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Time and frequency response of structures with frequency dependent, non-proportional linear damping

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Abstract

A method to compute the non-stationary time and frequency response of structures with a frequency-dependent non-proportional linear damping, called the resonance modes method, is presented in this paper. It consists of two main steps. The first step aims at spotting the structure resonance modes, which are the solutions of the matrix non-linear eigenvalue problem obtained using the finite element method in the complex plane. This step requires a complex eigensolver and an iterative scheme, a perturbation technique or a combination of both. The second step uses the computed resonance modes and an analytical expression of the inverse Laplace transform to deduce the time or frequency response of structures to general excitations. The response of an aluminum plate damped with an elastomer treatment to a point-force excitation, computed with the classical modal approach, the direct solution and the presented method shows its precision and efficiency. An acoustic power computation finally validates the implementation of a fast variant, based on the perturbation technique, for vibroacoustic applications.

Keywords: Damping, elastomer, vibroacoustic, complex modes

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1. Introduction

The modeling of damped structures is still a very active research field. The main reason is that most numerical methods available today have a number of limitations. Time approaches are excessively costly compared to frequency approaches, while frequency approaches are often reduced to stationary excitation types or are restricted to specific damping models. Some finite element codes can now take advantage of complex eigensolvers, adapted to non-proportionally damped systems, but they do not offer any consistent approach to deduce the response to dynamic loadings from the computed complex modes. A better handling of complex frequency-dependent eigenvalue problems, which stem from the viscoelastic constitutive equations written in the complex plane is however needed by the industry. In aeronautics, in particular, lighter damping treatments with enhanced efficiency need to be further developed. Five main types of methods with different levels of approximation are currently available to address the problem of viscoelasticity in dynamics: direct solutions, methods based on modal strain energy, perturbation-based methods, state space methods and finally methods introducing dissipation coordinates.

The direct solution is often the only option to model viscoelastic behavior in standard finite element codes with accuracy, but unfortunately at high numerical cost. A number of variants have been developed, though, which make the method become more practical over time. The Padé method, investigated by Chazot et al [1] for instance, allows an efficient reconstruction of the frequency response function with a substantial speed up gain.

Modal Strain Energy analysis (MSE), in its initial form, is based on the eigenvector basis of the undamped structure with no frequency dependence. It has been used by Finegan and Gibson [2], for instance, to characterize the damping loss factor of composite structures with coated fiber reinforcement; their approach takes the contribution of the fibers, fibers coating and
composite matrix damping separately to assess total damping. Using the MSE analysis, one needs to compute the modal strain energy of the different structure sub-systems in a first step; each energy is then weighted by the sub-system’s loss factor. Summing all weighted energies and dividing the result by the total modal strain energy finally yields the structure’s modal damping. MSE analysis requires the vectors of the undamped system to be very close to those of the damped system. Moreover, it is only valid when a uniform loss factor mean value can be associated with a sub-system. For this reason, non-local dissipations such as noise radiation energy loss can not be properly considered.

Damping non-proportionality is caused by inhomogeneous damping distributions. A complex eigensolver is needed in this case to compute the structure’s complex modes. Perturbation techniques, contrary to MSE analysis, can be used to handle non-proportionality, as pointed out by Woodhouse [3], or more recently by Cha [4] for arbitrarily damped nearly proportional systems. If the damping is slightly non-proportional, indeed, an approximation of the complex eigenvectors can be deduced from the undamped structure modal basis. Another alternative for solving complex eigenvalue problems is to use the state space method (Hurty and Rubinstein [5]). The method takes non-proportionality into account with no limiting assumption. It therefore gives accurate complex eigensolutions using a real eigensolver only. The main drawback of the method, though, is the matrix size doubling that results in prohibitive calculation times for large systems. The method does not handle frequency dependence either. Alternatively, Adhikari [6] proposed an approach that uses the undamped system normal modes obtained from a real eigensolver to deduce complex modes iteratively, and Cortés and Elejabarrieta [7] suggested an approximate method to compute them from the undamped solutions by finite increments using eigenvector derivatives and the Rayleigh quotient.
The last approach available currently uses analytical viscoelastic models to reshape any frequency-dependent complex problem into a real frequency-independent equivalent one. Dissipation coordinates that characterize an intermediate field need to be introduced. These coordinates are written explicitly with respect to other coordinates and a given complex viscoelastic model. The GHM method (Golla Hughes Mac Tavish, Golla and Hughes [8], Barbosa and Farage [9]) or the ADF method (Anelastic Displacement Field, Lesieutre and Bianchini [10]) are different variants of this approach. They offer a number of advantages: damping can be frequency-dependent, non-proportional; numerically, only a standard real eigensolver is needed. The main drawback, however, is numerical: matrices become larger and unsymmetric; also, the use of specific damping models restricts the approach generality.

The resonance modes method presented here, like GHM and ADF methods, not only handle frequency dependence and non-proportional damping but also remediates most of their disadvantages; damping, in particular, can remain general. It falls in the wide category of extended modal approaches such as the modal contribution functions superposition method proposed by Cortés and Elejabarrieta [11], the modal approach combined with fractional derivatives in the paper by Sorrentino and Fasana [12], or the fast frequency response analysis algorithm suggested by Kim [13], valid in the case of partially damped structures. Introduced initially to handle radiation frequency dependence in vibroacoustics (Filippi et al [14]), it is applied here to structural dynamics with viscoelasticity. The method is a mathematically founded generalization of the classical modal approach to frequency dependent cases. Based on the use of the inverse Laplace transform, it requires a modeling technique combined to iterative or perturbation techniques. The finite element method has been selected for this purpose but the choice, however, is not restricted; the finite difference method or even analytical expressions could also be used. The example of an elas-
tomer with strong viscoelastic characteristics has been chosen to illustrate the method.

2. Problem frequency formulation

2.1. Local equations

Assuming that the Laplace transform given by

\[ \mathcal{L} \left[ \tilde{f}_i (t) \right] = f_i (s) = \int_0^\infty \tilde{f}_i (t) e^{-st} dt \]  (1)

is applied to the local constitutive equations of an isotropic viscoelastic material, the following frequency representation is obtained:

\[ \sigma_{ij} (s) = \frac{E'(s) (1 + \frac{1}{2} \tan \delta (s))}{1 + \nu} \left[ \epsilon_{ij} (s) + \frac{\nu}{1 - 2\nu} \epsilon_{kk} (s) \delta_{ij} \right], \]  (2)

where \( \sigma_{ij} \) is the stress tensor, \( \epsilon_{ij} \) the strain tensor, \( \nu \) Poisson’s ratio and \( \delta_{ij} \) Kronecker symbol. \( E'(s) \) is the storage modulus, while \( \tan \delta (s) \) is associated to the material energy loss. Both can be expressed as polynomial curve-fittings of tabular experimental data, as displayed in Figures 1 and 2. In these figures, a frequency stiffening typical of elastomers can be observed, as well as a frequency region around 60000 Hz, around which damping is particularly large. Poisson’s ratio is set equal to a constant but it is not a restriction. Anisotropic viscoelastic properties could also be handled by the method.

The problem statement is completed by boundary conditions and the following local equation of motion:

\[ s^2 \rho u_i (s) - \sigma_{ij,j} (s) = f_i (s) + s \rho u_{i0} + \rho \dot{u}_{i0} \]  (3)

where \( u_i (s) \) is the local displacement, \( \rho \) the material density, \( f_i (s) \) a volume force, \( u_{i0} \) an initial displacement and \( \dot{u}_{i0} \) an initial velocity.
Figure 1: Elastomer frequency-dependent storage modulus.

Figure 2: Elastomer frequency-dependent $\tan \delta$. 
2.2. Finite element formulation

Local equations are multiplied by a conjugate weighting function and integrated over the structure volume (Ohayon and Soize [15]). Integrating by part and using Bubnov-Galerkin method (Hughes [16]) with homogeneous boundary conditions results in the following complex symmetric matrix problem:

\[
[s^2M + K^B(s) + K^A] u(s) = F(s) + s I_0 + V_0 \quad (4)
\]

where \( M \) is the mass matrix, \( u(s) \) the displacement solution vector, \( F(s) \) the excitation force vector, \( I_0 \) and \( V_0 \) the initial displacement and velocity vectors, respectively. The stiffness matrix has been split into a frequency-independent part \( K^A \) and a frequency-dependent one \( K^B(s) \). All matrices are generated by a 27-nodes fully quadratic solid element that gives a great flexibility for designing new damping treatments. A specific formulation with four pressure nodes per element has also been implemented to handle quasi-incompressibility. Bathe [17] emphasizes the good convergence qualities of this quasi-incompressible element. The main disadvantage of the current strategy is that it results in a bad conditioning of the matrix inversion problem that restricts the overall model size. As pointed out before, though, the resonance modes method is applicable to any kind of modeling technique.

The system is defined at a given value of the Laplace parameter \( s \) associated to a specific angular frequency \( \omega \) via \( s = i \omega \), where \( i \) is the complex imaginary number. Classical solving methods such as the direct solution, which inverts the left-hand side operator of Equation (4), or the modal resolution, which uses a biorthonormality relationship valid at fixed \( s \), need both a discretization of the frequency axis. As many matrix factorizations as points on the frequency axis need to be computed, which, in the case of broadband excitations, results in very long solving times.
3. Problem solution

3.1. The classical modal approach

The modal resolution is based on the eigenvalue problem deduced from the homogeneous form of Equation (4):

\[
\left[ s_k^2(s) M + K^B(s) + K^A \right] u_k(s) = 0,
\]

(5)

where \((s_k(s), u_k(s))\) is the \(k^{th}\) eigenelement. This eigenvalue problem is complex, nonlinear in frequency and results in complex mode shapes. Since matrices are symmetrical by construction, following biorthonormality relationship holds at fixed \(s\):

\[
\begin{align*}
    u^T_i(s) M u_j(s) &= \delta_{ij} \\
    u^T_i(s) \left[ sK^B(s) + K^A \right] u_j(s) &= -s^2_j(s) \delta_{ij}
\end{align*}
\]

(6)

Equation (4) solution \(u(s)\) is sought using the combination \(u(s) = \sum_i^N \alpha_i(s) u_i(s)\) of the modes \(u_i(s)\) computed at \(s\). Multiplying the equation with the transposed eigenvector \(u^T_j(s)\) yields:

\[
u^T_j(s) \left[ s^2 M + K^B(s) + K^A \right] \sum_i^N \alpha_i(s) u_i(s) = u^T_j(s) \left[ F(s) + sI_0 + V_0 \right].
\]

(7)

The frequency dependent coefficients \(\alpha_j(s)\) are easily determined using biorthonormality relationships (6). The displacement is finally given with respect to Laplace parameter \(s\) by

\[
u(s) = \sum_i^N \frac{u^T_i(s) \left[ F(s) + sI_0 + V_0 \right]}{s^2 - s^2_i(s)} u_i(s)
\]

(8)

It is worth pointing out that using relationship (8) is numerically inefficient. In this case, a linearized form of Equation (5) with constant system matrices is obtained at each point of the frequency axis by setting \(s\). The corresponding eigenvalue problem can be solved in a classical way, but its solution is valid at this specific frequency only. A considerable amount of
eigenvalue problems needs thus to be solved to compute the response over a wide frequency band. It results in longer computation times than with the direct solution.

3.2. Time solution

The time solution can be deduced from Equation (8) with Mellin-Fourier inverse integral transform or Bromvitch formula expressed as following:

\[
\mathcal{L}^{-1}[f(s)] = Y(t) f(t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} f(s) e^{st} ds
\]

(9)

where \(Y(t)\) is the Heaviside function, \(C\) the abscissa of the integration vertical axis. Details of similar calculations made in the Fourier domain can be found in the work by Filippi et al [14]. Let’s focus on the computation of the \(j^{th}\) term of Equation (8) sum only, associated to the dynamic loading \(F(s)\), for clarity reasons. This term is given by:

\[
U^j(s) = \frac{u_j^T(s) F(s) u_j(s)}{s^2 - s_j^2(s)} = \Psi(s) \frac{u_j^T(s) \phi u_j(s)}{s^2 - s_j^2(s)}
\]

(10)

The excitation vector \(F(s)\) has been split into a frequency-dependent part \(\Psi(s)\) and a spatial one \(\phi\). The inversion formula can thus be written as following:

\[
Y(t) u^j(t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Psi(s) \frac{u_j^T(s) \phi u_j(s)}{s^2 - s_j^2(s)} e^{st} ds
\]

(11)

If the denominator only consists of simple poles \(\hat{s}_j\), the residue theorem gives the time solution as the sum of following residues:

\[
\text{Res} (\phi, \hat{s}_j) = \lim_{s \to \hat{s}_j} \Psi(s) (s - s_j(s)) \frac{u_j^T(s) \phi u_j(s)}{s^2 - s_j^2(s)} e^{st}
\]

\[
= \lim_{s \to \hat{s}_j} \Psi(s) \frac{u_j^T(s) \phi u_j(s)}{s + s_j(s)} e^{st} = \Psi(\hat{s}_j) \frac{u_j^T(\hat{s}_j) \phi u_j(\hat{s}_j)}{2\hat{s}_j} e^{\hat{s}_j t}
\]

(12)

Determining the \(j^{th}\) residue requires to find the \(j^{th}\) resonance pair \((\hat{s}_j, \hat{u}_j)\) given by a limit calculation:

\[
\lim_{s \to s_j} \left[ s_j^2(s) M + K^B(s) + K^A \right] u_j(s) = \left[ s_j^2 \hat{M} + K^B(\hat{s}_j) + K^A \right] \hat{u}_j = 0
\]

(13)
This equation, called resonance value problem equation, is different from the eigenvalue problem Equation (5). It is worth noting that resonance pairs \((\hat{s}_j, \hat{u}_j)\), contrary to eigenpairs \((s_j(s), u_j(s))\), do not depend on frequency. Resonance pairs are computed with a linear complex eigensolver and a perturbation or an iterative scheme, which performs the limit calculation numerically.

Since the Laplace transform of a function \(f\) is such that \(f(\bar{s}) = \overline{f(s)}\), \(\overline{s}_j\) is also solution of the resonance value problem (13). A second residue, simply deduced from the first one by following relationship, needs thus to be considered:

\[ \text{Res}(\phi, \overline{s}_j) = \overline{\text{Res}(\phi, s_j)} \] (14)

The part of the time response associated to the \(j^{th}\) term of the sum is finally deduced by taking both residues into account:

\[ Y(t) u^j(t) = \Re \left[ \psi(s_j) \frac{u^T(s_j) \phi u_j(s_j)}{\bar{s}_j} e^{s_j t} \right] \] (15)

where \(\Re(z)\) stands for the real part of the complex number \(z\). The complete time solution has following final form (for \(t \geq 0\)):

\[ u(t) = \sum_{j=1}^{N} \Re \left[ \frac{u^T(s_j) \phi u_j(s_j)}{\bar{s}_j} \psi(t) * e^{s_j t} + u^T_j(\bar{s}_j) \left( I_0 + \frac{\nu}{\bar{s}_j} \right) u_j(\bar{s}_j) e^{s_j t} \right] \] (16)

where \(\psi(t)\) is the time representation of the excitation spectrum \(\psi(s)\) and the symbol \(*\) the convolution product. There are numerous advantages to using this analytical expression. Once resonance modes have been computed, Eq. (16) is easy to calculate and accurate. It is also valid for non-stationary excitation types. Any new excitation case can be considered with little additional computational cost compared to most transient methods. The resonance modes method can thus be seen as a generalization of the modal approach to the linear viscoelastic case. No hypothesis is made on the damping type; this one can be specified by a simple curve fitting of tabular data. The existence of the Laplace transform of an implicit viscoelastic
stiffness tensor defined in the time domain, which is however not needed in
the calculation, is the only hypothesis.

3.3. Frequency solution

The frequency response is finally obtained by computing the Laplace
transform of Equation (16). The displacement solution $u(s)$ is given by
following relationship:

$$u(s) = \frac{1}{2} \sum_{j=1}^{N} \left\{ \left[ u_j^T (\hat{s}_j) \frac{\phi \Psi(s) + \hat{s}_j f_0 + V_0}{\hat{s}_j (s-\hat{s}_j)} u_j (\hat{s}_j) \right] + \left[ \bar{u}_j^T (\hat{s}_j) \frac{\phi \Psi(s) + \hat{s}_j f_0 + V_0}{\hat{s}_j (s-\hat{s}_j)} \bar{u}_j (\hat{s}_j) \right] \right\}$$  (17)

This expression has to be compared with the expression obtained with
the classical modal approach in the non-proportional damping case (Equa-
tion (8)). The classical modal approach requires one to compute an eigen-
value problem per frequency axis point, while the resonance modes method
requires one to solve a number of eigenvalue problems that depends on the
selected strategy to spot resonance modes.

4. The numerical computation of resonance modes

Resonance modes can be computed with a linear complex eigensolver
combined with search techniques that make repeated calls to the solver.
Search techniques are presented in the forthcoming paragraph. The imple-
mented complex eigensolver is then described in the following paragraph.

4.1. Looking for the resonance modes

Several approaches can be used to spot resonance modes. Three of them
have been implemented and compared. The first one, based on matrix
Equation (5), is iterative, and focuses on each resonance mode separately.
Araújo et al [18] used a similar technique to track the modes of sandwich
laminated plates with viscoelastic core in order to obtain enhanced damp-
ing estimations. The authors, however, did not go forward by making use
of the resonance modes to compute the response and an inefficient direct
approach is used instead.

In the first step of the iterative approach, the viscoelastic matrix $K^B(s_0)$ is
constructed at a fixed parameter value $s_0$ based on an estimation of the first
resonance frequency. If no prior knowledge of this frequency is available,
as it will be assumed later in the numerical examples, one can simply set $s_0$ to zero, but it may deteriorate the convergence slightly. Setting a value
for $s$ makes the complex eigenproblem linear and solvable. Once the eigen-
solver has been called, the eigenvalue of the mode of interest that has just
been computed is used to create an updated viscoelastic matrix $K^B(s_1)$. The process is then reiterated. After repeated calls to the eigensolver, the
scheme converges toward a resonance mode as defined by Equation (13). The Arnoldi method, implemented in the eigensolver described in the next
paragraph, remains efficient when many eigenvalues are extracted simulta-
neously. This is why a few eigenvalues are actually computed to initialize
the algorithm properly when the next resonance modes are sought.

The chosen convergence criterion is based on the frequency difference ratio
between two successive iterations. Convergence is considered to be reached
when the ratio is lower than $10^{-3}\%$. It has always been observed in all the
numerical examples of the paper. Three iterations seem necessary to spot
the first resonance mode, since the program may start with a poor estimate
of the first resonance frequency, while two iterations are then required for
each subsequent mode, since a better estimation is available. The algo-

It could further be improved by adding a mode shape identification routine
to distinguish very closely spaced modes that might be mixed up during the
iteration process. The structural examples that will be considered in the
next paragraph, however, have not required this additional feature.
The second and third search approaches use a perturbation technique. This technique is based on an algebra theorem (Lascaux and Théodor [19]), which gives the analytical expression of new eigenvalues obtained when the stiffness matrix is perturbed by a matrix $C$ and a small parameter $\epsilon$. The theorem is formulated as following: Let $\lambda_k$ be a simple eigenvalue of the diagonalizable matrix $K$ and $\lambda_k(\epsilon)$ the corresponding one associated to $K + \epsilon C$, so that $\lim_{\epsilon \to 0} \lambda_k(\epsilon) = \lambda_k$. For a sufficiently small $\epsilon$, following results apply:

$$
\begin{align*}
\lambda_k(\epsilon) &= \lambda_k + \epsilon \left( v_k^H C u_k \right) + O(\epsilon^2) \\
u_k(\epsilon) &= u_k + \epsilon \left( \sum_{i=1, i \neq k}^N \frac{v_i^H C u_i}{\lambda_k - \lambda_i} u_i \right) + O(\epsilon^2)
\end{align*}
$$

(18)

where $v_k^H$ is the hermitian transpose of the $k^{th}$ left eigenvector. The same relationships hold for generalized eigenproblems after mass-matrix orthonormalization. As far as complex symmetric eigenproblems concerns, left eigenvectors are simply deduced from their right counterpart $u_k$ using $u_k = \overline{v}_k$.

This theorem can be used to determine a better estimation of the next resonance mode from a first eigenvalue computation. Let’s assume for instance that a first resonance mode of frequency $f_1$ associated to Laplace parameter $\hat{s}_1$ has been computed. The next resonance mode associated to Laplace parameter $\hat{s}_2$ is now targeted. An approximation $s_2(f_1)$ is already available, since the second eigenvalue of the system constructed at frequency $f_1$ has been extracted in a previous step. A perturbation matrix $\Delta K$, which takes the frequency effect into account, is built as following:

$$
\Delta K^B = K^B(s_2(f_1)) - K^B(\hat{s}_1)
$$

(19)

An improved estimation of $\hat{s}_2$ can then be deduced from the perturbation technique:

$$
\hat{s}_2^2 \approx s_2^2(f_1) + u_2^T(f_1) \Delta K^B u_2(f_1)
$$

(20)

where $u_2(f_1)$ is the second mode shape of the system constructed at frequency $f_1$. This expression can be used to accelerate the iterative scheme. It has been implemented in an algorithm combining perturbation and iteration, identified by label A2. In practice, two iterations only are needed to
determine the first resonance mode, instead of three with a pure iterative
approach. Only one iteration instead of two is necessary for the next res-
onance modes. The algorithm A2 thus needs to solve approximately one
eigenvalue problem per resonance mode, roughly half as many as in algo-
gorithm A1.

It is also possible to use the perturbation method from the results of a
single eigenvalue problem computation. The middle of the frequency range
is taken as reference to build up the viscoelastic matrix $K_V$. In doing so,
it is assumed that minimizing the maximum coefficient of all perturbation
matrices will increase the validity domain of the local approximation. Fur-
ther investigations in the field of perturbation analysis should be carried
out to determine an optimal reference frequency. All modes are computed
instead of just a few. The eigenvalues are then adjusted successively. This
approach has been implemented in a fast algorithm identified by label A3.

4.2. A complex eigensolver

A symmetric complex eigensolver has been implemented using ARPACK
library [20], based on the Implicitly Restarted Arnoldi Method. It can be
considered as an extrapolation of Lanczos method to general structured
matrices that have a certain degree of sparsity. ARPACK works by calling
user-supplied routines repeatedly via a reverse communication interface.
The user is thus free to choose any convenient data structure or matrix
inversion algorithm. Cholesky factorization could be used to reshape the
eigenvalue problem generalized form into a standard one, since the mass
matrix is symmetric positive-definite. However, the shift-invert mode for
generalized problems has been chosen. This mode is more efficient when
only a few localized eigenvalues are sought (algorithms A1 and A2). If a
spectral shift $\sigma$ is introduced in the eigenvalue problem, following equation
is obtained:

$$(K - \lambda M) u = (K - \sigma M - (\lambda - \sigma) M) u = 0$$

(21)
where $K$ is the stiffness matrix, $M$ the mass matrix and $(\lambda, u)$ an eigenpair. The problem can be further transformed into the following one:

$$
\left( \left( \frac{1}{\lambda - \sigma} \right) I - (K - \sigma M)^{-1} M \right) u = \left( \Lambda I - \tilde{K} \right) u = 0 \quad (22)
$$

The matrix $\tilde{K}$, defined by

$$
\tilde{K} = (K - \sigma M)^{-1} M \quad (23)
$$

has been introduced. The Arnoldi method can compute the spectrum dominant eigenpairs like $(\Lambda, u)$ from the transformed system $(\tilde{K}, I)$ efficiently. The corresponding eigenvalue $\lambda = \frac{1}{\Lambda} + \sigma$ of the initial matrix system $(K, M)$ can then be deduced. The convergence is quicker if $\sigma$ is close to $\lambda$. The system construction frequency thus gives an appropriate shift value $\sigma$, which is systematically updated during the iterative process and for each new resonance mode.

An inversion algorithm needs to be provided in order to carry out calculations such as $y = \tilde{K}x$, in which $x$ is a vector given by ARPACK and $y$ the vector requested by the reverse communication interface. Several iterative methods using Krylov-subspaces such as the Restarted Generalized Minimum Residual (RGMRES), the Conjugate Gradient Squared (CGS), the Polynomial Stabilized bi-conjugate gradient (Bi-CGSTAB(l)), or the TFQMR method (Transpose-Free Quasi-minimal Residual) have been tested. These algorithms, available in the Numerical Algorithms Group library (NAG [21]), can be used in combination with three preconditioners such as the Jacobi, SSOR (Symmetric Successive-Over-Relaxation), ILU (incomplete LU factorization) preconditioners. The option without preconditioning is also available. The only numerical drawback of the library algorithms is their non-symmetric storage scheme that requires a memory size doubling. Another solver computing the direct frontal solution has also been implemented. The program uses routines from the SPARSEPAK library that have been modified to handle complex numbers. It also uses a
routine from the software METIS [22] that carries out graph partitioning and fill-reducing orderings of sparse matrices.

<table>
<thead>
<tr>
<th>Material</th>
<th>Property</th>
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<tr>
<td>Aluminum</td>
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<td></td>
<td>damping</td>
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</tr>
<tr>
<td></td>
<td>Poisson ratio</td>
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</tr>
</tbody>
</table>

Table 1: Materials characteristics.

Numerical tests have been conducted to compare the performances of the different eigensolvers. Complex symmetric matrices have been generated by modeling a 35 cm $\times$ 40 cm $\times$ 2 mm aluminum plate with free boundary conditions and a 10 cm $\times$ 10 cm damping patch located in a corner. All materials characteristics are summarized in Table 1. Aluminum damping is considered negligible here. The patch is made of a 1 mm-thick elastomer layer (characteristics displayed in Figures 1 and 2) constrained by a 0.5 mm-thick aluminum layer. This type of damping treatment results in high levels of dissipation and is extensively used in the transportation industry. A unique eigenvalue problem constructed at 45 Hz, very close to the first structure non-rigid body mode, has been solved. This choice of frequency, although quite arbitrary, makes the inversion problem numerically hard to solve and seems well adapted to benchmark the algorithms. The whole procedure yields performance results in terms of CPU time and maximum memory usage that are useful to determine the best algorithmic option. It has been found, in particular, that only the ILU preconditioner makes the various iterative routines converge. It has also been found that all iterative routines behave in a similar way and give very close computation times and memory needs. Table 2 displays numerical comparisons between the two best solvers. The first one is the ILU/CGS iterative solver, while the second one computes the direct frontal solution. Each column of the table
represents a different system discretization and therefore a different matrix size. The table clearly shows that the algorithm implementing the direct

<table>
<thead>
<tr>
<th>Matrix size</th>
<th>387</th>
<th>891</th>
<th>1971</th>
<th>3051</th>
<th>4851</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time (s)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ILU/CGS</td>
<td>0.74</td>
<td>6.20</td>
<td>31.25</td>
<td>101.62</td>
<td>348.69</td>
</tr>
<tr>
<td>Direct</td>
<td>0.15</td>
<td>0.42</td>
<td>1.14</td>
<td>2.32</td>
<td>4.16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Max. memory (MB)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ILU/CGS</td>
<td>0.98</td>
<td>0.98</td>
<td>59</td>
<td>91</td>
<td>150</td>
</tr>
<tr>
<td>Direct</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td>23</td>
<td>36</td>
</tr>
</tbody>
</table>

Table 2: Comparison between the ILU/CGS and the direct solution complex eigensolvers for various matrix sizes.

method is much more efficient than algorithms based on iterative methods. The reason is related to the family type of interpolation functions used in the finite element model. Quadratic elements yield populated matrices with a bad conditioning for the iterative inversion problem.

5. Solvers comparison and validation

5.1. Response of a damped system

The response of a clamped plate to point-force excitation has been computed using algorithms A1, A2 and A3, which are different implementations of the resonance modes method. All three use the direct frontal algorithm, the fill-reducing ordering algorithm as well as the shift-invert strategy for generalized eigenvalue problems described in the previous paragraph. A1 uses the iterative technique presented in § 4.1, A2 the hybrid iterative/perturbation approach and A3 the perturbation method. Two additional algorithms that invert the left hand side operator of Equation (4) have also been programmed and can be considered as reference implementations of current standards for frequency-dependent dynamic problems. The first one, based on the direct frontal solution with fill-reducing ordering, is called **Direct**, while the second one, which uses the classical modal approach (Equation (8)), is called **Eigenmodes**. A resolution of 1 Hz has
been chosen to discretize the frequency axis. This choice has a tremendous impact on the computation time of these last two algorithms.

The modeled configuration is a 2mm-thick 35 cm × 40 cm clamped aluminum plate excited by a point force (x = 28 cm, y = 20 cm) of unit value from 0 Hz to 1000 Hz. The displacement has been computed at a single location (x = 28 cm, y = 32 cm). Two different treatment configurations have been studied: a slightly damped configuration with a 6.9 cm × 8.9 cm constrained elastomer patch located at the center of the plate and a very damped configuration with a patch covering it. The patches are made of a 1 mm-thick elastomer layer (characteristics displayed in Figures 1 and 2) constrained by a 0.5 mm-thick aluminum layer. The materials characteristics are given in Table 1, while Table 3 summarizes the numerical characteristics of both configurations.

<table>
<thead>
<tr>
<th>Case</th>
<th>Slightly damped</th>
<th>Very damped</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate mesh</td>
<td>20 × 20</td>
<td>20 × 20</td>
</tr>
<tr>
<td>Number of elements</td>
<td>480</td>
<td>1600</td>
</tr>
<tr>
<td>Matrix size</td>
<td>~ 16000</td>
<td>~ 44000</td>
</tr>
<tr>
<td>Number of matrix coefficients</td>
<td>~ 2.2 × 10^6</td>
<td>~ 7.4 × 10^6</td>
</tr>
</tbody>
</table>

Table 3: Numerical characteristics of the two studied plate configurations.

A number of indicators are output to analyze the performances of all five solvers. The CPU time has been estimated and normalized by dividing it to the five solvers minimum value. Other characteristics such as the maximum memory required, the number of system resolutions or the number of factorizations have also been measured.
5.2. The slightly damped case

The computation characteristics of all five algorithms are displayed in Table 4 for the slightly damped case. It can be noticed that the number of system constructions, factorizations and resolutions is equal to 999 in the case of the direct solution. It corresponds to the number of points on the frequency axis. A configuration with no frequency dependence would have required a unique factorization and the same number of resolutions, resulting in a much smaller computation time, since factorizations are numerically very intensive. The classical modal solution also needs a factorization per frequency but a larger number of resolutions, which makes it even less appropriate than the direct solution to solve problems with frequency dependent characteristics. Iterative versions of the resonance modes method, on the other hand, require a number of factorizations related to the number of modes located in the frequency band of interest (15 resonance modes in the current case). Only the implementation with perturbation needs a unique factorization. The observed CPU time, as shown in Table 4, are consistent with these figures. The hybrid iterative/perturbation solver A2 is roughly 20 times faster than the direct resolution, while the very fast perturbation solver A3 is about 200 times faster.

<table>
<thead>
<tr>
<th></th>
<th>Direct</th>
<th>Eigenmodes</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized CPU time</td>
<td>191</td>
<td>295</td>
<td>16</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>Max. memory (MB)</td>
<td>86</td>
<td>91</td>
<td>112</td>
<td>112</td>
<td>112</td>
</tr>
<tr>
<td>Factorizations number</td>
<td>999</td>
<td>999</td>
<td>31</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>Resolutions number</td>
<td>999</td>
<td>1921</td>
<td>1921</td>
<td>976</td>
<td>60</td>
</tr>
</tbody>
</table>

Table 4: Direct solution, modal resolution and resonance-based A1, A2 and A3 algorithms performance comparison. Frequency response computation of the slightly damped plate configuration to point-force excitation.
Figure 3: Plate response to point force excitation in the slightly damped case

Figure 4: Modal characteristics discrepancy between the perturbation algorithm A3 and the iterative/perturbation one A2, with respect to the resonance modes frequency. Slightly damped case.
The computed frequency response functions are displayed in Figure 3. Results associated to the algorithm **Eigenmodes** based on the classical modal approach are not displayed, for clarity reasons. The direct resolution curve has been plotted instead, since it can be considered as being the reference in terms of accuracy. Algorithms **A1** and **A2** give identical results because the same convergence criterion is used for both routines; therefore only **A2** results are displayed. The figure shows a very good agreement between the resonance modes method and the direct resolution. Resonance modes algorithms **A2** and **A3** also show very similar results.

Since algorithm **A3**, based on the perturbation method, yields approximate modal quantities, it is worth comparing them with those computed by solver **A2**. The error on frequency and damping, expressed as a percent with respect to frequency, is displayed in Figure 4. The damping parameter \( \alpha \), expressed in Hz, is computed from the imaginary part of the complex resonance angular frequency \( \Omega = \omega + i\alpha \), where \( \omega \) is the real angular frequency in rad/s\(^{-1}\). Classical measures of damping, such as the loss factor \( \eta \) or the inverse of the quality factor \( Q^{-1} \) can easily be deduced using the formula \( \eta = Q^{-1} = \alpha / \pi f \), where \( f \) is the frequency in Hz. The error on the resonance modes frequencies is found negligible for all modes. As far as damping concerns, a maximum error of 9% is observed for the very first mode, but it is less than 1% on the major part of the spectrum. The minimum error is found at 500 Hz where the reference system has been constructed. The error on damping has a direct impact on the maxima of the frequency response function, but it is barely noticeable when using a logarithmic scale like in Figure 3.

### 5.3. The very damped case

The numerical characteristics associated to the five algorithms are displayed in Table 5 for the configuration of a constrained elastomer patch covering the plate entirely. Apart from the patch size, all other model characteristics (plate dimensions, materials, excitation and measurement point...
location) remain unchanged.

<table>
<thead>
<tr>
<th></th>
<th>Direct</th>
<th>Eigenmodes</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized CPU time</td>
<td>404</td>
<td>486</td>
<td>32</td>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>Max. memory (MB)</td>
<td>413</td>
<td>425</td>
<td>454</td>
<td>455</td>
<td>457</td>
</tr>
<tr>
<td>Factorizations number</td>
<td>999</td>
<td>999</td>
<td>51</td>
<td>31</td>
<td>1</td>
</tr>
<tr>
<td>Resolutions number</td>
<td>999</td>
<td>26095</td>
<td>3289</td>
<td>1938</td>
<td>63</td>
</tr>
</tbody>
</table>

Table 5: Direct solution, modal resolution and resonance-based A1, A2 and A3 algorithms performance comparison. Frequency response computation of the very damped plate configuration to point-force excitation.

There are two main differences with previous computation. First, the system is heavily damped due to the large patch. Second, roughly three times more elements are needed to model the three layers sandwich (see Table 3). The eigenvalue problem is thus harder to solve, as one can notice by comparing the number of resolutions required by the algorithms Eigenmodes and Direct. Twice as many resonance modes as before (30 in total) are used to compute the response on the frequency band of interest. The 17th resonance mode is located around 1000 Hz while the 30th is very close to 1600 Hz. The solver A2 remains however very efficient compared to the direct solution with CPU times that are about 20 times smaller. Solver A3 is roughly 400 times faster than the direct solution solver. The frequency response functions obtained with algorithms A2, A3 and Direct are displayed in Figure 5. A 30 modes-basis has been selected for the first two algorithms and is compared with a 60 modes-basis. Results given by the resonance modes method are very similar and compare also well with the direct solution results. Maxima are consistently estimated by all three algorithms. A slight discrepancy between the curves can be observed at the responses minima, in particular around 0 Hz and 860 Hz. This is related
to the modal basis truncation, which alters the accuracy of the resonance modes method just as it is observed with the classical modal method. It is particularly detrimental to heavily damped structures, for which modes contribution to the overall response is prevailing below their resonance. One can notice that taking a larger modal basis constituted of 60 modes instead of 30 results in enhanced results.

![Figure 5: Plate response to point force excitation in the very damped case](image)

The error ratio on resonance modes frequency and damping between solvers A2 and A3 is displayed in Figure 6. The overall damping error ratio is small in the upper frequency range, close to zero at the construction frequency 500 Hz, but increases as frequency reduces. A maximum error of 16% is reached for the very first mode around 115 Hz. A possible refinement of algorithm A3 could be to include an additional construction point in the lower frequency range, for instance around 150 Hz. The selection of additional points should be adapted to the material frequency dependence displayed in Figures 1 and 2. The figures show that both storage modulus and tan δ
Figure 6: Modal characteristics discrepancy between the perturbation algorithm A3 and the iterative/perturbation one A2, with respect to the resonance modes frequency. Very damped case.

have a linear behavior between 300 Hz and 1000 Hz. This is why solver A3 is very accurate for both the slightly and very damped cases within this frequency band.

6. Vibroacoustic validation of the perturbation-based solver

6.1. The vibroacoustic system

A vibroacoustic validation of the fast perturbation-based solver A3 is presented now. The impact of the modal discrepancy between A2 and A3 on the predicted noise, indeed, is worth being estimated since damping treatments are often used to reduce noise. The acoustic power of the previous clamped plate configurations with two different patch sizes is computed by assuming baffled conditions of radiation. In the vibroacoustic case, the matrix system (4) associated to the structure can be written in an energy
form as follows:

$$w^T(s) \left[ s^2 M + K^B(s) + K^A \right] u(s) - P(w(s), u(s), s) = w(s)^T(F(s) + sI_0 + V_0)$$  \hfill (24)

where $w(s)$ is a displacement, $P(w, u, s)$ a vibroacoustic coupling operator that takes normal stress continuity at the solid/fluid interface into account.

In the case of a flat plate perpendicular to the vertical direction, it can be written as a combination of the pressure $p(M, s)$ and the normal displacement $w(M, s)$ on the top face $S_+$ and bottom face $S_-:

$$P(w, p, s) = \int_{S_+} w(M, s) p(M, s) \, dS_M - \int_{S_-} w(M, s) p(M, s) \, dS_M \quad \hfill (25)$$

Assuming baffled conditions of radiation, the acoustic pressure can be modeled using Rayleigh integral (Fahy [23]). On the top face, for instance, the pressure is expressed with respect to the normal velocity $u(M_0, s)$ as following:

$$p(M, s) = \rho \frac{s^2}{2} \int_{S_+} G(M_0, M, s) \, u(M_0, s) \, dS_{M_0} \quad \hfill (26)$$

where $G(M_0, M, s) = -\frac{e^{ikM_0}}{4\pi M_0} - \frac{e^{ikM_1}}{4\pi M_1}$ is the Green kernel for the Neumann problem of the Helmholtz equation in a semi-infinite medium limited by the plane occupied by the baffled structure, $M_1$ is the image of $M_0$ with respect to this plane; for a problem in which the only source is the radiating baffled structure, $M_1$ coincide with $M_0$ and one has $G(M_0, M, s) = -\frac{e^{ikM_0}}{2\pi M_0}$. $k$ is the wave number. On the bottom face, a negative sign, due to the negative orientation of the face, is added. The acoustic coupling operator is finally given by:

$$P(w, u, s) = \rho_f s^2 \left( \int_{S_+} \int_{S_+} w(M, s) G(M_0, M, s) \, u(M_0, s) \, dS_M \, dS_{M_0} \right. \left. + \int_{S_-} \int_{S_-} w(M, s) G(M_0, M, s) \, u(M_0, s) \, dS_M \, dS_{M_0} \right) \quad \hfill (27)$$

This formulation takes the normal displacement of both faces into account and is well adapted to a solid elements model able to assess strain along
the thickness. Numerically, the operator is evaluated by expressing the nor-
mal displacement from the finite element degrees of freedom and quadratic
functions to interpolate the values. When \( M_0 \) and \( M \) belong to the same
element, the Green function can become singular. A regularization of the
singularity is carried out over the element face by writing the displacement
in a local cylindrical coordinates system (de Lautour [24]). The Jacobian
of the transformation regularizes the final expression. After regularization,
a Gauss-Legendre integration scheme is used.

The vibroacoustic coupling has an impact on resonance modes: it creates
a mass effect, lowering modal frequencies. It also causes structural dissipa-
tion since vibration energy is extracted by acoustic radiation. These effects
are evaluated by solving the coupled vibroacoustic problem. The selected
method, which implements a perturbation technique, is presented in detail
in Appendix A.1. It makes repeated calls to a subroutine that evaluates the
coupling operator as described above. Once the vibroacoustic resonance
problem has been solved, the structural response can be evaluated in the
frequency domain using Equation (17) or in the time domain using Equa-
tion (16).

The acoustic power \( P_{ac} \) is computed by assuming steady state conditions,
for which the following time/frequency equivalence applies

\[
P_{ac} = \frac{1}{2} \int_0^T \int_S \text{Re} \left[ p(\omega, M)v(\omega, M) \right] \, dM \, d\omega
\]

(28)

where \( \omega \) is the angular frequency associated to \( s = i\omega \), \( p(t, M) \) and
\( v(t, M) \) are the acoustic pressure and velocity at point \( M \) in the time do-
main; \( p(\omega, M) \) and \( v(\omega, M) \) are their counterparts in the frequency domain.
The conjugate velocity at point \( M \) \( \overline{v}(\omega, M) = -i\omega u(\omega, M) \) is deduced from
the displacement \( u(\omega, M) \) given by expression (17) with the resonance so-
lutions of the complete vibroacoustic system.
Acoustic power spectral density levels can be deduced from the routine computing the acoustic coupling operator given by Equation (27) using following expression:

$$L_{P_{ac}} = 10 \log_{10} \frac{\Re(e^{-sP(u,u,s)})}{P_{ref}^{ac}}$$

(29)

where $P_{ref}^{ac} = 10^{-12} W$ (Pierce [25]).

6.2. Numerical comparison of the solvers

Acoustic power computations carried out with the reference hybrid iterative/perturbation solver A2 and the perturbation-based solver A3 are shown in Figure 7 for the slightly damped case and in Figure 8 for the very damped case. The construction frequency used by A3 remains 500 Hz as in the previous examples. Solvers A2 and A3 perform so similarly for either case that it is hard to distinguish any difference between the curves. It can thus be concluded that the very fast variant A3 is not only validated for dynamic but also for vibroacoustic cases.

7. Conclusions

A method for computing the non-stationary time and frequency response of viscoelastic structures, called the resonance modes method, has been presented. Based on the computation of the inverse Laplace transform, it requires a modeling technique such as the Finite Element Method, a complex solver able to solve linear complex eigenvalue problems and iterative/perturbation routines in order to spot the resonance modes, which are the free solutions of the system. The method, that can be seen as an extrapolation of the classical modal approach to complex frequency-dependent cases, is able to handle dissipative anisotropic materials characteristics expressed in raw data form. The case of an aluminum plate covered by either a small or a large constrained elastomer patch has been used to benchmark various methods: the classical modal method, the direct solution method,
Figure 7: Comparison of the level of acoustic power using the iterative algorithm $A_1$ (reference) and the perturbation-based algorithm $A_3$. Slightly damped case.

Figure 8: Comparison of the level of acoustic power using the iterative algorithm $A_1$ (reference) and the perturbation-based algorithm $A_3$. Very damped case.
considered as the reference one, and three implementations of the resonance modes method. The first implementation uses an iterative approach to look for the resonance modes, while the second one, also iterative, is being accelerated by perturbation. The third implementation is based on perturbation exclusively. The benchmark has shown that the resonance modes method results in much quicker computation times than the direct solution while keeping the same level of accuracy. The very quick variant, which provides computational times reduced by a factor of two in magnitude, has also been validated for vibroacoustic applications.

Appendix A. COMPUTATION OF THE VIBROACOUSTIC RESONANCE MODES

Appendix A.1. Computation of the resonance values

A perturbation technique has been used to compute the resonance modes of the damped vibroacoustic system. A weighting parameter $\epsilon$ is introduced, the value of which varies from $\epsilon = 0$ in the uncoupled case to $\epsilon = 1$ in the fully coupled case. The resonance values problem can be expressed as following:

$$w^T \left[ s_k(\epsilon)^2 M + K^B(s_k(\epsilon)) + K^A \right] u_k(\epsilon) = \epsilon P(w, u_k(\epsilon), s_k(\epsilon)) \quad (A.1)$$

The zero-order matrix equation is obtained by setting $\epsilon$ equal to zero:

$$\left[ s_k(0)^2 M + K^B(s_k(0)) + K^A \right] u_k(0) = 0, \quad (A.2)$$
where \((s_k(0), u_k(0))\) is the \(k^{th}\) resonance couple of the damped system in vacuum. The first derivative of (A.1) with respect to \(\epsilon\) is

\[
\begin{align*}
    w^T \left[ s_k(\epsilon)^2 M + K^B(s_k(\epsilon)) + K^A \right] \frac{\partial}{\partial \epsilon} (u_k(\epsilon)) \\
    + w^T \left[ \frac{\partial}{\partial \epsilon} (s_k(\epsilon)^2) M + \frac{\partial}{\partial \epsilon} (K^B(s_k(\epsilon))) \right] u_k(\epsilon)
\end{align*}
\]

(A.3)

The first-order matrix equation is obtained by letting \(\epsilon\) tend to zero and taking Equation (A.2) into account:

\[
    w^T \left[ \frac{\partial s_k(0)}{\partial \epsilon} s_k(0) M + \frac{\partial s_k(0)}{\partial \epsilon} \frac{\partial}{\partial s_k(0)} (K^B(s_k(0))) \right] u_k(0) = P(w, u_k(0), s_k(0))
\]

(A.4)

Choosing \(w = \overline{u}_k(0)\) and orthonormalizing resonance vectors with the mass matrix yields

\[
\frac{\partial s_k(0)}{\partial \epsilon} = P(\overline{u}_k(0), u_k(0), s_k(0)) / \left[ 2s_k(0) + u_k(0)^H \left( \frac{\partial}{\partial s_k(0)} K^B(s_k(0)) \right) u_k(0) \right]
\]

(A.5)

The first-order approximation of the perturbed Laplace parameter is given by \(s_k(\epsilon) = s_k(0) + \epsilon \frac{\partial s_k(0)}{\partial \epsilon}\). When \(\epsilon = 1\),

\[
    s_k(1) = s_k(0) + P(\overline{u}_k(0), u_k(0), s_k(0)) \\
    / \left[ 2s_k(0) + u_k(0)^H \frac{\partial}{\partial s_k(0)} (s_k(0)K^B(s_k(0))) u_k(0) \right]
\]

(A.6)

**Computation of the resonance vectors**

Similar developments can be written to assess the influence of the acoustic coupling on resonance vectors. Using Equation (A.3) and setting \(\epsilon\) equal to zero yields

\[
    w^T \left[ s_k(0)^2 M + K^B(s_k(0)) + K^A \right] \frac{\partial}{\partial \epsilon} (u_k(0))
\]

\[
    = P(w, u_k(0), s_k(0)) - w^T \left[ \frac{\partial}{\partial \epsilon} (s_k(0)^2) M + \frac{\partial}{\partial \epsilon} (K^B(s_k(0))) \right] u_k(0)
\]

(A.7)
The equation left side operator spans a N-1 dimensions space to which vector \( u_k(0) \) does not belong. A decomposition on this operator basis can be written as

\[
\frac{\partial}{\partial \epsilon} (u_k(0)) = \sum_{i=1, i \neq k}^{N} \alpha_i u_i(0), \quad (A.8)
\]

Equation (A.7) can thus be further transformed into the following one

\[
w^T \left[ s_k(0)^2 M + K^B(s_k(0)) + K^A \right] \frac{\partial}{\partial \epsilon} (u_k(0))
= w^T \sum_{i=1, i \neq k}^{N} \alpha_i \left[ (s_k(0)^2 - s_i(0)^2) M + K^B(s_k(0)) - K^B(s_i(0)) \right] u_i(0)
\]

\[(A.9)\]

Coefficients \( \alpha_i \) are deduced by selecting \( w = \pi_i(0) \) and by using the biorthonormality properties of the symmetric system:

\[
\alpha_i = \frac{P(\pi_i(0), u_k(0), s_k(0))}{(s_k(0)^2 - s_i(0)^2 + u_i^H(0) \left[ K^B(s_k(0)) - K^B(s_i(0)) \right] u_i(0))}
\]

\[(A.10)\]

The first order approximation of the \( k^{th} \) resonance vector of the damped vibroacoustic structure is given by setting \( \epsilon = 1 \),

\[
u_k(1) = u_k(0) + \sum_{i=1, i \neq k}^{N} \alpha_i u_i(0)
\]

\[(A.11)\]

The intermodal coupling terms of this series have been found negligible when the aluminum plate studied in this paper radiates in the air.