UNIQUENESS AND LONG TIME ASYMPTOTICS FOR THE PARABOLIC-PARABOLIC KELLER-SEGEL EQUATION

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Abstract. The present paper deals with the parabolic-parabolic Keller-Segel equation in the plane in the general framework of weak (or "free energy") solutions associated to initial data with finite mass $M < 8\pi$, finite second log-moment and finite entropy. The aim of the paper is twofold:

(1) We prove the uniqueness of the "free energy" solution. The proof uses a DiPerna-Lions renormalizing argument which makes possible to get the "optimal regularity" as well as an estimate of the difference of two possible solutions in the critical $L^{4/3}$ Lebesgue norm similarly as for the $2d$ vorticity Navier-Stokes equation.

(2) We prove a radially symmetric and polynomial weighted $H^1 \times H^2$ exponential stability of the self-similar profile in the quasi parabolic-elliptic regime. The proof is based on a perturbation argument which takes advantage of the exponential stability of the self-similar profile for the parabolic-elliptic Keller-Segel equation established by Campos-Dolbeault and Egana-Mischler.

December 21, 2016

Keywords: Keller-Segel system; uniqueness; regularisation; self-similar variables; long-time behaviour; stability.

AMS Subject Classification (2000): 35B45, 35B60, 35B65, 35K15, 35Q92, 92C17, 92B05

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1. Introduction

1.1. **The KS equation, motivation and main biological result.** The Keller-Segel (KS) system (or Patlak-Keller-Segel system) for chemotaxis describes the collective motion of cells that

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are attracted by a chemical substance that they are able to emit ([47, 35]). In this paper we are concerned with the parabolic-parabolic KS model in the plane which takes the form

$$\begin{aligned} \partial_t f &= \Delta f - \nabla(f\,\nabla u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \\ \varepsilon\partial_t u &= \Delta u + f - \alpha\, u \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \end{aligned} \tag{1.1}$$

and which is complemented with an initial condition

$$f(0,\cdot) = f_0 \geq 0 \quad \text{and} \quad u(0,\cdot) = u_0 \geq 0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.2}$$

Here $t \geq 0$ is the time variable, $x \in \mathbb{R}^2$ is the space variable, $f = f(t,x) \geq 0$ stands for the *mass density of cells* while $u = u(t,x) \geq 0$ is the *chemo-attractant concentration* and $\varepsilon > 0$, $\alpha \geq 0$ are constants. We refer to the work [10] as well as to the reviews [32, 53] and the references quoted therein for biological motivation and mathematical introduction.

In short, the KS equation models a cells population which is subject to two inverse mechanisms:

- a brownian motion (responsable to the diffusion term Δf in the first equation of (1.1)) modeling the fact that any cell change of direction and move in a completely erratic way and which global effect is to spread out the population all over the plane $\mathbb{R}^2$;

- an aggregation mechanism (responsable to the drift term $\nabla(-\nabla u\, f)$ in the first equation of (1.1)) modeling the fact that cells have a tendency to follow the gradient lines of the chemo-attractant, which is itself produced and diffused according to the second equation in (1.1). That mechanism has a concentration effect, which is quite strong due to the fact that the associated *interaction kernel* is singular.

From a mathematical point of view, both mechanisms are almost at the same order, and that makes the rigorous analysis of the model particularly difficult and interesting.

Let us first discuss the case when we make the strong modeling and mathematical simplification $\varepsilon = 0$, that we refer to as the *parabolic-elliptic KS model*, which corresponds to the situation when the diffusion of chemo-attractant occurs with infinite speed (quasistatic approximation).

The parabolic-elliptic KS system has been introduced by Patlak [47] in the 1950's and by Keller, Segel [35]. It is has latter been rigorously justified from a more microscopical level of description. In particular, the parabolic-elliptic KS model has been obtained as a mean-field limit of a system of a finite number of cells in interaction for regularized interaction kernel (at least at the level of the microscopic description) by Stevens [52], Haškovec, Schmeiser [28, 29] and Godinho, Quiñinao [26]. The true interaction kernel at the level of the microscopic description has been considered by Fournier, Jourdain [24], but they were only able to get a consistency result in the mean-field limit. The parabolic-elliptic KS has also been obtained as a diffusion limit of a run-and-tumble kinetic equation by Chalub et al. [15], that last equation describing the cells population at a mesoscopic (or statistical) level.

During the last decades, a huge literature concerning the mathematical analysis of the parabolic-elliptic KS has grown up. We only give below some part of the main aspects. Probably the most important feature of the parabolic-elliptic KS equation is the existence of a mass threshold, the mass M (or total number) of cells being conserved along time.

For a supercritical mass $M > 8\pi$, there does not exist global in time nonnegative function solution: *chemotactic collapse* occurs in finite time, or mathematically speaking, any solution blows up in finite time. Particular blowing up solutions have been exhibited by Herrero, Velázquez [30] and the universality of that phenomenon (the stability under small perturbation of these blowing up solutions) has been considered by Raphaël, Schweyer [49]. In other words, for large initial mass, the aggregation mechanism prevails on the spreading mechanism: the diffusion is not strong enough to prevent mass concentration.

On the other hand, for a subcritical mass $M < 8\pi$, function solutions are well-defined globally in time and behave more similarly as for a pure diffusion. More precisely, after self-similar change of variables, any solution converges to the unique self-similar profile with same mass. In other words, for small initial mass, the spreading mechanism prevails on the aggregation mechanism,

and no concentration occurs in finite nor infinite time. We refer to Blanchet et al. [7], Campos, Dolbeault [11] and Egaña, Mischler [18] for a detailed description of these results. For the critical mass $M = 8\pi$, Blanchet et al. [6] established that solutions are global in time and a complete concentration occurs in infinite time (any solution convergences to a Dirac distribution with mass 8π). That qualitative analysis is mainly based on the existence of a free-energy functional which behaves nicely along the flow of the parabolic-elliptic KS equation.

When the ratio of the speed of the cells over the speed of the chemo-attractant is very small, the parabolic-elliptic KS equation can be considered as a good approximation of (1.1). However, from a biological modeling point of view the simplification $\varepsilon = 0$ is not completely satisfying. It is clearly irrelevant in the case when the speed of cells and chemo-attractant are of comparable order, as it can be the case for a *Escherichia coli* population, see Saragosti et al. [50].

There are only very few works on the parabolic-parabolic KS system compared to the parabolic-elliptic KS system, and the parabolic-parabolic KS system is far from being well-understood. Let us however present some of available results.

The parabolic-parabolic KS system has been obtained as a diffusion limit from a kinetic equation by Erban, Hillen and Othmer [31, 46, 19, 20, 21]. But to our knowledge, no derivation as a mean-field limit of a microscopic cell-system has been performed. Concerning the qualitative behaviour of solutions, the threshold between blowing up solutions and solutions that spread out is not clear. It is known though that solutions are global in time for sub-critical masses $M \leq 8\pi$, see Calvez, Corrias [10]. It is also known that solutions are global for any mass when $\varepsilon > 0$ is large enough, see Biler et al. [5] and Corrias et al. [16], that regime corresponds to a small chemo-attractant production by cells, and thus to a small nonlinearity. Chemotactic collapsing solutions have been exhibited by Herrero, Velázquez [30] for supercritical masses. On the other hand, for any sub-critical mass (and for any mass if $\varepsilon > 0$ is large enough) unique self-similar solutions have been constructed by Biler et al. [4], see also Corrias et al. [16].

One of the major difficulty for performing a qualitative analysis of generic solutions to the parabolic-parabolic KS system is that the free-energy functional does not behave as nicely as for the parabolic-elliptic KS system, in particular it does not provide any Lyapunov functional after self-similar rescaling of variable.

Our work concerns the general parabolic-parabolic KS equation $\varepsilon > 0$ and the quasi-parabolic-elliptic regime, corresponding to the case when $\varepsilon > 0$ is small, for which a deeper mathematical analysis of the qualitative behavior of solutions can be performed. Our main result shows that for some class of initial data with sub-critical mass and in a quasi-parabolic-elliptic regime, the associated solution (f, u) to the parabolic-parabolic KS system (1.1) satisfies

$$f(t,x) \sim \frac{1}{t}\, G_\varepsilon\left(\frac{x}{t^{1/2}}\right), \quad u(t,x) \sim V_\varepsilon\left(\frac{x}{t^{1/2}}\right), \quad \text{as} \quad t \to \infty, \tag{1.3}$$

for some self-similar profile $(G_\varepsilon, V_\varepsilon)$. In other words, we prove, for the very first time, that the parabolic-parabolic KS system (1.1) behaves in that regime similarly as a diffusion: no concentration occurs (even in large time) and the diffusion phenomenon is really the dominant phenomenon for any time.

1.2. **Mathematical analysis of the KS equation.** The two fundamental identities associated to the Keller-Segel equation (1.1) are that any solution satisfies, at least formally, the conservation of *"mass"*

$$M(t) := \langle f(t,.)\rangle = \langle f_0\rangle =: M, \quad \text{with} \quad \langle g\rangle := \int_{\mathbb{R}^2} g(x)\, dx, \tag{1.4}$$

and the *"free energy-dissipation of the free energy identity"*

$$\mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\, ds = \mathcal{F}_0, \tag{1.5}$$

where the free energy $\mathcal{F}(t) = \mathcal{F}(f(t), u(t))$, $\mathcal{F}_0 = \mathcal{F}(f_0, u_0)$ is defined by

$$\mathcal{F} = \mathcal{F}(f,u) := \int_{\mathbb{R}^2} f \log f \, dx - \int_{\mathbb{R}^2} fu \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{\alpha}{2} \int_{\mathbb{R}^2} u^2 dx, \tag{1.6}$$

and the dissipation of free energy $\mathcal{D}_{\mathcal{F}}(s) = \mathcal{D}_{\mathcal{F}}(f(s), u(s))$ by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}(f,u) := \int_{\mathbb{R}^2} f \, |\nabla(\log f) - \nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\Delta u + f - \alpha \, u|^2 \, dx. \tag{1.7}$$

Following [10], throughout this paper, we shall assume that the initial data (f_0, u_0) satisfy

$$\begin{cases} f_0 \, (1 + \log\langle x \rangle^2) \in L^1(\mathbb{R}^2) \quad \text{and} \quad f_0 \log f_0 \in L^1(\mathbb{R}^2); \\ u_0 \in H^1(\mathbb{R}^2) \text{ if } \alpha > 0 \quad \text{or} \quad u_0 \in L^1(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2) \text{ if } \alpha = 0; \\ f_0 \, u_0 \in L^1(\mathbb{R}^2), \end{cases} \tag{1.8}$$

where here and below we define the weight function $\langle x \rangle := (1 + |x|^2)^{1/2}$ and the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^2)$ is defined by $\dot{H}^1(\mathbb{R}^2) := \{u \in L^1_{loc}(\mathbb{R}^2);\, \nabla u \in L^2(\mathbb{R}^2)\}$. We also make the important restriction of subcritical mass

$$M := \langle f_0 \rangle \in (0, 8\pi),$$

as a suitable global existence theory is available in that case (see [10, 42]), whereas for $M > 8\pi$ there exist solutions which blow up in finite time (see [30, 44, 43] and the discussion in [42, 1. Introduction]). We also refer to [5, 16] where a global existence theory is developed in the possible supercritical case $M > 8\pi$ and the condition that ε is large enough (which corresponds to a case where the nonlinearity in (1.1) is small).

As in [10], we consider the following definition of weak solution.

Definition 1.1. *For any initial datum (f_0, u_0) satisfying (1.8) with $M < 8\pi$, we say that the couple (f, u) of nonnegative functions satisfying*

$$\begin{gathered} f \in L^\infty(0,T; L^1(\mathbb{R}^2)) \cap C([0,T); \mathcal{D}'(\mathbb{R}^2)), \quad \forall \, T \in (0, \infty), \\ \begin{cases} u \in L^\infty(0,T; H^1(\mathbb{R}^2)) & \text{if } \alpha > 0; \\ u \in L^\infty(0,T; L^1(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) & \text{if } \alpha = 0; \end{cases} \\ fu \in L^\infty(0,T; L^1(\mathbb{R}^2)) \end{gathered} \tag{1.9}$$

is a global in time weak solution to the Keller-Segel equation associated to the initial condition (f_0, u_0) whenever (f, u) satisfies the mass conservation (1.4), the bound

$$\sup_{[0,T]} \mathcal{F}(t) + \sup_{[0,T]} \int_{\mathbb{R}^2} f \, \log\langle x \rangle^2 \, dx + \int_0^T \mathcal{D}_{\mathcal{F}}(t) \, dt \le C_T, \tag{1.10}$$

as well as the Keller-Segel system of equations (1.1)-(1.2) in the distributional sense, namely

$$\int_{\mathbb{R}^2} f_0(x) \, \varphi(0,x) \, dx = \int_0^T \int_{\mathbb{R}^2} f \left\{ (\nabla_x (\log f) - \nabla_x u) \cdot \nabla_x \varphi - \partial_t \varphi \right\} dx \, dt \tag{1.11}$$

$$\varepsilon \int_{\mathbb{R}^2} u_0(x) \, \psi(0,x) \, dx = \int_0^T \int_{\mathbb{R}^2} \{ u \, (-\Delta \psi + \alpha \psi - \varepsilon \partial_t \psi) - f(t,x) \psi \} \, dx \, dt, \tag{1.12}$$

for any $T > 0$ and $\varphi, \psi \in C^2_c([0,T) \times \mathbb{R}^2)$.

It is worth emphasizing that it is not assumed that the free energy-dissipation of the free energy identity (1.5) holds, but with (1.10), only that the feee energy and the dissipation of the free energy are bounded. Thanks to the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}^2} f \, |\nabla_x (\log f) - \nabla_x u| \, dx \le M^{1/2} \, \mathcal{D}_{\mathcal{F}}^{1/2},$$

and the RHS of (1.11) is then well defined thanks to (1.4) and (1.10).

This framework is well adapted for a global existence theory in the subcritical mass case.

Theorem 1.2. *([10, Theorem 1]) For any initial datum (f_0, u_0) satisfying (1.8) and $M < 8\pi$, there exists at least one global in time weak solution in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2).*

Our first main result establishes that this framework is also well adapted for the well-posedness issue.

Theorem 1.3. *For any initial datum (f_0, u_0) satisfying (1.8) with $M < 8\pi$, there exists at most one weak solution in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2). This one is furthermore a classical solution in the sense that*

$$f, u \in C_b^2((0,\infty)\times\mathbb{R}^2) \tag{1.13}$$

and satisfies the accurate small time estimate

$$\forall\, q \in (1,\infty), \qquad t^{1-\frac{1}{q}}\, \|f(t)\|_{L^q} \to 0 \quad as \quad t \to 0. \tag{1.14}$$

Finally, the free energy-dissipation of the free energy identity (1.5) holds.

Theorem 1.3 improves the uniqueness result proved in [14] in the class of solutions $f \in C([0,T]; L^1_2(\mathbb{R}^2)) \cap L^\infty((0,T)\times\mathbb{R}^2)$, $\forall\, T > 0$, which can be built under the additional assumption $f_0 \in L^\infty(\mathbb{R}^2)$ as well as the uniqueness part of the well-posedness results [3, 22, 16, 5] for solutions satisfying (1.14) which are established in some particular regimes (smallness assumption on the initial datum or on some parameters). We also refere to [25] where a uniqueness result is established for a related model. Our proof follows a strategy introduced in [23] for the 2D viscous vortex model and generalizes a similar result obtained in [18] for the parabolic-elliptic model (which corresponds to the case $\varepsilon = 0$). It is based on a DiPerna-Lions renormalization process (see [17]) which makes possible to get the optimal regularity of solutions for small time (1.14) and then to follow the uniqueness argument introduced by Ben-Artzi for the 2D viscous vortex model (see [2, 9]) and also used in [3, 22, 16, 5] for the parabolic-parabolic Keller-Segel equation. Ben-Artzi's argument consists in writing the mild formulation of the difference of two solutions and to establish a contraction estimate for the norm introduced in (1.14) for $q = 4/3$. The choice of the exponent $q = 4/3$ is crucial and it is made in order to handle the singularity of the force field (thanks to sharp estimates of the smoothing effect of the heat semigroup). It is worth emphasizing that such an argument is related to famous Kato's work on the Navier-Stokes equation (see e.g. [34]).

The smoothing effect and the free energy-dissipation of the free energy identity established in Theorem 1.3 are natural and physically relevant but new, even for stronger (and possibly local in time) notions of solutions.

From now on in this introduction, we definitively restrict ourselves to the case $\alpha = 0$ and we focus on the long time asymptotic of the solutions. For that last purpose it is convenient to work with self-similar variables. We introduce the rescaled functions g and v defined by

$$f(t,x) := R(t)^{-2} g(\log R(t), R(t)^{-1}x), \quad u(t,x) := v(\log R(t), R(t)^{-1}x), \tag{1.15}$$

with $R(t) := (1+t)^{1/2}$. For these new unknowns, the rescaled parabolic-parabolic Keller-Segel system reads

$$\partial_t g = \Delta g + \nabla(\frac{1}{2}\, x\, g - g\, \nabla v) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \tag{1.16}$$

$$\varepsilon \partial_t v = \Delta v + g + \frac{\varepsilon}{2}\, x\cdot\nabla v \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2. \tag{1.17}$$

We are interested in self-similar solutions to the Keller-Segel parabolic-parabolic equation (1.1), that is solutions which write as

$$f(t,x) = \frac{1}{t} G_\varepsilon(\frac{x}{t^{1/2}}), \quad u(t,x) = V_\varepsilon(\frac{x}{t^{1/2}}),$$

with

$$\int_{\mathbb{R}^2} f(t,x)\, dx = \int_{\mathbb{R}^2} G_\varepsilon(y)\, dy = M \in (0, 8\pi). \tag{1.18}$$

Such a couple of functions (f,u) is a solution to (1.1) if, and only if, the associated "self-similar profile" $(G_\varepsilon, V_\varepsilon)$ satisfies the elliptic system

$$\begin{aligned} &\Delta G_\varepsilon - \nabla(G_\varepsilon\,\nabla V_\varepsilon - \frac{1}{2}\,x\,G_\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^2, \\ &\Delta V_\varepsilon + \frac{\varepsilon}{2}\,x\cdot\nabla V_\varepsilon + G_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^2, \end{aligned} \tag{1.19}$$

and thus corresponds to a stationary solution to the rescaled parabolic-parabolic Keller-Segel system (1.16). It is known that, for any $\varepsilon \in (0,1/2)$ and any $M \in (0,8\pi)$, there exists a unique solution $(G_\varepsilon, V_\varepsilon)$ to (1.19) such that the mass of G_ε equals M, and which is furthermore radially symmetric and smooth (say $C^2(\mathbb{R}^2)$), see [45, 4, 16].

Our second main result concerns the exponential nonlinear stability of the self-similar profile for any given mass $M \in (0,8\pi)$ under the strong restriction of radial symmetry and closeness to the parabolic-elliptic regime. We define the norm

$$|||(g,v)||| := \|g\|_{H^1_k} + \|v\|_{H^2}, \quad k > 7,$$

where the weighted Lebesgue space $L^p_k(\mathbb{R}^2)$, for $1 \le p \le \infty$, $k \ge 0$, is defined by

$$L^p_k(\mathbb{R}^2) := \{f \in L^1_{loc}(\mathbb{R}^2);\ \|f\|_{L^p_k} := \|f\,\langle x\rangle^k\|_{L^p} < \infty\},$$

and the norm of the higher-order Sobolev spaces $W^{\ell,p}_k(\mathbb{R}^2)$ is defined by

$$\|f\|^p_{W^{\ell,p}_k} := \sum_{|\alpha|\le\ell} \|\langle x\rangle^k\,\partial^\alpha f\|^p_{L^p}.$$

Theorem 1.4. *For any given mass $M \in (0,8\pi)$, there exist $\varepsilon^* > 0$ and $\delta^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$ and any radially symmetric initial datum (g_0,v_0) satisfying*

$$|||(g_0,v_0) - (G_\varepsilon,V_\varepsilon)||| \le \delta^*, \quad \int_{\mathbb{R}^2} g_0\,dx = \int_{\mathbb{R}^2} G_\varepsilon\,dx = M,$$

the associated solution (g,v) to (1.16)-(1.17) satisfies

$$|||(g(t),v(t)) - (G_\varepsilon,V_\varepsilon)||| \le C_a\,e^{at} \quad \forall\, a \in (-1/2,\infty),\ \forall\, t \ge 0,$$

for some constant $C_a = C_a(g_0,v_0) \ge 1$.

Coming back to the original unknowns (f,u), this theorem asserts (1.3) holds.

That result extends to the parabolic-parabolic Keller-Segel equation similar results known on the parabolic-elliptic Keller-Segel equation, see [18]. To our knowledge, Theorem 1.4 is the first exponential stability result for the system (1.1) even under the two strong restrictions of radial symmetry and quasi parabolic-elliptic regime (we mean $\varepsilon > 0$ small), and, moreover, the rate obtained here can be taken as close as we want to the optimal rate $-1/2$ of the parabolic-elliptic case (see Theorem 4.8). However, we refer again to the recent work [16, Section 4] where some results of convergence (without rate) of some solutions to the associated self-similar profile are established. We also refer to that work for further discussion and additional references. We finally refer to [55] where an exponential convergence to the equilibrium for a somehow similar chemotaxis model is established.

The proof of Theorem 1.4 is based on the spectral analysis of the associated linearized operator and on growth estimates on the related linear semigroup. We take advantage of the similar analysis made on the linearized parabolic-elliptic Keller-Segel equation performed in [18] and we use a (quite singular) perturbation argument. The restriction to radially symetric functions is made in order to get sharp and convenient estimates on solutions to auxilliary elliptic equations (see Lemma C.2) used in the accurate spectral analysis performed in Theorem 4.7. We think that this additional assumption is only technical and can be circumvented, but we were not able to fix it.

Let us end the introduction by describing the plan of the paper. In Section 2 we present some functional inequalities which will be useful in the sequel of the paper and we establish several a posteriori bounds satisfied by any weak solution. Section 3 is dedicated to the proof of the

uniqueness result stated in Theorem 1.3. In Section 4, we prove the exponential stability of the linearized problem associated to (1.16)-(1.17). Finally, in Section 5, we prove the long-time behaviour result as stated in Theorem 1.4.

Acknowledgments. We thank J.-Y. Chemin, J. Dolbeault, I. Gallagher, G. Jankowiak and O. Kavian for fruitful discussions and for having pointed out some interesting references related to our work. The research leading to this paper was (partially) funded by the French "ANR blanche" project Kibord: ANR-13-BS01-0004. K. C. is supported by the Fondation Mathématique Jacques Hadamard.

2. Local in time a priori and a posteriori estimates

2.1. **A priori estimates.** In this short paragraph, we follow [10] and we explain how to obtain the basic estimates which lead to the notion of weak solution as presented in Definition 1.1. We first observe that the following space logarithmic moment control holds true

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} f\,(-\log H)dx &= \int_{\mathbb{R}^2} f\,\nabla(\log f - u)\cdot\nabla(\log H)\,dx\\ &\le \frac{\delta}{2}\,\mathcal{D}_{\mathcal{F}}(f,u) + \frac{1}{2\delta}\int_{\mathbb{R}^2} f\,|\nabla\log H|^2\,dx,\end{aligned}$$

where

$$H(x) := \frac{1}{\pi}\,\frac{1}{\langle x\rangle^4} \quad \text{and then} \quad |\nabla\log H(x)| \le 4,$$

which together with (1.5) implies that the modified free energy functional

$$\mathcal{F}_H = \mathcal{F}(f,u) - \int_{\mathbb{R}^2} f\,\log H$$

satisfies

$$\frac{d}{dt}\mathcal{F}_H(t) + \frac{1}{2}\mathcal{D}_{\mathcal{F}}(t) \le M. \tag{2.1}$$

On the one hand, introducing the Laplace kernel $\kappa_0(z) := -\frac{1}{2\pi}\log|z|$ and the Bessel kernel $\kappa_\alpha(z) := \frac{1}{4\pi}\int_0^\infty t^{-1}\exp(-|z|^2/(4t)-\alpha t)\,dt$ for $\alpha>0$, so that $\bar u = \bar u_\alpha := \kappa_\alpha * f$ is a solution to the Laplace type equation

$$-\Delta\bar u = f - \alpha\bar u \quad \text{in} \quad \mathbb{R}^2,$$

and introducing as well the *chemical energy* and the modified entropy

$$F_\alpha(f,u) := \frac{1}{2}\int|\nabla u|^2 + \frac{\alpha}{2}\int u^2 - \int f\,u, \quad \mathcal{H}_H(f) := \int f\log(f/H),$$

one can easily show (see e.g. [10, Lemma 2.2]) that

$$\mathcal{F}_H(f,u) = \mathcal{H}_H(f) + F_\alpha(f,\bar u_\alpha) + \frac{1}{2}\int|\nabla(u-\bar u_\alpha)|^2 + \frac{\alpha}{2}\int(u-\bar u_\alpha)^2 \tag{2.2}$$

and

$$F_\alpha(f,\bar u_\alpha) = -\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\kappa_\alpha(x-y)\,dxdy. \tag{2.3}$$

On the other hand, we know from the classical logarithmic Hardy-Littlewood-Sobolev inequality (see e.g. [1, 13]) or its generalization for the Bessel kernel (see [10, Lemma 4.2]) that

$$\begin{aligned}\forall\, f\ge 0, \quad \int_{\mathbb{R}^2} f(x)\log f(x)\,dx &- \frac{4\pi}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\kappa_\alpha(x-y)\,dxdy\\ &- \int_{\mathbb{R}^2} f(x)\log H(x)\,dx \ge -C_1(M),\end{aligned} \tag{2.4}$$

where here and below $C_i(M)$ denotes a positive constant which only depends on the mass M.

Then from (2.2), (2.3) and (2.4) together with the very classical functional inequality (see e.g. [10, Lemma 2.4])

$$\mathcal{H}^+ := \mathcal{H}^+(f) = \int f(\log f)_+ \le \mathcal{H}_H(f) - \frac{1}{4}\int f \log\langle x\rangle^2 + C_2(M), \tag{2.5}$$

one immediately obtains, for $M < 8\pi$,

$$\begin{aligned}\mathcal{F}_H(f,u) &\ge (1-\frac{M}{8\pi})\,\mathcal{H}_H(f) + \frac{M}{8\pi}\Big(\mathcal{H}_H(f) - \frac{4\pi}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\kappa_\alpha(x-y)\,dxdy\Big)\\ &\ge C_3(M)\,\mathcal{H}_+(f) + C_4(M)\int f\log\langle x\rangle^2 - C_5(M).\end{aligned}$$

One concludes that under the assumption (1.8) on the initial datum, the identity (1.4) and the inequality (2.1) provide a convenient family of a priori estimates in order to define weak solutions, namely

$$\begin{aligned}&C_3(M)\,\mathcal{H}^+(f(t)) + C_4(M)\int f(t)\log\langle x\rangle^2 + \frac{1}{2}\int_0^t \mathcal{D}_{\mathcal{F}}(f(s),u(s))\,ds\\ &\quad\le \mathcal{F}_H(0) + C_5(M) + M\,t,\end{aligned} \tag{2.6}$$

and one remarks that the RHS term is finite under assumption (1.8) on (f_0,u_0), because

$$\begin{aligned}\mathcal{F}_H(0) &= \mathcal{F}(f_0,u_0) - \int f_0\log H\\ &= \mathcal{F}(f_0,u_0) + M\log\pi + 2\int f_0\log\langle x\rangle^2 < +\infty.\end{aligned}$$

It is worth emphasizing that in order to get the bounds announced in Definition 1.1 in the case $\alpha > 0$ one may use the inequality

$$\mathcal{F}_H \ge C_6(M)\int|\nabla u|^2 + C_7(M)\alpha\int u^2 + C_8(M)\int f\,u - C_9(M)(1+1/\alpha) \tag{2.7}$$

which is established in [10, (3.5)].

2.2. **Local in time a posteriori estimates.** We start by presenting some elementary functional inequalities which will be of main importance in the sequel. The two first estimates are picked up from [23, Lemma 3.2] but are probably classical and the third one is a variant of the Gagliardo-Niremberg-Sobolev inequality.

Lemma 2.1. *For any $0 \le f \in L^1(\mathbb{R}^2)$ with finite mass M and finite Fisher information*

$$I = I(f) := \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f},$$

there holds

$$\forall\, p\in[1,\infty),\quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p\,M^{1/p}\,I(f)^{1-1/p}, \tag{2.8}$$

$$\forall\, q\in[1,2),\quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \le C_q\,M^{1/q-1/2}\,I(f)^{3/2-1/q}. \tag{2.9}$$

For any $0 \le f \in L^1(\mathbb{R}^2)$ with finite mass M, there holds

$$\forall\, p\in[2,\infty)\quad \|f\|_{L^{p+1}(\mathbb{R}^2)} \le C_p\,M^{1/(p+1)}\,\|\nabla(f^{p/2})\|_{L^2}^{2/(p+1)}. \tag{2.10}$$

We refer to [23, Lemma 3.2] and [18, Lemma 2.1] for a proof.

The proof of (1.13) in Theorem 1.3 is split into several steps that we present as some intermediate autonomous a posteriori bounds.

Proposition 2.2. *For any weak solution (f,u), we have*

$$I(f(t)) \in L^1(0,T),\quad \forall\, T>0.$$

Proof of Proposition 2.2. We write

$$\mathcal{D}_{\mathcal{F}}(f) \geq I(f) + 2\int f\,\Delta u. \tag{2.11}$$

Next, by Young's inequality, we have

$$\begin{aligned}\int f\,\Delta u &= \int f\,(\varepsilon\partial_t u - f + \alpha u)\\ &\geq -(1+\alpha/2+\varepsilon/2)\int f^2 - \varepsilon/2\int(\partial_t u)^2 - \alpha/2\int u^2.\end{aligned}$$

The second and third terms belong to $L^1(0,T)$ from (1.10), so we only need to estimate the first term.

For any $A>1$, using the Cauchy-Schwarz inequality and the inequality (2.8) for $p=3$, we have

$$\begin{aligned}\int f^2\,\mathbf{1}_{f\geq A} &\leq \Big(\int f\,\mathbf{1}_{f\geq A}\Big)^{1/2}\Big(\int f^3\Big)^{1/2}\\ &\leq \Big(\int f\,\frac{(\log f)_+}{\log A}\Big)^{1/2}\Big(C_3^3\,M\,I(f)^2\Big)^{1/2},\end{aligned}$$

from what we deduce for $A = A(M,\mathcal{H}^+)$ large enough, and more precisely taking A such that $\log A = 16\,\mathcal{H}_+\,C_3^3\,M(1+\alpha/2+\varepsilon/2)^2$,

$$\int f^2\,\mathbf{1}_{f\geq A} \leq C_3^{3/2}\,M^{1/2}\,\frac{\mathcal{H}_+(f)^{1/2}}{(\log A)^{1/2}}\,I(f) \leq \frac{(1+\alpha/2+\varepsilon/2)^{-1}}{4}\,I(f). \tag{2.12}$$

Denoting $\Phi(u) = \varepsilon\int(\partial_t u)^2 + \alpha\int u^2 \in L^1(0,T)$ and putting together the last estimate with (2.11), it follows

$$\begin{aligned}\frac12 I(f) &\leq \mathcal{D}_{\mathcal{F}} + C\int f^2\,\mathbf{1}_{f\leq A} + \Phi(u)\\ &\leq \mathcal{D}_{\mathcal{F}} + 2\,M\,\exp(C\,\mathcal{H}_+\,M) + \Phi(u),\end{aligned}$$

and we conclude thanks to (1.4)–(2.6). □

Remark 2.3. The logarithmic Hardy-Littlewood Sobolev inequality (2.4) in the supercritical case $M \geq 8\pi$ does not lead to a global estimate as for the subcritical case $M \in (0,8\pi)$. However, introducing the function $\mathscr{M} := MH$ of mass M and the modified free energy

$$\tilde{\mathcal{F}}_M(f,u) := \int_{\mathbb{R}^2}(f\,\log(f/\mathscr{M}) - f + \mathscr{M})\,dx - \int_{\mathbb{R}^2} f u\,dx + \frac12\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \frac{\alpha}{2}\int_{\mathbb{R}^2} u^2\,dx,$$

one shows that any solution (f,u) to the Keller-Segel equation (1.1) formally satisfies

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{F}}_M(f,u) &\leq -\frac12\mathcal{D}_{\mathcal{F}}(f,u) + M\\ &\leq -\frac12 I(f) - \int f\Delta u - \frac{\varepsilon}{2}\int(\partial_t u)^2 + M\\ &\leq -\frac12 I(f) - \frac{\varepsilon}{4}\int(\partial_t u)^2 + (1+\varepsilon)\int f^2 + \frac{\alpha}{2}\int u^2 + M,\end{aligned}$$

where we have just used (2.1), the estimate (2.11) and the estimates at the beginning of the proof of Proposition 2.2. We also observe that from (2.5) and (2.7), we may deduce

$$\mathcal{H}_+(f) + \int|\nabla u|^2 + \alpha\int u^2 \leq K_1\tilde{\mathcal{F}}_M(f,u) + K_2,$$

where K_i, $i = 1, 2$, are constants which may depend on $M > 0$ and $\alpha \geq 0$. Arguing then as in the proof of Proposition 2.2, we easily get

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{F}}_M(f,u) &\leq -\frac{1}{2}I(f) - \frac{\varepsilon}{4}\int(\partial_t u)^2 + (1+\varepsilon)\Big\{MA + C_3^{3/2}\,M^{1/2}\,\frac{(K_1\tilde{\mathcal{F}}_M(f,u))^{1/2}}{(\log A)^{1/2}}\,I(f)\Big\}\\ &\quad + \frac{K_1}{2}\tilde{\mathcal{F}}_M(f,u) + \frac{K_2}{2} + M \qquad (\forall\, A > 0)\\ &\leq -\frac{1}{4}I(f) - \frac{\varepsilon}{4}\int(\partial_t u)^2 + K_3\,\exp(K_4\,\tilde{\mathcal{F}}_M(f,u)) + K_5,\end{aligned}$$

by making the appropriate choice $\log A = K'\,\tilde{\mathcal{F}}_M(f,u)$ for A. This differential inequality provides a local a priori estimate on the modified free energy which can be used in order to prove local existence result for supercritical mass. Because we will prove in Theorem 1.3 that the above resulting bound is suitable in order to get the uniqueness of the solution, we can classically obtain the existence and uniqueness of maximal solutions (in the weak sense of Definition 1.1) $(f,u) \in C([0,T^*);\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}'(\mathbb{R}^2))$ such that

$$\sup_{[0,T)}\tilde{\mathcal{F}}_M(f(t),u(t)) + \int_0^T\Big\{I(f(t)) + \mathcal{D}_{\mathcal{F}}(f(t),u(t))\Big\}\,dt < \infty \quad \forall\, T \in (0,T^*)$$

and the alternative

$$T_* = +\infty \quad \text{or} \quad (T_* < \infty,\ \tilde{\mathcal{F}}_M(f(t),u(t)) \to \infty \ \text{ as } \ t \to T^*).$$

As an immediate consequence of Lemma 2.1 and Proposition 2.2, we have

Lemma 2.4. *For any $T > 0$, any weak solution (f,u) satisfies*

$$f \in L^{p/(p-1)}(0,T;L^p(\mathbb{R}^2)), \quad \forall\, p \in (1,\infty), \tag{2.13}$$

$$\nabla f \in L^{2p/(3p-2)}(0,T;L^p(\mathbb{R}^2)), \quad \forall\, p \in [1,2), \tag{2.14}$$

$$\Delta u \in L^2(0,T;L^2(\mathbb{R}^2)). \tag{2.15}$$

Proof of Lemma 2.4. The bound (2.13) is a direct consequence of (2.8) and Proposition 2.2. The bound (2.14) is a consequence of (2.9) and Proposition 2.2. From (1.10) we have $\mathcal{D}_{\mathcal{F}} \in L^1(0,T)$, which implies $\Delta u + f - \alpha u \in L^2(0,T;L^2(\mathbb{R}^2))$ and then, using (2.13), we obtain (2.15). □

Lemma 2.5. *Any weak solution (f,u) satisfies*

$$\begin{aligned}&\int_{\mathbb{R}^2}\beta(f_{t_1})\,dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta''(f_s)\,|\nabla f_s|^2\,dxds\\ &\qquad \leq \int_{\mathbb{R}^2}\beta(f_{t_0})\,dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^2}\{\beta(f_s) - f_s\beta'(f_s)\}\Delta u_s\,dxds,\end{aligned} \tag{2.16}$$

for any times $0 \leq t_0 \leq t_1 < \infty$ and any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}$ which is convex, piecewise of class C^1 and such that

$$|\beta(\xi)| \leq C\,(1 + \xi\,(\log\xi)_+), \quad |\beta(\xi) - \xi\beta'(\xi)| \leq C\,\xi \quad \forall\, \xi \in \mathbb{R}.$$

Proof of Lemma 2.5. We consider a weak solution (f,u) to the Keller-Segel equation, and we write, in the distributional sense,

$$\partial_t f = \Delta f - \nabla u\cdot\nabla f - (\Delta u)\,f.$$

We split the proof into three steps.

Step 1. Continuity. Consider a mollifier sequence (ρ_n) on $\mathbb{R}^2$, that is $\rho_n(x) := n^2\rho(nx)$, $0 \leq \rho \in \mathcal{D}(\mathbb{R}^2)$, $\int\rho = 1$, and introduce the mollified function $f_t^n := f_t *_x \rho_n$. Clearly, $f^n \in C([0,T);L^1(\mathbb{R}^2))$. Using (2.13) and (1.10), a variant of the commutation Lemma [17, Lemma II.1 and Remark 4] tells us that

$$\partial_t f^n = \Delta f^n - \nabla u\cdot\nabla f^n - (\Delta u)\,f^n + r^n, \tag{2.17}$$

with $r^n = r_1^n + r_2^n$ given by

$$r_1^n := \nabla u \cdot \nabla f^n - (\nabla u \cdot \nabla f) * \rho_n \to 0 \quad \text{in} \quad L^1(0,T;L^1_{loc}(\mathbb{R}^2)),$$
$$r_2^n := (\Delta u) f^n - [(\Delta u) f] * \rho_n \to 0 \quad \text{in} \quad L^1(0,T;L^1_{loc}(\mathbb{R}^2)).$$

The important point here is that $f \in L^2(0,T;L^2(\mathbb{R}^2))$ thanks to (2.13) and $\nabla u \in L^2(0,T;W^{1,2}(\mathbb{R}^2))$ thanks to (2.15), hence the commutation lemma holds true.

As a consequence, the chain rule applied to the smooth function f^n reads

$$\partial_t \beta(f^n) = \Delta \beta(f^n) - \beta''(f^n)\,|\nabla f^n|^2 - \nabla u \cdot \nabla \beta(f^n) - (\Delta u) f^n \beta'(f^n) + \beta'(f^n)\, r^n, \tag{2.18}$$

for any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Because the equation (2.17) with u fixed is linear, the difference $f^{n,k} := f^n - f^k$ satisfies (2.17) with r^n replaced by $r^{n,k} := r^n - r^k \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$, and then also (2.18) (with again f^n and r^n changed in $f^{n,k}$ and $r^{n,k}$). For any non-negative function $\chi \in C^2_c(\mathbb{R}^d)$ and any time $t \in (0,T]$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \beta(f^{n,k}(t))\,\chi \;&=\; \int_{\mathbb{R}^2} \beta(f^{n,k}(0))\,\chi - \int_0^t\!\!\int_{\mathbb{R}^2} \beta''(f^{n,k}(s))|\nabla f^{n,k}(s)|^2\,\chi \\
&+\; \int_0^t\!\!\int_{\mathbb{R}^2} \beta(f^{n,k}(s))\,\{\Delta\chi + \nabla u(s)\cdot\nabla\chi\} \\
&+\; \int_0^t\!\!\int_{\mathbb{R}^2} \{\beta(f^{n,k}(s)) - f^{n,k}(s)\beta'(f^{n,k}(s))\}\,\Delta u(s)\,\chi \\
&+\; \int_0^t\!\!\int_{\mathbb{R}^2} \beta'(f^{n,k}(s))\,r^{n,k}(s)\,\chi.
\end{aligned}$$

In that last equation, we choose $\beta(\xi) = \beta_1(\xi) = \xi^2/2$ for $|\xi| \le 1$, $\beta_1(\xi) = |\xi| - 1/2$ for $|\xi| \ge 1$. Using that $|\beta_1'| \le 1$ and $\beta_1'' \ge 0$, it follows

$$\begin{aligned}
\int_{\mathbb{R}^2} \beta_1(f^{n,k}(t))\,\chi \;&\le\; \int_{\mathbb{R}^2} \beta_1(f^{n,k}(0))\,\chi + \int_0^t\!\!\int_{\mathbb{R}^2} \beta_1(f^{n,k}(s))\,\{|\Delta\chi| + |\nabla u(s)||\nabla\chi|\} \\
&+\; \int_0^t\!\!\int_{\mathbb{R}^2} |\beta_1(f^{n,k}(s)) - f^{n,k}(s)\beta_1'(f^{n,k}(s))|\,|\Delta u(s)|\,\chi + \int_0^t\!\!\int_{\mathbb{R}^2} |r^{n,k}(s)|\,\chi.
\end{aligned}$$

Since $f_0 \in L^1$, we have $f^{n,k}(0) \to 0$ in $L^1(\mathbb{R}^2)$ and we deduce from the previous inequality and the following convergences: $r^{n,k} \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$; $\beta_1(f^{n,k})|\nabla u| \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$, because $\beta_1(\xi) \le |\xi|$, $f^{n,k} \to 0$ in $L^2(0,T,L^2(\mathbb{R}^2))$ by (2.13) with $p = 2$ and $\nabla u \in L^2(0,T;L^2(\mathbb{R}^2))$ by Definition 1.1; $\beta_1(f^{n,k})|\Delta u| \to 0$ in $L^1(0,T;L^1(\mathbb{R}^2))$, because $\beta_1(\xi) \le |\xi|$, $f^{n,k} \to 0$ in $L^2(0,T,L^2(\mathbb{R}^2))$ and $\Delta u \in L^2(0,T;L^2(\mathbb{R}^2))$ by (2.15); and $f^{n,k}\beta_1'(f^{n,k})|\Delta u| \to 0$ in $L^1(0,T;L^1(\mathbb{R}^2))$, because $|\beta_1'| \le 1$, $f^{n,k} \to 0$ in $L^2(0,T,L^2(\mathbb{R}^2))$ and $\Delta u \in L^2(0,T;L^2(\mathbb{R}^2))$, that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t,x))\,\chi(x)\,dx \underset{n,k\to\infty}{\longrightarrow} 0.$$

Since χ is arbitrary, we deduce that there exists $\bar f \in C([0,\infty);L^1_{loc}(\mathbb{R}^2))$ so that $f^n \to \bar f$ in $C([0,T];L^1_{loc}(\mathbb{R}^2))$, $\forall\, T > 0$. Together with the convergence $f^n \to f$ in $C([0,\infty);\mathcal{D}'(\mathbb{R}^2))$ and the bound (1.10), we deduce that $f = \bar f$ and

$$f^n \to f \quad \text{in} \quad C([0,T];L^1(\mathbb{R}^2)), \quad \forall\, T > 0. \tag{2.19}$$

Step 2. Linear estimates. We come back to (2.18), which implies, for all $0 \le t_0 < t_1$, all $\chi \in C^2_c(\mathbb{R}^2)$,

$$\begin{aligned}
\int_{\mathbb{R}^2} \beta(f^n_{t_1})\,\chi + \int_{t_0}^{t_1}\!\!\int_{\mathbb{R}^2} \beta''(f^n_s)\,|\nabla_x f^n_s|^2\,\chi &= \int_{\mathbb{R}^2} \beta(f^n_{t_0})\,\chi + \int_{t_0}^{t_1}\!\!\int_{\mathbb{R}^2} \beta(f^n_s)\,\{\Delta\chi + \nabla u\cdot\nabla\chi\} \\
&+ \int_{t_0}^{t_1}\!\!\int_{\mathbb{R}^2} \{\beta(f^n_s) - f^n_s\beta'(f^n_s)\}\,\Delta u_s\,\chi + \int_{t_0}^{t_1}\!\!\int_{\mathbb{R}^2} \beta'(f^n_s)\,r^n\,\chi.
\end{aligned} \tag{2.20}$$

Choosing $0 \le \chi \in C^2_c(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is non-negative and vanishes outside of a compact set, and passing to the limit as $n \to \infty$, we get

$$\int_{\mathbb{R}^2} \beta(f_{t_1})\,\chi + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \beta''(f_s)\,|\nabla_x f_s|^2\,\chi \le \int_{\mathbb{R}^2} \beta(f_{t_0})\,\chi + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \{\beta(f_s) - f_s\beta'(f_s)\}\,\Delta u_s\,\chi + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \beta(f_s)\,\{\Delta\chi + \nabla u\cdot\nabla\chi\}. \tag{2.21}$$

By approximating $\chi \equiv 1$ by the sequence (χ_R) with $\chi_R(x) = \chi(x/R)$, $0 \le \chi \in \mathcal{D}(\mathbb{R}^2)$, we see that the last term in (2.21) vanishes and we get (2.16) in the limit $R \to \infty$ for any renormalizing function β with linear growth at infinity.

Step 3. Super-linear estimates. Finally, for any β satisfying the growth condition as in the statement of the Lemma, we just approximate β by an increasing sequence of smooth renormalizing functions β_R with linear growth at infinity, and we pass to the limit in (2.16) in order to conclude. □

As a first application of the previous lemma we obtain the following estimate.

Lemma 2.6. *For any weak solution (f,u) there exists a constant $C := C(M,\mathcal{H}_0,\mathcal{F}_0,T,p)$ such that, for any $0 \le t_0 < t_1 \le T$, there holds*

$$\int_{\mathbb{R}^2} f_{t_1}(\log f_{t_1})^2_+ + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\,\log f\,\mathbf{1}_{f>e} \le \int_{\mathbb{R}^2} f_{t_0}(\log f_{t_0})^2_+ + C. \tag{2.22}$$

Proof of Lemma 2.6. Let $K \ge e^2$ and define the renormalizing function $\beta_K : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\beta_K(\xi) := \begin{cases} \frac{\xi^2}{e}, & \text{if } \xi \le e;\\ \xi(\log\xi)^2, & \text{if } e \le \xi \le K;\\ (2+\log K)\xi\log\xi - 2K\log K, & \text{if } \xi \ge K; \end{cases}$$

so that β_K is convex and piecewise C^1. Moreover it holds

$$\begin{aligned} |\beta_K(\xi) - \beta_K'(\xi)\xi| &\le \frac{\xi^2}{e}\,\mathbf{1}_{\xi<e} + 2\xi\log\xi\,\mathbf{1}_{e<\xi<K} + 4\,\xi\log K\,\mathbf{1}_{\xi>K}\\ &\le 4\{\xi\,\mathbf{1}_{\xi<e} + \xi\log\xi\,\mathbf{1}_{e<\xi<K} + \xi\log K\,\mathbf{1}_{\xi>K}\} =: 4\gamma_K(\xi), \end{aligned}$$

and

$$\beta_K''(\xi) \ge \frac{2}{e}\,\mathbf{1}_{\xi<e} + 2\frac{\log\xi}{\xi}\,\mathbf{1}_{e<\xi<K} + \frac{\log K}{\xi}\,\mathbf{1}_{\xi>K}.$$

We deduce from Lemma 2.5 that

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1}) + \frac{2}{e}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla f|^2\,\mathbf{1}_{f<e} + 2\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\log f\,\mathbf{1}_{e<f<K} + \log K\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\,\mathbf{1}_{f>K} \le \int_{\mathbb{R}^2} \beta_K(f_{t_0}) + 4\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \gamma_K(f)|\Delta u|. \tag{2.23}$$

On the one hand, for any $\delta > 0$, we have

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \gamma_K(f)|\Delta u| \le \delta\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \gamma_K(f)^2 + C(\delta)\|\Delta u\|^2_{L^2(0,T;L^2(\mathbb{R}^2))}.$$

On the other hand, defining $\widetilde{\log}_K\xi := \mathbf{1}_{\xi\le e} + \log\xi\,\mathbf{1}_{e<\xi\le K} + \log K\,\mathbf{1}_{\xi>K}$, we have

$$\int_{\mathbb{R}^2} \gamma_K(f)^2 \le \int_{\mathbb{R}^2} f^2 + \int_{\mathbb{R}^2} (f\widetilde{\log}_K f)^2,$$

and thanks to inequality (2.8) with $p = 2$

$$\begin{aligned}\int_{\mathbb{R}^2}(f\widetilde{\log}_K f)^2 &\le C_2^2\left(\int_{\mathbb{R}^2} f\widetilde{\log}_K f\right)\left(\int_{\mathbb{R}^2}\frac{|\nabla(f\widetilde{\log}_K f)|^2}{f\widetilde{\log}_K f}\right)\\ &\le C'(M+\mathcal{H}^+(f))\left(\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\widetilde{\log}_K f\,\mathbf{1}_{f\ge e}+I(f)\right). \qquad (2.24)\end{aligned}$$

Coming back to (2.23), Estimate (2.22) follows by taking $\delta > 0$ small enough, using Lemma 2.4 and Proposition 2.2, and then letting $K \to \infty$. □

Lemma 2.7. *For any weak solution (f,u), any $p \ge 2$ and any $t_0 \in [0,T)$ such that $f(t_0) \in L^p(\mathbb{R}^2)$, there exists a constant $C := C(M,\mathcal{H}_0,\mathcal{F}_0,T,p,\|f(t_0)\|_{L^p})$ such that, for all $t_0 < t_1 \le T$, there holds*

$$\|f(t_1)\|_{L^p}^p + \frac{1}{2}\int_{t_0}^{t_1}\|\nabla_x f^{p/2}\|_{L^2}^2\,dt \le C. \qquad (2.25)$$

Proof of Lemma 2.7. We split the proof into four steps.

Step 1. We define the renormalizing function $\beta_K : \mathbb{R}_+ \to \mathbb{R}_+$, $K \ge e^2$, by

$$\beta_K(\xi) := \frac{\xi^p}{p} \text{ if } \xi \le K, \quad \beta_K(\xi) := \frac{K^{p-1}}{\log K}(\xi\log\xi - \xi) - \frac{K^p}{p'} + \frac{K^p}{\log K} \text{ if } \xi \ge K,$$

so that β_K is convex, increasing and piecewise of class C^1. Moreover the following estimates hold

$$|\beta_K(\xi) - \beta_K'(\xi)\xi| \le \frac{1}{p'}\,\xi^p\,\mathbf{1}_{\xi<K} + 3K^{p-1}\,\xi\,\mathbf{1}_{\xi>K},$$

$$\beta_K''(\xi) = (p-1)\,\xi^{p-2}\,\mathbf{1}_{\xi<K} + \frac{K^{p-1}}{\log K}\,\frac{1}{\xi}\,\mathbf{1}_{\xi>K} \quad \text{and} \quad \beta_K(\xi) \ge 2^{1-p}p^{-1}K^{p-1}\,\xi\,\mathbf{1}_{\xi>K/2}.$$

Thanks to Lemma 2.5, we may write

$$\begin{aligned}&\int_{\mathbb{R}^2}\beta_K(f_{t_1}) + \frac{4}{pp'}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}|\nabla_x(f^{p/2})|^2\,\mathbf{1}_{f\le K} + \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\frac{|\nabla_x f|^2}{f}\,\mathbf{1}_{f\ge K}\\ &\quad\le \int_{\mathbb{R}^2}\beta_K(f_{t_0}) + \frac{1}{p'}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}|\Delta u|\,f^p\,\mathbf{1}_{f<K} + 3K^{p-1}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}|\Delta u|\,f\,\mathbf{1}_{f>K}. \qquad (2.26)\end{aligned}$$

Step 2. For the second term on the right-hand side of (2.26), using the Gagliardo-Niremberg-Sobolev inequality

$$\int_{\mathbb{R}^2} g^4\,dx \le C\int_{\mathbb{R}^2} g^2\,dx\int_{\mathbb{R}^2}|\nabla g|^2\,dx,$$

we have

$$\begin{aligned}\mathcal{T}_1 &:= \frac{1}{p'}\int_{\mathbb{R}^2}|\Delta u|\,f^p\,\mathbf{1}_{f<K}\\ &\le \frac{1}{p'}\|\Delta u\|_{L^2_x}\Big(\int_{\mathbb{R}^2}(f\wedge K)^{2p}\Big)^{1/2}\\ &\le C\,\|\Delta u\|_{L^2_x}\Big(\int_{\mathbb{R}^2}(f\wedge K)^p\int_{\mathbb{R}^2}|\nabla(f\wedge K)^{p/2}|^2\Big)^{1/2}\\ &\le C\,\|\Delta u\|_{L^2_x}^2\int_{\mathbb{R}^2}\beta_K(f) + \frac{1}{pp'}\int_{\mathbb{R}^2}|\nabla(f^{p/2})|^2\,\mathbf{1}_{f<K}.\end{aligned}$$

Step 3. Let us define the convex function Φ by $\Phi(\xi) := \xi^2(\widetilde{\log}\,\xi)^2$ for any $\xi \ge 0$, with $\widetilde{\log}\,\xi := \mathbf{1}_{\xi\le e} + \log\xi\,\mathbf{1}_{\xi>e}$. We observe that Φ is a N-function so that we may associate to Φ the Orlicz spaces L_Φ (see Appendix A.1). We have already obtained that $\|\Delta u\|_{L^2_x}^2 \in L^1(0,T)$ in Lemma 2.4 and we claim that we also have

$$\|\Phi(|\Delta u|)\|_{L^1_x} \in L^1(t_0,T). \qquad (2.27)$$

Indeed, passing to the limit $K \to \infty$ in Estimate (2.24), and using Proposition 2.2 and Lemma 2.6, we immediately deduce

$$f \in L_\Phi((t_0,T)\times\mathbb{R}^2). \tag{2.28}$$

We now consider the linear operator $f \mapsto U(f) := \Delta u$ where u is the solution to the linear parabolic equation $\varepsilon\partial_t u - \Delta u + \alpha u = f$. Thanks to standard results (see e.g. Theorem X.12 stated in [8] and the quoted references therein)

$$U : L^p((t_0,T)\times\mathbb{R}^2) \to L^p((t_0,T)\times\mathbb{R}^2)$$

is a bounded operator for any $p \in (1,\infty)$. Since $L_\Phi((t_0,T)\times\mathbb{R}^2)$ is an interpolation space between $L^{3/2}((t_0,T)\times\mathbb{R}^2)$ and $L^{9/2}((t_0,T)\times\mathbb{R}^2)$ (see Appendix A.2), we also get

$$U : L_\Phi((t_0,T)\times\mathbb{R}^2) \to L_\Phi((t_0,T)\times\mathbb{R}^2) \tag{2.29}$$

bounded. We then deduce (2.27) from (2.28), (2.29) and the fact that Φ satisfies the Δ_2-condition (see Appendix A.1).

Step 4. Now we estimate the last term in (2.26). We denote by Φ^* the conjugate function of Φ and we observe that $\Phi^*(\eta) \le C\,\eta^2(\widetilde{\log}\eta)^{-2}$ for any $\eta > 0$ and for some fixed constant $C \in (0,\infty)$ (see Appendix A.3). We introduce the notation $f_K := f\mathbf{1}_{f>K}$ and $A_K(f) := (\int_{\mathbb{R}^2} f\mathbf{1}_{f>K/2})^{1/2}$. Using Young's inequality $\xi\eta \le \Phi(\xi) + \Phi^*(\eta)$, we obtain

$$\begin{aligned}\mathcal{T}_2 &:= 2K^{p-1}\int_{\mathbb{R}^2} |\Delta u|\, f\,\mathbf{1}_{f\ge K}\\ &\le 2K^{p-1}\left\{\int_{\mathbb{R}^2}\Phi(A_K(f)|\Delta u|) + \int_{\mathbb{R}^2}\Phi^*(A_K(f)^{-1}f_K)\,\mathbf{1}_{f\ge K}\right\} =: \mathcal{T}_{2,1}+\mathcal{T}_{2,2}.\end{aligned}$$

For the term $\mathcal{T}_{2,1}$, using that $A_K(f) \le M^{1/2}$ and the elementary inequality $\widetilde{\log}(\xi\eta) \le \widetilde{\log}\,\xi + \widetilde{\log}\,\eta$, we may write

$$\begin{aligned}\mathcal{T}_{2,1} &:= 2K^{p-1}\int_{\mathbb{R}^2}\Phi(A_K(f)|\Delta u|)\\ &\le C\,K^{p-1}\int_{\mathbb{R}^2}|\Delta u|^2\,|A_K(f)|^2\,(\widetilde{\log}(A_K(f)|\Delta u|))^2\\ &\le C\,K^{p-1}\left(\int_{\mathbb{R}^2} f\,\mathbf{1}_{f>K/2}\right)\left(\int_{\mathbb{R}^2}|\Delta u|^2(\widetilde{\log}A_K(f))^2 + \int_{\mathbb{R}^2}|\Delta u|^2(\widetilde{\log}|\Delta u|)^2\right)\\ &\le C\left(\int_{\mathbb{R}^2}\beta_K(f)\right)\left(C(M)\|\Delta u\|^2_{L^2_x} + \|\Phi(|\Delta u|)\|_{L^1_x}\right).\end{aligned}$$

For the term $\mathcal{T}_{2,2}$, using that $f_K > K$ on $\{f > K\}$, we obtain, for $K > eM^{1/2}$,

$$\begin{aligned}
\mathcal{T}_{2,2} &:= 2K^{p-1} \int_{\mathbb{R}^2} \Phi^*(f_K/A_K(f))\, \mathbf{1}_{f\geq K} \\
&\leq C\, K^{p-1} \int_{\mathbb{R}^2} \frac{(f_K/A_K(f))^2}{(\widetilde{\log}(f_K/A_K(f)))^2}\, \mathbf{1}_{f\geq K} \\
&\leq C\, \frac{K^{p-1}\, |A_K(f)|^{-2}}{(\log \frac{K}{M^{1/2}})^2} \int_{\mathbb{R}^2} (f - K/2)_+^2 \\
&\leq C\, \frac{K^{p-1}\, |A_K(f)|^{-2}}{(\log \frac{K}{M^{1/2}})^2} \left(\int_{\mathbb{R}^2} |\nabla f|\, \mathbf{1}_{f\geq K/2} \right)^2 \\
&\leq C\, \frac{K^{p-1}\, |A_K(f)|^{-2}}{(\log \frac{K}{M^{1/2}})^2} \left(\int_{\mathbb{R}^2} f\, \mathbf{1}_{f\geq K/2} \right) \left(\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, \mathbf{1}_{f\geq K/2} \right) \\
&\leq C\, \frac{K^{p-1}}{(\log \frac{K}{M^{1/2}})^2} \left(\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, \mathbf{1}_{K/2\leq f\leq K/2} + \int_{\mathbb{R}^2} \frac{|\nabla f|}{f}\, \mathbf{1}_{f>K} \right) \\
&\leq C\, \frac{K^{p-1}}{(\log \frac{K}{M^{1/2}})^2} \left(K^{1-p} \int_{\mathbb{R}^2} |\nabla (f^{p/2})|^2\, \mathbf{1}_{f\leq K} + \int_{\mathbb{R}^2} \frac{|\nabla f|}{f}\, \mathbf{1}_{f>K} \right),
\end{aligned}$$

where we have used the elementary inequality $f_K^2 \leq C(f - K/2)_+^2$ in the second line, Sobolev's inequality in the third line and Cauchy-Schwarz inequality in the fourth line. Hence, for K large enough, we have

$$\begin{aligned}
\mathcal{T}_{2,2} &\leq C\, \frac{1}{(\log \frac{K}{M^{1/2}})^2} \int_{\mathbb{R}^2} |\nabla (f^{p/2})|^2\, \mathbf{1}_{f\leq K} + C\, \frac{K^{p-1}}{(\log \frac{K}{M^{1/2}})^2} \int_{\mathbb{R}^2} \frac{|\nabla f|}{f}\, \mathbf{1}_{f>K} \\
&\leq \frac{1}{pp'} \int_{\mathbb{R}^2} |\nabla (f^{p/2})|^2\, \mathbf{1}_{f\leq K} + \frac{1}{2}\, \frac{K^{p-1}}{\log K} \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, \mathbf{1}_{f>K}.
\end{aligned}$$

Gathering $\mathcal{T}_1$, $\mathcal{T}_{2,1}$ and $\mathcal{T}_{2,2}$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \beta_K(f_{t_1}) + \frac{2}{pp'} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla_x (f^{p/2})|^2\, \mathbf{1}_{f\leq K} \\
&\qquad \leq \int_{\mathbb{R}^2} \beta_K(f_{t_0}) + C \int_{t_0}^{t_1} \left(\|\Delta u\|_{L^2_x}^2 + \|\Phi(|\Delta u|)\|_{L^1_x} \right) \int_{\mathbb{R}^2} \beta_K(f).
\end{aligned} \tag{2.30}$$

Using the fact that $h(t) := \int_{\mathbb{R}^2} |\Delta u|^2 + \Phi(|\Delta u|) \in L^1(t_0, T)$ from Lemma 2.4 and Step 3, we can conclude to (2.25) by applying first the Gronwall's lemma and by passing then to the limit $K \to \infty$. □

Lemma 2.8. *Any weak solution (f, u) satisfies*

$$\partial_t f, \partial_x f, \partial^2_{x_i x_j} f, \partial_t u, \partial_x u, \partial^2_{x_i x_j} u \in C_b((0, T] \times \mathbb{R}^2),\ \forall T > 0,$$

so that it is a "classical solution" for positive time.

Proof of Lemma 2.8. For any time $t_0 \in (0, T)$ and any exponent $p \in (1, \infty)$, there exists $t_0' \in (0, t_0)$ such that $f(t_0') \in L^p(\mathbb{R}^2)$ thanks to (2.13), from what we deduce using (2.25) on the time interval (t_0', T) that

$$f \in L^\infty(t_0, T; L^p(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x f \in L^2((t_0, T) \times \mathbb{R}^2). \tag{2.31}$$

Since u satisfies the parabolic equation

$$\varepsilon \partial_t u - \Delta u + \alpha u = f,$$

the maximal regularity of the heat equation in L^p-spaces (see Theorem X.12 stated in [8] and the quoted references) and the fact that

$$u(t) = \gamma^\alpha_{t/\varepsilon} *_{t,x} f + \gamma^\alpha_{t/\varepsilon} *_x u_0 \quad \text{and} \quad \nabla u = \Gamma^\alpha_{t/\varepsilon} *_{t,x} f + \gamma^\alpha_{t,\varepsilon} *_x \nabla u_0,$$

where we denote $\gamma_s^\alpha = e^{-\alpha s}\,\gamma_s$ and similarly for Γ, γ_t is the heat kernel given by

$$\gamma_t(x) := \frac{1}{4\pi t}\,\exp\left(-\frac{|x|^2}{4t}\right) \in L^{z_1}(0,T;L^{z_2}(\mathbb{R}^2)), \quad \forall\, z_1,z_2\geq 1,\ 1/z_1+1/z_2>1,$$

and

$$\Gamma_t(x) := \nabla_x\gamma_t(x) \in L^{s_1}(0,T;L^{s_2}(\mathbb{R}^2)), \quad \forall\, s_1,s_2\geq 1,\ 1/s_1+1/s_2>3/2,$$

provide the bound

$$u\in L^\infty(t_0,T;L^p(\mathbb{R}^2)), \quad \nabla u\in L^\infty(t_0,T;L^p(\mathbb{R}^2)), \quad \partial_t u, D^2u\in L^p((t_0,T)\times\mathbb{R}^2), \tag{2.32}$$

for all $t_0\in(0,T)$ and $p\in(1,\infty)$. Since now f satisfies the parabolic equation

$$\partial_t f-\Delta f = -\nabla u\cdot\nabla f-(\Delta u)f =: Z$$

with $Z\in L^2(t_0,T;L^q(\mathbb{R}^2))$ for all $t_0\in(0,T)$ and all $q\in[1,2)$ from (2.31) and (2.32), the same maximal regularity of the heat equation in L^q-spaces (with the choice $s_1=s_2=(4/3)^-$) implies

$$\nabla f\in L^p(t_0,T;L^p(\mathbb{R}^2)),\ \forall\, p\in[2,4),$$

and then $Z\in L^p(t_0,T;L^p(\mathbb{R}^2))$, $\forall\, p\in[2,4)$. By a bootstrap argument of the regularity property of the heat equation, we easily get

$$f\in L^\infty(t_0,T;L^p(\mathbb{R}^2)), \quad \nabla f\in L^\infty(t_0,T;L^p(\mathbb{R}^2)), \quad \partial_t f, D^2f\in L^p((t_0,T)\times\mathbb{R}^2), \tag{2.33}$$

for all $t_0\in(0,T)$ and $p\in(1,\infty)$. The Morrey inequality implies then $f,\nabla f,u,\nabla u\in C^{0,\alpha}((t_0,T)\times\mathbb{R}^2)$ for any $0<\alpha<1$, and any $t_0>0$. Finally the classical Holderian regularity result for the heat equation (see Theorem X.13 stated in [8] and the quoted references) implies first $u\in C^{2,\alpha}((t_0,T)\times\mathbb{R}^2)$ and next $f\in C^{2,\alpha}((t_0,T)\times\mathbb{R}^2)$, which concludes the proof. □

We prove now the free energy-dissipation of the free energy identity (1.5) in Theorem 1.3.

Proof of the free energy identity in Theorem 1.3. We split the proof into four steps.

Step 1. We claim that the free energy functional $\mathcal{F}$ is lower semi-continuous (lsc) in the sense that for any sequences (f_n) and (u_n) of nonnegative functions such that (f_n) is bounded in $L^1\cap L^1(\log\langle x\rangle^2)$ with same mass $M<8\pi$, (u_n) is bounded in H^1 if $\alpha>0$ or in $L^1\cap\dot H^1$ if $\alpha=0$, $(f_n\,u_n)$ is bounded in L^1, $(\mathcal{F}(f_n,u_n))$ is bounded and $(f_n,u_n)\rightharpoonup(f,u)$ in $\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}'(\mathbb{R}^2)$, there holds

$$0\leq f\in L^1\cap L^1(\log\langle x\rangle^2) \quad\text{and}\quad \mathcal{F}(f,u)\leq\liminf_{n\to\infty}\mathcal{F}(f_n,u_n). \tag{2.34}$$

Indeed, because of (2.5) and (2.6), we have $\mathcal{H}^+(f_n)\leq C$ and we may apply the Dunford-Pettis lemma which implies that $f_n\rightharpoonup f$ weakly in $L^1(\mathbb{R}^2)$. We then rewrite the free energy functional as

$$\mathcal{F}(f_n,u_n)=\mathcal{H}(f_n)+F_\alpha(f_n,u_n),$$

with

$$\mathcal{H}(f_n):=\int f_n\log f_n \quad\text{and}\quad F_\alpha(f_n,u_n):=\frac12\int|\nabla u_n|^2+\frac{\alpha}{2}\int u_n^2-\int f_n\,u_n,$$

and we consider separately the case $\alpha>0$ and $\alpha=0$.

Step 2: Case $\alpha>0$. We denote $\bar u_n=\kappa_\alpha*f_n$ where $\kappa_\alpha(z)=\frac{1}{4\pi}\int_0^\infty\frac1t e^{-\frac{|z|^2}{4t}-\alpha t}\,dt$ is the Bessel kernel. Since $f_n\geq0$, $f_n\in L^1\cap L\log L$ and $u_n\in H^1$, [10, Lemma 2.2] implies that $\bar u_n\in H^1$ and also that the functional $F_\alpha(f_n,u_n)$ is finite and satisfies

$$F_\alpha(f_n,u_n)-F_\alpha(f_n,\bar u_n)=\frac12\|\nabla(u_n-\bar u_n)\|_{L^2}^2+\frac{\alpha}{2}\|u_n-\bar u_n\|_{L^2}^2.$$

Hence, we can write

$$\begin{aligned}\mathcal{F}(f_n,u_n)&=\mathcal{H}(f_n)-\frac12\iint f_n(x)f_n(y)\kappa_\alpha(x-y)+\frac12\|\nabla(u_n-\bar u_n)\|_{L^2}^2+\frac{\alpha}{2}\|u_n-\bar u_n\|_{L^2}^2\\&=:\mathcal{H}(f_n)+\mathcal{V}(f_n)+\mathcal{U}_1(u_n-\bar u_n)+\mathcal{U}_2(u_n-\bar u_n),\end{aligned}$$

where the functionals $\mathcal{U}_1$ and $\mathcal{U}_2$ are defined through the third and fourth term respectively. We clearly have that $\mathcal{U}_1+\mathcal{U}_2$ is lsc for the weak H^1 convergence and $\mathcal{H}$ is lsc for the weak L^1 convergence, so we investigate the functional $\mathcal{V}$. For any $\epsilon \in (0,1)$ we split $\mathcal{V} = \mathcal{V}_\epsilon + \mathcal{R}_\epsilon$ as

$$\mathcal{V}_\epsilon(g) := -\frac{1}{2}\iint g(x)g(y)\,\kappa_\alpha(x-y)\,\mathbf{1}_{|x-y|>\epsilon}$$
$$\mathcal{R}_\epsilon(g) := -\frac{1}{2}\iint g(x)g(y)\,\kappa_\alpha(x-y)\,\mathbf{1}_{|x-y|\le\epsilon}.$$

The Bessel kernel κ_α is a positive radial decreasing function with a singularity at the origin: $\kappa_\alpha(z) = -\frac{1}{2\pi}\log|z| + O(1)$ when $|z| \to 0$. Hence $\mathcal{V}_\epsilon$ is continuous for the weak L^1 convergence and for the rest term we obtain, for any $\epsilon \in (0,1)$ and $\lambda > 1$,

$$\begin{aligned}
|\mathcal{R}_\varepsilon(g)| &\le C\iint g(x)g(y)\mathbf{1}_{|x-y|\le\epsilon} + C\iint g(x)g(y)(\log|x-y|)_-\mathbf{1}_{|x-y|\le\epsilon}\\
&\le C\iint g(x)\mathbf{1}_{g(x)\le\lambda}g(y)\mathbf{1}_{|x-y|\le\epsilon} + C\iint g(x)\mathbf{1}_{g(x)>\lambda}g(y)\mathbf{1}_{|x-y|\le\epsilon}\\
&\quad + C\iint g(x)\mathbf{1}_{g(x)\ge\lambda}g(y)(\log|x-y|)_-\mathbf{1}_{|x-y|\le\epsilon} + C\iint g(x)\mathbf{1}_{g(x)>\lambda}g(y)\log(|x-y|^{-1})\mathbf{1}_{|x-y|\le\epsilon}\\
&\le C\lambda\int_y g(y)\left\{\int_{|z|\le\epsilon} dz\right\} + C\int_y g(y)\left\{\frac{1}{\log\lambda}\int_x g(x)\log g(x)\right\}\\
&\quad + C\lambda\int_y g(y)\left\{\int_{|z|\le\epsilon}(\log|z|)_-\,dz\right\} + C\int_x g(x)\mathbf{1}_{g(x)>\lambda}\int_y\{g(y)\log g(y) + |x-y|^{-1}\}\mathbf{1}_{|x-y|\le\epsilon}\\
&\le C\,M\lambda\epsilon^2 + C\,M\frac{\mathcal{H}(g)}{\log\lambda} + C\,M\lambda\epsilon^{3/2} + C\,\frac{\mathcal{H}(g)}{\log\lambda}\{\mathcal{H}(g)+\epsilon\},
\end{aligned}$$

where we have used the convexity inequality $uv \le u\log u + e^v$ for all $u > 0$, $v \in \mathbb{R}$ and the elementary inequality $u\log u \ge -u^{1/2}$ for all $u \in (0,1)$. Hence $\sup_n |\mathcal{R}_\varepsilon(f_n)| \to 0$ as $\epsilon \to 0$ and we deduce that $\mathcal{F}$ is lsc.

Step 3: Case $\alpha = 0$. We define $\bar{u}_n = \kappa_0(f_n - MH)$ where $H(x) = \langle x\rangle^{-4}/\pi$ and $\kappa_0(z) = -\frac{1}{2\pi}\log|z|$ is the Laplace kernel. Since $0 \le f_n \in L^1 \cap L^1(\log\langle x\rangle^2)$, $\mathcal{H}(f_n)$ is finite, $\int(f - MH) = 0$ and $u_n \in \dot{H}^1$, [10, Lemma 2.2] implies that $\bar{u}_n \in \dot{H}^1$ and also that the functional $F_0(f_n - MH, u_n)$ is finite and verifies

$$F_0(f_n - MH, u_n) - F_0(f_n - MH, \bar{u}_n) = \frac{1}{2}\|\nabla(u_n - \bar{u}_n)\|_{L^2}^2.$$

Now we argue as in the case $\alpha > 0$. First we write

$$\begin{aligned}
\mathcal{F}(f_n, u_n) &= \mathcal{H}(f_n) - \frac{1}{2}\iint f_n(x)f_n(y)\kappa_0(x-y) + M\iint f_n(x)H(y)\kappa_0(x-y)\\
&\quad + \frac{1}{2}\|\nabla(u_n - \bar{u}_n)\|_{L^2}^2 - M\int Hu_n - \frac{M^2}{2}\iint H(x)H(y)\kappa_0(x-y)\\
&=: \mathcal{H}(f_n) + \mathcal{V}(f_n) + \mathcal{W}(f_n) + \mathcal{U}_1(u_n - \bar{u}_n) + \mathcal{U}_0(u_n) + \mathcal{Z}(H).
\end{aligned}$$

The functional $\mathcal{U}_1$ is lsc for the weak $\dot{H}^1$ convergence and $\mathcal{H}$ is lsc for the weak L^1 convergence. For $\mathcal{V}$ we just argue as in the preceding case $\alpha > 0$. In the same (even simpler) way we obtain that $\mathcal{W}$ is lsc for the weak L^1 convergence. Finally we conclude that $\mathcal{F}$ is lsc.

Step 4. Now, we easily deduce that the free energy identity (1.5) holds. Indeed, since (f,u) is smooth for positive time, for any fixed $t > 0$ and any given sequence (t_n) of positive real numbers which decreases to 0, we clearly have

$$\mathcal{F}(f(t_n), u(t_n)) = \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_{\mathcal{F}}(f(s), u(s))\,ds.$$

Then, thanks to the Lebesgue convergence theorem, the lsc property of $\mathcal{F}$ and the fact that $f(t_n) \rightharpoonup f_0$ and $u(t_n) \rightharpoonup u_0$ weakly in $\mathcal{D}'(\mathbb{R}^2)$, we deduce from the above free energy identity for positive time that

$$\begin{aligned}\mathcal{F}(f_0,u_0) &\le \liminf_{n\to\infty} \mathcal{F}(f(t_n),u(t_n)) \\ &\le \lim_{n\to\infty} \Big\{ \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_{\mathcal{F}}(f(s),u(s))\,ds \Big\} = \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(f(s),u(s))\,ds.\end{aligned}$$

Together with the reverse inequality (1.10) we conclude to (1.5). □

3. Uniqueness - Proof of Theorem 1.3

In this section we prove the uniqueness part of Theorem 1.3. In order to do so, we first prove some estimates in Lemmas 3.1 and 3.2.

Lemma 3.1. *Any weak solution (f,u) to the Keller-Segel equation satisfies that for any $p \in (1,\infty)$, $T \in (0,\infty)$, there exists a constant $K = K(f_0,p,T)$ such that*

$$t^{p-1}\|f(t)\|_{L^p}^p \le K \qquad \forall\, t \in (0,T). \tag{3.1}$$

Proof of Lemma 3.1. Recall that we already know that $\|f\|_{L^p} \in C^1(0,T)$ for any $p > 1$ and $\|f\|_{L^p} \in L^\infty(t_0,T)$ for any $0 < t_0 < T$ and any $p \in [1,\infty]$. For $p > 1$, we have

$$\begin{aligned}\frac{d}{dt}\int f^p &= -4(1-1/p)\int |\nabla(f^{p/2})|^2 + (p-1)\int f^{p+1} - (p-1)\int (\partial_t u + \alpha u) f^p \\ &=: T_1 + T_2 + T_3.\end{aligned}$$

Using the splitting $f = \min(f,A) + (f-A)_+$, for some $A > 0$, and denoting $h(u) := \partial_t u + \alpha u \in L^2(0,T;L^2(\mathbb{R}^2))$, we have

$$|T_3| \le C\int |h(u)| \min(f,A)^p + C\int |h(u)|(f-A)_+^p =: T_{31} + T_{32}.$$

For the term T_{31}, we have

$$|T_{31}| \le CA^{p-1/2}\int |h(u)| f^{1/2} \le CA^{p-1/2}(\|h(u)\|_{L^2}^2 + M).$$

For T_{32}, using Gagliardo-Niremberg-Sobolev inequality $\|g\|_{L^4(\mathbb{R}^2)}^4 \le C\|g\|_{L^2(\mathbb{R}^2)}^2 \|\nabla g\|_{L^2(\mathbb{R}^2)}^2$ with $g = (f-A)_+^{p/2}$, we have

$$\begin{aligned}|T_{32}| &\le C\int |h(u)|(f-A)_+^p \\ &\le C\|h(c)\|_{L^2}\left(\int (f-A)_+^{2p}\right)^{1/2} \\ &\le C\|h(c)\|_{L^2}\left(\int (f-A)_+^p\right)^{1/2}\left(\int |\nabla(f-A)_+^{p/2}|^2\right)^{1/2} \\ &\le C_\delta \|h(u)\|_{L^2}^2 \int (f-A)_+^p + \delta\int |\nabla(f^{p/2})|^2\,\mathbf{1}_{f\ge A},\end{aligned}$$

for any $\delta > 0$.

For the second term T_2, we have

$$|T_2| \le C\int \min(f,A)^{p+1} + C\int (f-A)_+^{p+1}.$$

At the right-hand side, the first term is easily bounded by CA^pM, and using (2.10), the second one is bounded as follows

$$\int (f-A)_+^{p+1} \le C\left(\int (f-A)_+\right)\left(\int |\nabla(f-A)_+^{p/2}|^2\right) \le C\,\frac{\mathcal{H}^+(f)}{\log A}\,\left(\int |\nabla(f^{p/2})|^2\,\mathbf{1}_{f\ge A}\right).$$

Gathering all the previous estimates, choosing $\delta > 0$ small enough and A large enough, we get

$$\frac{d}{dt}\int f^p \le -C_0 \int |\nabla(f^{p/2})|^2 + C_1\|h(u)\|_{L^2}^2 \int f^p + C_1(M + \|h(u)\|_{L^2}^2). \tag{3.2}$$

Thanks to the following Hölder and Gagliardo-Niremberg-Sobolev inequalities

$$(\|f\|_{L^p}^p)^{p/(p-1)} \le C\|f\|_{L^1}^{1/(p-1)}\|f\|_{L^{p+1}}^{p+1} \quad \text{and} \quad \|f\|_{L^{p+1}}^{p+1} \le C\|f\|_{L^1}\|\nabla(f^{p/2})\|_{L^2}^2,$$

we obtain from (3.2) the differential inequality

$$\frac{d}{dt}X(t) \le -C_0 X(t)^{\frac{p}{p-1}} + C_1 H(t)X(t) + C_2(1+H(t)), \qquad t \in (0,T),$$

where we denote $X(t) := \|f(t)\|_{L^p}^p$ and $H(t) := \|h(u)\|_{L^2}^2(t) \in L^1(0,T)$. By standard arguments (see e.g. [10, Proof of Theorem 5.1]) we conclude to (3.1). □

We (crucially) improve the preceding estimate by showing

Lemma 3.2. *For any $q \in (1,\infty)$, any weak solution (f,u) to the Keller-Segel equation satisfies:*

$$t^{1-\frac{1}{q}}\|f(t,.)\|_{L^q} \to 0 \quad \text{as } t \to 0. \tag{3.3}$$

Proof of Lemma 3.2. We prove (3.3) from (3.1) and an interpolation argument. On the one hand, denoting $\widetilde{\log}_+ f := 2 + \log_+ f$, we use the Cauchy-Schwarz inequality in order to get

$$\int f^q \quad \le \quad \Big(\int f \widetilde{\log}_+ f\Big)^{1/2}\Big(\int f^{2q-1}(\widetilde{\log}_+ f)^{-1}\Big)^{1/2},$$

or in other words

$$\|f\|_{L^q} \le C(M, \mathcal{H}_+(f))\Big(\int f^{2q-1}(\widetilde{\log}_+ f)^{-1}\Big)^{1/(2q)}. \tag{3.4}$$

On the other hand, we observe that

$$\begin{aligned} t^{2q-2}\int f^{2q-1}(\widetilde{\log}_+ f)^{-1} \quad &\le \quad t^{2q-2}\frac{R^{2q-1}}{\widetilde{\log}_+ R}\int_{f\le R} f + \frac{t^{2q-2}}{\widetilde{\log}_+ R}\int_{f\ge R} f^{2q-1} \quad \forall\, R>0 \\ &\le \quad t^{2q-2}\frac{MR^{2q-1}}{\widetilde{\log}_+ R} + \frac{K}{\widetilde{\log}_+ R}, \end{aligned}$$

for any $R > 0$, where we have used that $s \mapsto s^{2q-1}(\widetilde{\log}_+ s)^{-1}$ is an increasing function in the first line, the mass conservation of the solution of the Keller-Segel equation and the estimate (3.1) in the second line. Choosing $R := t^{-1}$, we deduce

$$t^{2q-2}\int f^{2q-1}(\widetilde{\log}_+ f)^{-1} \le \frac{M+K}{\widetilde{\log}_+ t}, \quad \forall\, t \le 1. \tag{3.5}$$

We conclude to (3.3) by gathering (3.4) and (3.5). □

We are now able to prove the uniqueness of solutions.

Proof of the uniqueness part in Theorem 1.3. We consider two weak solutions (f_1,u_1) and (f_2,u_2) to the Keller-Segel equation (1.1) that we write in the mild form

$$f_i(t) = e^{t\Delta}f_i(0) - \int_0^t \nabla e^{(t-s)\Delta}(f_i(s)\nabla u_i(s))\,ds$$

and

$$u_i(s) = e^{-\frac{\alpha}{\varepsilon}s}e^{\frac{s}{\varepsilon}\Delta}u_i(0) + \frac{1}{\varepsilon}\int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)}e^{\frac{(s-\sigma)}{\varepsilon}\Delta}f_i(\sigma)\,d\sigma,$$

from which we also obtain

$$\nabla u_i(s) = e^{-\frac{\alpha}{\varepsilon}s}e^{\frac{s}{\varepsilon}\Delta}(\nabla u_i(0)) + \frac{1}{\varepsilon}\int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)}(\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta})f_i(\sigma)\,d\sigma.$$

When we assume $f_1(0) = f_2(0)$ and $u_1(0) = u_2(0) = u_0$, the difference $F := f_2 - f_1$ satisfies

$$\begin{aligned} F(t) &= -\int_0^t \nabla e^{(t-s)\Delta} \left\{ F(s) \left[e^{-\frac{\alpha}{\varepsilon} s} e^{\frac{s}{\varepsilon}\Delta} (\nabla u_0) \right] \right\} ds \\ &\quad - \int_0^t \nabla e^{(t-s)\Delta} \left\{ F(s) \left[\frac{1}{\varepsilon} \int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)} \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\, d\sigma \right] \right\} ds \\ &\quad - \int_0^t \nabla e^{(t-s)\Delta} \left\{ f_1(s) \left[\frac{1}{\varepsilon} \int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)} \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma)\, d\sigma \right] \right\} ds \\ &=: I_1(t) + I_2(t) + I_3(t). \end{aligned} \tag{3.6}$$

For any $t > 0$, we define

$$Z_p^i(t) := \sup_{0<s\le t} s^{\frac{1}{2}-\frac{1}{2p}} \left\| f_i(s) \right\|_{L^{\frac{2p}{p+1}}}, \quad \delta(t) := \sup_{0<s\le t} s^{\frac{1}{4}} \left\| F(s) \right\|_{L^{4/3}}.$$

We recall the explicit formula for the heat semigroup

$$e^{t\Delta} g = \gamma(t,\cdot) *_x g, \qquad \gamma(t,x) := \frac{1}{4\pi t} \exp\left(-\frac{|x|^2}{4t} \right),$$

and the following well-known inequalities that will be useful in the sequel

$$\|K * g\|_{L^r} \le \|K\|_{L^q} \|g\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \le p, q, r \le \infty, \tag{3.7}$$

and

$$\|\gamma(t,\cdot)\|_{L^q(\mathbb{R}^2)} \le C_q\, t^{\frac{1}{q}-1}, \quad \|\nabla\gamma(t,\cdot)\|_{L^q(\mathbb{R}^2)} \le C_q\, t^{\frac{1}{q}-\frac{3}{2}}.$$

We fix $p > 2$ and we compute the quantity $t^{\frac{1}{4}} \|\cdot\|_{L^{4/3}}$ for each term of (3.6).

For the second term, we compute

$$\begin{aligned} t^{\frac{1}{4}} \|I_2(t)\|_{L^{4/3}} &\le C(\alpha,\varepsilon)\, t^{\frac{1}{4}} \int_0^t \left\| \nabla e^{(t-s)\Delta} \left\{ F(s) \int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\, d\sigma \right\} \right\|_{L^{4/3}} ds \\ &\le C\, t^{\frac{1}{4}} \int_0^t \|\nabla\gamma(t-s)\|_{L^{4/3}} \left\| F(s) \int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\, d\sigma \right\|_{L^1} ds \\ &\le C\, t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|F(s)\|_{L^{4/3}} \int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\|_{L^4}\, d\sigma\, ds, \end{aligned} \tag{3.8}$$

where we have used Young's inequality for convolution (3.7) in the second line and Hölder's inequality in the third line. Now we can estimate the integral over $d\sigma$ using again Young's inequality (3.7) with $1/4 + 1 = 1/a + (p+1)/(2p)$, i.e. $1/a = 3/4 - 1/(2p)$, by

$$\begin{aligned} \int_0^s \left\| \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma) \right\|_{L^4} d\sigma &\le \int_0^s \|\nabla\gamma(\tfrac{t-s}{\varepsilon})\|_{L^a} \left\| f_2(\sigma) \right\|_{L^{\frac{2p}{p+1}}} d\sigma \\ &\le C \int_0^s (s-\sigma)^{\frac{3}{4}-\frac{1}{2p}-\frac{3}{2}} \left\| f_2(\sigma) \right\|_{L^{\frac{2p}{p+1}}} d\sigma \\ &\le C\, Z_p^2(s) \int_0^s (s-\sigma)^{-\frac{3}{4}-\frac{1}{2p}} \sigma^{-\frac{1}{2}+\frac{1}{2p}}\, d\sigma \\ &\le C\, Z_p^2(s)\, s^{-\frac{1}{4}}, \end{aligned}$$

since the last integral is bounded thanks to $-\frac{3}{4} - \frac{1}{2p} > -1$ from $p > 2$.

Gathering that last estimate with (3.8), it follows

$$\begin{aligned} t^{\frac{1}{4}} \|I_2(t)\|_{L^{4/3}} &\le C\, Z_p^2(t)\, \delta(t) \int_0^t (t-s)^{-\frac{3}{4}} t^{\frac{1}{4}} s^{-\frac{1}{2}}\, ds \\ &\le C\, Z_p^2(t)\, \delta(t). \end{aligned} \tag{3.9}$$

For the term I_3, we have

$$
\begin{aligned}
t^{\frac14}\|I_3(t)\|_{L^{4/3}} &\le C\,t^{\frac14}\int_0^t \left\|\nabla e^{(t-s)\Delta}\left\{f_1(s)\int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}F(\sigma)\,d\sigma\right\}\right\|_{L^{4/3}} ds \\
&\le C\,t^{\frac14}\int_0^t \|\nabla\gamma(t-s)\|_{L^{4/3}}\left\|f_1(s)\int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}F(\sigma)\,d\sigma\right\|_{L^1} ds \\
&\le C\,t^{\frac14}\int_0^t (t-s)^{-\frac34}\,\|f_1(s)\|_{L^{\frac{2p}{p+1}}}\int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}F(\sigma)\|_{L^{\frac{2p}{p-1}}}\,d\sigma\,ds.
\end{aligned}
\tag{3.10}
$$

We compute the integral over $d\sigma$ in the following way

$$
\begin{aligned}
\int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}F(\sigma)\|_{L^{\frac{2p}{p-1}}}\,d\sigma &\le \int_0^s \|\nabla\gamma(\tfrac{s-\sigma}{\varepsilon})\|_{L^{\frac{4p}{3p-2}}}\|F(\sigma)\|_{L^{4/3}}\,d\sigma \\
&\le C\,\delta(s)\int_0^s (s-\sigma)^{-\frac34-\frac{1}{2p}}\,\sigma^{-\frac14}\,d\sigma \\
&\le C\,\delta(s)\,s^{-\frac{1}{2p}},
\end{aligned}
$$

since the last integral is bounded because $p>2$. Putting together this estimate with (3.10), we obtain

$$
\begin{aligned}
t^{\frac14}\|I_3(t)\|_{L^{4/3}} &\le C\,Z_p^1(t)\,\delta(t)\int_0^t (t-s)^{-\frac34}\,t^{\frac14}\,s^{-\frac12} \\
&\le C\,Z_p^1(t)\,\delta(t).
\end{aligned}
\tag{3.11}
$$

For the term I_1, we compute

$$
\begin{aligned}
t^{\frac14}\|I_1(t)\|_{L^{4/3}} &\le C\,t^{\frac14}\int_0^t \left\|\nabla e^{(t-s)\Delta}\left\{F(s)e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\right\}\right\|_{L^{4/3}} ds \\
&\le C\,t^{\frac14}\int_0^t \|\nabla\gamma(t-s)\|_{L^{4/3}}\left\|F(s)e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\right\|_{L^1} ds \\
&\le C\,t^{\frac14}\int_0^t (t-s)^{-\frac34}\,\|F(s)\|_{L^{4/3}}\left\|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\right\|_{L^4} ds,
\end{aligned}
\tag{3.12}
$$

where we have used Young's and Hölder's inequalities. Let $K>0$ to be chosen later, we estimate

$$
\|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\|_{L^4} \le \|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\le K\}}\|_{L^4} + \|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\ge K\}}\|_{L^4}. \tag{3.13}
$$

Using Young's inequality, we have

$$
\|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\le K\}}\|_{L^4} \le \|\gamma(\tfrac{s}{\varepsilon})\|_{L^1}\|\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\le K\}}\|_{L^4} \le CK^{\frac12}\,\|\nabla u_0\|_{L^2}^{1/2}.
$$

Using Young's inequality again for the second term in (3.13), we have

$$
\|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\ge K\}}\|_{L^4} \le \|\gamma(\tfrac{s}{\varepsilon})\|_{L^{4/3}}\|\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\ge K\}}\|_{L^2} \le Cs^{-\frac14}\,\varphi(K),
$$

where

$$
\varphi(K) := \|\nabla u_0\,\mathbf{1}_{\{|\nabla u_0|\ge K\}}\|_{L^2} \to 0 \quad \text{as} \quad K\to+\infty,
$$

by the dominated convergence theorem. Putting together that last estimates in (3.13) and choosing $K = s^{-\frac14}$, it follows

$$
\|e^{\frac{s}{\varepsilon}\Delta}\nabla u_0\|_{L^4} \le C\,s^{-\frac14}\,\alpha(s) \quad \text{with} \quad \alpha(s) := s^{\frac18} + \varphi(s^{-\frac14}) \xrightarrow[s\to0]{} 0.
$$

Coming back to (3.12), we obtain

$$
\begin{aligned}
t^{\frac14}\,\|I_1(t)\|_{L^{4/3}} &\le C\,t^{\frac14}\int_0^t (t-s)^{-\frac34}\,\|F(s)\|_{L^{4/3}}\,s^{-\frac14}\,\alpha(s)ds \\
&\le C\left(\sup_{0<s\le t}\alpha(s)\right)\delta(t)\int_0^t t^{\frac14}(t-s)^{-\frac34}\,s^{-\frac12}\,ds \\
&\le C\left(\sup_{0<s\le t}\alpha(s)\right)\delta(t).
\end{aligned}
\tag{3.14}
$$

Gathering (3.9), (3.11) and (3.14) and using Lemma 3.2, we conclude to

$$
\delta(t)\le C\left[\sup_{0<s\le t}\alpha(s)+Z_p^1(t)+Z_p^2(t)\right]\delta(t)\le\frac12\,\delta(t),
$$

for $t\in(0,T)$, $T>0$ small enough. That in turn implies $\delta(t)\equiv 0$ on $[0,T)$. We may then repeat the argument for later times and conclude to the uniqueness of the solution. □

4. Self-similar solutions and linear stability

4.1. **Convergence of the stationary solutions.** First, for a given mass $M\in(0,8\pi)$ and a given parameter $\varepsilon\in(0,1/2)$, we consider the self-similar profile $(G_\varepsilon,V_\varepsilon)$ which is the unique solution of the system of elliptic equations (1.18)-(1.19). We also consider the unique positive solution (G,V) to the system of equations corresponding to the limit case $\varepsilon=0$

$$
\begin{aligned}
&\Delta G-\nabla(G\,\nabla V-\frac12\,x\,G)=0\quad\text{in}\quad\mathbb{R}^2,\qquad\int_{\mathbb{R}^2}G\,dx=M,\\
&\Delta V+G=0\quad\text{in}\quad\mathbb{R}^2.
\end{aligned}
\tag{4.1}
$$

It is worth emphasizing that (G,V) is the unique self-similar profile associated to the parabolic-elliptic Keller-Segel equation, see [12, 18].

Lemma 4.1. *There exists a constant C such that for any $\varepsilon\in(0,1/4]$*

$$
0\le G_\varepsilon(x)\le C\,e^{-|x|^2/4},
\tag{4.2}
$$

$$
\sup_{x\in\mathbb{R}^2}\big(\frac{1}{|x|}+\langle x\rangle\big)\,|\nabla V_\varepsilon(x)|\le C,
\tag{4.3}
$$

and

$$
\sup_{x\in\mathbb{R}^2}|\Delta V_\varepsilon(x)|\le C.
\tag{4.4}
$$

Proof of Lemma 4.1. We split the proof into three steps.

Step 1. The estimate (4.2) has been proved in [4]. More precisely it is a consequence of equations (26) and (49) in [4], and

$$
G(0)=b,\quad 0\le M(\varepsilon,b)\le 4\pi\,\min(2,b).
$$

Here the parametrization of G is made in function of ε and $b=G(0)$ instead of ε and M because this dependence is more tractable. Observe that the above estimate guarantees that the mass is subcritical, i.e. $M(\varepsilon,b)\le 8\pi$.

Step 2. Since V_ε and G_ε are radially symmetric functions, the equation on V_ε writes

$$
V_\varepsilon''+\big(\frac1r+\frac12\varepsilon\,r\big)\,V_\varepsilon'+G_\varepsilon=0\quad\forall\,r>0,
\tag{4.5}
$$

where we abuse notation in writing $V_\varepsilon(r)=V_\varepsilon(x)$, $G_\varepsilon(r)=G_\varepsilon(|x|)$, $r=|x|$. The function V_ε is smooth and the equation is complemented with the boundary conditions $V_\varepsilon'(0)=V_\varepsilon'(\infty)=0$. Defining $w:=(rV_\varepsilon')^2$, we find

$$
\frac{w'}{2}\;=\;-\frac12\varepsilon\,r\,w-G_\varepsilon\,V_\varepsilon'\,r^2\le C\,\sqrt{w}\,\frac{r}{\langle r\rangle^3},\quad C:=\sup_{r>0}G_\varepsilon\,\langle r\rangle^3.
$$

As a consequence

$$\frac{d}{dr}\sqrt{w} \le C\,\frac{r}{\langle r\rangle^3},$$

and then

$$\sqrt{w} \le C\,(1\wedge r)^2,$$

from which the inequality $\sup_x[\langle x\rangle\,|\nabla V_\varepsilon(x)|] \le C$ of (4.3) follows.

Step 3. We rewrite (4.5) as

$$\frac{1}{r}(V_\varepsilon' r)' = w := -G_\varepsilon - \frac{1}{2}\varepsilon r V_\varepsilon' \in L^\infty,$$

which implies

$$|V_\varepsilon'(r) r| = \left|\int_0^r s w(s)\,ds\right| \le C\,r^2.$$

This completes the estimate (4.3). Coming back to (4.5), we also obtain

$$|V_\varepsilon''(r)| \le C,$$

which gives (4.4) and ends the proof. □

Corollary 4.2. *As $\varepsilon \to 0$, there hold*

$$G_\varepsilon \to G \quad in \quad W^{2,p} \quad \forall\, p \in (1,\infty),$$

and

$$\nabla V_\varepsilon \to \nabla V \quad in \quad L^\infty_1, \quad (\Delta V_\varepsilon)_{\varepsilon>0} \text{ uniformly bounded in } L^\infty \text{ and } \Delta V_\varepsilon \to \Delta V \quad a.e..$$

Proof of Corollary 4.2. Coming back to (1.19) and using Lemma 4.1, for any $p \in (1,\infty)$, we have

$$LG_\varepsilon = \nabla G_\varepsilon \cdot \nabla V_\varepsilon + G_\varepsilon \Delta V_\varepsilon \in L^p,$$

where L denotes the operator $LG_\varepsilon := \Delta G_\varepsilon - \nabla\cdot(\frac{1}{2} x G_\varepsilon)$. By elliptic regularity we obtain that G_ε is uniformly bounded (with respect to $\varepsilon \in (0,1/2)$) in $W^{2,p}$. Thanks to previous estimates and Lemma 4.1, there exists $(\bar G, \bar V)$ and a subsequence (still denoted as $(G_\varepsilon, V_\varepsilon)$) such that $G_\varepsilon \to \bar G$, $V_\varepsilon \to \bar V$. We may pass to the limit (in the weak sense) in the system of equations, and we find

$$\bar V'' + \frac{1}{r}\bar V' + \bar G = 0, \quad \bar V'(0) = \bar V'(\infty) = 0.$$

We conclude that $(\bar G, \bar V)$ is a solution to the stationary equation (4.1), so that $(\bar G, \bar V) = (G, V)$. □

4.2. **Splitting structure for the linearized operator.** The evolution equation in self-similar variables writes (see (1.16) and (1.17))

$$\begin{cases} \partial_t g = \Delta g + \nabla(\dfrac{1}{2}\,x\,g - g\,\nabla v), \\ \partial_t v = \dfrac{1}{\varepsilon}(\Delta v + g) + \dfrac{1}{2}\,x\cdot\nabla v, \end{cases} \tag{4.6}$$

and the associated linearized equation around the self-similar profile $(G_\varepsilon, V_\varepsilon)$ is given by

$$\begin{cases} \partial_t f = \Lambda_{1,\varepsilon}(f,u) := \Delta f + \nabla(\dfrac{1}{2}\,x\,f - f\,\nabla V_\varepsilon - G_\varepsilon\,\nabla u), \\ \partial_t u = \Lambda_{2,\varepsilon}(f,u) := \dfrac{1}{\varepsilon}(\Delta u + f) + \dfrac{1}{2}\,x\cdot\nabla u, \end{cases} \tag{4.7}$$

which we also denote $\partial_t(f,u) = \Lambda_\varepsilon(f,u) = \big(\Lambda_{1,\varepsilon}(f,u), \Lambda_{2,\varepsilon}(f,u)\big)$. From now on, we restrict ourselves to a radially symmetric setting.

We introduce some classical notation of operator theory. Let X be a Banach space and consider a linear operator $\Lambda : X \to X$. We denote by $S_\Lambda(t) = e^{t\Lambda}$ the semigroup of operators generated by Λ, by $\Sigma(\Lambda)$ its spectrum and by $\Sigma_d(\Lambda)$ its discrete spectrum. Moreover, for two Banach spaces X, Y we denote by $\mathscr{B}(X,Y)$ the space of bounded linear operators from X to Y and by $\|\cdot\|_{\mathscr{B}(X,Y)}$

its norm, with the usual shorthand $\mathscr{B}(X) = \mathscr{B}(X, X)$. We also define the subset $\mathcal{C}_a \subset \mathbb{C}$ for any $a \in \mathbb{R}$ by

$$\mathcal{C}_a := \{z \in \mathbb{C} \ | \ \Re e z > a\}. \tag{4.8}$$

Let us denote by L^2_{rad} the L^2 space of radially symmetric functions and by $L^2_{k,j}$, $j < k$, the following space

$$L^2_{k,j} := \left\{ g \in L^2_k \ | \ \int x^\alpha g = 0, \ \forall\, \alpha \in \mathbb{N}^2, \ |\alpha| \le j \right\}.$$

We fix $k > 7$ and we introduce the Hilbert space

$$X := X_1 \times X_2, \quad X_1 := L^2_{rad} \cap L^2_{k,0} \subset L^2_{k,1}, \quad X_2 = L^2_{rad}, \tag{4.9}$$

associated to the norm

$$\|(f,u)\|^2_X := \|f\|^2_{L^2_k} + \|u\|^2_{L^2}. \tag{4.10}$$

We now state a property of the spectrum of Λ_ε in X that is the main result of this subsection.

Proposition 4.3. *Fix some $a^* > -1/2$. There exist $\varepsilon^*, r^* > 0$ such that in X*

$$\forall\, \varepsilon \in (0, \varepsilon^*) \qquad \Sigma(\Lambda_\varepsilon) \cap \mathcal{C}_{a^*} \subset \Sigma_d(\Lambda_\varepsilon) \cap B(0, r^*).$$

We define the bounded operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : X \to X$ by

$$\mathcal{A}_1(f,u) := N\chi_R[f] := N(\chi_R f - \chi_1 \langle \chi_R f \rangle), \quad \mathcal{A}_2(f,u) := 0, \tag{4.11}$$

for some constants $N, R > 0$ to be chosen later and a smooth non-negative radially symmetric cut-off function $\chi_R(x) := \chi(x/R)$ with $\chi \equiv 1$ on $B_{1/2}$, $\mathrm{Supp}\, \chi \subset B_2$ and $\langle \chi_1 \rangle = 1$. We can split the operator $\Lambda_\varepsilon = \mathcal{A} + \mathcal{B}_\varepsilon$ and we shall investigate some properties of $\mathcal{A}$ and $\mathcal{B}_\varepsilon$ in the next lemmas before proving Proposition 4.3.

Lemma 4.4. *In the above splitting, we may choose N_* and R_* large enough in such a way that for any $N \ge N_*$, $R \ge R_*$, the operator $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in X for any $a \in (-1/2, 0)$, in the sense that*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X)} \le C_a\, e^{at}, \quad \forall\, t \ge 0,$$

for some constant $C_a > 0$.

Proof of Lemma 4.4. First of all, thanks to Lemma B.1 and using the notation of Appendix B, we see that in X the norm of $L^2_k \times L^2$ is equivalent to the norm defined by

$$\|(f,u)\|^2_{X_*} := \|f\|^2_{L^2_k} + \eta\, \|u - \kappa_f\|^2_{L^2}, \tag{4.12}$$

for any fixed $\eta > 0$. We also observe that, thanks to Lemma B.2, we have

$$\|\nabla \kappa_f\|_{L^2} = \|\mathcal{K} * f\|_{L^2} \le C \|f\|_{L^2_\ell}, \quad \forall\, \ell > 2,$$

and

$$\|\nabla \kappa_f\|_{L^2_1} = \|\mathcal{K} * f\|_{L^2_1} \le C \|f\|_{L^2_\ell}, \quad \forall\, \ell > 3,$$

thus we fix some $\ell \in (3, k)$ from now on.

We consider the equation

$$\partial_t (f,u) = \mathcal{B}_\varepsilon(f,u) = \Lambda_\varepsilon(f,u) - \mathcal{A}(f,u) \tag{4.13}$$

and split the proof into three steps.

Step 1. We write the equation satisfied by f as

$$\partial_t f = \Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V_\varepsilon - G_\varepsilon\, \nabla u) - N\chi_R[f].$$

Using that $\langle f\rangle = 0$ and the notation $\chi_R^c = 1 - \chi_R$, we compute

$$\begin{aligned}\frac12\frac{d}{dt}\int f^2\langle x\rangle^{2k} &= \int \Delta f\, f\langle x\rangle^{2k} + \frac12\int \nabla\cdot(xf)\, f\langle x\rangle^{2k} - \int \nabla\cdot(f\nabla V_\varepsilon)\, f\langle x\rangle^{2k}\\ &\quad - \int \nabla\cdot(G_\varepsilon\nabla u)\, f\langle x\rangle^{2k} - \int N\chi_R[f] f\langle x\rangle^{2k}\\ &= -\int |\nabla f|^2\langle x\rangle^{2k} + \int \{\varphi(x) - N\chi_R(x)\}\, f^2\langle x\rangle^{2k}\\ &\quad - \int \nabla\cdot(G_\varepsilon\nabla u)\, f\langle x\rangle^{2k} - N\int f\,\chi_1\,\langle x\rangle^{2k}dx\,\langle\chi_R^c f\rangle,\end{aligned} \tag{4.14}$$

using that $\langle f\rangle = 0$ in order to replace χ_R with χ_R^c in last line, and where

$$\begin{aligned}\varphi(x) &= \left(\frac12\Delta\langle x\rangle^{2k} - \frac12 x\cdot\nabla\langle x\rangle^{2k} + \frac14\nabla\cdot(x\langle x\rangle^{2k}) - \frac12\nabla\cdot(\nabla V_\varepsilon\langle x\rangle^{2k}) + \nabla V_\varepsilon\cdot\nabla\langle x\rangle^{2k}\right)\langle x\rangle^{-2k}\\ &= -\frac12(k-1) + k(2k+1/2)\langle x\rangle^{-2} - k(2k-2)\langle x\rangle^{-4} - \frac12\Delta V_\varepsilon + k(\nabla V_\varepsilon\cdot x)\langle x\rangle^{-2}.\end{aligned}$$

We observe that, thanks to Lemma 4.1, we have $(\nabla V_\varepsilon\cdot x)\langle x\rangle^{-2}\to 0$ as $|x|\to\infty$. From (1.19), we also have that

$$-\frac12\Delta V_\varepsilon = \frac12 G_\varepsilon + \frac{\varepsilon}{4}\, x\cdot\nabla V_\varepsilon$$

with $G_\varepsilon\to 0$ as $|x|\to\infty$ from (4.2) and $|x\cdot\nabla V_\varepsilon|\le C_{V_\varepsilon}$ from (4.3). All together, it follows

$$\limsup_{|x|\to\infty}\varphi(x) \le -\frac12(k-1-\varepsilon\, C), \tag{4.15}$$

where $C>0$ is the constant exhibited in (4.3).

For the third term in (4.14), for any $\delta>0$, thanks to Hölder's inequality and using that $G_\varepsilon(x)\le C\langle x\rangle^{-\alpha}$ from (4.2), we get

$$\begin{aligned}-\int \nabla\cdot(G_\varepsilon\nabla u)\, f\,\langle x\rangle^{2k} &= \int \nabla f\cdot\nabla u\, G_\varepsilon\,\langle x\rangle^{2k} + \int G_\varepsilon\nabla u\cdot\nabla(\langle x\rangle^{2k})\, f\\ &\le \delta\|\nabla f\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}^2 + C\|f\|_{L^2}^2\\ &\le \delta\|\nabla f\|_{L^2}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla\kappa_f\|_{L^2}^2 + C\|f\|_{L^2}^2\\ &\le \delta\|\nabla f\|_{L^2}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2 + C\|f\|_{L^2}^2,\end{aligned}$$

where we recall that we have fixed some $\ell\in(3,k)$. For the fourth term in (4.14), we have

$$\begin{aligned}-N\int f\,\chi_1\,\langle x\rangle^{2k}dx\,\langle\chi_R^c f\rangle &\le N\,\|\chi_1\|_{L^2_{2k-\ell}}\,\|\chi_R^c\,\langle x\rangle^{-\ell}\|_{L^2}\,\|f\|_{L^2_\ell}^2\\ &\le N\,R^{1-\ell}\,C_\ell\,\|f\|_{L^2_\ell}^2.\end{aligned}$$

We conclude this step by gathering the previous estimates to obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|f\|_{L^2_k}^2 &\le -(1-\delta)\|\nabla f\|_{L^2_k}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2\\ &\quad + \int\left\{\varphi(x) + (C(\delta) + C\,N\,R^{1-\ell})\langle x\rangle^{2(\ell-k)} - N\chi_R(x)\right\} f^2\langle x\rangle^{2k}.\end{aligned} \tag{4.16}$$

Step 2. From the second equation in (4.7), we have

$$\begin{aligned}\frac12\frac{d}{dt}\int(u-\kappa_f)^2 &= \int(u-\kappa_f)\left\{\frac1\varepsilon(\Delta u + f) + \frac12 x\cdot\nabla(u-\kappa_f) + \frac12 x\cdot\nabla\kappa_f - \partial_t\kappa_f\right\}\\ &= -\frac1\varepsilon\int|\nabla(u-\kappa_f)|^2 - \frac12\int(u-\kappa_f)^2 + \int(u-\kappa_f)\left\{\frac12 x\cdot\nabla\kappa_f - \partial_t\kappa_f\right\},\end{aligned}$$

and we shall estimate the last integral. Since $\partial_t \kappa_f = \kappa * \partial_t f$, we may write

$$\int (u-\kappa_f)\left\{\frac{1}{2}x\cdot\nabla\kappa_f - \partial_t\kappa_f\right\} = \frac{1}{2}\int (u-\kappa_f)\big\{x\cdot\nabla\kappa_f\big\} - \int (u-\kappa_f)\kappa * \left\{\Delta f + \frac{1}{2}\nabla(xf) - \nabla(f\nabla V_\varepsilon) - \nabla(G_\varepsilon\nabla u) - N\chi_R[f]\right\},$$

and we estimate each of these terms separately. First, the first and third terms together gives

$$I_1 := \frac{1}{2}\int (u-\kappa_f)\big\{x\cdot\nabla\kappa_f - \kappa * \nabla(xf)\big\} = 0,$$

because

$$\begin{aligned} x\cdot\nabla\kappa_f - \kappa * \nabla(xf) &= x\cdot\mathcal{K} * f - \mathcal{K} * \nabla(xf) \\ &= -\frac{1}{2\pi}\int \frac{(x-y)}{|x-y|^2}\big\{x\,f(y) - y\,f(y)\big\}\,dy = -\frac{1}{2\pi}\int f(y)\,dy = 0. \end{aligned}$$

Next, we have

$$\begin{aligned} I_2 &:= -\int (u-\kappa_f)\kappa * \{\Delta f\} = \int (u-\kappa_f) f \\ &\le \delta\,\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\,\|f\|_{L^2}^2. \end{aligned}$$

Furthermore, since $f\nabla V_\varepsilon \in L^2_{k,0}$, we can apply Lemma B.2 and use that $\nabla V_\varepsilon \in L^\infty$ to obtain

$$\begin{aligned} I_3 &:= \int (u-\kappa_f)\kappa * \nabla(f\nabla V_\varepsilon) = \int (u-\kappa_f)\,\mathcal{K}_i * (f\partial_i V_\varepsilon) \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\nabla V_\varepsilon\|_{L^2_\ell}^2 \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2. \end{aligned}$$

For the next term, using Lemma B.2, since $G_\varepsilon\nabla u \in L^2_{k,0}$, and the bound $G_\varepsilon \in L^\infty_k$, we have

$$\begin{aligned} I_4 &:= \int (u-\kappa_f)\kappa * \nabla(G_\varepsilon\nabla u) = \int (u-\kappa_f)\,\mathcal{K}_i * (G_\varepsilon\,\partial_i u) \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\mathcal{K} * (G_\varepsilon\nabla u)\|_{L^2}^2 \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|G_\varepsilon\nabla u\|_{L^2_k}^2 \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}^2 \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla\kappa_f\|_{L^2}^2 \\ &\le \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2. \end{aligned}$$

For the last term and thanks to Lemma B.1, we finally have

$$\begin{aligned} I_5 &:= N\int (u-\kappa_f)\kappa * \{\chi_R[f]\} \le C\,\|u-\kappa_f\|_{L^2}\,N\,\|\chi_R[f]\|_{L^2_{\ell,1}} \\ &\le \delta\|u-\kappa_f\|_{L^2}^2 + C(\delta)N^2\|f\|_{L^2_\ell}^2. \end{aligned}$$

Putting together all the estimates of this step, we deduce

$$\frac{1}{2}\frac{d}{dt}\|u-\kappa_f\|_{L^2}^2 \le -\Big(\frac{1}{\varepsilon} - C(\delta)\Big)\|\nabla(u-\kappa_f)\|_{L^2}^2 - \Big(\frac{1}{2}-\delta\Big)\|u-\kappa_f\|_{L^2}^2 + C(\delta)N^2\|f\|_{L^2_\ell}^2. \tag{4.17}$$

Step 3. Conclusion. Gathering (4.16) and (4.17), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(f,u)\|_{X_*}^2 \le \int \left\{\varphi(x) + \Big[(1+\eta N^2)C(\delta) + \frac{CN}{R^{\ell-1}}\Big]\langle x\rangle^{2(\ell-k)} - N\chi_R(x)\right\}|f|^2\langle x\rangle^{2k} \\ - \eta\Big(\frac{1}{2}-\delta\Big)\|u-\kappa_f\|_{L^2}^2 - (1-\delta)\|\nabla f\|_{L^2_k}^2 - \eta\Big(\frac{1}{\varepsilon} - C(\delta)\Big)\|\nabla(u-\kappa_f)\|_{L^2}^2. \end{aligned} \tag{4.18}$$

Taking then first $\delta \in (0,1)$ small enough and next $\varepsilon \in (0,1)$ small enough, it follows that for $\eta = N^{-3}$ and $R = N$

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{X_*}^2 \leq \int \{\bar\varphi_N(x) - N\chi_R(x)\}\,|f|^2\langle x\rangle^{2k} + a\|\nabla f\|_{L^2}^2 + a\,\eta\,\|u-\kappa_f\|_{L^2}^2 + a\,\eta\,\|\nabla(u-\kappa_f)\|_{L^2}^2,$$

for any $a > -1/2$, where $\bar\varphi_N(x) = \varphi(x) + C(1 + N^{-1} + N^{2-\ell})\langle x\rangle^{2(\ell-k)}$ has the same asymptotic behaviour as $\varphi(x)$ when $|x| \to \infty$ and $\bar\varphi_N$ decreases as N increases. We can choose N large enough such that

$$\bar\varphi_N(x) - N\chi_R(x) \leq a, \quad \forall\, x \in \mathbb{R}^2,$$

which yields that $\mathcal{B}_\varepsilon$ is a-hypo-dissipative for any $a > -1/2$. □

We introduce the space

$$Y := Y_1 \times Y_2, \quad Y_1 := H^1_k \cap L^2_{k,0} \cap L^2_{rad}, \quad Y_2 := H^1 \cap L^2_{rad}, \tag{4.19}$$

endowed with the norm

$$\|(f,u)\|_Y^2 := \|(f,u)\|_X^2 + \|\nabla f\|_{L^2_k}^2 + \|\nabla u\|_{L^2}^2. \tag{4.20}$$

For the operator $\mathcal{A}$ defined in (4.11), the following result holds true.

Lemma 4.5. *There hold $\mathcal{A} \in \mathscr{B}(X)$ and $\mathcal{A} \in \mathscr{B}(Y)$.*

Proof of Lemma 4.5. The proof is straightforward so we omit it. □

Lemma 4.6. *We can choose N and R large enough such that $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in Y for any $a \in (-1/2, 0)$, i.e.*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(Y)} \leq Ce^{at}, \quad \forall\, t \geq 0.$$

Moreover, we also have

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Y)} \leq C\,t^{-1/2}\,e^{at}, \quad \forall\, t \geq 0.$$

Proof of Lemma 4.6. We introduce the following norm

$$\|(f,u)\|_{Y_*}^2 := \|(f,u)\|_{X_*}^2 + \eta_1\|\nabla f\|_{L^2_k}^2 + \eta_1\|\nabla(u-\kappa_f)\|_{L^2}^2, \tag{4.21}$$

which is equivalent to (4.20) for any $\eta_1 > 0$ thanks to Lemma B.2. We recall that, as in Lemma 4.4, we have

$$\|\nabla u\|_{L^2} \leq \|\nabla(u-\kappa_f)\|_{L^2} + \|\nabla\kappa_f\|_{L^2} \leq \|\nabla(u-\kappa_f)\|_{L^2} + C\|f\|_{L^2_\ell},$$

for the same fixed $\ell \in (3,k)$, and moreover we observe that

$$\|\nabla^2 u\|_{L^2} \leq \|\nabla^2(u-\kappa_f)\|_{L^2} + \|\nabla^2\kappa_f\|_{L^2} \leq \|\nabla^2(u-\kappa_f)\|_{L^2} + \|f\|_{L^2}.$$

We consider now the equation (4.13) and we split the proof of the announced results into three steps.

Step 1. L^2 differential inequality. For $i = 1, 2$, $\partial_i u$ verifies

$$\partial_t(\partial_i u) = \frac{1}{\varepsilon}\Delta(\partial_i u) + \frac{1}{\varepsilon}\partial_i f + \frac{1}{2}x\cdot\nabla(\partial_i u) + \frac{1}{2}\partial_i u.$$

We have then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_i(u-\kappa_f)\|_{L^2}^2 &= \int \partial_i(u-\kappa_f)\partial_t\{\partial_i u - \partial_i\kappa_f\} \\
&= \frac{1}{\varepsilon}\int \partial_i(u-\kappa_f)\Delta\{\partial_i(u-\kappa_f)\} + \frac{1}{2}\int \partial_i(u-\kappa_f)\partial_i u \\
&\quad + \frac{1}{2}\int \partial_i(u-\kappa_f)x\cdot\nabla\{\partial_i(u-\kappa_f)\} + \frac{1}{2}\int \partial_i(u-\kappa_f)x\cdot\nabla\{\partial_i\kappa_f\} \\
&\quad - \int \partial_i(u-\kappa_f)\mathcal{K}_i * (\partial_t f) \\
&=: T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned}$$

For the first term, we easily have

$$T_1 = -\frac{1}{\varepsilon}\|\nabla\{\partial_i(u-\kappa_f)\}\|_{L^2}^2.$$

For the second term, we have

$$T_2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2_\ell}^2.$$

We also easily see that

$$T_3 = -\frac{1}{2}\|\partial_i(u-\kappa_f)\|_{L^2}^2 \le 0,$$

and, for the fourth term, that

$$T_4 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|D^2\kappa_f\|_{L^2_1}^2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2_1}^2.$$

For the last term T_5, we use the equation satisfied by f and we get

$$\begin{aligned} T_5 &= -\int \partial_i(u-\kappa_f)\mathcal{K}_i * \left\{\Delta f + \frac{1}{2}\nabla(xf) - \nabla(f\nabla V_\varepsilon) - \nabla(G_\varepsilon\nabla u) - N\chi_R[f]\right\} \\ &=: T_{51} + T_{52} + T_{53} + T_{54} + T_{55}. \end{aligned}$$

We estimate each term separately. We have

$$T_{51} \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla f\|_{L^2}^2$$

and

$$T_{52} \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * (xf)\|_{L^2}^2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2_1}^2.$$

Using that $\nabla V_\varepsilon \in L^\infty$, we have

$$\begin{aligned} T_{53} &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * (f\nabla V_\varepsilon)\|_{L^2}^2 \\ &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\nabla V_\varepsilon\|_{L^2}^2 \\ &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2}^2, \end{aligned}$$

and arguing as above with $G_\varepsilon \in L^\infty$, we also obtain

$$\begin{aligned} T_{54} &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * (G_\varepsilon\nabla u)\|_{L^2}^2 \\ &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|G_\varepsilon\nabla u\|_{L^2}^2 \\ &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\ &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2_\ell}^2. \end{aligned}$$

For the last term, using Lemma B.2, we have

$$\begin{aligned} T_{55} &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + CN^2\|\mathcal{K} * \chi_R[f]\|_{L^2}^2 \\ &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + CN^2\|f\|_{L^2_\ell}^2. \end{aligned}$$

Gathering these previous estimates, we finally obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla(u-\kappa_f)\|_{L^2}^2 \le -\frac{1}{\varepsilon}\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C(1+N^2)\|f\|_{L^2_\ell}^2 \\ + C\|\nabla f\|_{L^2}^2 + C\|\nabla(u-\kappa_f)\|_{L^2}^2. \end{aligned} \tag{4.22}$$

Step 2. H^1 differential inequality. We write the equation satisfied by $\partial_i f$ which is nothing but

$$\partial_t(\partial_i f) = \mathcal{B}_{\varepsilon,1}(\partial_i f, \partial_i u) - \frac{1}{2}\partial_i f - \nabla(f\nabla(\partial_i V_\varepsilon)) - \nabla(\partial_j G_\varepsilon\nabla u) - N(\partial_i\chi_R)f + N\langle\chi_R f\rangle\partial_i\chi_1,$$

and then we can write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_i f\|_{L^2_k}^2 &= \int \mathcal{B}_{\varepsilon,1}(\partial_i f, \partial_i u)\,\partial_i f\,\langle x\rangle^{2k} - \frac{1}{2}\|\partial_i f\|_{L^2_k}^2 - \int \nabla(f\nabla(\partial_i V_\varepsilon))\,\partial_i f\,\langle x\rangle^{2k} \\ &\quad - \int \nabla(\partial_j G_\varepsilon\nabla u)\,\partial_i f\,\langle x\rangle^{2k} - N\int (\partial_i\chi_R)f\partial_i f\langle x\rangle^{2k} + N\langle\chi_R f\rangle\int (\partial_i\chi_1)\,\partial_i f\,\langle x\rangle^{2k} \\ &=: A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \end{aligned}$$

Arguing as in the first step of Lemma 4.4, for any $\delta > 0$, we have

$$\begin{aligned} A_1 &\le -(1-\delta)\|\nabla(\partial_i f)\|_{L^2_k}^2 + C(\delta)\|\nabla^2(u-\kappa_f)\|_{L^2}^2 \\ &\quad + \int \Big\{\varphi(x) + [C(\delta) + C\,N\,R^{1-\ell}]\langle x\rangle^{2(\ell-k)} - N\chi_R(x)\Big\}\,|\partial_i f|^2\langle x\rangle^{2k}. \end{aligned}$$

We next compute

$$\begin{aligned} A_3 &:= \int f\nabla(\partial_i V_\varepsilon)\cdot\nabla(\partial_i f)\langle x\rangle^{2k} + \int f\nabla(\partial_i V_\varepsilon)\cdot\nabla\langle x\rangle^{2k}\,\partial_i f \\ &\le \varepsilon C(\delta)\|f\|_{L^2_k}^2 + \delta\|\nabla f\|_{L^2_k}^2 + \delta\|\nabla^2 f\|_{L^2_k}^2, \end{aligned}$$

using that $\Delta V_\varepsilon = -G_\varepsilon - (\varepsilon/2)x\cdot\nabla V_\varepsilon$ and Lemma 4.1. We also have

$$\begin{aligned} A_4 &= \int \partial_i G_\varepsilon \nabla u\cdot\nabla(\partial_i f)\langle x\rangle^{2k} + \int \partial_i G_\varepsilon\nabla u\cdot\nabla\langle x\rangle^{2k}\,\partial_j f \\ &\le C(\delta)\|\nabla u\|_{L^2}^2 + \delta\|\nabla f\|_{L^2_{k-1/2}}^2 + \delta\|\nabla^2 f\|_{L^2_k}^2 \\ &\le C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2 + \delta\|\nabla f\|_{L^2_{k-1/2}}^2 + \delta\|\nabla^2 f\|_{L^2_k}^2, \end{aligned}$$

and we easily get

$$A_5 \le N\frac{C}{R}\int \mathbf{1}_{R/2\le|x|\le 2R}\,f^2\langle x\rangle^{2k} + N\frac{C}{R}\int \mathbf{1}_{R/2\le|x|\le 2R}\,|\partial_i f|^2\langle x\rangle^{2k}.$$

For the last term, we have

$$\begin{aligned} A_6 &\le N\langle\chi_R f\rangle\int(\partial_i\chi_1)\,\partial_i f\,\langle x\rangle^{2k} \\ &\le CN\,\|f\|_{L^2}\,\|\partial_i f\|_{L^2_k} \le C(\delta)N^2\,\|f\|_{L^2}^2 + \delta\|\partial_i f\|_{L^2_k}^2. \end{aligned}$$

Finally, putting together all the above estimates, we obtain

$$\begin{aligned} \frac12\frac{d}{dt}\|\nabla f\|_{L^2_k}^2 &\le -(1-\delta)\|\nabla^2 f\|_{L^2_k}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla^2(u-\kappa_f)\|_{L^2}^2 \\ &\quad + \int\psi^0(x)\,|f|^2\langle x\rangle^{2k} + \int\psi^1(x)\,|\nabla f|^2\langle x\rangle^{2k}, \end{aligned} \tag{4.23}$$

where

$$\psi^0(x) := \varepsilon C(\delta) + N\frac{C}{R}\mathbf{1}_{R/2\le|x|\le 2R} + C(\delta)\langle x\rangle^{2(\ell-k)} + C(\delta)N^2\langle x\rangle^{-2k}$$

and

$$\psi^1(x) := \varphi(x) - \frac12 + \delta + N\frac{C}{R}\mathbf{1}_{R/2\le|x|\le 2R} + [C(\delta) + CNR^{1-\ell}]\langle x\rangle^{2(\ell-k)} - N\chi_R.$$

Step 3. Conclusion. We gather estimates (4.22), (4.23) and (4.18), and we get

$$\begin{aligned} \frac12\frac{d}{dt}\|(f,u)\|_{Y_*}^2 &\le \int\Big\{\varphi(x) + \eta_1\psi^0(x) - N\chi_R \\ &\quad + [\eta_1 C(\delta) + \eta_1 C(1+N^2) + C(\delta)(1+\eta N^2) + C\frac{N}{R^{\ell-1}}]\langle x\rangle^{2(\ell-k)}\Big\}f^2\langle x\rangle^{2k} \\ &\quad + \eta_1\int\psi^1(x)\,|\nabla f|^2\langle x\rangle^{2k} - (1-\delta)\|\nabla f\|_{L^2_k}^2 \\ &\quad - \eta\left(\frac12 - \delta\right)\|u-\kappa_f\|_{L^2}^2 - \eta_1(1-\delta)\|\nabla^2 f\|_{L^2_k}^2 \\ &\quad - \eta\left(\frac1\varepsilon - C(\delta) - \frac{\eta_1}{\eta}C(\delta)\right)\|\nabla(u-\kappa_f)\|_{L^2}^2 - \eta_1\left(\frac1\varepsilon - C(\delta)\right)\|\nabla^2(u-\kappa_f)\|_{L^2}^2. \end{aligned} \tag{4.24}$$

Now we conclude as in Step 3 of the proof of Lemma 4.4. We choose first $\delta \in (0,1)$ small enough, next $\varepsilon \in (0,1)$ small enough and then $\eta_1 = \eta = N^{-3}$ and $R = N$ in the above inequality. For any $a > -1/2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2 \le & \int \{\varphi_N^1(x) - N\chi_R\} f^2 \langle x\rangle^{2k} + \eta_1 \int \left\{\varphi_N^2(x) - N\chi_R\right\} |\nabla f|^2 \langle x\rangle^{2k} \\ & + a\|\nabla f\|_{L^2_k}^2 + a\eta\|u - \kappa_f\|_{L^2}^2 + a\eta_1\|\nabla^2 f\|_{L^2_k}^2 \\ & + a\eta_1\|\nabla(u-\kappa_f)\|_{L^2}^2 + a\eta_1\|\nabla^2(u-\kappa_f)\|_{L^2}^2,\end{aligned} \tag{4.25}$$

where

$$\varphi_N^1(x) := \varphi(x) + CN^{-3} + CN^{-1}\mathbf{1}_{R/2\le|x|\le 2R} + (C + CN^{-1} + CN^{-3} + CN^{2-\ell})\langle x\rangle^{2(\ell-k)} + CN^{-1}\langle x\rangle^{-2k}$$

and

$$\varphi_N^2(x) := \varphi(x) + C\mathbf{1}_{R/2\le|x|\le 2R} + (C + CN^{2-\ell})\langle x\rangle^{2(\ell-k)} + C\langle x\rangle^{-2k}$$

have the same asymptotic behaviour as $\varphi(x)$ when $|x| \to \infty$ and are decreasing as a function of N. Picking N large enough such that

$$\varphi_N^i(x) - N\chi_R(x) \le a, \quad \forall\, x \in \mathbb{R}^2,$$

we deduce from (4.25) that, for some constant $K > 0$,

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2 \le a\|(f,u)\|_{Y_*}^2 - K\big(\|\nabla^2 f\|_{L^2_k}^2 + \|\nabla^2(u-\kappa_f)\|_{L^2}^2\big). \tag{4.26}$$

We first conclude that $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in Y. Moreover, using the interpolation inequality

$$\|g\|_{H^1_k}^2 \le C\|g\|_{H^2_k}\|g\|_{L^2_k},$$

it follows from (4.26) that

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2 \le a\|(f,u)\|_{Y_*}^2 - K\|(f,u)\|_{Y_*}^4\,\|(f,u)\|_{X_*}^{-2}.$$

By standard arguments, we get the estimate

$$\|S_{\mathcal{B}_\varepsilon}(t)(f,u)\|_Y \le C\,t^{-1/2}\,e^{at}\|(f,u)\|_X,$$

concluding the proof. □

Proof of Proposition 4.3. The domain $\mathrm{D}(\Lambda_\varepsilon)$ of the operator $\Lambda_\varepsilon : \mathrm{D}(\Lambda_\varepsilon) \subset X \to X$ is given by

$$\mathrm{D}(\Lambda_\varepsilon) = H^2_k \cap L^2_{k,0} \cap L^2_{rad} \times H^2 \cap L^2_{rad}$$

and we recall that $X = L^2_{k,0} \cap L^2_{rad} \times L^2_{rad}$ and $Y = H^1_k \cap L^2_{k,0} \cap L^2_{rad} \times H^1 \cap L^2_{rad}$ (see equations (4.9) and (4.19)). We define a family of interpolation spaces

$$X^\eta := H^{2\eta}_k \cap L^2_{k,0} \cap L^2_{rad} \times H^{2\eta} \cap L^2_{rad}, \quad \eta \in [0,1],$$

so that $X^0 = X$, $X^1 = \mathrm{D}(\Lambda_\varepsilon)$ and $X^{1/2} = Y$. Thanks to classical interpolation results we have $Y = X^{1/2} \subset \mathrm{D}(\Lambda_\varepsilon^\eta)$ for any $\eta \in [0,1/2)$, see [36, 38, 41]. Now we fix some $\eta \in (0,1/2)$ and we have $Y \subset \mathrm{D}(\Lambda_\varepsilon^\eta) \subset X$.

Recalling the results from Lemma 4.4, Lemma 4.5, Lemma 4.6 and (4.11) we have, for any $a > -1/2$,

$$\begin{aligned} S_{\mathcal{B}_\varepsilon}(t) : X \to X, &\quad \text{with} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X)} \le C\,e^{at}, \\ S_{\mathcal{B}_\varepsilon}(t) : Y \to Y, &\quad \text{with} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(Y)} \le C\,e^{at}, \\ S_{\mathcal{B}_\varepsilon}(t) : X \to Y, &\quad \text{with} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Y)} \le C\,t^{-1/2}\,e^{at}, \end{aligned}$$

moreover $\mathcal{A} \in \mathscr{B}(X) \cap \mathscr{B}(Y)$ and

$$\mathcal{A}S_{\mathcal{B}_\varepsilon}(t) : X \to Y, \quad \text{with} \quad \|\mathcal{A}S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Y)} \le C\,t^{-1/2}\,e^{at}.$$

First, from the previous estimates, we immediately obtain, for any $a > -1/2$,

$$\forall\, \ell \ge 0, \quad \|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}S_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{\mathscr{B}(X)} \le C\,e^{at}. \tag{4.27}$$

Moreover, from [27, Lemma 2.17] there exists $n \in \mathbb{N}$ such that

$$\|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{*(n)}(t)\|_{\mathscr{B}(X,Y)} \leq C\, e^{at},$$

which together with the fact $S_{\mathcal{B}_\varepsilon}(t) : \mathrm{D}(\Lambda_\varepsilon^\eta) \to \mathrm{D}(\Lambda_\varepsilon^\eta)$ with $\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(\mathrm{D}(\Lambda_\varepsilon^\eta))} \leq C\, e^{at}$ (by interpolation of the same results in X and Y), yield

$$\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{\mathscr{B}(X,\mathrm{D}(\Lambda_\varepsilon^\eta))} \leq C\, e^{at}. \tag{4.28}$$

Recall that we have fixed some $a^* > -1/2$. Gathering that last estimate with (4.27), we can apply [41, Theorem 2.1] which yields, for some $r^* > 0$,

$$\Sigma(\Lambda_\varepsilon) \cap \mathcal{C}_{a^*} \subset B(0, r^*) \quad \text{on} \quad X.$$

From the previous estimates together with the fact that $\mathcal{A} \in \mathscr{B}(X, L^2_{k+1} \times L^2_{k+1})$, we also obtain

$$\int_0^\infty \|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{*(n+1)}(t)\|_{\mathscr{B}(X,\overline{Y})}\, e^{-at}\, dt \leq C, \tag{4.29}$$

where $\overline{Y} := Y \cap (L^2_{k+1} \times L^2_{k+1}) \subset X$ with compact embedding. Hence, thanks to (4.27)-(4.28)-(4.29), we are able to apply [41, Theorem 3.1] that implies

$$\Sigma(\Lambda_\varepsilon) \cap \mathcal{C}_{a^*} \subset \Sigma_d(\Lambda_\varepsilon) \quad \text{on} \quad X,$$

and that concludes the proof. □

4.3. **Localization of the spectrum for the linearized operator in a radially symmetric setting.** We recall that we consider a radially symmetric setting and we have already defined the space X in (4.9). We establish in this subsection the following localization of the spectrum of Λ_ε.

Theorem 4.7. *Fix some $a^* > -1/2$. There exists $\varepsilon^* > 0$ such that in X there holds*

$$\Sigma(\Lambda_\varepsilon) \cap \mathcal{C}_{a^*} = \emptyset \quad \textit{for any} \quad \varepsilon \in (0, \varepsilon^*).$$

As a consequence, for any $a > a^*$, there exists $C_a \geq 1$ such that

$$\|S_{\Lambda_\varepsilon}(t)\|_{\mathscr{B}(X)} \leq C_a\, e^{at} \quad \forall\, t \geq 0,\ \forall\, \varepsilon \in (0, \varepsilon^*).$$

The difficulty is that Λ_ε is not a perturbation of some fixed operator Λ and we cannot apply directly the perturbation theory developed in [40, 54]. However, we are able to identify the limit of $\mathcal{R}_{\Lambda_\varepsilon}$ as $\varepsilon \to 0$ which is enough to conclude.

We introduce the notations

$$Af := \Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V_\varepsilon), \quad Bu := -\nabla(G_\varepsilon\, \nabla u)$$
$$Cu := \Delta u, \quad Du := \frac{1}{2}\, x \cdot \nabla u,$$

so that the linearized equation writes

$$\partial_t f = Af + Bu, \qquad \partial_t u = \frac{1}{\varepsilon}(Cu + f) + Du. \tag{4.30}$$

The important point is that at a very formal level, the limit system (as $\varepsilon \to 0$) is the linearized parabolic-elliptic system

$$\partial_t f = A_0 f + B_0 u, \qquad Cu = -f, \tag{4.31}$$

where

$$A_0 f := \Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V), \quad B_0 u := -\nabla(G\, \nabla u),$$

with G and V defined in (4.1), which simplifies into a single equation

$$\partial_t f = (A_0 + B_0(-C)^{-1})f =: \Omega f. \tag{4.32}$$

Observe that the last equation is nothing but the linearized equation associated to parabolic-elliptic Keller-Segel equation which has been studied in [11, 12, 18] and for which it has been proved therein that the associated semigroup is exponentially stable in several weighted Lebesgue spaces.

In the sequel we explain why the linearized parabolic-parabolic system inherits that exponential stability at least for $\varepsilon > 0$ small enough.

We recall the following result which is an immediate consequence of [12, Section 6.1] and [18, Theorem 4.3].

Theorem 4.8. *There exists a constant C such that*

$$\forall\, t \geq 0,\ \forall\, h \in L^2_{k,0}, \quad \|e^{\Omega t} h\|_{L^2_k} \leq C\, e^{-t/2}\, \|h\|_{L^2_k}.$$

As a consequence, there holds

$$\mathcal{R}_\Omega \in \mathcal{H}(\mathcal{C}_{-1/2}; \mathscr{B}(X_1)) \quad \textit{and then} \quad \Sigma(\Omega) \cap \mathcal{C}_{-1/2} = \emptyset \textit{ in } X_1,$$

where $X_1 = L^2_{rad} \cap L^2_{k,0}$ is defined in (4.9).

Remark. The optimal decay rate -1 stated in [12, 18] becomes $-1/2$ here, because of the different normalization choice in the definition of the rescaled functions g and v in (1.16) and (1.17).

In order to formalize the link between the linearized parabolic-parabolic equation and the linearized parabolic-elliptic equation, we write the linearized parabolic-parabolic system (4.30) into the matrix form

$$\frac{d}{dt} \begin{pmatrix} f \\ u \end{pmatrix} = \Lambda_\varepsilon \begin{pmatrix} f \\ u \end{pmatrix}, \quad \Lambda_\varepsilon := \begin{pmatrix} A & B \\ \varepsilon^{-1} I & \varepsilon^{-1} C + D \end{pmatrix}.$$

For the analysis of the spectrum of Λ_ε, for any $z \in \mathbb{C}$, we have

$$\Lambda_\varepsilon - z = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with

$$a := A(z) = A - z, \quad b := B, \quad c := \varepsilon^{-1} I, \quad d := \varepsilon^{-1} C + D(z), \quad D(z) = D - z.$$

One can readily verify that for $z \in \mathbb{C}$ such that $D - z$ and its Schur's complement

$$s_\varepsilon = s_\varepsilon(z) := a - b d^{-1} c = A(z) - B(C + \varepsilon D(z))^{-1}$$

are invertible, the resolvent of Λ_ε is given by

$$\mathcal{R}_{\Lambda_\varepsilon}(z) = (\Lambda_\varepsilon - z)^{-1} = \begin{pmatrix} s_\varepsilon^{-1} & -s_\varepsilon^{-1} b d^{-1} \\ -d^{-1} c s_\varepsilon^{-1} & d^{-1} + d^{-1} c s_\varepsilon^{-1} b d^{-1} \end{pmatrix} =: \begin{pmatrix} \mathcal{R}^{\Lambda_\varepsilon}_{11}(z) & \mathcal{R}^{\Lambda_\varepsilon}_{12}(z) \\ \mathcal{R}^{\Lambda_\varepsilon}_{21}(z) & \mathcal{R}^{\Lambda_\varepsilon}_{22}(z) \end{pmatrix}.$$

Then, at least formally, we see that

$$\mathcal{R}_{\Lambda_\varepsilon}(z) \underset{\varepsilon \to 0}{\longrightarrow} \begin{pmatrix} \mathcal{R}_\Omega(z) & 0 \\ -C^{-1} \mathcal{R}_\Omega(z) & 0 \end{pmatrix}. \tag{4.33}$$

Indeed, on the one hand, we have

$$\begin{aligned} \mathcal{R}^{\Lambda_\varepsilon}_{11} = s_\varepsilon^{-1} &= \{A(z) - B(\varepsilon^{-1} C + D(z))^{-1} \varepsilon^{-1} I\}^{-1} \\ &= \{A - z - B(C + \varepsilon D - \varepsilon z)^{-1}\}^{-1} \underset{\varepsilon \to 0}{\longrightarrow} (A_0 - B_0 C^{-1} - z)^{-1} = \mathcal{R}_\Omega(z). \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}^{\Lambda_\varepsilon}_{21}(z) = -d^{-1} c s_\varepsilon^{-1} &= -(\varepsilon^{-1} C + D(z))^{-1} \varepsilon^{-1} I \{A(z) - B(C + \varepsilon D(z))^{-1}\}^{-1} \\ &= -(C + \varepsilon D - \varepsilon z)^{-1} \{A - z - B(C + \varepsilon(D - z))^{-1}\}^{-1} \\ &\underset{\varepsilon \to 0}{\longrightarrow} -C^{-1} (A_0 - B_0 C^{-1} - z)^{-1} = -C^{-1} \mathcal{R}_\Omega(z). \end{aligned}$$

In the same way, we have

$$\mathcal{R}^{\Lambda_\varepsilon}_{12}(z) = -s_\varepsilon^{-1} b d^{-1} = -\varepsilon \{A - z - B(C + \varepsilon D - \varepsilon z)^{-1}\}^{-1} B (C + \varepsilon D - \varepsilon z)^{-1}$$

as well as

$$\mathcal{R}^{\Lambda_\varepsilon}_{22}(z) = d^{-1} + d^{-1} c s_\varepsilon^{-1} b d^{-1} \quad = \varepsilon (C + \varepsilon D(z))^{-1} + \varepsilon (C + \varepsilon D(z))^{-1} \{A(z) - B(C + \varepsilon D(z))^{-1}\}^{-1} B (C + \varepsilon D(z))^{-1},$$

and then both last terms vanish in the limit $\varepsilon \to 0$.

In fact, we will not try to prove that convergence (4.33) rigorously holds, but we will just prove the following result. We define, for a given $\rho > 0$ and some fixed $a^* > -1/2$,

$$\mathcal{O}_\rho := \mathcal{C}_{a^*} \cap B(0,\rho).$$

Proposition 4.9. *For any $\rho > 0$, there exists $\varepsilon_\rho^* > 0$ such that in X there holds*

$$\mathcal{R}_{\Lambda_\varepsilon} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X)) \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_\rho^*).$$

Before proving Proposition 4.9, we establish some estimates on the terms involved in $\mathcal{R}_{\Lambda_\varepsilon}$.

Lemma 4.10. *Define*

$$\tilde{s}(z) := B(C + \varepsilon D(z))^{-1}(-D(z))\, C^{-1}.$$

For any $\rho > 0$, there exists $\varepsilon_\rho^* > 0$ such that

$$\sup_{z\in\mathcal{O}_\rho} \sup_{\varepsilon\in(0,\varepsilon_\rho^*)} \|\tilde{s}(z)\|_{\mathscr{B}(X_1)} \le C.$$

Proof of Lemma 4.10. On the one hand, from Lemma B.1 and Lemma B.2 we have

$$-D(z)\, C^{-1} : L^2_{k,0} \cap L^2_{rad} \to L^2_{rad}$$

is bounded uniformly for $z \in \mathcal{O}_\rho$. More precisely, for $f \in L^2_{k,0}$ we write

$$f = f_0 + f_1 + f_2; \quad f_i = \lambda_i F_i,\ i = 1,2,$$

where we define the coefficients $\lambda_i \in \mathbb{R}$ by

$$\lambda_1 = \int_0^\infty f(r)\, r^2\, r\, dr, \quad \lambda_2 = \int_0^\infty f(r)\, r^4\, r\, dr,$$

and the functions F_i by

$$F_1(r) = \left(-\frac{1}{8}r^4 + \frac{5}{4}r^2 - \frac{3}{2}\right) e^{-r^2/2}, \quad F_2(r) = \left(\frac{1}{64}r^4 - \frac{1}{8}r^2 + \frac{1}{8}\right) e^{-r^2/2},$$

so that it holds

$$\int_0^\infty F_1(r)\,(1, r^2, r^4)\, r\, dr = (0,1,0), \quad \int_0^\infty F_2(r)\,(1, r^2, r^4)\, r\, dr = (0,0,1),$$

and hence $f_0 \in L^2_{k,5}$.

We may then solve the equation

$$Cu = \Delta u = f, \quad u \text{ radially symmetric}, \quad u'(0) = u'(\infty) = 0,$$

by writing

$$u = u_0 + u_1 + u_2, \quad \Delta u_i = f_i, \quad u_i'(0) = u_i'(\infty) = 0,$$

where u_i, $i = 1,2$, is defined by the relation

$$r u_i'(r) := \int_0^r \sigma f_i(\sigma)\, d\sigma,$$

so that $\langle r\rangle |u_i'| \le C$, $|u| \le C \log(1 + \langle r\rangle)$, and $u_0 \in L^2_{rad} \cap L^2_4$, $\nabla u_0 \in L^2_5$ is the unique solution to the above Poisson equation as given by Lemma B.1 and Lemma B.2. As a consequence, $g_0 := D(z)u_0 \in L^2_{4,1}$ and $g_i := r u_i' + z u_i$ satisfy the estimates $g_i\, e^{-r/2} \in L^\infty$ for $i = 1,2$. Thanks to Lemmas C.1 and C.2 and using the notation of Appendix C, we have $v_i := L_\varepsilon^{-1} g_i$ which satisfy $\|v_0\|_{\dot{H}^1\cap\dot{H}^2} \le C\, \|g_0\|_{L^2_{4,1}}$, while for $i = 1,2$, v_i satisfy the estimates

$$\|v_i\, e^{-(1+\varepsilon|z|)r}\|_{L^\infty} + \|v_i'\, e^{-(1+\varepsilon|z|)r}\|_{L^\infty} \le C \|g_i\, e^{-r/2}\|_{L^\infty}.$$

Finally, we solve the equation $w_i := B v_i$, which means

$$w_i = G_\varepsilon \Delta v_i + \nabla G_\varepsilon \cdot \nabla v_i = G_\varepsilon (v_i'' + \frac{1}{r}\, v_i') + r\, G_\varepsilon' v_i',$$

and the previous estimates together with the bound (4.2) yield $w = w_0 + w_1 + w_2 \in L^2_{k,0} \cap L^2_{rad} = X_1$. □

Lemma 4.11. *With the above notation, for any $\rho > 0$, there exists C_ρ such that*

$$\sup_{z\in\mathcal{O}_\rho} \|d^{-1}(z)\|_{\mathscr{B}(X_2)} \le C_\rho. \tag{4.34}$$

Proof of Lemma 4.11. Consider the equation

$$d(z)v = \varepsilon^{-1}\Delta v + \frac{1}{2}x\cdot\nabla v - z\,v = u, \tag{4.35}$$

for $z \in \mathcal{O}_\rho$. Multiplying the equation by $\bar v$ and the conjugated equation by v, we find

$$\begin{aligned} \frac{1}{\varepsilon}\int |\nabla v|^2 + \big(\frac{1}{2} + \Re e z\big)\int |v|^2 &= -\frac{1}{2}\int v\bar u - \frac{1}{2}\int \bar v u \\ &\le \|u\|_{L^2}\,\|v\|_{L^2}, \end{aligned} \tag{4.36}$$

and then

$$\big(\frac{1}{2} + \Re e z\big)\,\|v\|_{L^2} \le \|u\|_{L^2},$$

which is nothing but (4.34). □

Lemma 4.12. *With the above notations, for any $\rho > 0$, there exists $C_{k,a,\rho} > 0$ such that*

$$\sup_{z\in\mathcal{O}_\rho} \|bd^{-1}(z)\|_{\mathscr{B}(X_2,X_1)} \le C_{k,a,\rho}\,\sqrt{\varepsilon} \to 0. \tag{4.37}$$

Proof of Lemma 4.12. Consider the equation (4.35) again. Coming back to (4.36), we have

$$\|\nabla v\|_{L^2} \le \sqrt{\frac{\varepsilon}{\frac{1}{2} + \Re e z}}\|u\|_{L^2}. \tag{4.38}$$

Next, multiplying the equation (4.35) by $x\cdot\nabla\bar v$ and the conjugated equation by $x\cdot\nabla v$, we find

$$\int |x\cdot\nabla v|^2 = \int (u - zv)(x\cdot\nabla\bar v) + (\bar u - \bar z\bar v)(x\cdot\nabla v),$$

which in turn implies

$$\|x\cdot\nabla v\|_{L^2} \le C_{a,r}.$$

Coming back to (4.35) and together with (4.38), we have proved

$$\|\Delta v\|_{L^2} + \|\nabla v\|_{L^2} \le C_{a,r}\sqrt{\varepsilon}\|u\|_{L^2}. \tag{4.39}$$

We then immediately conclude to (4.37). □

Lemma 4.13. *With the above notation, for any $\rho > 0$, there exists $C_{k,\rho} > 0$ such that*

$$\sup_{z\in\mathcal{O}_\rho} \|cd^{-1}\|_{\mathscr{B}(X_1,X_2)} \le C_{k,\rho}. \tag{4.40}$$

Proof of Lemma 4.13. We just have to use appendix C. □

Proof of Proposition 4.9. We split the proof into four steps.

Step 1. We prove that

$$\forall\,\varepsilon\in(0,\varepsilon^*_\rho)\quad \mathcal{R}^{\Lambda_\varepsilon}_{11} = s_\varepsilon^{-1} \in \mathcal{H}(\mathcal{O}_\rho;\mathscr{B}(X_1)). \tag{4.41}$$

We write

$$s_\varepsilon(z) = A(z) - BC^{-1} - [B(C+\varepsilon D(z))^{-1} - BC^{-1}] =: s_0(z) - \varepsilon\tilde s(z),$$

and then

$$s_\varepsilon(z) - \Omega(z) = [s_0(z) - \Omega(z)] - \varepsilon\tilde s(z),$$

with

$$s_0(z) = A(z) - BC^{-1},\quad \Omega(z) = \Omega - z = A_0(z) - B_0C^{-1},\quad \tilde s(z) := B(C+\varepsilon D(z))^{-1}(-D(z))\,C^{-1}.$$

We remark that

$$s_0(z) - \Omega(z) = s_0 - \Omega = -\nabla(\cdot\nabla(V_\varepsilon - V)) - \nabla((G_\varepsilon - G)\nabla(\Delta^{-1}\cdot))$$

does not depend on z, and thanks to Corollary 4.2, we easily get

$$\|s_0 - \Omega\|_{\mathscr{B}(Y_1,X_1)} \leq \eta(\varepsilon) \quad \text{with} \quad \eta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

where we recall that $Y_1 = H^1_k \cap L^2_{k,0} \cap L^2_{rad}$ is defined in (4.19). Moreover, using Lemma 4.10, we get

$$\sup_{z\in\mathcal{O}_\rho} \sup_{\varepsilon\in(0,\varepsilon^*_\rho)} \|\tilde{s}\|_{\mathscr{B}(X_1)} \leq C,$$

from which we deduce, for any $z \in \mathcal{O}_\rho$ and $\varepsilon \in (0, \varepsilon^*_\rho)$, the bound

$$\|s_\varepsilon(z) - \Omega(z)\|_{\mathscr{B}(Y_1,X_1)} \leq \eta(\varepsilon) + C\varepsilon.$$

Then, arguing as is [54, Lemma 2.16], the operator

$$\mathcal{T}_\varepsilon(z) := (-1)^n (s_\varepsilon - \Omega)\mathcal{R}_\Omega(z) \left(\mathcal{A}_{11}\mathcal{R}^{\mathcal{B}_\varepsilon}_{11}(z)\right)^n$$

satisfies

$$\|\mathcal{T}_\varepsilon(z)\|_{\mathscr{B}(X_1)} \leq \eta'(\varepsilon) \quad \forall\, z \in \mathcal{O}_\rho, \quad \eta'(\varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0,$$

where

$$\mathcal{R}_{\mathcal{B}_\varepsilon} =: \begin{pmatrix} \mathcal{R}^{\mathcal{B}_\varepsilon}_{11} & \mathcal{R}^{\mathcal{B}_\varepsilon}_{12} \\ \mathcal{R}^{\mathcal{B}_\varepsilon}_{21} & \mathcal{R}^{\mathcal{B}_\varepsilon}_{22} \end{pmatrix}, \quad \mathcal{A} =: \begin{pmatrix} \mathcal{A}_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

and the integer n is defined in the proof of Proposition 4.3. As a consequence, the operators $I + \mathcal{T}_\varepsilon(z)$ and $s_\varepsilon(z)$ are invertible for any $z \in \mathcal{O}_\rho$, and furthermore

$$\mathcal{R}^{\Lambda_\varepsilon}_{11}(z) = s_\varepsilon(z)^{-1} = \mathcal{U}_\varepsilon(z)(1 + \mathcal{T}_\varepsilon(z))^{-1},$$

where

$$\mathcal{U}_\varepsilon(z) = \sum_{j=0}^{n-1} (-1)^j \mathcal{R}^{\mathcal{B}_\varepsilon}_{11}(z)(\mathcal{A}_{11}\mathcal{R}^{\mathcal{B}_\varepsilon}_{11}(z))^j + (-1)^n \mathcal{R}_\Omega(z)(\mathcal{A}_{11}\mathcal{R}^{\mathcal{B}_\varepsilon}_{11}(z))^n.$$

We immediately conclude to (4.41), because $\Sigma(\Omega) \cap \mathcal{C}_a = \emptyset$ on X_1, and $\|R_\Omega(z)\|_{\mathscr{B}(X_1)} \leq C$ for any $z \in \mathcal{O}_\rho$ from Theorem 4.8.

Step 2. We have

$$\forall\, \varepsilon \in (0, \varepsilon^*_\rho) \quad \mathcal{R}^{\Lambda_\varepsilon}_{12} := -s_\varepsilon^{-1} b d^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_2, X_1)), \tag{4.42}$$

as an immediate consequence of Lemmas 4.10 and 4.12.

Step 3. We also have

$$\forall\, \varepsilon \in (0, \varepsilon^*_\rho) \quad \mathcal{R}^{\Lambda_\varepsilon}_{21} := -d^{-1} c s_\varepsilon^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_1, X_2)), \tag{4.43}$$

as a consequence of Lemmas 4.10 and 4.13.

Step 4. We finally have

$$\forall\, \varepsilon \in (0, \varepsilon^*_\rho) \quad \mathcal{R}^{\Lambda_\varepsilon}_{22} := d^{-1} + d^{-1} c s_\varepsilon^{-1} b d^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_2)), \tag{4.44}$$

as an immediate consequence of Step 1 together with Lemmas 4.12 and 4.13. □

Proof of Theorem 4.7. The proof is a consequence of Proposition 4.3, Proposition 4.9 and Theorem 4.8 together with [41, Theorem 2.1]. □

5. Nonlinear exponential stability of self-similar solutions

5.1. **Linear stability in higher-order norms.** Define the space

$$Z := Z_1 \times Z_2, \quad Z_1 := H^1_k \cap L^2_{k,0} \cap L^2_{rad}, \quad Z_2 := H^2 \cap L^2_{rad}, \tag{5.1}$$

associated to the norm

$$\|(f,u)\|^2_Z := \|(f,u)\|^2_Y + \|\nabla^2 u\|^2_{L^2}. \tag{5.2}$$

We shall prove that the same linear stability estimate in X established in Theorem 4.7 also holds in Z, as stated in the following result.

Proposition 5.1. *Let $a^* > -1/2$ be fixed. There exists $\varepsilon^* > 0$ such that there holds in Z*

$$\Sigma(\Lambda_\varepsilon) \cap \mathcal{C}_{a^*} = \emptyset, \quad \forall\, \varepsilon \in (0, \varepsilon^*).$$

As a consequence, we have

$$\|S_{\Lambda_\varepsilon}(t)\|_{\mathscr{B}(Z)} \le Ce^{at}, \quad \forall\, t \ge 0,\ \forall\, \varepsilon \in (0, \varepsilon^*),\ \forall\, a > a^*.$$

Before proving Proposition 5.1, we state and prove an auxiliary technical result.

Lemma 5.2. *(1) $\mathcal{A} \in \mathscr{B}(Z)$.*

(2) There exist N, R large enough such that the operator $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in Z for any $a > -1/2$, i.e.

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(Z)} \le Ce^{at}, \quad \forall\, t \ge 0.$$

Moreover, we have the following estimate

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Z)} \le C\, t^{-1}\, e^{at}, \quad \forall\, t \ge 0,\ \forall\, a > -1/2.$$

Proof. Point (1) is straightforward from (4.11) and we omit the proof. For proving point (2), we consider a solution (f, u) to the equation $\partial_t(f, u) = \mathcal{B}_\varepsilon(f, u)$. First of all, observe that the norm $\|\cdot\|_Z$ is equivalent to

$$\|(f, u)\|^2_{Z_*} := \|(f, u)\|^2_{Y_*} + \eta_2 \|\nabla^2(u - \kappa_f)\|^2_{L^2}, \tag{5.3}$$

for any $\eta_2 > 0$. We write

$$\partial_t(\partial_{ij} u) = \frac{1}{\varepsilon}\Delta(\partial_{ij}(u - \kappa_f)) + \partial_{ij} u + \frac{1}{2} x \cdot \nabla(\partial_{ij} u),$$

and then we compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\partial_{ij}(u - \kappa_f)|^2 &= \int \partial_{ij}(u - \kappa_f)\partial_t\{\partial_{ij}(u - \kappa_f)\} \\
&= \int \partial_{ij}(u - \kappa_f)\frac{1}{\varepsilon}\Delta(\partial_{ij}(u - \kappa_f)) + \int \partial_{ij}(u - \kappa_f)\partial_{ij} u \\
&\quad + \int \partial_{ij}(u - \kappa_f)\frac{1}{2} x \cdot \nabla(\partial_{ij}(u - \kappa_f)) + \int \partial_{ij}(u - \kappa_f)\frac{1}{2} x \cdot \nabla(\partial_{ij}\kappa_f) \\
&\quad - \int \partial_{ij}(u - \kappa_f)\partial_{ij}\kappa * (\partial_t f) \\
&=: B_1 + \cdots + B_5.
\end{aligned}$$

We estimate each term separately. We easily obtain

$$B_1 = -\frac{1}{\varepsilon}\|\nabla\{\partial_{ij}(u - \kappa_f)\}\|^2_{L^2},$$

moreover

$$B_2 \le C\|\nabla^2(u - \kappa_f)\|^2_{L^2} + C\|\nabla^2 u\|^2_{L^2} \le C\|\nabla^2(u - \kappa_f)\|^2_{L^2} + C\|f\|^2_{L^2},$$

by integration by parts $B_3 \le 0$ and also

$$B_4 \le C\|\nabla^2(u - \kappa_f)\|^2_{L^2} + C\|\nabla^3\kappa * f\|^2_{L^2_1} \le C\|\nabla^2(u - \kappa_f)\|^2_{L^2} + C\|\nabla f\|^2_{L^2_1}.$$

For the last term, we get

$$\begin{aligned}
B_5 &= -\int \partial_{ij}(u - \kappa_f)\partial_{ij}\kappa * \left\{\Delta f + \frac{1}{2}\nabla(xf) - \nabla(f\nabla V_\varepsilon) - \nabla(G_\varepsilon \nabla u) - N\chi_R[f]\right\} \\
&=: B_{51} + \cdots + B_{55}.
\end{aligned}$$

We have

$$B_{51} \le C(\delta)\|\nabla^2(u - \kappa_f)\|^2_{L^2} + \delta\|\nabla^2 f\|^2_{L^2}$$

and

$$B_{52} \le C\|\nabla^2(u - \kappa_f)\|^2_{L^2} + C\|\nabla^2\kappa * \nabla(xf)\|^2_{L^2} \le C\|\nabla^2(u - \kappa_f)\|^2_{L^2} + C\|\nabla f\|^2_{L^2_1} + C\|f\|^2_{L^2}.$$

Moreover, using that $\nabla V_\varepsilon, \Delta V_\varepsilon \in L^\infty$, we get

$$\begin{aligned} B_{53} &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * \nabla(f\nabla V_\varepsilon)\|_{L^2}^2 \\ &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla(f\nabla V_\varepsilon)\|_{L^2}^2 \\ &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla f\|_{L^2}^2 + C\|f\|_{L^2}^2, \end{aligned}$$

and arguing as above with $G_\varepsilon, \nabla G_\varepsilon \in L^\infty$, we also obtain

$$\begin{aligned} B_{54} &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * \nabla(G_\varepsilon\nabla u)\|_{L^2}^2 \\ &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla(G_\varepsilon\nabla u)\|_{L^2}^2 \\ &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\ &\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2_\ell}^2, \end{aligned}$$

where we recall that $\ell \in (3,k)$ is fixed in Lemma 4.4. For the last term, we easily obtain

$$B_{55} \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|N\chi_R[f]\|_{L^2}^2 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + CN^2\|f\|_{L^2}^2.$$

All the above estimates yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla^2(u-\kappa_f)\|_{L^2}^2 \le -\frac{1}{\varepsilon}\|\nabla^3(u-\kappa_{\tilde f})\|_{L^2}^2 + C(\delta)\|\nabla^2(u-\kappa_{\tilde f})\|_{L^2}^2 \\ + C\|f\|_{L^2_\ell}^2 + CN^2\|f\|_{L^2}^2 + C\|\nabla f\|_{L^2_1}^2 + \delta\|\nabla^2 f\|_{L^2}^2. \end{aligned} \tag{5.4}$$

Putting together that last equation with (4.24), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Z_*}^2 \\ &\le \int \Big\{\varphi(x) + \eta_1\varepsilon C(\delta) + \eta_1 C\frac{N}{R}\mathbf{1}_{R/2\le|x|\le 2R} + \eta_1 C(\delta)N^2\langle x\rangle^{-2k} + \eta_2 CN^2\langle x\rangle^{-2k} \\ &\qquad + [\eta_1 C(\delta) + \eta_1 C(1+N^2) + C(\delta)(1+\eta N^2) + C\frac{N}{R^{\ell-1}} + \eta_2 C]\langle x\rangle^{2(\ell-k)} - N\chi_R\Big\} f^2\langle x\rangle^{2k} \\ &\quad + \eta_1\int \Big\{\varphi(x) - \frac{1}{2} + \delta + C\frac{N}{R}\mathbf{1}_{R/2\le|x|\le 2R} + [C(\delta) + C\frac{N}{R^{\ell-1}}]\langle x\rangle^{2(\ell-k)} \\ &\qquad + C\langle x\rangle^{-2k} + \frac{\eta_2}{\eta_1}C\langle x\rangle^{2(1-k)} - N\chi_R\Big\}|\nabla f|^2\langle x\rangle^{2k} \\ &\quad - (1-\delta)\|\nabla f\|_{L^2_k}^2 - \eta\left(\frac{1}{2}-\delta\right)\|u-\kappa_f\|_{L^2}^2 - \eta_1\left(1-\delta-\frac{\eta_2}{\eta_1}\delta\right)\|\nabla^2 f\|_{L^2_k}^2 \\ &\quad - \eta\left(\frac{1}{\varepsilon} - C(\delta) - \frac{\eta_1}{\eta}C(\delta)\right)\|\nabla(u-\kappa_f)\|_{L^2}^2 - \eta_1\left(\frac{1}{\varepsilon} - C(\delta) - \frac{\eta_2}{\eta_1}C(\delta)\right)\|\nabla^2(u-\kappa_f)\|_{L^2}^2 \\ &\quad - \frac{1}{\varepsilon}\|\nabla^3(u-\kappa_{\tilde f})\|_{L^2}^2. \end{aligned}$$

We can now conclude exactly as in the proof of Lemma 4.4 and Lemma 4.6, and we obtain that for any $a > -1/2$,

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Z_*}^2 \le a\|(f,u)\|_{Z_*}^2 - K(\|\nabla^2 f\|_{L^2_k}^2 + \|\nabla^3(u-\kappa_f)\|_{L^2}^2), \tag{5.5}$$

for some constant $K > 0$, from which $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in Z.

From (5.5) and the interpolation inequalities

$$\|f\|_{H^1_k}^2 \le C\|f\|_{H^2_k}\|f\|_{L^2_k}, \quad \|u\|_{H^2}^2 \le C\|u\|_{H^3}^{4/3}\|u\|_{L^2}^{2/3}$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Z_*}^2 \le a\|(f,u)\|_{Z_*}^2 - K\|(f,u)\|_{X_*}^{-1}\|(f,u)\|_{Z_*}^3.$$

We obtain by standard arguments

$$\|S_{\mathcal{B}_\varepsilon}(t)(f,u)\|_Z \le C\,t^{-1}\,e^{at}\,\|(f,u)\|_X,$$

which concludes the proof. □

Proof of Proposition 5.1. From Lemma 4.4, Lemma 4.5, Lemma 5.2 and [27, Lemma 2.17], it follows that there is $n \in \mathbb{N}$ such that

$$\|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{*n}(t)\|_{\mathscr{B}(X,Z)} \le C\, e^{at}.$$

Then the proof of the linear stability result in Z is a consequence of last estimate, Lemma 5.2, Theorem 4.7 and the "extension theorem" [39, Theorem 1.1]. □

5.2. **Dissipative norm.** We define the new norm

$$|||(f,u)|||_Z^2 := \eta\|(f,u)\|_Z^2 + \int_0^\infty \|S_{\Lambda_\varepsilon}(\tau)(f,u)\|_Z^2\, d\tau$$

for some $\eta > 0$. Thanks to Proposition 5.1, the norm $|||\cdot|||_Z$ is equivalent to $\|\cdot\|_Z$ for any $\eta > 0$. Moreover, considering a solution (f,u) to the linearized equation $\partial_t(f,u) = \Lambda_\varepsilon(f,u)$, we obtain from Proposition 5.1, Lemma 5.2 and arguing as in [27, 18], that

$$\frac{d}{dt}|||(f,u)|||_Z^2 \le -K|||(f,u)|||_Z^2 - K\{\|\nabla^2 f\|_{L^2_k}^2 + \|\nabla^3 u\|_{L^2}^2\} =: -K\|(f,u)\|_{\widetilde{Z}}^2, \tag{5.6}$$

for $\eta > 0$ small enough and some constant $K > 0$.

5.3. **The nonlinear problem : Proof of theorem 1.4.** We focus now on the nonlinear parabolic-parabolic Keller-Segel system (1.16)-(1.17) in self-similar variables and we prove Theorem 1.4. Consider a solution (g,v) to (1.16)-(1.17) and define $f := g - G_\varepsilon$ and $u := v - V_\varepsilon$, which satisfy

$$\begin{aligned} \partial_t f &= \Lambda_{\varepsilon,1}(f,u) - \nabla\cdot(f\nabla u) \\ \partial_t u &= \Lambda_{\varepsilon,2}(f,u), \end{aligned} \tag{5.7}$$

together with the initial condition $(f,u)_{|t=0} = (f_0,u_0) := (g_0,v_0) - (G_\varepsilon, V_\varepsilon) \in Z$.

We split the proof into three parts.

5.3.1. *A priori estimate.*

Lemma 5.3. *The solution (f_t,u_t) to (5.7) satisfies, at least formally, the following differential inequality*

$$\frac{d}{dt}|||(f,u)|||_Z^2 \le (-K + C|||(f,u)|||_Z)\,\|(f,u)\|_{\widetilde{Z}}^2, \tag{5.8}$$

where $\|\cdot\|_{\widetilde{Z}}$ is defined in (5.6).

Proof of Lemma 5.3. Because

$$\|(f,u)\|_Z^2 = \|f\|_{H^1_k}^2 + \|u\|_{H^2}^2,$$

and denoting $Q(f,u) = (-\nabla\cdot(f\nabla u), 0)$, we obtain from (5.7) that,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|||(f,u)|||_Z^2 &= \eta\langle(f,u),\Lambda_\varepsilon(f,u)\rangle + \int_0^\infty \langle S(\tau)(f,u), S(\tau)\Lambda_\varepsilon(f,u)\rangle\, d\tau \\ &\quad + \eta\langle(f,u),Q(f,u)\rangle + \int_0^\infty \langle S(\tau)(f,u), S(\tau)Q(f,u)\rangle\, d\tau =: I_1 + I_2. \end{aligned}$$

For the first (linear) term, we have already obtained in (5.6) that

$$\begin{aligned} I_1 &:= \eta\langle(f,u),\Lambda_\varepsilon(f,u)\rangle + \int_0^\infty \langle S(\tau)(f,u), S(\tau)\Lambda_\varepsilon(f,u)\rangle\, d\tau \\ &\le -K\left\{|||(f,u)|||_Z^2 + \|\nabla^2 f\|_{L^2_k}^2 + \|\nabla^3 u\|_{L^2}^2\right\} =: -K\|(f,u)\|_{\widetilde{Z}}^2. \end{aligned}$$

For the second (nonlinear) term, we use the linear stability in Z from Proposition 5.1 to obtain

$$\begin{aligned} I_2 &:= \eta\langle (f,u), Q(f,u)\rangle + \int_0^\infty \langle S(\tau)(f,u), S(\tau)Q(f,u)\rangle \, d\tau \\ &\le \eta \|(f,u)\|_Z \, \|Q(f,u)\|_Z + \int_0^\infty \|S(\tau)(f,u)\|_Z \, \|S(\tau)Q(f,u)\|_Z \, d\tau \\ &\le \eta \|(f,u)\|_Z \, \|Q(f,u)\|_Z + C\int_0^\infty e^{at}\|(f,u)\|_Z \, e^{at}\|Q(f,u)\|_Z \, d\tau \\ &\le C\|(f,u)\|_Z \, \|Q(f,u)\|_Z. \end{aligned}$$

Now, we have to compute

$$\|Q(f,u)\|_Z^2 = \|(-\nabla\cdot(f\nabla u),0)\|_Z^2 = \|\nabla\cdot(f\nabla u)\|_{L^2_k}^2 + \|\nabla\cdot(f\nabla u)\|_{\dot H^1_k}^2.$$

We split $\nabla\cdot(f\nabla u) = f\Delta u + \nabla f\cdot\nabla u$ and compute

$$\begin{aligned} \|f\Delta u\|_{L^2_k}^2 &= \int f^2|\Delta u|^2\langle x\rangle^{2k} \le C\|\nabla^2 u\|_{L^2}^2 \, \|\langle x\rangle^k f\|_{L^\infty}^2 \\ &\le C\|\nabla^2 u\|_{L^2}^2 \, \|f\|_{H^2_k}^2 \le C\|(f,u)\|_Z^2 \, \|(f,u)\|_{\widetilde Z}^2, \end{aligned}$$

where we have used the embedding $H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$. Moreover, we have

$$\begin{aligned} \|\nabla f\cdot\nabla u\|_{L^2_k}^2 &= \int |\nabla f|^2|\nabla u|^2\langle x\rangle^{2k} \le C\|\nabla u\|_{L^\infty}^2 \, \|\nabla f\|_{L^2_k}^2 \\ &\le C\|u\|_{H^3}^2 \, \|\nabla f\|_{L^2_k}^2 \le C\|(f,u)\|_{\widetilde Z}^2 \, \|(f,u)\|_Z^2. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \|\nabla(f\Delta u)\|_{L^2_k}^2 &\le C\int f^2|\nabla^3 u|^2\langle x\rangle^{2k} + C\int |\nabla f|^2|\nabla^2 u|^2\langle x\rangle^{2k} \\ &\le C\|\nabla^3 u\|_{L^2}^2 \, \|\langle x\rangle^k f\|_{L^\infty}^2 + C\|\langle x\rangle^k\nabla f\|_{L^4}^2 \, \|\nabla^2 u\|_{L^4}^2 \\ &\le C\|\nabla^3 u\|_{L^2}^2 \, \|f\|_{H^2_k}^2 + C\|f\|_{L^2_k} \, \|f\|_{H^2_k} \, \|\nabla u\|_{L^2} \, \|\nabla^3 u\|_{L^2}^2 \\ &\le C\|(f,u)\|_{\widetilde Z}^4, \end{aligned}$$

where we have used the Gagliardo-Niremberg-Sobolev inequality $\|h\|_{L^4(\mathbb{R}^2)}^2 \le C\|h\|_{L^2(\mathbb{R}^2)} \, \|\nabla h\|_{L^2(\mathbb{R}^2)}$. For the last term, we have

$$\begin{aligned} \|\nabla(\nabla f\cdot\nabla u)\|_{L^2_k}^2 &\le C\int |\nabla f|^2|\nabla^2 u|^2\langle x\rangle^{2k} + C\int |\nabla^2 f|^2|\nabla u|^2\langle x\rangle^{2k} \\ &\le C\|\langle x\rangle^k\nabla f\|_{L^4}^2 \, \|\nabla^2 u\|_{L^4}^2 + C\|\nabla u\|_{L^\infty}^2 \, \|\nabla^2 f\|_{L^2_k}^2 \\ &\le C\|f\|_{L^2_k} \, \|f\|_{H^2_k} \, \|\nabla u\|_{L^2} \, \|\nabla^3 u\|_{L^2}^2 + C\|u\|_{H^3}^2 \, \|\nabla^2 f\|_{H^2_k}^2 \\ &\le C\|(f,u)\|_{\widetilde Z}^4. \end{aligned}$$

Finally, gathering the above estimates, the solution (f,u) of (5.7) verifies

$$\frac{d}{dt}|||(f,u)|||_Z^2 \le -K\|(f,u)\|_{\widetilde Z}^2 + C\|(f,u)\|_Z \, \|(f,u)\|_{\widetilde Z}^2 \le (-K + C|||(f,u)|||_Z) \, \|(f,u)\|_{\widetilde Z}^2,$$

which concludes the proof. □

5.3.2. *Regularity.* We prove that starting close enough to the self-similar profile, the solution to the Keller-Segel equation (1.1) satisfies some strong and uniform in time estimates.

Proposition 5.4. *There is $\delta > 0$ such that, if $|||(f_0,u_0)|||_Z \le \delta$, there exists a solution $(f,u)\in C([0,\infty);Z)$ to (5.7) (and thus to the Keller-Segel equation (1.1)) that verifies*

$$\forall\, t\ge 0, \qquad |||(f,u)(t)|||_Z^2 + \frac{K}{2}\int_0^t \|(f,u)(\tau)\|_{\widetilde Z}^2 \, d\tau \le 2\delta^2. \tag{5.9}$$

Proof of Proposition 5.4. At least formally, taking $\delta := K/(2C)$ in (5.8), we see that $t \mapsto |||(f,u)(t)|||_Z$ is decreasing if $|||(f_0,u_0)|||_Z \le \delta$, and then the a priori estimate (5.9) immediately follows from (5.8). We refer to [27, Theorem 5.3] for a completely rigorous proof based on an iterative scheme. □

5.3.3. *Sharp exponential convergence to the equilibrium.* We now complete the proof of Theorem 1.4 following [27]. Applying Lemma 5.3 to the solution (f,u) constructed above and using the estimate (5.9), we obtain

$$\begin{aligned}\frac{d}{dt}|||(f,u)|||_Z^2 &\le (-K + C|||(f,u)|||_Z)\,\|(f,u)\|_{\widetilde{Z}}^2 \\ &\le (-K + C'\delta^2)\,\|(f,u)\|_{\widetilde{Z}}^2 \le (-K + C'\delta^2)\,C''\,|||(f,u)|||_Z^2.\end{aligned}$$

If $\delta > 0$ is small enough so that $-K + C'\delta^2 \le -K/2$, this differential inequality implies the exponential decay

$$|||(f,u)(t)|||_Z \le e^{-\frac{KC''}{4}t}\,|||(f_0,u_0)|||_Z.$$

Finally, we can recover the optimal decay rate $O(e^{at})$ of the linearized semigroup in Proposition 5.1 by performing a bootstrap argument as in [27, Proof of Theorem 5.3], and that concludes the proof.

Appendix A. Orlicz space, interpolation and a convex function

We describe here some classical and technical results on Orlicz spaces and interpolation spaces. We then apply these results to the function $\xi \mapsto \xi^2(\widetilde{\log}\,\xi)^2$ used during the proof of Lemma 2.7.

A.1. **Basic notions.** We recall some basic notions of the theory of Orlicz spaces, and we refer to [37, 48] for more details. We say that a function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a N-function if it is continuous, convex, $\Phi(t) > 0$ for any $t > 0$ and satisfies

$$\lim_{t\to 0}\frac{\Phi(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty.$$

For such a function Φ, we define the *Orlicz space* L_Φ by

$$f \in L_\Phi \quad \Leftrightarrow \quad \exists \lambda > 0 \text{ such that } \int \Phi\left(\frac{f}{\lambda}\right) < \infty,$$

and we endow L_ϕ with the Luxembourg norm

$$\|f\|_{L_\Phi} := \inf\{\lambda > 0 \ : \ \int \Phi\left(\frac{f}{\lambda}\right) \le 1\}.$$

Moreover, when Φ satisfies the following Δ_2-condition,

$$\exists\, c, s > 0, \quad \forall\, t \ge s, \quad \Phi(2t) \le c\,\Phi(t),$$

we have

$$\|f\|_{L_\Phi} < \infty \quad \Leftrightarrow \quad \forall\, \lambda > 0, \ \int \Phi\left(\frac{f}{\lambda}\right) < \infty.$$

A.2. **Interpolation.** We state an interpolation result on Orlicz spaces from [51] (see also [33] and the references therein) which is used in Lemma 2.7.

Theorem A.1. *Consider an N-function Φ such that*

$$1 \le p < \inf_{s>0}\frac{s\Phi'(s)}{\Phi(s)} \le \sup_{s>0}\frac{s\Phi'(s)}{\Phi(s)} < q < \infty.$$

Then the Orlicz space L_Φ is an interpolation space between L^p and L^q. In other words, if $U : L^r \to L^r$ is bounded for $r = p, q$, then $U : L_\Phi \to L_\Phi$ is bounded.

A.3. **The function $s \mapsto s^2(\widetilde{\log} s)^2$.** With the notation of the proof of Lemma 2.7, we define $\Phi(s) := s^2(\widetilde{\log} s)^2$, as well as Φ^* the conjugate function of Φ by

$$\Phi^*(t) = \sup_s \{ts - \Phi(s)\} \quad \forall\, t \geq 0.$$

We show below that there exists a constant $C > 0$ such that

$$\Phi^*(t) \leq C\, t^2 (\widetilde{\log} t)^{-2}. \tag{A.1}$$

In order to prove such a claim, we first observe that, for any $t > 0$, the above supremum is reached at some unique point $s_t \in (0, \infty)$, which is implicitly given by the equation $t = \Phi'(s_t)$. In particular, $s_t \to \infty$ when $t \to \infty$ (because Φ' is increasing and bijective from $\mathbb{R}_+^*$ onto $\mathbb{R}_+^*$) so that there exists $t^* > 0$ such that $s_t > e$ for any $t > t^*$. For $t > t^*$, we have

$$4s_t(\log s_t)^2 \geq t = \Phi'(s_t) = 2s_t(\log s_t)^2 + 2s_t^2(\log s_t)^2 \geq 2s_t(\log s_t)^2.$$

We then compute

$$\frac{t}{(\log t)^2} \geq \frac{2s_t(\log s_t)^2}{(\log[4s_t(\log s_t)^2])^2} \geq \frac{s_t}{2}$$

for $t > t^{**}$, t^{**} large enough. We conclude that

$$\Phi^*(t) = ts_t - \Phi(s_t) \leq 2\frac{t^2}{(\log t)^2}$$

for any $t > t^{**}$. We then easily conclude to (A.1) by observing that $\Phi^*(t) = 4t^2$ for t small enough.

Appendix B. Estimates on the solutions to the Poisson equation

In this section, we give some technical (and we believe standard) estimates on the solutions to the Poisson equation in $\mathbb{R}^2$ that are useful in the paper. We recall the notation

$$\kappa(z) = -\frac{1}{2\pi}\log|z|, \quad \mathcal{K}(z) := \nabla\kappa(z) = -\frac{1}{2\pi}\,\frac{z}{|z|^2}$$

and

$$\kappa_f := \kappa * f, \quad \mathcal{K}_f := \mathcal{K} * f,$$

so that there holds

$$\kappa_f \in C^2(\mathbb{R}^2), \quad |\kappa_f| \leq C\,(1 + \log\langle x\rangle), \quad -\Delta\kappa_f = f.$$

Lemma B.1. *For any integer $j \geq 1$ and real number $k > j + 2$, there exists $C_{k,j}$ such that*

$$\|\kappa * f\|_{L^2_{j-1}} \leq C_{k,j}\|f\|_{L^2_k} \quad \forall\, f \in L^2_{k,j}. \tag{B.1}$$

Proof of Lemma B.1. Consider $f \in L^2_{k,j}$, so that $\hat f \in C^{j+1}_b$ and $\partial^\alpha_\xi \hat f(0) = 0$ for any $|\alpha| \leq j$ thanks to the moments condition. For a multi-index $\alpha \in \mathbb{N}^2$, $|\alpha| \leq j - 1$, we may thus write the Taylor expansion

$$\partial^\alpha_\xi \hat f(\xi) = \int_0^1 (1-s) D^2_\xi \partial^\alpha_\xi \hat f(s\,\xi) : \xi^{\otimes 2}\, ds.$$

In Fourier variables the Laplace equation writes $|\xi|^2\hat\kappa_f(\xi) = \hat f(\xi)$, and then for a multi-index $\alpha \in \mathbb{N}^2$

$$\int |x^\alpha|^2\, |\kappa_f|^2 = \int |\partial^\alpha_\xi \hat\kappa_f|^2 = \int \frac{|\partial^\alpha_\xi \hat f|^2}{|\xi|^4}.$$

All together, we have

$$\begin{aligned}\int |x|^{2(j-1)}\, |\kappa_f|^2 &\leq \int_{B_1^c} \frac{|D^{j-1}\hat\kappa_f|^2}{|\xi|^4}\, d\xi + \int_{B_1} \sup_{\eta\in B_1} |D^{j+1}\hat f(\eta)|^2\, d\xi \\ &\leq \|f\|^2_{L^2_{j-1}} + C\|f\|^2_{L^1_{j+1}} \leq C\|f\|^2_{L^2_k},\end{aligned}$$

where we have used $\|g\|_{L^1} \leq C\|g\|_{L^2_r}$ for $r > 1$ with $g = \langle x\rangle^{j+1} f$ in the last line, which gives $k > j + 2$. □

Lemma B.2. *For any integer $j \geq 0$ and real number $k > j+2$, there exists $C_{k,j}$ such that*

$$\|\mathcal{K} * f\|_{L^2_j} \leq C_{k,j}\|f\|_{L^2_k} \quad \forall\, f \in L^2_{k,j}. \tag{B.2}$$

Proof of Lemma B.2. The proof is similar to Lemma B.1. In the case $j = 0$ for instance, we write in Fourier variables $|\xi|^2\hat{\kappa}_f(\xi) = \hat{f}(\xi)$, we observe that

$$\int |\nabla \kappa_f|^2 = \int |\xi|^2 |\hat{\kappa}_f|^2 = \int \frac{|\hat{f}|^2}{|\xi|^2},$$

and we use the moments conditions to conclude. □

Appendix C. Estimates on the $c^{-1}d$ operator

With the notations of Section 4.3, we consider the equation

$$L_\varepsilon u := c^{-1}du = \Delta u + \frac{\varepsilon}{2} x \cdot \nabla u - \varepsilon z u = f \tag{C.1}$$

with $z \in \mathcal{C}_{-1/2}$.

Lemma C.1. *For any $f \in L^2_{k,0}$, $k > 2$, $z \in \mathcal{C}_{-1/2}\backslash\{0\}$, there exists a solution $u \in H^2_{loc}$ to the equation $L_\varepsilon u = f$ and there exists a constant C (which does not depend on $\varepsilon > 0$ and $z \in \mathcal{C}_{-1/2}\backslash\{0\}$) such that*

$$\|\nabla u\|_{L^2} + \|\Delta u\|_{L^2} \leq C\,\|f\|_{L^2_k}.$$

Proof of Lemma C.1. Multiplying equation (C.1) by $\bar{u}$ and $\Delta\bar{u}$ and its conjugate by u and Δu, we find

$$\int |\nabla u|^2 + \varepsilon\big(\frac{1}{2} + \Re e z\big) \int |u|^2 = -\frac{1}{2}\int (f\bar{u} + \bar{f}u) \tag{C.2}$$

and

$$\int |\Delta u|^2 + (\varepsilon\,\Re e z) \int |\nabla u|^2 = \frac{1}{2}\int f\Delta\bar{u} + \frac{1}{2}\int \bar{f}\Delta u. \tag{C.3}$$

Writing

$$u(x) = u(y) + \int_0^1 \nabla u(z_s)\,(x-y)\,ds, \quad z_s := y + s\,(y-x),$$

we have

$$u(x) = \langle u\rangle_1 + \int_{B(0,1)}\int_0^1 \nabla u(z_s)\,(x-y)\,dsdy, \quad \langle u\rangle_1 := \int_{B(0,1)} u(y)\,dy.$$

From the above equation (C.2) and the moment condition, we obtain

$$\begin{aligned}
\int |\nabla u|^2 &\leq -\frac{1}{2}\int_{\mathbb{R}^2}\int_{B(0,1)}\int_0^1 \{\bar{f}(x)\nabla u(z_s) + f(x)\nabla\bar{u}(z_s)\}\cdot(x-y)\,dsdydx \\
&\leq \frac{1}{2\alpha}\int_{\mathbb{R}^2}\int_{B(0,1)}\int_0^1 |f(x)|^2\,\langle x\rangle^{2\ell}\,|x-y|^2\,dsdydx \\
&\quad + \frac{\alpha}{2}\int_{\mathbb{R}^2}\int_0^{1/2}\Big\{\int_{B(0,1)} |\nabla u(sx+(1-s)y)|^2\,dy\Big\}\,ds\frac{dx}{\langle x\rangle^{2\ell}} \\
&\quad + \frac{\alpha}{2}\int_{B(0,1)}\int_{1/2}^1\Big\{\int_{\mathbb{R}^2} |\nabla u(sx+(1-s)y)|^2\,dx\Big\}\,dsdy \\
&\leq \frac{1}{\alpha}\Big\{\int_{B(0,1)} dy\Big\}\int_{\mathbb{R}^2} |f(x)|^2\,\langle x\rangle^{2(\ell+1)}\,dx \\
&\quad + \frac{\alpha}{2}\int_{\mathbb{R}^2}\frac{dx}{\langle x\rangle^{2\ell}}\int_0^{1/2}\frac{ds}{(1-s)^2}\int_{\mathbb{R}^2}|\nabla u(z)|^2\,dz \\
&\quad + \frac{\alpha}{2}\Big\{\int_{B(0,1)} dy\Big\}\int_{1/2}^1\frac{ds}{s^2}\int_{\mathbb{R}^2}|\nabla u(z)|^2\,dz,
\end{aligned}$$

and we deduce that

$$\int |\nabla u|^2 \le C \int_{\mathbb{R}^2} |f(x)|^2 \langle x\rangle^{2k}\, dx,$$

by choosing $\ell = k-1$ and $\alpha > 0$ small enough. From (C.3), we have

$$\|\Delta u\|_{L^2}^2 \le \frac{\varepsilon}{2}\|\nabla u\|_{L^2}^2 + \|f\|_{L^2}\|\Delta u\|_{L^2},$$

and we conclude the proof thanks to the above estimate on $\|\nabla u\|_{L^2}$ and Young's inequality. □

Lemma C.2. *There exists a constant C such that for any $\varepsilon \in (0,1)$, any $z \in \mathcal{C}_{-1/2}$ and any radially symmetric function f the equation*

$$L_\varepsilon u = f, \quad u \text{ radially symmetric}, \quad u(0) = u'(0) = 0,$$

has a unique solution which furthermore satisfies

$$\|u\, e^{-(1+\varepsilon|z|)r}\|_{L^\infty} + \|u'\, e^{-(1+\varepsilon|z|)r}\|_{L^\infty} \le C\, \|f\, e^{-r/2}\|_{L^\infty}. \tag{C.4}$$

Proof of Lemma C.2. We may write the equation as

$$u'' + \big(\frac{1}{r} + \varepsilon\, r\big)\, u' + \varepsilon z u = f, \quad \forall\, r > 0, \tag{C.5}$$

with the additional boundary conditions $u(0) = u'(0) = 0$. Defining $U := |u|^2 + |u'|^2$, we have

$$\begin{aligned} U' &= u\bar{u}' + u'\bar{u} + u'\bar{u}'' + u''\bar{u}' \\ &\le 2(1+\varepsilon|z|)\,|u|\,|u'| - \big(\frac{1}{r} + \varepsilon\, r\big)\,|u'|^2 + 2\,|u'|\,|f| \\ &\le (2+\varepsilon|z|)\,U + |f|^2, \end{aligned}$$

from which we immediately get (C.4) thanks to Gronwall's lemma. □

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