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Borel chromatic number of closed graphs

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Abstract. We construct, for each countable ordinal ξ, a closed graph with Borel chromatic number two and Baire class ξ chromatic number ℵ₀.

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1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in \(\{1, 2, 3, \aleph_0\}\). They also provide a minimum graph \(G_0\) of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the \(G_0\)-dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of \(G_0\) to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the \(\Delta^0_\xi\) chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1] and was motivated by the \(G_0\)-dichotomy. More precisely, let \(B\) be a Borel binary relation, on a Polish space \(X\), having a Borel countable coloring (i.e., a Borel map \(c : X \to \omega\) such that \(c(x) \neq c(y)\) if \((x, y) \in B\)). Is there a relation between the Borel class of \(B\) and that of the coloring? In other words, is there a map \(k : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\}\) such that any \(\Pi^0_\xi\) binary relation having a Borel countable coloring has in fact a \(\Delta^0_\xi\) measurable countable coloring, for each \(\xi \in \omega_1 \setminus \{0\}\)?

In [L-Z2], the authors give a negative answer: for each countable ordinal \(\xi \geq 1\), there is a partial injection with disjoint domain and range \(i : \omega^\omega \to \omega^\omega\), whose graph
- is \(D_2(\Pi^0_1)\) (i.e., the difference of two closed sets),
- has Borel chromatic number two,
- has no \(\Delta^0_\xi\)-measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring \(c\) has also a \(\Delta^0_\xi\)-measurable finite coloring (consider the differences of the \(c^{-1}\{\{n\}\}\)'s, for \(n\) in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is \(\Delta^0_2\)-measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal \(\xi \geq 1\), a closed binary relation with a Borel finite coloring but no \(\Delta^0_\xi\)-measurable finite coloring. This is indeed the case:

**Theorem** Let \(\xi \geq 1\) be a countable ordinal. Then there exists a partial injection with disjoint domain and range \(f : \omega^\omega \to \omega^\omega\) whose graph is closed (and thus has Borel chromatic number two), and has no \(\Delta^0_\xi\)-measurable finite coloring (and thus has \(\Delta^0_\xi\) chromatic number \(\aleph_0\)).

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

**Notation.** The symbol \(\tau\) denotes the usual product topology on the Baire space \(\omega^\omega\).
Definition 2.1 We say that a partial map \( f : \omega^\omega \to \omega^\omega \) is nice if its graph \( \text{Gr}(f) \) is a \((\tau \times \tau)\)-closed subset of \( \omega^\omega \times \omega^\omega \).

The construction of \( P_\xi \) and \( \tau_\xi \), and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

Lemma 2.2 Let \( 1 \leq \xi < \omega_1 \). Then there are \( P_\xi \subseteq \omega^\omega \), and a topology \( \tau_\xi \) on \( \omega^\omega \) such that

1. \( \tau_\xi \) is zero-dimensional perfect Polish and \( \tau \subseteq \tau_\xi \subseteq \Sigma^0_1(\tau) \).
2. \( P_\xi \) is a nonempty \( \tau_\xi \)-closed nowhere dense set,
3. if \( S \in \Sigma^0_1(\omega^\omega, \tau) \) is \( \tau_\xi \)-nonmeager in \( P_\xi \), then \( S \) is \( \tau_\xi \)-nonmeager in \( \omega^\omega \).
4. if \( V, W \) are nonempty \( \tau_\xi \)-open subsets of \( \omega^\omega \), then we can find a \( \tau_\xi \)-dense \( G_\delta \) subset \( H \) of \( \omega \setminus P_\xi \), a \( \tau_\xi \)-dense \( G_\delta \) subset \( L \) of \( \omega \setminus P_\xi \), and a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism from \( H \) onto \( L \).

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology \( T_\xi \) that we will use in the sequel.

Lemma 2.3 Let \( 1 \leq \xi < \omega_1 \). Then there is a disjoint countable family \( G_\xi \) of subsets of \( \omega^\omega \) and a topology \( T_\xi \) on \( \omega^\omega \) such that

1. \( T_\xi \) is zero-dimensional perfect Polish and \( \tau \subseteq T_\xi \subseteq \Sigma^0_1(\tau) \).
2. for any nonempty \( T_\xi \)-open sets \( V, V' \), there are disjoint \( G, G' \in G_\xi \) with \( G \subseteq V \), \( G' \subseteq V' \), and there is a nice \((T_\xi, T_\xi)\)-homeomorphism from \( G \) onto \( G' \), and, for every \( G \in G_\xi \),
   3. \( G \) is nonempty, \( T_\xi \)-nowhere dense, and in \( \Pi^0_3(T_\xi) \).
4. if \( S \in \Sigma^0_1(\omega^\omega, \tau) \) is \( T_\xi \)-nonmeager in \( G \), then \( S \) is \( T_\xi \)-nonmeager in \( \omega^\omega \).

The construction of \( G_\xi \) and \( T_\xi \) ensures that \( T_\xi \) is \((\omega^\omega, \tau_\xi)\), where \( \tau_\xi \) is as in Lemma 2.2. This topology is on \((\omega^\omega, \tau)\), identified with \( \omega^\omega \). We will need the following consequence of the construction of \( G_\xi \) and \( T_\xi \).

Lemma 2.4 Let \( 1 \leq \xi < \omega_1 \), and \( V \) be a nonempty \( T_\xi \)-open set. Then \( \overline{V} \) is not \( \tau \)-compact.

Proof. The fact that \( T_\xi \) is \((\omega^\omega, \tau_\xi)\) gives a finite sequence \( U_0, \ldots, U_n \) of nonempty \( \tau_\xi \)-open subsets of \((\omega^\omega, \tau_\xi)\) with \( U_0 \times \cdots \times U_n \subseteq V \). Thus \( \overline{V} \) contains the \( \tau \)-closed set \( \overline{U_0 \times \cdots \times U_n} \subseteq (\omega^\omega, \tau) \), and it is enough to see that this last set is not \( \tau \)-compact. This comes from the fact that the Baire space \((\omega^\omega, \tau)\) is not compact.

3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets \( A, B \) of a product \( Y \times Z \) of Polish spaces can be separated by an open rectangle. We set \( A_0 := \{(\omega^\omega, n) \mid n \in \omega \} \),

\[
B_0 := \{(0^{m+1}(n+1)\omega, (m+1)^{n+1}0^\omega) \mid m, n \in \omega \}
\]

and \( B_1 := \{(m+1)^{n+1}0^\omega, 0^{m+1}(n+1)^\omega) \mid m, n \in \omega \} \). Then \( A \) is not separable from \( B \) by an open rectangle exactly when there are \( \varepsilon \in 2 \) and continuous maps \( g : \omega^\omega \to Y \), \( h : \omega^\omega \to Z \) such that \( A \subseteq (g \times h)^{-1}(A) \) and \( B \subseteq (g \times h)^{-1}(B) \).
Example. Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite $\Delta_0^\xi$ chromatic number $n$. There is an example with $\xi=1$ and $n=3$ where $\mathbb{B}_0$ is involved. We set $C:=\{(2m)^\infty, (2m+1)^\infty) \mid m \in \omega\} \cup \mathbb{B}_0$, 
$$D:=\{(2m)^\infty \mid m \in \omega\} \cup \{0^{m+1}(n+1)^\infty \mid m, n \in \omega\},$$
$$R:=\{(2m+1)^\infty \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\infty \mid m, n \in \omega\},$$
$$f((2m)^\infty):=(2m+1)^\infty \text{ and } f(0^{m+1}(n+1)^\infty):=(m+1)^{n+1}0^\infty.$$  
This defines $f:D \to R$ whose graph is $C$. The first part of $C$ is discrete, and thus closed. Assume that $(\alpha_k, \beta_k):=(0^{m_k+1}(n_k+1)^\infty, (m_k+1)^{n_k+1+10^\infty}) \in \mathbb{B}_0$ and converges to $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ as $k$ goes to infinity. We may assume that $(m_k)$ is constant, and $(n_k)$ too, so that $(\alpha, \beta) \in \mathbb{B}_0$, which is therefore closed. This shows that $C$ is closed. Note that $D, R$ are disjoint and Borel, so that $C$ has Borel chromatic number two. Let $\Delta$ be a clopen subset of $\omega^\omega$. Let us prove that $C \cap \Delta^2$ or $C \cap (\Delta^2)$ is not empty. We argue by contradiction. Then $\Delta$ or $\neg \Delta$ has to contain $0^\infty$. Assume that it is $\Delta$, the other case being similar. Then $0^{m+1}(n+1)^\infty \in \Delta$ if $m$ is big enough. Thus $(m+1)^{n+1}0^\infty \in \Delta$ if $m$ is big enough. Therefore $(m+1)^\infty \notin \Delta$ if $m$ is big enough. Thus $((2m)^\infty, (2m+1)^\infty) \in C \cap (\Delta^2)$ if $m$ is big enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip $\omega^m$ with the discrete topology $\tau_d$, for each $m \geq 0$.

**Lemma** Let $\xi \geq 1$ be a countable ordinal, $n \geq 1$ be a natural number, and $X:=\omega \times \omega^\omega$. Then we can find a partial injection $f:X \to X$ and a disjoint countable family $\mathcal{F}$ of subsets of $X$ such that

(a) $f$ has disjoint domain and range,
(b) $\text{Gr}(f)$ is $(\tau_d \times \tau) \cap (\tau_d \times \tau)$-closed,
(c) there is no sequence $(\Delta_i)_{i<n}$ of $\Delta_0^\xi$ subsets of $(X, \tau_d \times \tau)$ such that 
(i) $\forall i < n \quad \text{Gr}(f) \cap \Delta_i^2 = \emptyset$,
(ii) $\bigcup_{i<n} \Delta_i$ is $(\tau_d \times T_\xi)$-comeager in $X$,
(d) $\mathcal{F}$ has the properties (b)-(d) in Lemma 2.3, where $\mathcal{G}_c$, $\omega^\omega$, $T_\xi$ and $\tau$ are respectively replaced with $\mathcal{F}$, $X$, $\tau_d \times T_\xi$ and $\tau_d \times \tau$.
(e) $(\bigcup \mathcal{F}) \cap (\text{Domain}(f) \cup \text{Range}(f)) = \emptyset$.

**Proof.** We argue by induction on $n$.

**The basic case $n=1$**

Let $\mathcal{G}_c$ be the family given by Lemma 2.3. We split $\mathcal{G}_c$ into two disjoint subfamilies $\mathcal{G}_0^\xi$ and $\mathcal{G}_1^\xi$ having the property (b) in Lemma 2.3. This is possible since the elements of $\mathcal{G}_c$ are $T_\xi$-nowhere dense. Let $G_0, G_1 \in \mathcal{G}_0^\xi$ be disjoint, and $\varphi$ be a nice $(T_\xi, T_\xi)$-homeomorphism from $G_0$ onto $G_1$. We then set $f(0, \alpha):=(0, \varphi(\alpha))$ if $\alpha \in G_0$, and $\mathcal{F}:=\{\{n\} \times G \mid n \in \omega \wedge G \in \mathcal{G}_1^\xi\}$. It remains to check that the property (c) is satisfied. We argue by contradiction, which gives $\Delta_0 \in \Delta_0^\xi$. By property (d) in Lemma 2.3, $\Delta_0 \cap \{\{0\} \times G\}$ is $(\tau_d \times T_\xi)$-comeager in $\{\{0\} \times G\}$ for each $\varepsilon \in 2$. As $f$ is a $(\tau_d \times T_\xi, \tau_d \times T_\xi)$-homeomorphism, $\Delta_0 \cap \{\{0\} \times G\} \cap f^{-1}(\Delta_0 \cap \{\{0\} \times G_1\})$ is $(\tau_d \times T_\xi)$-comeager in $\{\{0\} \times G_0\}$, which contradicts the fact that $\text{Gr}(f) \cap \Delta_0^\xi = \emptyset$. 

4
The induction step from $n$ to $n+1$

The induction assumption gives $f$ and $F$. Here again, we split $F$ into two disjoint subfamilies $F^0$ and $F^1$ having the property (b) in Lemma 2.3, where $G_{\xi}$, $\omega^\omega$, $T_{\xi}$ and $\tau$ are respectively replaced with $F$, $X$, $\tau_d \times T_{\xi}$ and $\tau_d \times \tau$. Let $(V_p)$ be a basis for the topology $\tau_d \times T_{\xi}$ made of nonempty sets. Fix $p \in \omega$. By Lemma 2.4, there is a countable family $(W^p_{q})_{q \in \omega}$, with $(\tau_d \times \tau)$-closed union, and made of pairwise disjoint $(\tau_d \times \tau)$-clopen subsets of $X$ intersecting $V_p$.

- Let $b : \omega \to \omega^2$ be a bijection. We construct, for $\tilde{\nu} = (p, q) \in \omega^2$ and $\varepsilon \in 2$, and by induction on $b^{-1}(\tilde{\nu})$,

  - $G^\varepsilon_{\tilde{\nu}} \in F^0$,
  - a nice $(\tau_d \times T_{\xi}, \tau_d \times T_{\xi})$-homeomorphism $\varphi^\varepsilon : G^\varepsilon_{0} \to G^\varepsilon_{1}$.

We want these objects to satisfy the following:

- $G^\varepsilon_{0} \subseteq (V_p \cap W^p_{q}) \setminus \left( \bigcup_{m < b^{-1}(\tilde{\nu})} G^b(m) \cup G^b(1) \right)$,
- $G^\varepsilon_{1} \subseteq V_{q} \setminus \left( G^\varepsilon_{0} \cup \bigcup_{m < b^{-1}(\tilde{\nu})} G^b(m) \right)$.

- We now define the desired partial map $\tilde{f} : \omega \times \omega \times \omega^\omega \to \omega \times \omega \times \omega^\omega$, as well as $\tilde{F} \subseteq 2^{\omega \times \omega \times \omega^\omega}$, as follows:

  $\tilde{f}(l, x) := \begin{cases} (p+1, \varphi^{p,q}(x)) & \text{if } l = 0 \land x \in G^p_{0,q}, \\ (l, f(x)) & \text{if } l > 0 \land x \in \text{Domain}(f). \end{cases}$

and $\tilde{F} := \{ \{ l \} \times G \mid l \in \omega \land G \in F^1 \}$. Note that $\tilde{f}$ is well-defined and injective, by disjointness of the $(G^p_{0} \cup G^p_{1})$'s. Identifying $X$ with $\omega \times \omega \times \omega^\omega$, we can consider $\tilde{f}$ as a partial map from $X$ into itself and $\tilde{F}$ as a family of subsets of $X$ (this identification is based on the identification of $\omega$ with $\omega \times \omega$).

(a), (d) and (e) are clearly satisfied.

(b) Assume that $((l_k, x_k), (m_k, y_k)) \in \text{Gr}(\tilde{f})$ tends to $(l, x), (m, y) \in (\omega \times X)^2$ as $k$ goes to infinity. We may assume that $(l_k)$ and $(m_k)$ are constant.

If $l = 0$, then there is $p$ such that $p+1 = m$ and $(x_k, y_k) \in G^p_{0,q} \times G^p_{1,q}$. As $G^p_{0,q} \subseteq W^p_{q}$, we may also assume that $(q_k)$ is also constant and equals $q$. As $\varphi^{p,q}$ is nice, $((l, x), (m, y)) \in \text{Gr}(\tilde{f})$.

If $l > 0$, then $(x_k, y_k) \in \text{Gr}(f)$. As $\text{Gr}(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$-closed, $(l, x), (m, y) \in \text{Gr}(\tilde{f})$.

(c) We argue by contradiction, which gives $(\Delta_n)_{n \leq n}$. We may assume, without loss of generality, that $(\{ 0 \} \times \omega \times \omega^\omega) \cap \Delta_n$ is not meager in $(\{ 0 \} \times \omega \times \omega^\omega, \tau_d \times T_{\xi})$. This gives $p \in \omega$ such that $(\{ 0 \} \times V_p) \cap \Delta_n$ is $(\tau_d \times T_{\xi})$-comeager in $V_p := \{ 0 \} \times V_p$. As $V_p \setminus \Delta_n \in \Sigma^0_\xi(\tau_d \times \tau)$, $(\{ 0 \} \times G^p_{0,q}) \cap \Delta_n$ is $(\tau_d \times T_{\xi})$-comeager in $\{ 0 \} \times G^p_{0,q}$ for each $q \in \omega$. 

5
As \( \text{Gr}(\tilde{f}) \cap \Delta_n^2 = \emptyset \) and the \( \varphi_i \)'s are \( (\tau_d \times T_\xi, \tau_d \times T_\xi) \)-homeomorphisms, \( \{p+1\} \times G_1^{p,q} \cap \Delta_n \)
is \( (\tau_d \times T_\xi) \)-meager in \( \{p+1\} \times G_1^{p,q} \), for each \( q \).

As \( (\omega \times \omega \times \omega^\omega) \setminus (\bigcup_{i \leq n} \Delta_i) \) is \( (\tau_d \times T_\xi) \)-meager in \( \omega \times \omega \times \omega^\omega \) and \( \Delta_0(\tau_d \times \tau) \),
\[
\{p+1\} \times G_1^{p,q} \setminus \left( \bigcup_{i \leq n} \Delta_i \right)
\]
is \( (\tau_d \times T_\xi) \)-meager in \( \{p+1\} \times G_1^{p,q} \), for each \( q \). Thus \( \{p+1\} \times G_1^{p,q} \cap (\bigcup_{i \leq n} \Delta_i) \) is \( (\tau_d \times T_\xi) \)-comeager in \( \{p+1\} \times G_1^{p,q} \), for each \( q \).

**Claim** The set \( \{p+1\} \times \omega \times \omega^\omega \) \( \cap (\bigcup_{i \leq n} \Delta_i) \) is \( (\tau_d \times T_\xi) \)-comeager in \( \{p+1\} \times \omega \times \omega^\omega \).

Indeed, we argue by contradiction. This gives \( W \subset (\tau_d \times T_\xi) \setminus \{0\} \) such that
\[
\{p+1\} \times W \cap \left( \bigcup_{i \leq n} \Delta_i \right)
\]
is \( (\tau_d \times T_\xi) \)-meager in \( W' := \{p+1\} \times W \). Let \( q \in \omega \) be such that \( V_q \subset W \). Then \( G_1^{p,q} \subset W \) and \( \{p+1\} \times G_1^{p,q} \subset W' \). As \( W' \cap \left( \bigcup_{i \leq n} \Delta_i \right) \in \Sigma_1^\omega(\tau_d \times \tau) \) and \( \{p+1\} \times G_1^{p,q} \cap W' \cap \left( \bigcup_{i \leq n} \Delta_i \right) \) is \( (\tau_d \times T_\xi) \)-comeager in \( \{p+1\} \times G_1^{p,q} \), \( W' \cap \left( \bigcup_{i \leq n} \Delta_i \right) \) is not \( (\tau_d \times T_\xi) \)-meager in \( W' \), which is absurd.

Now we set \( \Delta'_i := \{p+1\} \times \omega \times \omega^\omega \cap \Delta_i \), if \( i \leq n \). Note that \( \Delta'_i \in \Delta_0(\{p+1\} \times \omega \times \omega^\omega, \tau_d \times \tau) \), \( \text{Gr}(\tilde{f}) \cap (\Delta'_i)^2 = \emptyset \), and \( \bigcup_{i \leq n} \Delta'_i \) is \( (\tau_d \times T_\xi) \)-comeager in \( \{p+1\} \times \omega \times \omega^\omega \), which contradicts the induction assumption. \( \square \)

In order to get our main result, it is enough to apply the main lemma to each \( n \geq 1 \). This gives
\[ f_n : \omega \times \omega^\omega \to \omega \times \omega^\omega. \]
It remains to define \( f : \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega) \to \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega) \) by \( f(n, x) := f_n(x) \) (we identify \( (\omega \setminus \{0\}) \times \omega \times \omega^\omega \) with \( \omega^\omega \)).

4 References


[Za] R. Zamora, Separation of analytic sets by rectangles of low complexity, *manuscript (see arXiv)*