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Abstract

This paper concerns observer design for Lipschitz nonlinear systems with sampled output. Using reachability analysis, an upper approximation of the attainable set is given. When this approximation is formulated in terms of a convex combination of linear mappings, a sufficient condition is given in terms of linear matrix inequalities (LMIs) which can be solved employing an LMIs solver. This novel approach seems to be an efficient tool to solve the problem of observer synthesis for a class of Lipschitz systems of small dimensions.

Index Terms

Continuous discrete-time observers, reachable sets, Pontryagin Maximum Principle, LMI.

I. Introduction

In this paper, we consider the state estimation problem for a class of continuous time Lipschitz systems with discrete time measurements. More precisely, the aim of our study is to design a continuous-discrete state observer. The use of this type of algorithm to estimate the state of nonlinear systems has already been investigated in the literature. It can be traced back to Jazwinski who introduced the continuous-discrete Kalman filter to solve a filtering problem for

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stochastic continuous-discrete time systems (see [10]). Inspired by this approach, the popular high-gain observer introduced in [9] has been adapted to the continuous-discrete context in [7]. Since then, different approaches have been investigated by different authors. The robustness of an observer with respect to time discretization was studied in [4]. In [11] observers were designed from an output predictor (see also related works in [1, 12]). Some other approaches based on time delayed techniques have also been considered in [16].

In our study we focus on the preliminary work of [7], and consider the case in which the continuous-discrete observer is obtained in two steps:

i) when no measurement is available, the state estimate is computed by integrating the model;
ii) when a measurement occurs, the observer makes an impulsive correction on the state estimate.

Note that in [7] the correction gain of this impulsive correction is obtained by integrating a continuous-discrete time Riccati equation. However, in the following, inspired by [15], we will consider a constant correction term.

In most of the above cited works, the asymptotic convergence of the estimate to the state is obtained by dominating the Lipschitz nonlinearities with high-gain techniques. However, as this is now well understood, there is a trade-off between the high-gain parameter and the measurement step size. This can lead to restrictive design conditions on the sampling measurement time (see for instance [2]) which may prevent the use of such technique in practice.

Recently, a new observer design methodology for Lipschitz nonlinear systems with continuous time measurements has been introduced in [20] (see also [21]). In this approach, it is shown that the differential equation satisfied by the estimation error can be rewritten in the form of a linear parameter varying system (LPV). Hence, the convergence to zero of the estimation error can be obtained by solving some specific linear matrix inequalities (see Section II for a brief summary of this approach). The aim of our paper is to extend the approach presented in [20] to the discrete time measurement case. In the adopted strategy, the main problem is decomposed into two subproblems:

i) *Computation of an upper approximation of the reachable set for a bilinear system.* This set characterizes the possible expansion of the estimation error between two measurements when the estimate is given by integrating the model.
ii) *Construction of a correction term K ensuring the convergence to zero of a quadratic error*

Lyapunov function. As in [20], this step is performed through linear matrix inequalities (LMIs) techniques.

In this paper, we address the first problem by considering systems with specific structure: upper triangular nonlinearities (as in the preliminary version of this work in [3]), and in observability canonical form. This allows us to obtain a constructive approach to the synthesis of observers for a wide class of Lipschitz nonlinear systems while avoiding standard high-gain approach. Consequently, we hope to design observers with larger sampling period than those obtained by usual techniques. However, the large size of the linear matrix inequality restricts the use of this approach only to low dimensional systems.

The paper is organized as follows. In Section II, after having defined the considered class of systems (in Subsection II-A), some preliminary results and in particular the approach of [20] for continuous time measurements are recalled (in Subsection II-B). An approach of observer design based on reachability set analysis is presented in Subsection II-C. Section III concerns uniformly observable systems. Simulations of an academic example is given in Section IV to illustrate the methodology proposed in this paper.

II. Preliminaries

A. Problem statement

The class of nonlinear systems under consideration is described by the following differential equation

$$\dot{x}(t) = Ax(t) + \phi(x(t), u(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u : \mathbb{R} \to \mathbb{R}^p$ is a known input, A is a matrix in $\mathbb{R}^{n\times n}$ and $\phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a C^1 globally Lipschitz (uniformly in the input) function. In other words, the following assumption is made.

Assumption 1. *For every i, j in $\{1, \ldots n\}$, there exists a real number $c_{ij} \geqslant 0$ such that*

$$\left|\frac{\partial \phi_i}{\partial x_j}(x, u)\right| \leqslant c_{ij}, \quad \forall\ (x, u) \in \mathbb{R}^n \times \mathbb{R}^p.$$

The state x of system (1) is accessible via discrete time measurements given as a sequence of m dimensional real vectors $(y_k)_{k\in\mathbb{N}}$ of the form

$$y_k = Cx(t_k), \tag{2}$$

where C is a real matrix in $\mathbb{R}^{m\times n}$ and $(t_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers defined by $t_{k+1} = t_k + \delta$, $\delta > 0$ representing the sampling measurement time.

The main objective of this work is to design a global observer for system (1) which gives an estimate $\hat{x}$ that converges asymptotically to x from the knowledge of the output y_k given in (2).

Inspired by [7] and [15], the analysis is restricted to a specific class of continuous-discrete time observers defined by the hybrid system

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \phi(\hat{x}(t), u(t)), \quad t \in [t_k, t_{k+1}), \\ \hat{x}(t_k) = \hat{x}(t_k^-) + K(y_k - C\hat{x}(t_k^-)), \end{cases} \tag{3}$$

where $\quad \hat{x}(t_k^-) = \lim_{t\to t_k,\ t<t_k} \hat{x}(t). \qquad (4)$

The estimation problem consists in determining a gain K such that the estimation error $e(t) = x(t) - \hat{x}(t)$ converges asymptotically toward zero. The proposed approach is based on a result obtained in [20] which is recalled in the following section.

B. An approach for continuous time measurements

In [20], the authors addressed the problem of observer design for system (1) where the output is considered as a continuous time function and given by

$$y(t) = Cx(t), \quad \forall\ t > 0. \tag{5}$$

Let $\mathcal{R} \subset \mathbb{R}^{n\times n}$ be the set of matrices defined as

$$\mathcal{R} = \left\{ R \in \mathbb{R}^{n\times n} \mid R_{ij} = A_{ij} \pm c_{ij}, \quad \forall\ i,j = 1,\dots,n \right\}.$$

Note that $\mathcal{R}$ is composed of 2^ρ elements where ρ is the number of $c_{ij} \neq 0$.

One of the results obtained in [20] is the following.

Theorem II.1 ([20])**.** *Assume that Assumption 1 is satisfied for system (1). If there exist a symmetric positive definite (SPD) matrix P in $\mathbb{R}^{n\times n}$ and a matrix L in $\mathbb{R}^{n\times m}$ such that the following matrix inequalities hold:*

$$R'P + PR - C'L' - LC < 0, \quad \forall\ R \in \mathcal{R}, \tag{6}$$

then the system

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \phi(\hat{x}(t), u(t)) + P^{-1}L(y(t) - C\hat{x}(t)), \tag{7}$$

is an asymptotic observer for system (1), where y is given by (5), i.e. $\lim_{t \to +\infty} |\hat{x}(t) - x(t)| = 0.$

In the rest of the paper, we extend this procedure when the measurement is discrete in time.

C. An approach based on reachability analysis

The approach of observer design proposed in this work, is based on a reachable set computation. To develop this approach, let us consider system (1) and observer (3). The estimation error $e = \hat{x} - x$ is solution to

$$\begin{cases} \dot{e}(t) &= Ae(t) + \Delta\phi(\hat{x}(t), u(t), e(t)), \quad t \in [k\delta, (k+1)\delta), \\ e(k\delta) &= (\mathrm{Id} - KC)e(k\delta^-), \end{cases} \tag{8}$$

where the notation (4) is used and $\Delta\phi$ is the C^0 function

$$\Delta\phi(\hat{x}, u, e) = \phi(\hat{x}, u) - \phi(\hat{x} - e, u).$$

Employing the mean value theorem, it yields the existence of n functions $z_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, n$ such that the components of the function $\Delta\phi$ satisfy

$$\Delta\phi_i(\hat{x}, u, e) = \frac{\partial \phi_i}{\partial x}(z_i(\hat{x}, e), u)e, \quad i = 1, \ldots, n. \tag{9}$$

Hence, the error $e(t)$ satisfies the following equation

$$\dot{e}(t) = Ae(t) + V(t)e(t), \tag{10}$$

where $t \in [k\delta, (k+1)\delta)$ and the elements $V_{ij}(t) = \frac{\partial \phi_i}{\partial x_j}(z_i(\hat{x}(t), e(t)), u(t))$ of the matrix $V(t)$ satisfy

$$|V_{ij}(t)| \leqslant c_{ij}, \qquad \forall\, i, j \in \{1, \ldots, n\}. \tag{11}$$

In our approach, we consider the error equation (10) as a bilinear control system where the elements V_{ij} are bounded control inputs. The observer should converge for all values of V_{ij} satisfying (11). One way to formalize this problem is to introduce the notion of attainable set in finite time.

Let $\mathcal{A}_\delta(e_0)$ denote the attainable set from e_0 at time $\delta \geqslant 0$ of system (10), i.e.,

$$\mathcal{A}_\delta(e_0) = \{e(\delta) \mid \delta \geqslant 0,\ e(\cdot) \text{ is solution to (10) with } e(0) = e_0\}.$$

The following theorem uses the set $\mathcal{A}_\delta(e)$ to give a condition (formulated as LMIs) guaranteeing the convergence of a continuous-discrete observer (3) for system (1)-(2).

Theorem II.2 (Sufficient condition for observer design)**.** *Let Assumption 1 hold for system (1) and let δ be the sampling measurement time. Assume that there exist a finite set $\mathscr{M}$ of matrix functions (mappings from $\mathbb{R}_+$ into $\mathbb{R}^{n\times n}$), a matrix $P > 0 \in \mathbb{R}^{n\times n}$ and a vector $W \in \mathbb{R}^{n\times m}$ such that*[1]

$$\mathcal{A}_\delta(e) \subset \mathrm{Conv}\{M(\delta)e,\ M \in \mathscr{M}\}, \quad \forall\ e \in \mathbb{R}^n, \tag{12}$$

and

$$\begin{bmatrix} P & M(\delta)'(P - C'W') \\ (P - WC)M(\delta) & P \end{bmatrix} > 0, \quad \forall\ M \in \mathscr{M}. \tag{13}$$

Then, for $K = P^{-1}W$, the estimation error given by the observer (3) converges asymptotically to zero.

The proof of this result may be found in [3].

D. Some remarks on the approach of Theorem II.2

The first step of the proposed approach is the computation of an approximation of the attainable set $\mathcal{A}_\delta(e)$ for a bilinear control system. Reachability analysis has received numerous attentions in the literature; for instance, in [5], the author analyzes the geometry of the reachable set of bilinear systems. In [18], the author gave sufficient conditions guaranteeing that the reachable set of a bilinear controllable system is convex. If some results on the characterization of the reachable set are now available for low dimension systems (see for instance the recent result in [14]), its characterization is still an open problem in general.

However, the novelty of the studied problematic is that the exact computation of this set is not needed. As a matter of fact, only an upper approximation in terms of the matrix functions $M \in \mathscr{M}$ as expressed in (12) is required. As it will be seen in the remaining part of the paper, for uniformly observable systems, an upper approximation can be explicitly given. Hence, for these two classes of systems our observer design strategy can be performed.

Given the matrix function set $\mathscr{M}$, the second step of the design is to solve the linear matrix inequality (13). In fact, the usual detectability property is embedded in this inequality. For instance, it is a necessary condition that the pair $(\exp(A\delta), C)$ is detectable for this inequality

[1] Conv denotes the convex closure.

to have a solution. Note however that inequality (13) is much stronger than this detectability condition since all Lipschitz nonlinearities have to be taken into account.

In fact, it can be shown that if Theorem II.1 applies for the continuous time measurement case, then for a small sampling measurement time, the proposed approach can be applied provided that the elements of the matrix functions set $\mathscr{M}$ satisfy some local properties. Indeed, the link between the two matrix inequalities (6) and (13) can be expressed as follows.

Proposition II.1 (Local properties of matrices in $\mathscr{M}$)**.** *Assume that there exist P and L such that the matrix inequality (6) holds for a given set of matrices $\mathcal{R}$. If the set $\mathscr{M}$ of matrix functions is such that every function $M \in \mathscr{M}$ is C^1 at time $t = 0$ and satisfies*

$$M(0) = \mathrm{Id}, \qquad \dot{M}(0) \in \mathcal{R}, \tag{14}$$

then, for all δ small enough, the matrix inequality (13) is satisfied with the same matrix P and for $W = \delta L$.

The proof of this result may be found in [3]. This approach has been employed in [3] to the particular case of systems with an upper triangular structure. In the following section, we apply this approach to a more standard context of systems in observability canonical form.

III. Case of system in observability canonical form

In this section, we assume the following assumption.

Assumption 2 (Obsevability canonical form)**.** *The matrix A and the positive real number c_{ij} are such that $Ae = (e_2, \dots, e_n, 0)$ and $c_{ij} = 0$, for all $(i, j) \in \{1, \dots, n-1\} \times \{1, \dots, n\}$.*

In this particular context, we have $V'(t) = (0, \dots, 0, v(t))$, where $v = (v_1, \dots, v_n)$. Consequently, we may rewrite system (10) with $c_j := c_{nj}$ as

$$\begin{cases} \dot{e}_j = e_{j+1}, \quad j = 1, \dots, n-1 \\ \dot{e}_n = \langle v, e \rangle, \quad v \in \displaystyle\prod_{j=1}^{n} [-c_j, c_j]. \end{cases} \tag{15}$$

1) Statement of the main result: If N is a positive integer, we denote by $C_0^1(\mathbb{R}_+, \mathbb{R}^N)$ the set of mappings from $\mathbb{R}_+$ to $\mathbb{R}^N$ which are C^1 at zero. In this framework, the following result is established.

Theorem III.1 (Systems in observability canonical form)**.** *Let Assumptions 1 and 2 hold for system (1). Then, there exist $\delta^* > 0$ and a set $\mathscr{M} \subset C^1_0(\mathbb{R}_+, \mathbb{R}^{n\times n})$ of matrix functions such that for all $\delta \in [0, \delta^*]$, the inclusion (12) is satisfied. Moreover, the conditions (14) are satisfied for all M in $\mathscr{M}$.*

Theorem III.1 allows to construct an observer which estimates asymptotically the system state if the matrix inequality (13) is satisfied. Since conditions (14) are satisfied for all M in $\mathscr{M}$ we know from Proposition II.1 that this matrix inequality has a solution provided that the LMI (6) has one and δ is sufficiently small.

Note that when $C = [1, \dots, 0]$, (6) has a solution which is the well-known high-gain observer (see [9]). Hence, Theorem III.1 gives a continuous-discrete version of the high-gain observer. This result is not new since the system is in observable form and, based on the results presented in [15], we know that there exists an observer for a small enough sampling time δ. Moreover, not all systems considered in [15] can be addressed since general lower triangular nonlinearities are not allowed.

The interest of Theorem III.1 lies in its constructive proof and in the fact that the approach is not based on high-gain technics. Hence, we expect that δ may be chosen larger than the one allowed employing the high-gain approach of [15] (see [2] for a study of the limitation of the usual high-gain approach).

However, this approach may be difficult to apply due to the fact that for systems of high dimension (n large), the set $\mathscr{M}$ may be composed of 2^{n^2} elements.

2) Preliminaries for the proof of Theorem III.1: The idea of the proof of Theorem III.1 is to approximate the attainable set $\mathcal{A}_\delta(e)$ by an n-dimensional rectangle (a direct product of n intervals). This approximation is obtained solving some optimal control problems.

Indeed, the next proposition which is proved employing Pontryagin's maximum principle gives a way to build such a rectangle from the solutions of the two following Lipschitz dynamical systems

$$\dot{e} = F^+(e), \quad F^+(e) = \left[e_2, \dots, e_n, \quad \sum_{j=1}^{n} c_j |e_j|\right]', \tag{16}$$

$$\dot{e} = F^-(e), \quad F^-(e) = \left[e_2, \dots, e_n, -\sum_{j=1}^{n} c_j |e_j|\right]'. \tag{17}$$

Let $e^+(e_0, \cdot)$ (resp. $e^-(e_0, \cdot)$) denote the solution of (16) (resp. (17)) emanating from a point e_0 at time 0 and let $T(e_0, \delta)$ be the n-dimensional rectangle defined by

$$T(e_0, \delta) = \prod_{j=1}^{n} [e_j^-(e_0, \delta), e_j^+(e_0, \delta)] \subset \mathbb{R}^n.$$

Proposition III.1. *There exists $\delta^* > 0$ such that for every $e_0 \in \mathbb{R}^n$ and every $\delta \in [0, \delta^*]$, $\mathcal{A}_\delta(e_0)$ is contained in $T(e_0, \delta)$.*

The proof of Proposition III.1 is obtained solving $2n$ Mayer problems of the form

$$\begin{cases} \text{Minimize } \langle a, x(T) \rangle \\ \text{Subject to } \dot{x} = f(x, v), \quad v \in \Omega \\ x(0) = x_0, \end{cases} \tag{18}$$

where $a \in \mathbb{R}^n$ is a constant vector, f is a smooth functions, and $\Omega \subset \mathbb{R}^n$ is compact.

The main tool to solve the Mayer problem (18) is the Pontryagin's Maximum Principle (PMP) arising in optimal control. We refer the reader to [19] for a general version of PMP adapted to the resolution of Mayer problems. The next theorem is a version of PMP that we state in our own context only.

Theorem III.2 (PMP for problem (18))**.** *Consider the control system (15). Let $H : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ denote the control dependent Hamiltonian function defined by $H(x, p, v) = \langle p, f(x, v) \rangle$. If the pair $(x(\cdot), v(\cdot)) : [0, T] \to \mathbb{R}^n \times \Omega$ is a (local) minimizer for problem (18), then there exist a Lipschitz covector $p(\cdot) : t \in [0, T] \mapsto p(t) \in \mathbb{R}^n$ and a constant $\lambda \leqslant 0$ such that the pair $(p(\cdot), \lambda)$ is never trivial and for a.e. $t \in [0, T]$:*

i. $\dot{x}(t) = \dfrac{\partial H}{\partial p}(x(t), p(t), v(t)),$
ii. $\dot{p}(t) = -\dfrac{\partial H}{\partial x}(x(t), p(t), v(t)),$
iii. $H(x(t), p(t), v(t)) = \max_{v \in \Omega} H(x, p, w),$
iv. $p(T) = \lambda a.$

Remark III.2.
- *The PMP is just a necessary condition for optimality. A trajectory $x(\cdot)$ (resp. a pair $(x(\cdot), p(\cdot))$) satisfying the conditions given by the PMP is said to be an extremal (resp. an extremal pair). An extremal corresponding to $\lambda = 0$ is said to be an abnormal extremal, otherwise we call it a normal extremal.*

- *Notice that a Mayer problem of the form (18) do not admit abnormal extremals. Indeed, otherwise, due to the transversality condition iv and equation ii (which is linear), the pair* $(p(t), \lambda)$ *would be trivial (everywhere) which contradicts the PMP. Consequently,* $p(t)$ *is never zero as soon as the constant vector* $a \neq 0$.

We are now ready to prove Proposition III.1.

Proof of Proposition III.1. First of all, notice that the result is obvious for $e_0 = 0$ since for every δ the attainable set of system (15) at $e_0 = 0$ is $\mathcal{A}_\delta(0) = \{0\}$. Suppose from now that $e_0 \neq 0$.

Let $a_1, \ldots, a_n$ denote the canonical basis vectors of $\mathbb{R}^n$, and let $\Omega = \prod_{j=1}^n [-c_j, c_j]$.

The proof is based upon the resolution of the $2n$ Mayer's problems $(\mathbf{P}_{k,\epsilon,\delta})$ with $k \in \{1, \ldots, n\}$, $\epsilon \in \{-, +\}$, $\delta > 0$ defined as follows

$$(\mathbf{P}_{k,\epsilon,\delta}) \quad \begin{cases} \text{Minimize } \ \langle \epsilon a_k, e(\delta) \rangle \\ \text{Subject to } (15) \\ e(0) = e_0. \end{cases}$$

We note that the solution of each of these problems allow us to find the minimum and maximum of each component of the solutions of the error system (15). Consequently, to show Proposition III.1, it suffices to prove the existence of a $\delta^* > 0$ such that, for every $\delta < \delta^*$, and every $k = 1, \ldots, n$ we have

$$\min_{v(\cdot) \in \mathbb{L}^\infty(\mathbb{R}_+, \Omega)} \langle \epsilon a_k, e(\delta) \rangle = e_k^\epsilon(\delta), \tag{19}$$

where $e_k^\epsilon(\delta)$ is the solution of (16) or (17) depending on the value of ϵ. The rest of the proof is devoted to the resolution of the $2n$ optimal control problems $(\mathbf{P}_{k,\epsilon,\delta})$ to prove the existence of such a δ^*.

Let us apply Theorem III.2 to problem $(\mathbf{P}_{k,\epsilon,\delta})$. The control dependent Hamiltonian associated to (15) reads

$$H(e, p, v) = \sum_{j=1}^{n-1} p_j e_{j+1} + p_n \sum_{j=1}^{n} e_j v_j, \quad p \in \mathbb{R}^n.$$

Assume that $(e(\cdot), v(\cdot))$ is an extremal pair associated with the minimization problem $(\mathbf{P}_{k,\epsilon,\delta})$, then, according to Theorem III.2, for a.e. $t \in [0, \delta]$: the adjoint system reads

$$\begin{cases} \dot{p}_1 = -p_n v_1 \\ \dot{p}_j = -p_{j-1} - p_n v_j, \quad j = 2, \ldots, n, \end{cases} \tag{20}$$

and the maximality reads

$$\begin{aligned} H(e(t),p(t),v(t)) &= \max_{w\in\Omega} H(e(t),p(t),w) \\ &= \sum_{j=1}^{n-1} p_j(t)e_{j+1}(t) + \sum_{j=1}^{n} c_j|p_n(t)e_j(t)|. \end{aligned} \tag{21}$$

Moreover, the following transversality condition holds

$$p(\delta) = -\epsilon a_k. \tag{22}$$

Notice that, according to Remark III.2, the adjoint vector $p(\cdot)$ is nontrivial. Hence, $p_n(\cdot)$ cannot vanish on an interval. Indeed, otherwise, with (20), we get also $p_{n-1}(t) = -\dot{p}_n(t) = 0$ for every t in this interval. Then iteratively $p(\cdot) = 0$ in this interval.

Similarly, for $j = 1, \ldots, n$, $e_j(\cdot)$ cannot vanish on an interval. Indeed, otherwise, from (15) we get also $e_{j+1}(t) = \dot{e}_j(t) = 0$ for t in this interval and iteratively, we obtain $e_\ell(\cdot) = 0$ for $\ell = j, \ldots, n$. If $j \geqslant 2$, we have $e_1^{(j)}(\cdot) = 0$ and therefore $e_1, \ldots, e_{j-1}$ are polynomials of degree $j-2, \ldots, 0$ respectively. On the other hand, in this interval we have $\dot{e}_n(t) = \sum_{\ell=1}^{j-1} v_\ell(t)e_\ell(t) = 0$. Thus, with (21) we get that, for almost all t in this interval

$$p_n(t)\sum_{\ell=1}^{j-1} e_\ell(t)v_\ell(t) = |p_n(t)|\sum_{\ell=1}^{j-1} c_\ell\,|e_\ell(t)| = 0.$$

Now, since the $e_j(\cdot)$s are polynomials of different degrees and $p_n(\cdot)$ cannot vanish on an interval, we conclude that $e_1(t) = \cdots = e_{j-1}(t) = 0$ for a.e. t. Consequently, $e(t) = 0$, which is a contradiction with $e_0 \neq 0$ since the system (15) is linear.

Summing up, we obtain that $p_n(\cdot)e_j(\cdot)$, $j = 1, \ldots, n$ cannot vanish on an interval, which implies that for almost all t in $\mathbb{R}_+$ $v_j(t) = |c_j|\,\mathrm{sign}(p_n(t)e_j(t))$.

Note that if we show that there exists a time $\delta^* > 0$ such that p_n does not change its sign if $t < \delta^*$, then with the transversality condition we get that

$$v_j(t) = -\epsilon|c_j|\,\mathrm{sign}(e_j(t)), \quad \forall\ t\ \in\ [0,\delta^*]. \tag{23}$$

In this case equation (19) is thus obtained and the proof of Proposition (III.1) is complete. Consequently, to complete the proof it remains to show that $p_n(\cdot)$ does not change its sign if t is small enough. This property is obtained from the following technical lemma, the proof of which is given in [8].

Lemma III.3. *There exist two positive real numbers δ_1 and d such that for every t in $[0, \delta_1]$ and for every $\delta > 0$, we have $|p_n(\delta - t)| < d\, t^{n-k}$.*

From Lemma III.3, (20) and using the fact that $|v_1(t)| \leqslant c_1$ we get, $|\dot{p}_1(\delta - t)| \leqslant c_1 d\, t^{n-k}, \quad \forall\, t \;\in\; [0, \delta_1]$. Integrating this inequality, we obtain

$$|p_1(\delta - t)| \leqslant \frac{c_1 d}{n-k+1} t^{n-k+1} + |p_1(\delta)|, \quad \forall\, t \;\in\; [0, \delta_1].$$

By integrating successively the previous inequality, and using the transversality conditions (22), it yields that there exists a positive number d_{k-1} such that

$$|p_{k-1}(\delta - t)| \leqslant d_{k-1} t^{n-k+1}, \quad \forall\, t \;\in\; [0, \delta_1]. \tag{24}$$

On the other hand, we have the following inequality for all t in $[0, \delta_1]$

$$p_k(\delta - t) \geqslant p_k(\delta) - \int_0^t |p_{k-1}(\delta - s)| + c_k |p_n(\delta - s)| ds.$$

From (22) and (24), it follows that for t in $[0, \delta_1]$

$$p_k(\delta - t) \geqslant 1 - d_{k-1}(\delta - t)^{n-k+2} - c_k d\, t^{n-k} \geqslant 1 - d_k\, t^{n-k},$$

where d_k is positive. Now, by integrating the previous inequality, we obtain for all t in $[0, \delta_1]$:

$$p_{k+1}(\delta - t) \geqslant t - d_k\, t^{n-k+1} - c_{k+1} d\, t^{n-k} \geqslant t - d_{k+1} t^{n-k},$$

where d_{k+1} is positive. Proceeding in the same manner successively, we obtain a positive constant d_{n-1} such that

$$p_{n-1}(\delta - t) \geqslant t^{n-k-1} - d_{n-1} t^{n-k}.$$

Comparing the two degrees of monomials of the last equation, we obtain the existence of a time δ_2 in $[0, \delta_1]$ such that $p_{n-1}(\delta - t) > 0, \quad \forall\, t \;\in\; [0, \delta_2]$.

Finally, we have $\dot{p}_n \geqslant v_n p_n + p_{n-1}$ and $p_n(\delta) = 0$. Thus, we get for every t in $[0, \delta_2]$,

$$p_n(\delta - t) = \int_\delta^{\delta - t} \exp\left(\int_s^{\delta - t} v_n(r) dr \right) p_{n-1}(s) ds > 0.$$

Proceeding in the same way, one infers that there exists a sufficiently small time δ^*, such that for every t in $[0, \delta^*]$ and every $k \in \{1, \ldots, n\}$, $p_n(\delta - t)$ is positive (resp. negative) if $\epsilon = -$ (resp. $\epsilon = +$). From the structure of minimizing control given in (23), it is concluded that $\delta < \delta^*$. Hence, $v_k(t) = -\epsilon |e_k(t)|$. ■

3) Proof of Theorem III.1: With the approximation of the attainable set $\mathcal{A}_\delta(e_0)$ by the rectangle $T(e_0,\delta)$ (see Proposition III.1), we can now give the proof of Theorem III.1. In order to do that, let us define the Clarke's gradient in the direction of x_j (denoted by $\partial_{\mathrm{C}}^j f(x)$) of a vector function f to be the *generalized gradient* [2] of the function $x_j \mapsto f(x_1,\ldots,x_n)$.

Since (16) and (17) are globally Lipschitz, $e^+(e_0,\delta)$ and $e^-(e_0,\delta)$ are globally Lipschitz [3]. Consequently, the Clarke gradients $\partial_{\mathrm{C}}^j e_i^+(e,\delta)$ and $\partial_{\mathrm{C}}^j e_i^-(e,\delta)$ exist for every $i,j=1,\ldots,n$. Let $\mathbb{S}$ denote the unit sphere of $\mathbb{R}^n$. Introduce the two functions

$$m_{ij}^-(\delta) = \min \bigcup_{\nu\in\mathbb{S}} \min\left\{\partial_{\mathrm{C}}^j e_i^+(\nu,\delta), \partial_{\mathrm{C}}^j e_i^-(\nu,\delta)\right\},$$

$$m_{ij}^+(\delta) = \max \bigcup_{\nu\in\mathbb{S}} \max\left\{\partial_{\mathrm{C}}^j e_i^+(\nu,\delta), \partial_{\mathrm{C}}^j e_i^-(\nu,\delta)\right\}.$$

Let $\mathscr{M}$ be the set of 2^{n^2} matrices M given by

$$M_{ij}(\delta) \in \left\{m_{ij}^-(\delta), m_{ij}^+(\delta)\right\}. \tag{25}$$

To complete the proof of Theorem III.1 we have to show the following two properties:

1) the conditions (14) are satisfied for all M in $\mathscr{M}$;
2) there exists $\delta^* > 0$ such that for all $0 \leqslant \delta \leqslant \delta^*$, the inclusion (12) is satisfied.

Due to space limitation this proof has been removed and may be found in [8].

IV. Illustrative examples

We consider the following simple model of a pendulum $\dot{x}_1 = x_2, \quad \dot{x}_2 = \sin x_1$, where x_1, x_2 denote the angle between the pendulum and the vertical axis and the pendulum speed, respectively, $y_k = x_1(t_k)$ and $t_k = t_{k-1} + \delta$. The associate estimation error equation is given

[2] Following F. Clarke [6], we define the generalized gradient of a scalar function f at x_0 as the convex envelop of all possible limits of derivatives of f at points $x_n \in \mathbb{R}^n$, $x_n \to x_0$. Note that, in general, $\partial_{\mathrm{C}} f(x_0)$ is a set.

[3] To see this, note that for (e_a,e_b) in $\mathbb{R}^n$, we have

$$\begin{aligned}
\left|e^+(e_a,\delta) - e^+(e_b,\delta)\right| &\leqslant \int_0^\delta \left|F^+(e^+(e_a,s)) - F^+(e^+(e_b,s))\right| ds \\
&\leqslant \int_0^\delta c_{\max} \left|e^+(e_a,s) - e^+(e_b,s)\right| ds \\
&\leqslant \exp(c_{\max}\,\delta)\left|e_a - e_b\right|.
\end{aligned}$$

by $\dot{e}_1 = e_2, \quad \dot{e}_2 = ve_1, \quad v \in [-1,1]$. Following the proposed approach, we get that the approximation by the rectangle denoted by $T(e_0,\delta)$ is possible for $\delta^* = \sqrt{3}-1$. By integrating solutions of systems (16) and (17), we obtain a set $\mathscr{M}$ of 64 matrix functions M satisfying the assumptions of Theorem III.1, where

$$\begin{aligned} &M_{11} \in \{\cos\delta, \cosh\delta\}, \quad M_{21} \in \{-\sin\delta, \sinh\delta\}, \\ &M_{12} \in \{\sin\delta, \sinh\delta, \cosh\delta\tan\delta, \tanh\delta\cos\delta\}, \\ &M_{22} \in \left\{\cos\delta, \cosh\delta, \tfrac{1+\sinh\delta\sin\delta}{\cos\delta}, \tfrac{1+\sin\delta\sinh\delta}{\cosh\delta}\right\}. \end{aligned}$$

Note that by considering the minimum and maximum of each element, we can reduce the number of matrices to 16.

We also note that the obtained set of matrices satisfies the local properties (14) of Proposition II.1. The system is in uniformly observable form, so we know that for small values of δ, the assumptions of Theorem II.2 are satisfied.

Employing the *Yalmip* package ([13]) in *Matlab* in combination with the solver *Sedumi* ([17]), it can be checked that the LMI (13) is satisfied [4] for $\delta \leqslant 0.668$. The observer gain for $\delta = 0.668$ is $K = [-1, \ -1.8361]'$. It is interesting to notice that the bound obtained is much larger then the one obtained employing the usual high-gain approach as exposed in [2].

V. Conclusion

In this paper, the problem of designing an observer for nonlinear systems with discrete time measurements and globally Lipschitz nonlinearities is addressed. A solution based on the synthesis of an upper approximation of a reachable set have been presented. When this approximation is given in terms of a convex combination of linear mappings, a sufficient condition of the global convergence of the proposed observer is obtained in terms of a linear matrix inequality. The good performances obtained on an illustrative example demonstrate that the proposed approach is an efficient tool.

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