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Hybrid High-Order Methods for Variable-Diffusion Problems on General Meshes

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Abstract
We extend the Hybrid High-Order method introduced by the authors for the Poisson problem to problems with heterogeneous/anisotropic diffusion. The cornerstone is a local discrete gradient reconstruction from element- and face-based polynomial degrees of freedom. Optimal error estimates are proved.

Résumé
Méthodes hybrides d'ordre élevé pour des problèmes à diffusion variable sur des maillages généraux. Nous étendons la méthode hybride d'ordre élevé conçue par les auteurs pour le problème de Poisson à des problèmes de diffusion hétérogène/anisotrope. La pierre angulaire est une reconstruction locale du gradient discret à partir des degrés de liberté polynomiaux sur les éléments et les faces. On établit des estimations d’erreur optimales.

1. Introduction
Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote an open, bounded, polytopic domain. Let $f \in L^2(\Omega)$ and, for a subset $X \subset \Omega$, denote by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ the inner product and norm in $L^2(X)$, respectively. We focus on the following variable-diffusion problem: Find $u \in U_0 := H^1_0(\Omega)$ such that

$$(\kappa \nabla u, \nabla v)_{\Omega} = (f, v)_{\Omega} \quad \forall v \in U_0, \quad (1)$$

where $\kappa$ is a bounded, tensor-valued function in $\Omega$, taking symmetric values with lowest eigenvalue uniformly bounded from below away from zero. Owing to the Lax–Milgram Lemma, problem (1) is well-posed.

The approximation of diffusive problems on general polytopic meshes has received an increasing attention lately. Several low-order methods have been developed; see, e.g., \cite{1,2} and references therein. Recently, high-order methods have also become available; we mention the high-order Mimetic Finite Difference (MFD) schemes \cite{3,4}, the Virtual Element Method \cite{5}, the Mixed High-Order method \cite{6}, and the Hybrid High-Order (HHO) methods \cite{7,8}. For the latter, the degrees of freedom (DOFs) are scalar-valued polynomials at mesh elements and faces up to some degree $k \geq 0$ (as for the MFD schemes in \cite{4}), and the construction hinges on (i) a local discrete gradient reconstruction of order $k$ and (ii) a least-squares local penalty that weakly enforces the matching between element- and face-based DOFs while preserving the order of the gradient reconstruction. This design leads to optimal energy- and $L^2$-norm error estimates; cf. \cite{7} for the Poisson problem ($\kappa$ being the identity tensor in (1)) and \cite{8} for (quasi-incompressible) linear elasticity.

The purpose of the present work is to extend the HHO method of \cite{7} to the variable-diffusion problem (1). The key idea is to modify the gradient reconstruction so as to account for the diffusion tensor $\kappa$. Then, adapting the ideas of \cite{7}, we prove stability of the discrete problem and derive optimal error estimates. We make the reasonable assumption that there is a partition $P_0$ of $\Omega$ so that $\kappa$ is piecewise Lipschitz. For simplicity of exposition, we also assume that $\kappa$ is a piecewise polynomial; otherwise, an additional quadrature error has to be accounted for. In applications from the geosciences, $\kappa$ can often be taken piecewise constant.

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2. Discrete setting and local gradient reconstruction

We consider admissible mesh sequences in the sense of [2] Sect. 1.4. Each mesh \( T_h \) in the sequence is a finite collection \( \{ T \} \) of nonempty, disjoint, open, polytopic elements such that \( \Omega = \bigcup_{T \in T_h} T \) and \( h = \max_{T \in T_h} h_T \) (with \( h_T \) the diameter of \( T \)), and there is a matching simplicial submesh of \( T_h \) with locally equivalent mesh size and which is shape-regular in the usual sense. For all \( T \in T_h \), the faces of \( T \) are collected in the set \( \mathcal{F}_T \). In an admissible mesh sequence, \( \text{card}(\mathcal{F}_T) \) is uniformly bounded, the usual discrete and multiplicative trace inequalities hold on element faces, and the \( L^2 \)-orthogonal projector onto polynomial spaces enjoys optimal approximation properties on each mesh element. Let a polynomial degree \( k \geq 0 \) be fixed. For all \( T \in T_h \), we define the local space of DOFs as \( U_T^k := \mathbb{P}_d^k(T) \times \{ x \in \mathbb{C} \} \), where \( \mathbb{P}_d^k(T) \) (resp., \( \mathbb{P}_d^{k-1}(T) \)) is spanned by the restrictions to \( T \) (resp., \( F \)) of \( d \)-variate (resp., \( (d-1) \)-variate) polynomials of total degree \( \leq k \). In what follows, we denote by \( \kappa_T^2 \) the lowest and \( \kappa_T^4 \) the largest eigenvalue of \( \kappa \) in \( T \), respectively, and we introduce the local heterogeneity/anisotropy ratio \( \rho_T := \kappa_T^2 / \kappa_T^4 \geq 1 \). In what follows, we explicitly track the dependency of the bounds on the ratio \( \rho_T \). To avoid the proliferation of symbols, we assume that for all \( T \in T_h \), the Lipschitz constant of \( \kappa \) in \( T \), say \( L^\kappa \), satisfies \( L^\kappa_T \leq \kappa_T^4 \).

For all \( T \in T_h \), we define the local gradient reconstruction operator \( G_T^k : U_T^k \rightarrow \nabla \mathbb{P}_d^{k+1}(T) \) such that, for all \( v := (v_T, (v_F)_{F \in \mathcal{F}_T}) \in U_T^k \) and all \( w \in \mathbb{P}_d^{k+1}(T) \),

\[
(\kappa G_T^k v, \nabla w)_T = (\kappa v_T, \nabla w)_T + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \nabla w : \kappa \eta_{TF})_F,
\]

where \( \kappa_T \) denotes the mean-value of \( \kappa \) in \( T \). Note that the right-hand side vanishes if \( \kappa \) is piecewise constant. In the general case, owing to the assumptions on \( \kappa \) and using the approximation properties of the \( L^2 \)-orthogonal projectors along with a discrete trace inequality for \( \| \kappa^{1/2} \nabla w \|_T \), we infer that

\[
\| \kappa^{1/2} (\nabla (v - p_T^{k+1} v), \nabla w)_T \|_T \leq L_T^\kappa h_T \| \nabla^{k+1} v \|_{H^{k+1}(T)} \| \nabla w \|_T \leq L_T^\kappa h_T \| \nabla^{k+1} v \|_{H^{k+1}(T)} \| \nabla w \|_T.
\]

We now observe that

\[
\| \kappa^{1/2} (\nabla (v - p_T^{k+1} v), \nabla w)_T \|_T^2 = (\kappa (\nabla (v - p_T^{k+1} v), \nabla (v - p_T^{k+1} v)_T) + (\kappa (\nabla (v - p_T^{k+1} v), \nabla (v - p_T^{k+1} v)))_T.
\]

Denote by \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) the addends on the right-hand side of (5). Using the Cauchy–Schwarz inequality and the approximation properties of \( p_T^{k+1} \), we obtain \( |\mathcal{S}_1| \subseteq \| \kappa^{1/2} (\nabla (v - p_T^{k+1} v))_T (\kappa T^{1/2} h_T^{k+1} \| \| \nabla^{k+1} v \|_{H^{k+1}(T)} \| \nabla w \|_T.\)
When $\kappa$ is piecewise constant, $\mathcal{S}_2$ vanishes, so that using Young’s inequality yields $\|\nabla(v - p_T^h[v])\|_T \leq (\kappa_T^{-1/2}\|\kappa_T^{1/2}\nabla(v - p_T^h[v])\|_T \leq p_T^h \|v\|_{H^{1/2}(\Omega)}$. In the general case, using $w = (\pi_T^h v - p_T^h[v])$ and since $\|\nabla((\pi_T^h v - p_T^h[v]))\|_T = \|\nabla(v - p_T^h[v])\|_T$, we infer that $\|\mathcal{S}_2\|_T \leq \rho_T^h \|\kappa_T^{1/2}(\kappa_T^{1/2} h_T^{-1})\|_{H^{1/2}(\Omega)} \|\kappa_T^{1/2}\nabla(v - p_T^h[v])\|_T$, which leads to the estimate on $\|\nabla(v - p_T^h[v])\|_T$ in [4]. The other terms in [4] are then bounded as in [7] Lemma 3.

Remark 1 ($\alpha = 0$). It is also possible to take $\alpha = 0$ whenever, for all $T \in \mathcal{T}_h$, the eigenvectors of $\kappa_T$ are constant and its eigenvalues satisfy, with obvious notation, $|\lambda(x) - \lambda_T| \leq h_T \lambda(x)$ for all $x \in T$.

3. Discrete problem and stability

For all $T \in \mathcal{T}_h$, we introduce the local bilinear forms $a_T$ and $s_T$ on $U^k_T \times U^k_T$ such that

$$a_T(u, v) := (\kappa^k G^k_T u, G^k_T v)_T + s_T(u, v), \quad s_T(u, v) := \sum_{F \in \mathcal{F}_T} \frac{K_F}{h_F} (\pi_F(uF_P^T u) - \pi_F(vF_P^T v))_F,$$

with $K_F := \|n_{TF} \cdot \kappa n_{TF} \|_{L^\infty(F)}$ and the local potential reconstruction $P^h_T : U^k_T \rightarrow \mathbb{P}^{k+1} (T)$ such that $P^h_T v := v_T + (p^h_T - P^h_T v)_T$. We define the global space of DOFs by patching interface values, so that $U^k_h := \{v \in \bigoplus_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \} \times \{v \in \mathcal{F}_h, v_{F_0^c}(F) \equiv 0 \}$, where, for all $T \in \mathcal{T}_h$, we denote by $L_T : U^k_T \rightarrow U^k_T$ the restriction operator that maps the global DOFs in $U^k_h$ to the corresponding local DOFs in $U^k_T$. The discrete problem consists in seeking $v_h \in U^k_{h, 0} := \{v_h = (v_{TF})_{T \in \mathcal{T}_h}, (vF_P^T)_{F \in \mathcal{F}_h} \} \subset U^k_h \告诉你 

$$a_T(u_h, v_h) := \sum_{T \in \mathcal{T}_h} a_T(l_T u_h, l_T v_h) = \sum_{T \in \mathcal{T}_h} (f, vF_P^T)_T :=: l_h(u_h) \forall u_h \in U^k_{h, 0}.$$ (7)

To analyze the stability of the discrete problem, we introduce the following seminorm on $U^k_T$:

$$\|v\|_{k, T} := \|\kappa^{1/2} \nabla v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{K_F}{h_F} \|vF_P^T - v\|_F^2,$$

and we set $\|v\|_{k, h, T}^2 := \sum_{T \in \mathcal{T}_h} \rho_T^{-1} \|L_T v_h\|_{k, T}^2$ for all $v_h \in U^k_h$. Observe that $\|\cdot\|_{k, h, T}$ is a norm on $U^k_{h, 0}$.

**Lemma 3.1 (Stability).** *The following inequalities hold for all $v \in U^k_T$:

$$\rho_T^{-1} \|v\|^2_{k, h, T} \leq a_T(v, v) \leq \rho_T \|v\|_{k, T}^2.$$ (9)

Consequently, $\|v\|_{k, h, T}^2 \leq a_T(v_h, v_h)$ for all $v_h \in U^k_h$ and problem (7) is well-posed.*

**Proof.** We adapt the proof of [7] Lemma 4. Concerning the face terms, we obtain

$$\sum_{F \in \mathcal{F}_T} \frac{K_F}{h_F} \|vF_P^T - v\|_F^2 \leq s_T(v, v) + \rho_T \|\kappa^{1/2} v_T\|_T^2,$$

and we set $\|v\|_{k, h, T}^2 := \sum_{T \in \mathcal{T}_h} \rho_T^{-1} \|L_T v_h\|_{k, T}^2$ for all $v_h \in U^k_h$. Observe that $\|\cdot\|_{k, h, T}$ is a norm on $U^k_{h, 0}$.

To compare $\|\kappa^{1/2} G^k_T v\|_T$ and $\|\kappa^{1/2} \nabla v_T\|_T$, we observe that, for all $w \in \mathbb{P}^{k+1}(T)$ and all $F \in \mathcal{F}_T$,

$$\|\nabla(w \cdot \kappa_{TF})\|_T^2 \leq \left(\|n_{TF} \cdot \kappa_{TF}\|_{L^\infty(F)} \|\nabla w \cdot \kappa_{TF}\|_F \right)^2 \|\kappa_T^{1/2} \nabla w\|_T^2,$$

where we have used the Cauchy–Schwarz inequality for $\kappa$, the definition of $K_F$, and a discrete trace inequality. Taking $w = v_T$ in the definition [2] of $G^k_T v$ yields $\|\kappa^{1/2} \nabla v_T\|_T^2 = (\kappa_T^{1/2} v_T, \nabla v_T) - \sum_{F \in \mathcal{F}_T} (vF_P^T - v_T, \nabla v_T \cdot \kappa_{TF})$. Hence, using [1], a discrete trace inequality for $\|\kappa^{1/2} \nabla v_T\|_T$, the first bound in (10), $\rho_T \geq 1$, and Young’s inequalities yield

$$\|\kappa^{1/2} \nabla v_T\|_T^2 \leq \|\kappa^{1/2} G^k_T v\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{K_F}{h_F} \|vF_P^T - v\|_F^2 \leq \rho_T \|\kappa^{1/2} G^k_T v\|_T^2 + s_T(v, v).$$

Moreover, since $\|\kappa^{1/2} G^k_T v\|_T = \sup_{w \in \mathbb{P}^{k+1}(T)} \|\kappa_T^{1/2} \nabla w\|_T$, and proceeding similarly leads to $\|\kappa^{1/2} G^k_T v\|_T \leq \|v\|_{k, T}$. Combining the above bounds yields (9), and the rest of the proof is straightforward. \qed
4. Error analysis

**Theorem 4.1** (Energy-error estimate). Let \( u \in U_0 \) solve \( \int \) and let \( u_h \in U_{k,0}^h \) solve \( \int \). Assume that \( u_T \in H^{k+2}(T) \) for all \( T \in T_h \). Then, letting \( \tilde{u}_h := ((\pi^k_T u)_{T \in T_h}, (\pi^k_F u)_{F \in F_h}) \in U_{k,0}^h \) and, recalling the definition of \( \alpha \) from Lemma 2.1, the following holds with consistency error \( \mathcal{E}_h(v_h) := a_h(\tilde{u}_h,v_h) - l_h(v_h) \):

\[
\|\tilde{u}_h - u_h\|_{k,h} \lesssim \sup_{v_h \in U_{k,0}^h, \|v_h\|_{k,h} = 1} \mathcal{E}_h(v_h) \lesssim \left\{ \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}.
\] (12)

**Proof.** We adapt the proof of Theorem 8. The first inequality in (12) is an immediate consequence of Lemma 3.1 and \( v_h \in U_{k,0}^h \) with \( \|v_h\|_{k,h} = 1 \) leads to

\[
\mathcal{E}_h(v_h) = \sum_{T \in T_h} \left( |\kappa| \nabla (\tilde{u}_T - u), \nabla v_T \right) + \sum_{T \in T_h} \sum_{F \in F_T} (v_F - v_T, (\nabla \tilde{u}_T - \nabla u) \cdot n)_{TF} + \sum_{T \in T_h} s_T (L_T \tilde{u}_h, L_T v_h).
\]

Denote by \( \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \) the three terms on the right-hand side. Combining the results of Lemmas 2.1 and 3.1, we infer that \( \|\mathcal{T}_1 + \mathcal{T}_2\|^2 \lesssim \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \). Moreover, since \( s_T (L_T \tilde{u}_h, L_T v_h) \leq s_T (L_T u_h, L_T v_h) \|\tilde{u}_h - u_h\|_{k,h} \), we get

\[
\mathcal{T}_3 \lesssim \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2.
\]

Finally, adapting the proof of Theorem 10 leads to the following \( L^2 \)-error estimate.

**Theorem 4.2** (L²-error estimate). Assume elliptic regularity for problem \( \int \) in the form \( \|z\|_{H^2(\Omega)} \lesssim \|g\|_{\Omega} \) for all \( g \in L^2(\Omega) \) and \( z \in U_0 \) solving \( \int \), with data \( g \). Assume \( f \in H^{k+2}(\Omega) \) with \( \delta = 0 \) for \( k \geq 1 \) and \( \delta = 1 \) for \( k = 0 \). Then, using the same notation as in Theorem 4.1 and defining the piecewise polynomial functions \( \tilde{u}_h \) and \( u_h \) such that \( \tilde{u}_h \big|_T = \pi_T u \) and \( u_h \big|_T = u_T \) for all \( T \in T_h \), the following holds:

\[
\|\tilde{u}_h - u_h\|_{\Omega} \lesssim \left( \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right)^{1/2} + h^{k+2} \|f\|_{H^{k+2}(\Omega)}.
\]

where \( \|g\|_{\Omega} := \max_{T \in T_h} (\kappa_T^k \rho_T^{1+2\alpha} h_T)^{1/2} \).

**References**


