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Abstract

Purpose   The purpose of this paper is to apply the method of separation of variables to obtain the current distribution in the slot of an electrical machine, taking into account the skin effect.

Design/methodology/approach   A slot in an electrical machine, filled with a solid conductor, and fed with an externally imposed density current, presents a current distribution that depends on the skin effect. The magnetic potential vector is formulated and solved using a separate representation as a finite sum of unidimensional (space and time) functions, taking into account the boundary conditions. The proposed method obtains the transient and permanent distribution of the current in the interior of the slot, both in transient and steady regime, and the results at the end of the transient are compared with the analytic ones in permanent regime.

Findings   The magnetic potential vector in the interior of a slot filled with a solid conductor can be expressed as a finite sum of just 16 modes, which capture the evolution of the field during the transient and permanent regime. These modes are expressed as the product of space and time functions, which have been obtained automatically by the separation of variables algorithm. Instead of solving multiple field problems, one for each time instant, the proposed method just solves two single variable differential equations, one in the time domain and other in the spatial one.

Research limitations/implications   The application of the proposed method to non sinusoidal currents, such as those generated by variable speed drives, would allow to compute the field taking into account both the very small time scale of the pulse width modulation pulses, in the range of kiloHz, and the wide time scale of the currents envelope, in the range of 0 100 Hz. Extension to 2D and 3D spatial configurations is also under consideration.

Originality/value   Using the method of separation of variables to solve electromagnetic problems provides a new method which can simplify and speed up the computation of transient fields in multidimensional time and space domains.

Keywords Modelling, Computer applications, Electric current, Electromagnetism, Transient voltages, Simulation

I. Introduction

Separated representations based on finite sums can help to reduce drastically the computer resources and the calculation costs that the functional approximation requires in multidimensional problems (Ammar *et al.*, 2006, 2007, n.d.; Chinesta *et al.*, 2008; Gonzalez *et al.*, 2010). For example, the simulation of transient currents in motor slots exhibits an exponential growing, characteristic of mesh-based discretization strategies, as we introduce new dimensions (either spatial one or the time) in the model. Using the method of separation of variables, on the contrary, the number of degrees of freedom scales just linearly with the dimension of the space in which the model is defined. This method, also known as finite sums decomposition or proper generalized decomposition, expresses an N-dimensional function $u(x_1, x_2, \ldots, x_N)$ as:

$$u(x_1, x_2, \ldots, x_N) \approx \sum_{i\,1}^{i\,r} F_1^i(x_1) \cdot F_2^i(x_2) \cdots F_N^i(x_N) \tag{1}$$

where the dimensions can be either spatial ones (x, y, z), time, etc. If function u is the unknown field involved in a partial differential equation (PDE), expressing it as equation (1) can reduce drastically the size of the mesh required to obtain it: instead of N-dimensional mesh, the problem can be solved with N meshes of a much lower size. The problem with this approach is to be able to obtain the decomposition of the unknown field with an easy to apply numerical procedure. This procedure exists (Chinesta *et al.*, 2008) and it is presented here applied to the problem of computing the current distribution inside of a machine slot with an externally imposed current distribution. This problem is solved in a 2D domain (one spatial dimension and time), but it can readily extended to a 3D or even to a 4D (x, y, z, t) domain.

II. Analytical solution of the current in a rectangular slot in permanent regime

The method of separation of variables is used to obtain the total current in the slot of an electrical machine filled with a solid copper conductor, fed with an externally imposed density current (Figure 1). First, the analytical solution in permanent regime will be obtained. Second, the transient solution will be obtained with the proposed method, and compared with the analytical solution once reached the permanent regime.

The slot has dimensions $b \times L$ and it is occupied by a material with constant properties μ $4\pi10^{-7}$ Henries/m and σ 59×10^6 Siemens/m. The slot has a total depth of L 30 mm. An external density current $\mathbf{J_0}$ $J_{0\,max}\cos(\omega t)$ is imposed in the conductor, in the z direction, with ω $2\pi50$ rad/s and J_{0max} 10^4 A/m^2. The magnetic field $\mathbf{H}$ and the magnetic flux density $\mathbf{B}$ verify:

$$\begin{cases} \text{rot } \mathbf{H} \mathbf{J} & \text{Ampere's law} \\ \text{rot } \mathbf{E} \frac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\ \mathbf{B} \mu\mathbf{H}, \ \mathbf{J} \sigma\mathbf{E} & \text{Constitutive equations} \end{cases} \tag{2}$$

As all the currents have only z-component, so does the magnetic vector potential (MVP). The total current density in the conductor will be the sum of the externally imposed one and the induced current due to the time-variation of the magnetic flux density:
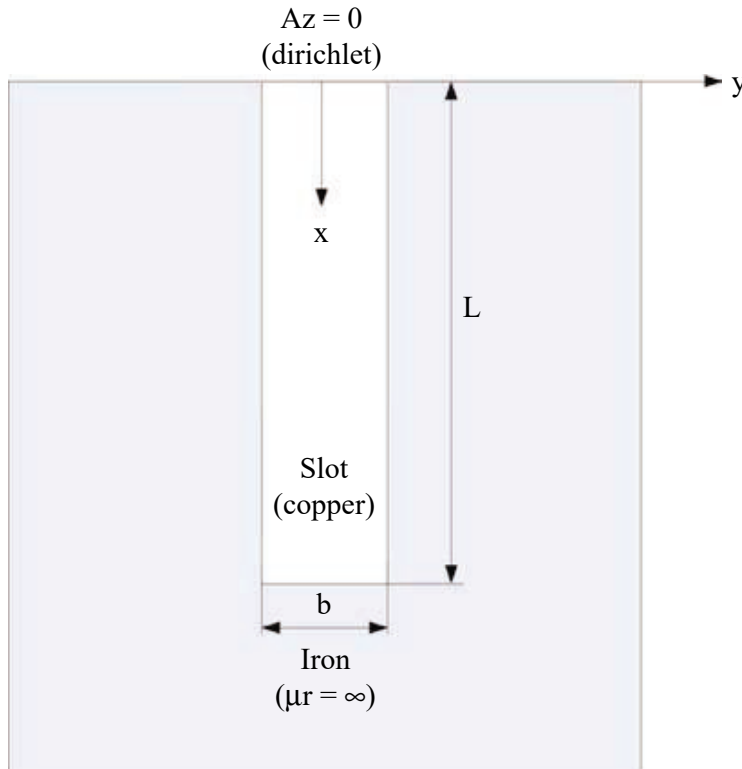
Az = 0
(dirichlet)

Figure 1.
Slot with a solid copper
conductor

$$\mathbf{J} = \mathbf{J}_{\text{ext}} + \mathbf{J}_{\text{ind}} = \mathbf{J}_{\text{ext}} + \sigma\mathbf{E}_{\text{ind}} \tag{3}$$

Using the MVP $\mathbf{A}$, so that $\mathbf{B} = \text{rot}(\mathbf{A})$, Faraday's law gives:

$$\mathbf{J} = \mathbf{J_0} - \sigma\frac{\partial\mathbf{A}}{\partial t} \tag{4}$$

Both the currents and the MVP have only a z-components. Besides, as it is assumed infinite magnetic permeability of the iron (magnetic flux density normal to the walls), and due to the symmetry of the problem, this component has no variation in the y direction, so applying Ampère law to equation (4) gives:

$$\frac{\partial^2 A}{\partial x^2} - \mu\sigma\frac{\partial A}{\partial t} = -\mu J_0 \tag{5}$$

Assuming that the all the magnitudes vary sinusoidally with time (permanent regime), equation (5) can be expressed by means of complex phasors as:

$$\frac{\partial^2 A}{\partial x^2} - j\omega\mu\sigma A = -\mu J_0 \tag{6}$$

what constitutes the diffusion equation of the magnetic potential vector in the slot.

Assuming that the top of the slot is a flux line, and that the magnetic flux density at the bottom of the slot is null the following boundary conditions hold:

$$A(0) = 0 \qquad \frac{\partial A}{\partial x}\bigg|_{x=L} = 0 \tag{7}$$

3

Equation (7) is a second order inhomogeneous differential equation with constant coefficients, so:

$$A = \underbrace{C_1 e^{\left(\sqrt{j\omega\mu\sigma}\right)x} + C_2 e^{-\left(\sqrt{j\omega\mu\sigma}\right)x}}_{\text{General Solution}} + \underbrace{\frac{J_0}{j\omega\sigma}}_{\text{Particular Solution}} \tag{8}$$

The constants C_1 and C_2 are found applying to equation (9) the boundary conditions (8), which gives:

$$A = \frac{J_0}{j\omega\sigma}\left(1 - \frac{e^{\left(\sqrt{j\omega\mu\sigma}\right)(L-x)} + e^{-\left(\sqrt{j\omega\mu\sigma}\right)(L-x)}}{e^{\left(\sqrt{j\omega\mu\sigma}\right)L} + e^{-\left(\sqrt{j\omega\mu\sigma}\right)L}}\right) \tag{9}$$

And, applying the definition of the depth of penetration factor δ (which only depends on the material properties and on the frequency):

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi50\cdot4\pi10^{-7}\cdot59.6\cdot10^6}} = 9.21\cdot10^{-3}\ \text{m} \tag{10}$$

$$\begin{aligned}A &= \frac{J_0}{j\omega\sigma}\left(1 - \frac{e^{(L-x)/\delta}e^{j(L-x)/\delta} + e^{-(L-x)/\delta}e^{-j(L-x)/\delta}}{e^{L/\delta}e^{jL/\delta} + e^{-L/\delta}e^{-jL/\delta}}\right)\\ &= \frac{J_0}{j\omega\sigma}\left(1 - \frac{\cosh((1+j)(L-x)/\delta)}{\cosh((1+j)L/\delta)}\right)\end{aligned} \tag{11}$$

The current density in the slot is given by equation (6):

$$J = J_0\frac{e^{(L-x)/\delta}e^{j(L-x)/\delta} + e^{-(L-x)/\delta}e^{-j(L-x)/\delta}}{e^{L/\delta}e^{jL/\delta} + e^{-L/\delta}e^{-jL/\delta}} = J_0\frac{\cosh((1+j)(L-x)/\delta)}{\cosh((1+j)L/\delta)} \tag{12}$$

The variation of the magnitude and phase of equation (12) with the depth in the slot is shown in Figure 2.

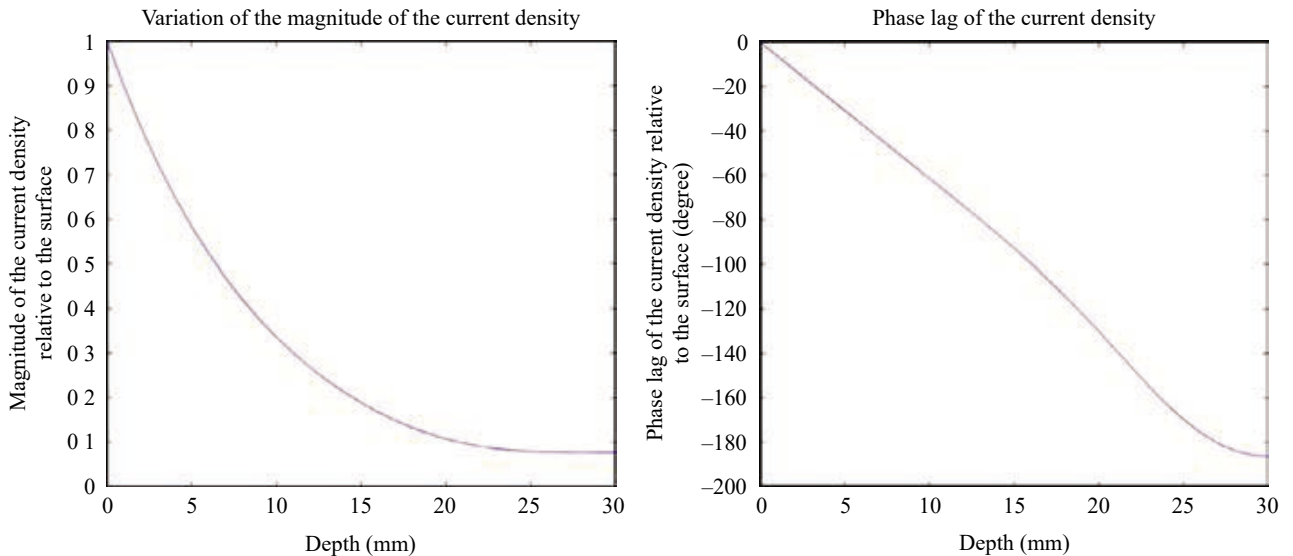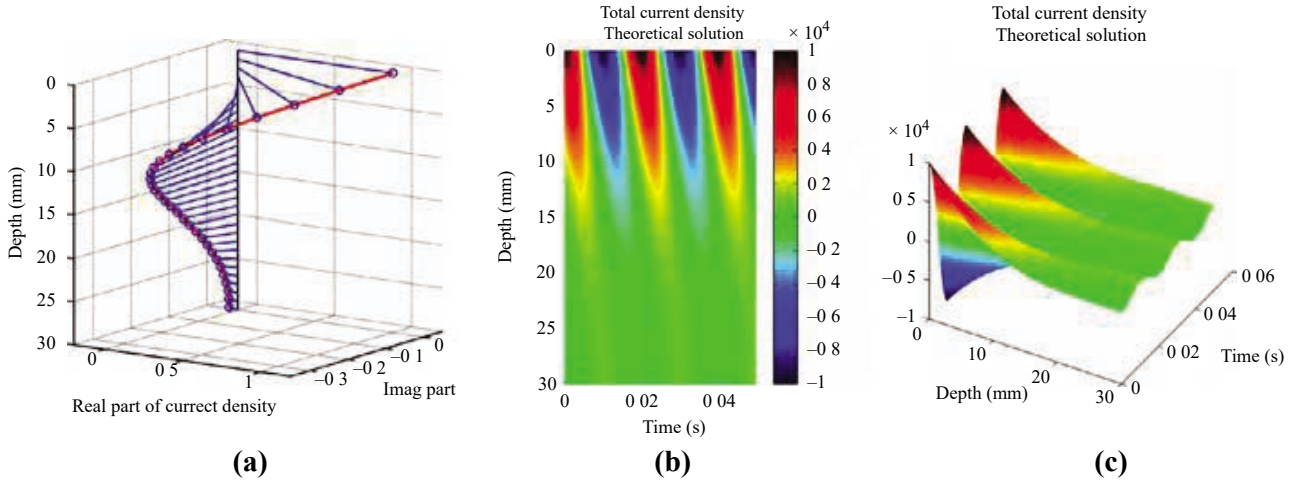Figure 3 shows the current in a phasor view, and also a 2D and 3D sinusoidal waves.

Figure 2. Variation of the magnitude of the current with the depth (left) and phase lag of the current (right), with respect to the current at the top of the slot

Notes: (a) Phasor view; (b) 2D view; (c) 3D view

Figure 3. Variation of the current density with depth, relative to the current at the top of the slot

III. Transient solution of the current in a rectangular slot in transient regime

In transient regime, the total current density in the slot J can be found by solving equation (5), which is a 1D diffusion problem. In the case of homogeneous boundary conditions, the diffusion PDE is given by:

$$
\begin{cases}
K \cdot \dfrac{\partial u(x,t)}{\partial t} \quad D \cdot \dfrac{\partial^2 u(x,t)}{\partial x^2} \quad b(x,t) & \text{in } \Omega \quad (0,\mathrm{L}), \ \mathrm{I} \quad (0,\mathrm{T}) \\[2mm]
u(0,t) \quad u(L,t) \quad 0 \text{ on } \Gamma \equiv \partial\Omega & \text{Essential boundary conditions (EBC)} \\[2mm]
u(x,0) \text{ at } t \quad 0 & \text{Initial conditions}
\end{cases}
\tag{13}
$$

The field $u(x,t)$ and the second member $b(x,t)$ are represented as separate variables products:

$$
u(x,t) \quad \sum_{i\ 1}^{n} X_i(x)T_i(t) \quad b(x,t) \quad X_b(x)T_b(t)
\tag{14}
$$

So equation (14) can be expressed (the independent variables are no longer shown for simplicity) as:

$$
K \cdot \sum_{i\ 1}^{n} X_i \frac{\partial T_i}{\partial t} \quad D \cdot \sum_{i\ 1}^{n} \frac{\partial^2 X_i}{\partial x^2} T_i \quad X_b T_b
\tag{15}
$$

Modes X_i and T_i can be computed numerically, using an iterative non-linear procedure. Suppose that the first $(n\text{-}1)$ modes are known. The nth mode is obtained via a Galerkin method using equation (13):

$$
u \quad \sum_{i\ 1}^{n\ 1} X_i T_i + X_n T_n \Rightarrow u^* \quad X_n^* T_n + X_n T_n^*
\tag{16}
$$

$$
\int_{x\ 0}^{x\ L} \int_{t\ 0}^{t\ T} u^* \quad K \cdot \sum_{i\ 1}^{n} X_i \frac{\partial T_i}{\partial t} \quad D \cdot \sum_{i\ 1}^{n} \frac{\partial^2 X_i}{\partial x^2} T_i \quad X_b T_b \Bigg) \cdot dt \cdot dx \quad 0
\tag{17}
$$

5

To find mode n an iterative procedure is followed. Suppose that the mode T_n is known at a given iteration, $(u^* \quad X_n^* \cdot T_n)$ so by substitution of equation (16) in equation (17) we get:

$$\int_x^{x\ L} X_n^* \left(K \sum_{i\ 1}^n \underbrace{\left(\int_{t\ 0}^{t\ T} T_n \frac{\partial T_i}{\partial t}\, dt \right)}_{\alpha_i} X_i \quad D \sum_{i\ 1}^n \underbrace{\left(\int_{t\ 0}^{t\ T} T_n T_i dt \right)}_{\beta_i} \frac{\partial^2 X_i}{\partial x^2} \right.$$

$$\left. \underbrace{\left(\int_{t\ 0}^{t\ T} T_n T_b dt \right)}_{\gamma} X_b \right) dx \quad 0 \tag{18}$$

This corresponds to the PDE:

$$K \sum_{i\ 1}^n \alpha_i \cdot X_i \quad D \sum_{i\ 1}^n \beta_i \cdot \frac{\partial^2 X_i}{\partial x^2} \quad \gamma \cdot X_b \quad 0 \tag{19}$$

And passing all the known terms to the second member we get:

$$K \cdot \alpha_n \cdot X_n \quad D \cdot \beta_n \cdot \frac{\partial^2 X_n}{\partial x^2} \quad \sum_{i\ 1}^{n\ 1} \left(\quad K \cdot \alpha_i \cdot X_i + D \cdot \beta_i \cdot \frac{\partial^2 X_i}{\partial x^2} \right) + \gamma \cdot X_b \tag{20}$$

This equation is solved in the spatial domain with a simple finite differences or finite element method (FEM), giving the value of X_n in the present iteration. With this value, the computation of a new value of T_n is performed:

$$\int_{t\ 0}^{t\ T} T_n^* \left(K \sum_{i\ 1}^n \underbrace{\left(\int_{x\ 0}^{x\ L} X_n X_i dx \right)}_{\beta_i'} \frac{\partial T_i}{\partial t} \quad D \sum_{i\ 1}^n \underbrace{\left(\int_{x\ 0}^{x\ L} X_n \frac{\partial^2 X_i}{\partial x^2}\, dx \right)}_{\alpha_i'} T_i \right.$$

$$\left. \underbrace{\left(\int_{x\ 0}^{x\ L} X_n X_b dx \right)}_{\gamma'} T_b \right) dt \quad 0 \tag{21}$$

This corresponds to the PDE:

$$K \sum_{i\ 1}^n \beta_i' \cdot \frac{\partial T_i}{\partial t} \quad D \sum_{i\ 1}^n \alpha_i' \cdot T_i \quad \gamma' \cdot T_b \quad 0 \tag{22}$$

And passing all the known terms to the second member we get:

$$K \cdot \beta_n' \cdot \frac{\partial T_n}{\partial t} \quad D \cdot \alpha_n' \cdot T_n \quad \sum_{i\ 1}^{n\ 1} \left(\quad K \cdot \beta_i' \cdot \frac{\partial T_i}{\partial t} + D \cdot \alpha_i' \cdot T_i \right) + \gamma' \cdot T_b \tag{23}$$

This equation is solved in the spatial domain with a simple implicit or Runge-Kutta method, giving the value of T_n in the present iteration. At each iteration the new values of X_n and T_n are compared with the previous ones, and if the absolute value of their difference falls below a predefined threshold, the iterations are finished, and the solution is updated with the new values. This process begins again for computing the next n + 1 mode. When the absolute value of the new mode falls below a predefined threshold, the solution is considered valid and the process finishes.

IV. Non-homogeneous boundary conditions

To obtain the MVP in the slot, the non-homogeneous boundary conditions at the bottom of the slot (7) must be satisfied. This kind of conditions can be imposed by adding to the solution of equation (13) a new function ψ, continuous in Ω, that verifies them (Gonzalez *et al.*, 2010). So, the solution of equation (13) can be obtained in the form $u \quad \psi + z$, and the problem becomes:

$$
\begin{cases}
K \cdot \dfrac{\partial z(x,t)}{\partial t} \quad D \cdot \dfrac{\partial^2 z(x,t)}{\partial x^2} \quad b(x,t) \quad K \cdot \dfrac{\partial \psi(x,t)}{\partial t} \quad D \cdot \dfrac{\partial^2 \psi(x,t)}{\partial x^2} \quad \text{in } \Omega \quad (0,L), I \quad (0,T) \\
z(0,t) \quad z(L,t) \quad 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{EBC} \\
z(x,0) \quad u(x,0) \quad \psi(x,0) \qquad\qquad\qquad\qquad\qquad\quad \text{IC}
\end{cases}
$$

$$(24)$$

The problem (24) gets simplified if function ψ is chosen such as:

$$
\frac{\partial^2 \psi(x,t)}{\partial x^2} \quad 0 \quad \text{in } \Omega \quad (0,L), \ I \quad (0,T) \tag{25}
$$

In 1D, this goal can be accomplished by defining a linear function such as:

$$
\psi(x,t) \quad u(0,t) + \left(\frac{u(L,t) \quad u(0,t)}{L} \right)x \quad x \in [0,L] \tag{26}
$$

V. Current in the slot solved with the separation of variables method

In the case of the slot of Figure 1, the method of separation of variables converges in 16 modes, as shown in Figure 4.

The solution obtained by the proposed method is shown in Figure 5. After a short transient, the current reaches its steady state value, equal to the one obtained analytically (Figure 3). The norm of the relative error with respect to the theoretical solution (12), computed at the last time interval, has been shown in Figure 6.

VI. Extension to higher dimensions

The proposed method can be easily extended to higher dimensions: instead of solving the spatial equations in a 2D or 3D grid, the unknown field can be expressed as a product of one-dimensional modes in the form:
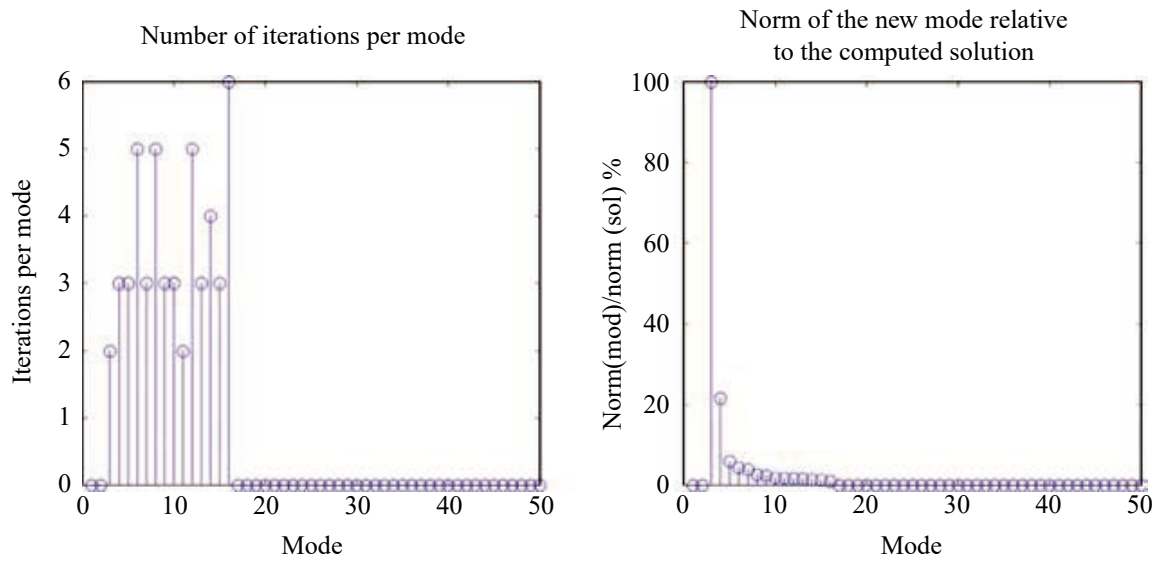
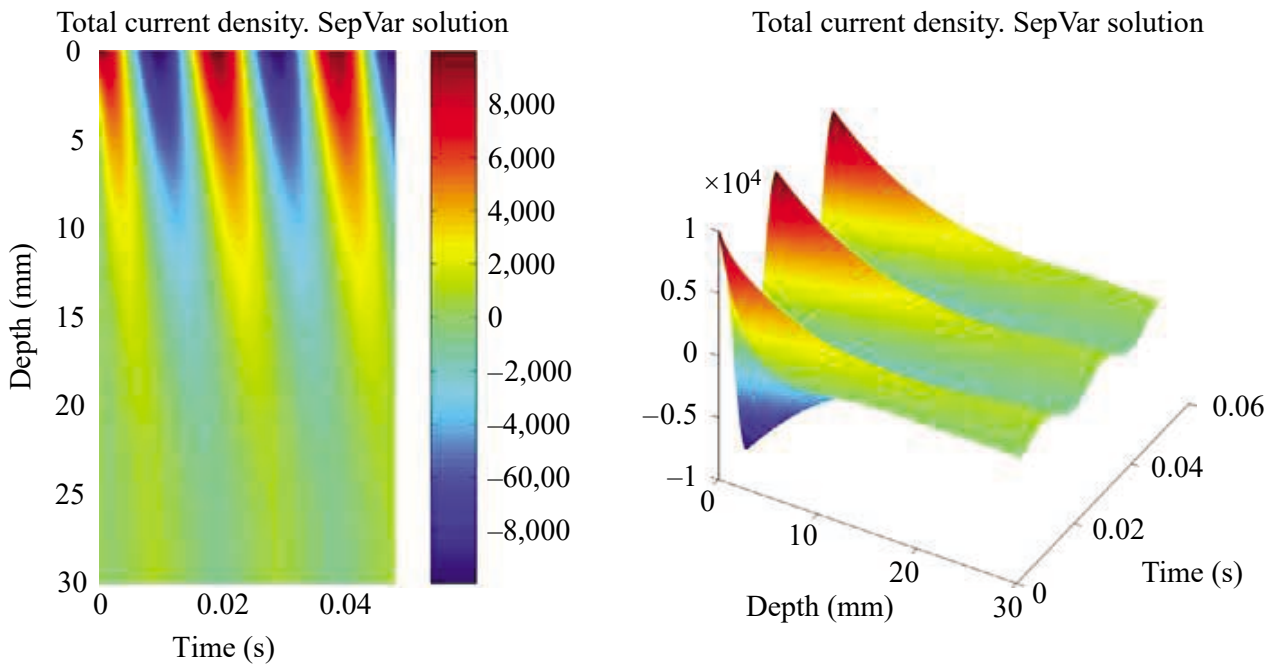**Figure 4.** Number of iterations per mode (left) and norm of each mode (right)

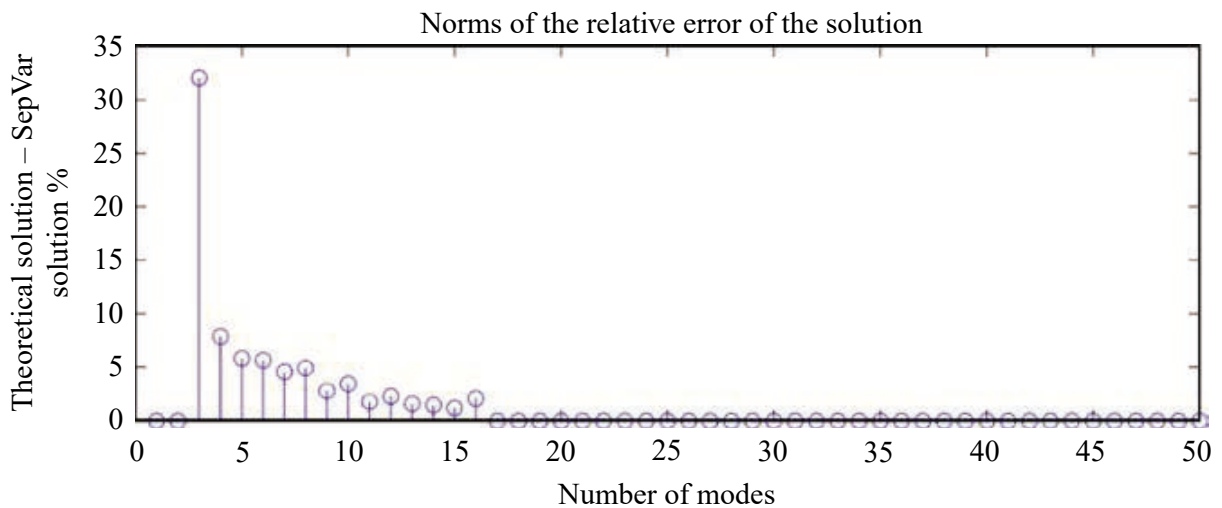**Figure 5.** Total current in the slot: 2D view (left) and 3D view (right)

Figure 6. Difference between the theoretical solution and the solution found by the variables separation method at the last time interval

$$u(x,y,z,t) \quad \sum_{i\ 1}^{n} X_i(x)Y_i(y)Z_i(z)T_i(t) \qquad (27)$$

One of the difficulties of applying this technique is the imposition of EBCs on arbitrarily-shaped spatial domains Ω. For example, a slot defined on R^2 can have a shape like that in Figure 7, left. A work in progress is to treat these general domains with the method of the characteristic function (Rvachev and Sheiko, 1995). The field u is multiplied by a real function Φ that acts as a "distance" function: it has continuous derivatives, and it is zero on the boundary and outside the irregular shape surface or volume of the slot. In Figure 6, right, such a function is shown for a slot in 2D. In this case, the method of variables separation is applied to the function:

$$u(x,y,t) \quad \Phi \cdot \sum_{i\ 1}^{n} X_i(x)Y_i(y)T_i(t) \qquad (28)$$

VII. Conclusions

In this paper, the method of separation of variables has been presented, and has been applied to solve the currents distribution in an electrical motor slot in transient regime. The numerical technique presented in this paper allows the numerical construction of the low dimensional functions whose product approximates the MVP in the slot. This approximation has been compared with the analytical solution to verify the accuracy of the method. The great advantage of this method is that it scales linearly with an increase in the dimensionality of the problem, instead of exponentially, as traditional mesh-based methods. Nevertheless, some issues like the application of the method to non-rectangular domains in high dimensions are an open research field.
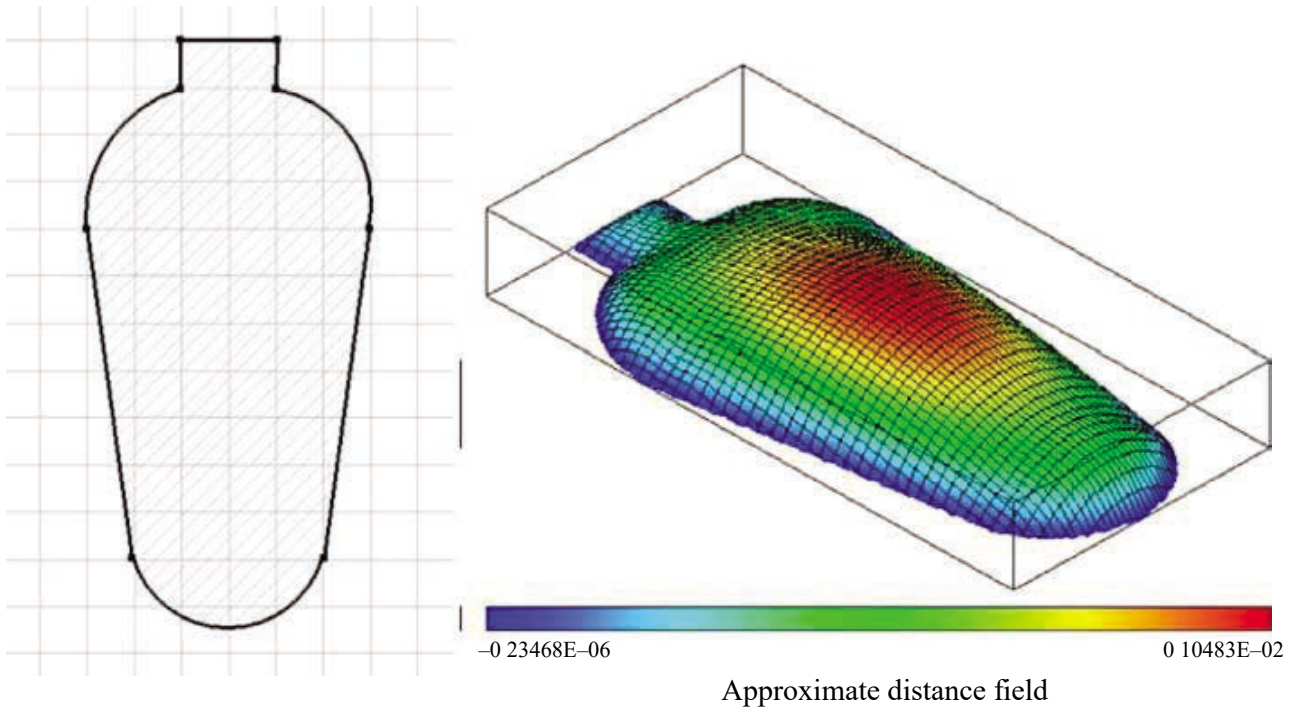

−0 23468E–06   0 10483E–02

Approximate distance field

Figure 7. Slot with a complex 2D shape (left) and distance like characteristic function Φ associated to it (right)

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M. Pineda Sanchez was born in 1962 in Albacete (Spain). He received his Dipl. Ing. and Dr Ing. degrees in electrical engineering from the Universidad Politécnica de Valencia (Spain) in 1985 and 2004, respectively. He joined the faculty of the Universidad Politécnica de Valencia in 1987 as Professor in the Department of Electrical Engineering, in the area of theory and control of electrical machines. His research interests include electrical machines and drives, induction motor diagnostics, numerical simulation of electromagnetic fields, and software development. M. Pineda Sanchez is the corresponding author and can be contacted at: mpineda@die.upv.es

F. Chinesta, born in 1966 in Valencia (Spain), is currently Professor of Computational Mechanics at the Ecole Centrale of Nantes (France) and titular of the EADS Corporate Foundation International Chair on Advanced Modeling of Composites Manufacturing Processes. His main research contribution concerns the proposal and development of efficient model reduction strategies based on the proper orthogonal decomposition and more particularly the development of separated representations based strategies making possible the solution of models suffering the so called "curse of dimensionality". F. Chinesta is author of more than 500 scientific works, with more than 140 papers in peer review journals (120 in international journals), 31 contributions to research books and more than 300 contributions to conference proceedings.

J. Roger Folch obtained his MSc degree in electrical engineering in 1970 from the Universidad Politécnica de Cataluña and his PhD in 1980 from the Universidad Politécnica de Valencia, Spain. From 1971 to 1978 he worked in the electrical industry as Project Engineer. He joined the Universidad Politécnica de Valencia in 1978 and is currently Professor of Electrical Installations and Machines. His main research areas are numerical methods (FEM and others) applied to the design and maintenance of electrical machines and equipment.

M. Riera Guasp received an MSc degree in industrial engineering and the PhD degree in electrical engineering from the Universidad Politécnica de Valencia (Spain) in 1981 and 1987, respectively. Currently he is an Associate Professor in the Department of Electrical Engineering of the Universidad Politécnica de Valencia. His research interests include condition monitoring of electrical machines, applications of the wavelet theory to electrical engineering, and efficiency in electric power applications.

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