Growth of Cuspate Spits
Frédéric Bouchette, Miguel Manna, Pablo Montalvo, Alexis Nutz, Mathieu Schuster, Jean-François Ghienne

To cite this version:
Frédéric Bouchette, Miguel Manna, Pablo Montalvo, Alexis Nutz, Mathieu Schuster, et al.. Growth of Cuspate Spits. 13th International Coastal Symposium, Apr 2014, Durban, South Africa. pp.047-052. hal-01006569

HAL Id: hal-01006569
https://hal.archives-ouvertes.fr/hal-01006569
Submitted on 15 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
INTRODUCTION

A wide range of large-scale long-standing geomorphic features occur in shallow water environments, from tens of metres of water depth to the shoreline, either in the open sea or on continental settings. Ripples, megaripples, dunes and sandwaves develop in rhythmic or isolated patterns at metre to kilometre scale (Braun, 1954; Bakker, 1968; Lomsdale and Malfait, 1974; McBride and Moslow, 1991; Reynaud et al., 1999; Lykousis, 2001; Todd, 2005; Raynal et al., 2009; Bouchette et al., 2010; Raynal et al., 2010). Sandbanks are a part of this family of bedforms and include features such as mega-dunes, bars and ridges (Dyer and Huntley, 1999). Some sandbanks, termed shoreface-connected ridges and headland-associated banks, correspond to features that develop seaward from high points connected to the coast (McBride and Moslow, 1991; Drakens, 2005). They are prograding down-drift and they usually extend down to deep waters. Obviously these local shoreline perturbations are associated with an accumulation of sand.

Zenkovitch (1959) first described cuspate spits (Figure 1) as a limited category of shore-connected features that result from symmetrical wind/wave forcings and/or peculiar initial shore configuration (Bird, 1994; Coco and Murray, 2007). Asthon et al. (2001) and Asthon and Murray (2006) proposed that cuspate spits, flying spits and other shoreline features are derived from instabilities inherent in the relationship between alongshore sediment transport and local shoreline orientation. They presented a comprehensive weakly non-linear theory for cuspate and spit dynamics, and gave a striking numerical solution to the problem.

The present work focuses on cuspate spits, also termed foreland or v-notches (Gilbert, 1885; Gulliver, 1896; Fisher, 1955; Zenkovitch, 1959), which are slightly symmetrical shoreline-connected features that grow along the shoreline of shallow water environments. Cuspate spits belong to the class of self-similar pattern. That is to say, as the time proceeds, the shoreline varies whilst remaining geometrically similar. From this point, we develop a new formulation for the dynamics of cuspate spits. Our approach relies on classical paradigms such as a conservation law to the shoreface scale and an explicit formula for alongshore sediment transport. We derive a non-linear diffusion equation and a fully explicit solution for the growth of cuspate spits. From this general expression, we found interesting applications to quantify shoreline dynamics in the presence of cuspate spits. In particular, we point out a simple method for the datation of a cuspate spit given a limited number of input parameters. Furthermore, we develop a method to quantify the mean alongshore diffusivity along a shoreline perturbed by well-defined cuspate spits of known sizes. Finally, we introduce a formal relationship between the geometric characteristics (amplitude, length) of cuspate spits, which reproduce the self-similarity of these geomorphic features.

The Non-Linear Pelnard-Considère Equation

In this work, we make the assumption that seabed and shoreline changes driven by strict cross-shore dynamics smooth and counterbalance over time. We consider that the consequence for the net change in the shoreline position over years is weak (Ruessink and Terwindt, 2000; Marino-Tapia et al., 2007). Indeed, at a long time scale, mean cross-shore transport equals zero. The significant contribution to the long term shoreline change is thus from longshore dynamics...
This assumption is at the origin of the formulation proposed here for the development of cuspate spits. Having this in mind, a basic mass balance equation states that the volume of sand required to move a profile cross shore is the shift of shoreline times the height of the active profile. Let be $y = S(x, t)$ in the equation of shoreline position in a fixed $(x, y)$ coordinate system with the $x$-axis oriented alongshore, the $y$-axis oriented offshore and t the time (Figure 2A). $S(x, t)$ satisfies:

$$\frac{dS}{dt} + \frac{1}{h_0 + B} \frac{dQ_L}{dx} = 0 \quad (1)$$

where $h_0$ is the closure water depth (seaward of which no significant transport occurs), $B$ is the active berm height, $h_0 + B$ is the height of the active profile (Figure 2B). The total amount of sediment transported alongshore $Q_L$ is related to the alongshore flux of energy available for the nearshore per unit length along the shoreline (Inman and Bagnold, 1963):

$$Q_L(x) = \frac{KF_L(x)}{(\rho_s - \rho)g(1 - p)} \quad (2)$$

where $\rho_s$ and $\rho$ are densities of sediment and water respectively, $p$ is the porosity, $g$ is the acceleration of gravity. The dimensionless parameter $K$ is an empiric constant. The energy flux to the beach $F_L$ is defined by:

$$F_L = C_g F_0 \cos(\delta_0 - \theta) \sin(\delta_0 - \theta) \quad (3)$$

where $\cos(\delta_0 - \theta)$ is the ratio of incoming energy that flows from the closure water depth through the nearshore to the shoreline. In other words, it is the ratio of energy between two infinitely (dL) close wave rays that acts on an infinitely small dx shoreline segment (Figure 2A). Hence, $\sin(\delta_0 - \theta)$ is the longshore contribution of the total incoming energy. At the closure water depth $h_0$, the incoming energy is classically defined with the expression derived from linear wave theory:

$$F_0 = \frac{1}{8} \rho g H_0$$

with $H_0$ representing the wave height. This energy propagates at the group velocity $C_g$. This velocity must be calculated at the point where the energy flows into the active domain, that is at the closure water depth $h_0$. In this case, the linear wave theory provides the simple formulation:

$$C_g = \frac{g}{4\pi T_0}$$

The longshore transport rate $Q_L$ is thus:

$$Q_L = 2CL \cos(\delta_0 - \theta) \sin(\delta_0 - \theta) \quad (4)$$

with:

$$C_L = \frac{Kppg^2T_0}{64\pi(\rho_s - \rho)(1 - p)} \quad (5)$$

Several formulations for the alongshore transport rate were successively derived from Eq. (3) (e.g. Komar and Inman, 1970; Komar, 1971; Bailard, 1984). Reviews and compared analyses of alongshore transport formulae were also performed (Bayram et al., 2001). Here the sediment transport is strictly controlled by $F_L$, the
latter works concerned more specifically rythmic shoreline
axis (Figure 2C) satisfies:

connected features including foreland spits originate in
patterns like beach cusps. These works argued that shoreline –
truncated series of eigenfunctions of the linear problem” (Calvete
with an “expansion of the flow and the bottom perturbations in a
spits. Another striking idea was to introduce some non-linearity
et al. (2001) focussed on rythmic foreland
introduced a diffusion coefficient that depends on θ (and may be
but not equal) to that of Dean and Dalrymple (2002) and
solved the problem numerically with an expression of Q similar
recently, Asthon et al. (2001) and Asthon and Murray (2006)

Figure 2. Sketches for the design of the mathematical model (A) a plan view of the nearshore and shoreline Wherein the x-axis is oriented
longshore and the y-axis is oriented seaward. From deep water, waves (example of Ray 1 and Ray 2) propagate from the top to the
bottom of the figure and refract depending upon bathymetric contours (B) Definition of the normal n to the shoreline, the berm height B,
and of the closure water depth h₂. (C) Relative orientation of vectors and angles used in the paper.

longshore portion of flux of energy ε₀ per shoreline unit length.
No matter what type of wave transformation occurs in the
nearshore, the only significant information is the fact that δ₀ = θ
varies along the shoreline and that the energy that flows in the
nearshore up dip of the shoreline depends upon this. This point of
view is quite different from that chosen by other authors (for more
explanations, see Ashton et al., 2001).

The combination of Eqs (4) and (1) is a model of long-term
shoreline changes S(x, t) under mean wave forcings and mean
sediment texture conditions. To date, several strategies have been
tested to find solutions for this kind of problem. First, Pelnard-
Considère (1956) (and a significant amount of subsequent
literature) linearized the problem so that a single linear diffusi on
equation describes the planform evolution of S(x, t). More
recently, Asthon et al. (2001) and Asthon and Murray (2006)
solved the problem numerically with an expression of Q similar
(but not equal) to that of Dean and Dalrymple (2002) and
introduced a diffusion coefficient that depends on θ (and may be
thus negative). Asthon et al. (2001) focussed on rythmic foreland
spits. Another striking idea was to introduce some non-linearity
with an “expansion of the flow and the bottom perturbations in a
truncated series of eigenfunctions of the linear problem” (Calvete
et al., 2002; Falquès et al., 2008) which is not discussed here. The
latter works concerned more specifically rythmic shoreline
patterns like beach cusps. These works argued that shoreline-
connected features including foreland spits originate in
instabilities. We provide a new solution to the problem.

The angle θ between the local normal to the shoreline and the y-
axis (Figure 2C) satisfies:

\[ \sin \theta = \frac{dS}{dx} \]
\[ \cos \theta = \frac{1}{\sqrt{1 + (dS/dx)^2}} \]

(6) and (7)

Eq. (4) can be rewritten in the following Eq (8):

\[ Q_L = C_L \left[ \sin 2\delta_0 (\cos^2 \theta - \sin^2 \theta) - 2 \cos 2\delta_0 \sin \theta \cos \theta \right] \]

Developing Eqs (6), (7) in Taylor series until order two in \( \partial S/\partial x \),
and combining Eqs (1) and (8) results in

\[ \frac{dS}{dt} = G_0 \cos 2\delta_0 \frac{d^2 S}{dx^2} + 2G_0 \sin 2\delta_0 \frac{dS}{dx} \frac{d^2 S}{dx^2} \]

This is a nonlinear diffusion equation. When waves are directed
along the x-axis (alongshore wind/wave forcings) \sin 2\delta_0 is zero
and Eq. (9) reduces to a classical diffusion equation (Pelnard-
Considère, 1956) with G₀ the longshore diffusivity. Another way
to obtain Pelnard-Considère is to linearize Eq. (9). For this reason,
we could name Eq. (9) the “non-linear Pelnard-Considère
equation”. In such a formulation, G₀ is given by:

\[ G_0 = \frac{C_L}{h_0 + B} \]

(10) and one will notice that

\[ G_0 = \mathcal{G}(H_0, T_0, \delta_0, \rho, \rho_s, p, h_0 + B) \]

(11)

which means that G₀ is a function of wave properties, sediment
properties and basic geometrical informations.

Derivation of the Cuspate Equation

From Eq. (11) we know that G₀ depends upon most of the
‘environmental’ variables of the problem, i.e., those relative to the
geometrical context and the forcings. As long term dynamics are
mostly driven by mean values averaged to the historical/
geological time scale, we can consider that H₀, T₀, \rho, \rho_s, p are
constant through time or that they vary very slowly. In the same
manner, h₀ + B may not change as sea bottom is always in
equilibrium (Short, 1999, p. 45, Fig 3).

Let us consider the following particular scenario in the frame (0,
x, y) (Figure 2A). At \( t = 0 \), we have a non perturbed shoreline for \( x \)
in \([-\infty, +\infty]\). At time \( t_0 \), an x-symmetric and positively defined
perturbation appears that develops on both sides of the origin O
and extends in \([-x_f, +x_f]\), being zero beyond. The building of
such a cuspate spit supposes that the longshore sediment
transport results from two main dominant forcings varying close
enough to \( \delta_0 = \pm\pi/4 \). Under these conditions Eq. (11) splits in
two equations with solutions \( S_R \) (for \( \delta_0 = +\pi/4 \)) and \( S_L \) (for \( \delta_0 = -\pi/4 \))
satisfying:
\[ \frac{dS_{R/L}}{dt} = 2G_0 \sin(\pm \pi/2) \frac{dS_{R/L}}{dx} \frac{d^2S_{R/L}}{dx^2} \] (12)

As one can consider that the two forcings compete through time, we can substitute the real system represented by the two Eqs. (12) by a model based on a distributed solution satisfying:

\[ \frac{dS}{dt} = \begin{cases} 0 & \text{for } x \leq -x_f \\ 2G_0 \frac{dS_R}{dx} \frac{d^2S_R}{dx^2} & \text{for } -x_f < x \leq 0 \\ -2G_0 \frac{dS_L}{dx} \frac{d^2S_L}{dx^2} & \text{for } 0 \leq x \leq x_f \\ 0 & \text{for } x \geq x_f \end{cases} \] (13)

We already recalled that cuspate spits possess self-similar patterns. Thus it is obvious to take into account the \((x,t)\) dependence of \(S\) through a self-similar variable \(\xi\) so that:

\[ (x,t) \leftrightarrow (\xi, \tau) \quad \text{with} \quad \tau = \frac{x}{t^{1/3}} \] (14)

Applying this variable substitution to the operators \(d/dx\) and \(d/dt\), we derive a new writing of Eq. (13):

\[ 2G_0 S_{R\xi} - \frac{1}{3} \xi = 0, \xi \in \left[ -\xi_0, 0 \right] \] (15)

Integrating twice, we obtain another expression with 4 distinct constants to be determined by the geometrical behavior of the cuspate spit. We impose to \(S\) to be continuous and positively defined at \(\xi=0\). In addition, we impose discontinuity of the derivative of \(S\) at \(\xi=0\). And we impose to \(S\) to be zero at the points \(\xi_0\) where \(S_\xi=0\). We obtain a set of equations with a single unknown parameter \(a\). Going back to the original coordinates \((x,y)\), we get a new equation. This equation is an exact solution to the problem developed in Eq. (9) adapted to the growth of any cuspate spit. Figure 3 displays some examples of plots of the expression \(S(x,\tau)\) derived here at various arbitrary times and for various values of the control parameters. Each curve could be cuspate spits like those in Figure 1. The expression of the solution in the original coordinates is given by equation (16) given below.

\[ \frac{dS}{dt} = \begin{cases} 0 & \text{for } x \leq -\sqrt[3]{6at^{1/3}} \\ \frac{1}{2G_0} \left( \frac{2a \sqrt{6a}}{3} + \frac{ax^3}{t^{1/3}} - \frac{x^3}{18t} \right) & \text{for } -\sqrt[3]{6at^{1/3}} \leq x \leq 0 \\ \frac{1}{2G_0} \left( \frac{2a \sqrt{6a}}{3} - \frac{ax^3}{t^{1/3}} + \frac{x^3}{18t} \right) & \text{for } 0 \leq x \leq \sqrt[3]{6at^{1/3}} \\ 0 & \text{for } x \geq \sqrt[3]{6at^{1/3}} \end{cases} \]

At this stage the model must be developed further. Indeed, unlike the longshore diffusivity \(G_0\) which affects the ability of the system to transport sediment alongshore, the parameter \(a\) has no clear physical meaning as it simply results from an integration process. The plots in the Figure 3 are consistent with highly symmetric geomorphic features; but, at this stage, we have no way to use the model for applications.

### Using the Cuspate Model

From Eq. (27), the length \(\lambda(t)\) of the foreland spit is:

\[ \lambda(t) = 2 \sqrt{6at^{1/3}} \] (17)

For \(x=0\), Eq. (16) results in:

\[ S(0) = \sqrt{6a} \sqrt[3]{t^{1/2}} / (3G_0) \] (18)

Making power three Eq. (28), and deleting \(a\) to the power of 3/2 from equations, we get the expression:

---

**Figure 3.** Various cuspate spits calculated from the mathematical model proposed in Equation (16). The different plots correspond to the configurations given below. Parameters to be set are \(G_0\) and \(a\). \(G_0\) is given by \(K = 0.77\), \(\rho = 1025\) kg.m\(^{-3}\), \(\rho_s = 2400\) kg.m\(^{-3}\), \(H_0=1.5\) m, \(T_0 = 10\) s, \(\kappa = 0.55, h_0 = 10\) m. The definition of the parameter \(a\) (arising from the integration) changes in the various plots: (A) parameter \(a\) is a constant set to 20; (B) parameter \(a\) ranges from 16 to 20; (C) parameter \(a\) ranges from 10 to 14; (D) parameter \(a\) ranges from 5 to 11; the various curves represent cuspate geometry from 1 year to 450 years.
If \( t_p \) is the present time, one may say that
\[
\Delta t = t_p - t_0
\]
is the time required to nucleate and to develop the cuspate spit from an initial moment \( t_0 \). For convenience, let us rename \( S(0) \) as \( A_p \). With this formalism, Eq. (19) can be rewritten as:
\[
\Delta T = \frac{\lambda_p^2}{2G_0} \cdot \frac{s}{72A_p}
\]
where \( A_p \) and \( \lambda_p \) are the amplitude and the alongshore length of the cuspate spit at the present day, respectively. These parameters can be easily measured on an aerial photograph or by satellite imagery (Figure 4) following the simple protocol described below.

Figure 4. Sketch of two cuspate spits along the Carolina Coast, from Ashton et al. (2001). The two cuspates (1 and 2) display well defined wave lengths \( \lambda \) and wave amplitudes \( A_p \). Those parameters can be estimated with the methodology described in the text.

To quantify \( A_p \) and \( \lambda \), for a given cuspate, the following method is applied: a) draw a line at the tangent to the shoreline; it intersects the shoreline in two points. This defines a segment which is the cuspate length \( \lambda \); (b) from the head of the cuspate spit, draw the perpendicular line to the segment defined in (a); the point at their intersection and the point at the head of the cuspate form a new segment whose length is the amplitude \( A_p \) of the cuspate spit.

Eq. (21) can be considered as a method for the datation of a cuspate spit knowing the mean alongshore diffusivity and the geometrical features of the cuspate in the present day. Alternatively, the cuspate model can be thought as a tool to quantify the mean alongshore diffusivity \( G_0 \) having information on the geometry of the cuspate spit and its age. Indeed, considering that the present day geometry of a cuspate is known, as well as its age \( \Delta t \) (e.g. dated by its occurrence in an artificial continental water body dammed at a given period; or directly dated by any well-adapted datation method). This information can be robustly retrieved through field work by geologists. From Eq. (21) it reads directly:
\[
2G_0 = \frac{\lambda_p^3}{72A_p} \cdot \Delta T
\]
The calculation of such a mean alongshore diffusivity from parameters expressed on a geological/historical time scale is interesting for the classification of the paleo-system concerned with respect to well-known existing littoral systems (such as described in Dean & Dalrymple (2002)).

Another obvious application is to calculate the mean active profile (or the mean wave height or period) for the system at a given time knowing the present day geometry and all the other long-term mean forcings. Combining Eqs (5), (10) and (21), we have:
\[
h_0 + B = 9K_0 H_0^2 T_0 A_p \Delta T
\]
The benefit of this last application could be debated for recent cuspate spits. Indeed, one will claim that there exist other ways of calculating an active profile directly from data in the field (e.g. considering the analysis of wave data at a buoy, and the subsequent extraction of a closure depth). Nevertheless, for ancient cuspate spits it seems obvious that the calculation of \( H_0^2 T_0 \) directly from Eq. (23) would provide significant semi-quantitative restraints for the reconstruction of paleo-wave regimes. Indeed, the parameters \( \rho, \rho_s \) and \( \rho_p \) could be determined by analysis of sand properties within the cuspate deposits. \( \lambda_p \) and \( A_p \) could be determined by direct surface observation (or seismic investigation if the cuspate is buried), \( h_0 \) could be determined by the identification of the location where wave ripples vanish and \( B \) by the identification of the position of the shoreline and the highest location concerned by wave impact.

**DISCUSSION**

The cuspate model offers the opportunity to calculate useful parameters for coastal engineering and geosciences. But applications under real conditions are beyond the objective of this paper, although they could be presented in future works.

Nevertheless, the cuspate model highlights a more fundamental result. In their numerical analysis of the growth of cuspate spits, Ashton & Murray (2006) suggested that there exists an approximate relationship between the age and the length of a cuspate feature. They wrote (Eq. (10) in their paper):
\[
\Delta t \approx \frac{K_0}{D_{sf}} \left[ H_0^{12/5} T_0^{-1/5} \right] \Delta x^2
\]
where \( \Delta t \) is our \( \Delta T \) (the age of the cuspate), \( \Delta x^2 = \lambda_p^2 \) the cuspate length squared, and \( D_{sf}, K_0 \) and other variables can be considered as constant and are ignored in the following. The authors derived this approximate relationship from the equation at the origin of their numerical model and from an analysis of numerical simulations.

In this paper we demonstrate with Eq. (21) that a formal relationship between the age and the length of the cuspate is mathematically correct, and that this relationship is definitively controlled by a power law as Ashton & Murray (2006) suggested. Although the term to the power is different (3 in our case; 2 in the case of Ashton & Murray), these two totally independent proofs give more confidence in the reality of such a relationship. The occurrence of a power law suggests also that more underlying physics remain to be analyzed.

Finally, Ashton & Murray (2006) sustained the idea that, contrary to a traditional point of view, the wave angle in deep water strongly controls shoreline perturbations through their so-called anti-diffusional high wave angle instability. In this paper we demonstrate that deep water wave features \( H_0 \) and \( T_0 \) drive non-linear shoreline instabilities in a simple way with absolutely no dependance on what occurs to waves in the nearshore. This is also
a formal confirmation of what was suggested by Ashton & Murray (2006). Fourth, we demonstrate that there is no need to transform – by quite complex and obscure operations – shallow water variables into deep water ones (Equations (1), (4) and (5) of Ashton & Murray, 2006) to get this result.

CONCLUSION

From what can be called the non-linear Pelnard-Considère equation, we develop an explicit formulation of the growth of a symmetrical cuspate spit through time. This formulation can be applied (a) to the calculation of the age of a cuspate spit, (b) to the determination of a mean alongshore diffusivity in the vicinity of a cuspate spit, or (c) to the calculation of a paleo-active profile or information on paleo-wave regimes after some simple geological field data has been acquired. More substantially, the paper confirms the works of Ashton & Murray (2006) in the sense that it provides an alternative formal proof of what was suggested. In a near future the cuspate model will be more deeply explored, tentatively extended to other geomorphic features and engineering/geological applications will be engaged to confirm its relevancy.

ACKNOWLEDGEMENT

This work was funded by NUCLEASPIT (CRNS Mathematics and Physics) and by KUNSHEN (ANR international program). The authors thank GLADYS (www.gladys-littoral.org) and SO LTC (www.solte.org) for the comments on the original work as well as the reviewer of the previous version of the document.

LITERATURE CITED


Pelnard-Considère, R., 1956. Essai de théorie de l’évolution des formes de CONTINUUM. From what can be called the non-linear Pelnard-Considère equation, we develop an explicit formulation of the growth of a symmetrical cuspate spit through time. This formulation can be applied (a) to the calculation of the age of a cuspate spit, (b) to the determination of a mean alongshore diffusivity in the vicinity of a cuspate spit, or (c) to the calculation of a paleo-active profile or information on paleo-wave regimes after some simple geological field data has been acquired. More substantially, the paper confirms the works of Ashton & Murray (2006) in the sense that it provides an alternative formal proof of what was suggested. In a near future the cuspate model will be more deeply explored, tentatively extended to other geomorphic features and engineering/geological applications will be engaged to confirm its relevancy.

ACKNOWLEDGEMENT

This work was funded by NUCLEASPIT (CRNS Mathematics and Physics) and by KUNSHEN (ANR international program). The authors thank GLADYS (www.gladys-littoral.org) and SO LTC (www.solte.org) for the comments on the original work as well as the reviewer of the previous version of the document.

LITERATURE CITED


