) = 0$. If $v \in T$, then $\gamma(v) = -3$ by $\gamma 1$. Since M is distance-3, none of $\delta 4$ and $\delta 5$ applies to v, and v is incident with $d(v) - 1$ faces. From $\delta 6$ we deduce that $\delta(v) = -\frac{5}{2}(d(v) - 1)$. Since $d(v) \geq \Delta - 3 \geq 13$, it follows that $3d(v) - 6 - 3 - \frac{5}{2}(d(v) - 1) = \frac{1}{2}d(v) - \frac{13}{2} \geq 0$, and hence (9) is satisfied again. Next assume that $v \in U$, and so $\gamma(v) = 0$. The fact that M is distance-3 ensures that $\delta 6$ applies to at least $d(v) - 1$ faces incident with v. This implies that $\delta(v) \geq -(4 + \frac{5}{2}(d(v) - 1))$. Combined with the assumption that $\Delta \geq 15$, this is always enough to satisfy (9).

Finally, suppose that $d(v) = \Delta$. In this case $\beta(v) = -2$. If $v \in T$, then the distance condition on M ensures that $\gamma(v) = -3$ and $\delta(v) = -\frac{5}{2}(\Delta - 1)$. Since $\Delta \geq 17$ this confirms (9).

So we are left with the case where $v \in U$. Since M is a distance-3 matching, at most one of $\gamma 2$, $\gamma 3$, $\delta 4$ and $\delta 5$ applies. Moreover, if $\gamma 2$ does apply, then $\gamma(v) = -3$, the vertex v is incident with $\Delta - 1$ faces and for all those faces f we have $\delta_f(v) = -\frac{5}{2}$. If $\gamma 3$ does apply, then $\gamma(v) = -2$, the vertex v is incident with Δ faces and for all those faces f we have $\delta_f(v) = -\frac{5}{2}$. If $\delta 4$ or $\delta 5$ does apply, then $\gamma(v) = 0$, the vertex v is incident with Δ faces, for all those faces f but one we have $\delta_f(v) = -\frac{5}{2}$, while for the final face f we have $\delta_f(v) \in \{-4, -3\}$. Finally, if none of $\gamma 2$, $\gamma 3$, $\delta 4$ or $\delta 5$ applies, then $\gamma(v) = 0$, the vertex v is incident with Δ faces and for all those faces f we have $\delta_f(v) = -\frac{5}{2}$. Using that $\Delta \geq 20$, we can check that (9) is satisfied in all cases.

This confirms (9) for all vertices and completes the proof of the theorem. □

6 Avoiding Prescribed Colours on a Matching

In this section, we show the following statement, which directly implies Theorem 12.

Theorem 21.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$, and let M_1 and M' be two disjoint matchings in G. Suppose that each edge e of G is assigned a list $L(e) \subseteq [\Delta(G) + \mu(G)]$ of colours such that

- $L(e) = \{1\}$ *if* $e \in M_1$*;*
- $L(e) = \{2, \ldots, \Delta(G) + \mu(G)\}$ *if* $e \in M'$*; and*
- $L(e) = [\Delta(G) + \mu(G)]$ *if* $e \in E(G) \setminus (M_1 \cup M')$*.*

Then there exists a proper edge-colouring ψ of G such that $\psi(e) \in L(e)$ for every $e \in E(G)$.

To establish Theorem 21, we use a result mentioned just after Conjecture 3.

Theorem 22 (Berge and Fournier [7])**.**
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$, and let M be a matching in G. Then there exists a proper edge-colouring of G using the palette $[\Delta(G) + \mu(G)]$ such that every edge of M receives the same colour.

Proof of Theorem 21. We may assume without loss of generality that M_1 is a maximal matching in $G \setminus M'$. We set

$$B = \{\, e' \in M' \;:\; e' \cap e = \varnothing \text{ for all } e \in M_1 \,\}.$$

Let ψ to be a partial proper edge-colouring of G using colours in $[\Delta(G) + \mu(G)]$ such that

(i) $\psi(e) = 1$ for every $e \in M_1$;

(ii) $\psi(e') \neq 1$ for every $e' \in M'$;

(iii) every edge of $E(G) \setminus B$ receives a colour under ψ; and

(iv) the number of edges of B that receive a colour under ψ is maximal.

To show that ψ is well defined, we need to prove the existence of a partial proper edge-colouring of $G - B$ using the palette $[\Delta(G) + \mu(G)]$ that satisfies (i) – (iii).

To this end, let $G' = G - B$. By Theorem 22, there is a proper edge-colouring ϕ of G' using colours in $[\Delta(G)+\mu(G)]$ such that every edge in M_1 receives colour 1. By the definition of B, each edge in $M' \setminus B$ is incident to at least one edge in M_1. Each edge in M_1 receives colour 1 under ϕ and therefore ϕ does not map any edge of $M' \setminus B$ to colour 1. Thus ϕ ensures that ψ exists.

We now show that every edge of B receives a colour under ψ, which completes the proof. Suppose, on the contrary, that $xy \in B$ is an edge that is not coloured by ψ. We start by making the following observations.

Claim 6. *For every $e \in E(G)$, we have $\psi(e) = 1$ if and only if $e \in M_1$.*

Indeed, if e is an edge that is coloured 1, then $e \notin M'$ and e is not adjacent to an edge in M_1, since all such edges are also coloured 1. Consequently, $e \in M_1$, as M_1 is a maximal matching of $G - M'$.

Claim 6 and the definition of B ensure the following.

Claim 7. *Neither x nor y is incident with an edge in M_1.*

For each vertex $v \in V(G)$ let $A_v \subseteq [\Delta(G)+\mu(G)]$ be the set of colours that do not appear on edges incident to v. Claim 7 states that A_x and A_y both contain the colour 1.

Claim 8. *If $v \in N_G(x) \setminus \{y\}$, then v is incident to an edge in M_1 and so A_v does not contain the colour* 1.

Indeed, for if v is not incident to an edge in M_1, then by Claim 7 the edge xv could be added to M_1 to form a larger matching in $G - M'$, thereby contradicting the maximality of M_1.

We know that the edge xy is not yet coloured so both A_x and A_y must contain some colour different from 1 and we shall from now on redefine A_y to be $A_y \setminus \{1\}$, which is non-empty. We consider the following iterative procedure.

Initially ($t = 0$), we set $D_0 = \{y\}$. At each step $t \geq 1$, we form the set

$$D_t = \{\, v \in N_G(x) \setminus \cup_{i \leq t-1} D_i \;:\; xv \text{ has colour } c \text{ for some } c \in \cup_{w \in D_{t-1}} A_w \,\}.$$

Since $\cup_{i \geq 0} D_i \subseteq N_G(x)$, there exists t_0 such that $D_{t_0+1} = \varnothing$. We define $D = \cup_{i \leq t_0} D_i$. We consider now two cases.

Case 1. Assume that there exist a vertex $w \in D$ and a colour $c \in A_w \cap A_x$. Let t_1 be the least integer such that $w \in D_{t_1}$. There is a sequence $y = w_0, w_1, w_2, \ldots, w_{t_1} = w$ of vertices, such that $w_i \in D_i$ and $\psi(xw_i) \in A_{w_{i-1}}$, whenever $1 \leq i \leq t_1$.

We may then define a partial proper edge-colouring ψ' of $E(G)$, using colours in $[\Delta(G) + \mu(G)]$, with

- $\psi'(e) = \psi(e)$ if $e \notin \{\, xw_i \;:\; 1 \leq i \leq t_1 \,\}$;
- $\psi'(xw_i) = \psi(xw_{i+1})$ for each $i \in \{0, \ldots, t_1 - 1\}$; and
- $\psi'(xw = xw_{t_1}) = c$.

One can check that ψ' satisfies (i)−(iii) and colours one more edge of B than ψ does, which contradicts the choice of ψ.

For the second case, we require the following two observations.

Claim 9. *For every $z \in N_G(x)$, it holds that $\mu(G) \leq |A_z|$.*

The only case that is not trivial is when $z = y$, due to our redefinition of A_y. However, as the edge xy is not coloured, the vertex y sees at most $\Delta(G) - 1$ different colours, which implies the statement.

Let H be the bipartite subgraph of G induced by the bipartition $(\{x\}, D)$. (In particular, the edges of G between vertices in D are not in H.) The next statement follows directly from the fact that the number of coloured edges between x and y is at most $\mu(G) - 1$.

Claim 10. *The bipartite graph H contains fewer than $|D|\mu(G)$ coloured edges.*

We can now proceed with the second case.

Case 2. For every vertex $w \in D$ and every colour $c \in A_w$, there exists a vertex $z \in D$ such that $\psi(xz) = c$. By Claims 9 and 10, we know that the number of colours appearing in the bipartite graph H is less than $|D| \cdot \mu(G)$, which is at most $\sum_{w \in D} |A_w|$. This implies that there are two distinct vertices v_1 and v_2 in $N_G(x)$ with $A_{v_1} \cap A_{v_2} \neq \varnothing$. Let $c_1 \in A_{v_1} \cap A_{v_2}$ and note that $c_1 \neq 1$ by Claim 8. Let $c_2 \in A_x \setminus \{1\}$. Then $c_2 \notin A_{v_1} \cup A_{v_2}$ and $c_1 \notin A_x$. (And hence $c_1 \neq c_2$.)

For $i \in \{1, 2\}$, let P_i be the maximal alternating path with colours c_1 and c_2 beginning at v_i. Note that x cannot belong to both paths. But if x does not belong to P_i, then we may swap c_1 and c_2 along the edges of P_i. This leads us back to Case 1 because then c_2 belongs to $A_x \cap A_{v_i}$. (Note that such a swap affects neither the colours of the edges inside H nor those of edges in M_1.)

We have shown that in each case there exists a partial proper edge-colouring using colours in $[\Delta(G) + \mu(G)]$ and satisfying (i) – (iii) that assigns colours to more edges of B than ψ does, a contradiction. □

7 Conclusion

During the preparation of this manuscript, we learned of a related work in the context of graph limits [13], in which is proposed the following conjecture that has a similar flavour to our Conjecture 3.

Conjecture 23 (Csóka, Lippner and Pikhurko [13])**.**
Let G be a graph such that every vertex is of degree at most d, except one of degree $d + 1$. Using the palette $\mathcal{K} = [d + 1]$, suppose that at most $d - 1$ pendant edges are precoloured. This precolouring can be extended to a proper edge-colouring of all of G.

The authors of Conjecture 23 proved the weaker statement with $\mathcal{K} = [d + 9\sqrt{d}]$ instead of $\mathcal{K} = [d + 1]$.

With respect to Question 1, rather than imposing conditions on the matching M, we could instead constrain the precolouring. In the light of Theorem 14 and the result of Berge and Fournier [7], the following is a natural strengthened version of Conjecture 3.

Conjecture 24.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Using the palette $\mathcal{K} = [\Delta(G) + \mu(G)]$, any precoloured matching such that no two edges precoloured differently are within distance 2 can be extended to a proper edge-colouring of all of G.

We may rephrase Theorem 12 in the language of list colouring as follows: for any multigraph G, any matching M in G, and any list assignment $L : E(G) \to 2^{[\Delta(G)+\mu(G)]}$ such that $|L(e)| = \Delta(G) + \mu(G) - 1$ if $e \in M$ and $L(e) = [\Delta(G) + \mu(G)]$ otherwise, there is a proper L-edge-colouring of G. Theorem 14 still leaves open the possibility that the following holds.

Conjecture 25.
Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$ and let M be a matching in G. Let $L : E(G) \to 2^{[\Delta(G)+\mu(G)]}$ be a list assignment such that $|L(e)| = 2$ if $e \in M$ and $L(e) = [\Delta(G) + \mu(G)]$ otherwise. Then there is a proper L-edge-colouring of G.

It would also be interesting if either of Conjectures 24 and 25 could be confirmed with the constant 2 replaced by any larger fixed integer.

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