Uniform time-decay of semigroups of contraction
Xue Ping Wang

To cite this version:
Xue Ping Wang. Uniform time-decay of semigroups of contraction. Integral Equations and Operator Theory, Springer Verlag, 2012, 73 (1), pp.3- 4. <hal-01006429>

HAL Id: hal-01006429
https://hal.archives-ouvertes.fr/hal-01006429
Submitted on 16 Jun 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Uniform time-decay of semigroups of contractions

Xue Ping Wang

Abstract. We discuss the uniform time-decay of semi-groups generated by dissipative Schrödinger operators in the semiclassical regime.

Mathematics Subject Classification (2010). 35J10, 35P15, 47A55.

Keywords. Dissipative Schrödinger operators, semiclassical analysis.

High frequency analysis of propagation of waves in media with variable absorption index leads to the following dissipative Schrödinger equation:

\[
\begin{cases}
  i\hbar \frac{\partial}{\partial t} u^h(x, t) = P(h)u^h(x, t), \\
  u^h(x, 0) = u^h_0(x),
\end{cases}
\]

where \( P(h) = -\hbar^2 \Delta + V_1(x) - i\hbar V_2(x), x \in \mathbb{R}^n, h [0, h_0] \) is a small parameter proportional to wave length and \( V_j, j = 1, 2, \) are real functions with \( V_2 \geq 0 \) and \( V_2 \neq 0 \). Assume that \( V_j \) is smooth, satisfying for some \( \rho > 0 \)

\[
|\partial_x^\alpha V_j(x)| \leq C_\alpha \langle x \rangle^{-\rho - |\alpha|}, \quad j = 1, 2;
\]

here \( \langle x \rangle = (1 + |x|^2)^{1/2} \). Let \( S_h(t) = e^{-itP(h)/\hbar}, t \geq 0, \) be the associated semigroup of contractions in \( L^2(\mathbb{R}^n) \). Then \( ||S_h(t)|| \leq 1 \) for all \( t \geq 0 \) and \( h \in [0, h_0] \). The interplay between propagation along the flow of the Hamiltonian \( p_1(x, \xi) = \xi^2 + V_1(x) \) and the dissipation governs the long-time behavior of solutions. A natural question in this connection is the following

**Question.** Can one establish a uniform time-decay estimate for the semigroup \( S_h(t) \) in the form

\[
||\langle x \rangle^{-s}S_h(t)\langle x \rangle^{-s}|| \leq w(t), \quad t > 0,
\]

uniformly in \( h \in [0, h_0] \)? Here \( s > 0 \) and \( w(t) \) is independent of \( h \) and such that \( w(t) \to 0 \) as \( t \to \infty \).

When \( n = 3 \) and \( \rho > 2 \), one can show that for each fixed \( h > 0 \), one can take \( w(t) = C_h(t)^{-r} \) for any \( r < s \) and \( r \in [0, 3/2] \). In the regime \( h \to 0 \), long-time behaviors of the quantum evolution are closely related to the classical dynamics. Let \( (x(t; y, \eta), \xi(t; y, \eta)) \) denote the classical Hamiltonian flow of
$p_1(x, \xi)$ with initial data $(y, \eta)$. Making use of Egorov’s theorem, one can deduce that a necessary condition for (1) to be true is that

$$\left| \langle x(t; y, \eta) \rangle - e^{-s} \int_0^t V_2(x(\tau; y, \eta)) d\tau \langle y \rangle - s \right| \leq w(t)$$

for all $(y, \eta) \in \mathbb{R}^{2n}$. Since $w(t)$ tends to zero, estimate (2) implies that each bounded classical trajectory should pass through the open set $\{x; V_2(x) > 0\}$. In addition, one derives from (2) that $w(t) \geq C_s(t)^{-\sigma}$ with $\sigma = \min\{1, \tau_0\}$, where $\tau_0$ is the divergence rate in $t$ of non-trapping trajectories with energy 0. Is the condition (2) sufficient for a uniform time-decay estimate of the form (1)? If yes, can one take $w(t) = C_s(t)^{-\min\{1/2, \sigma\}}$? The restriction on the decay rate by $1/2$ comes from the threshold behavior of the semigroup for fixed $h$.

Recall that in the selfadjoint case ($V_2 = 0$) and with a localization in energies away from $\mathbb{R}^-$, the result is true with $w(t) = C_s(t)^{-s}$ for any $s > 0$. More precisely, let $U(t, h) = e^{-itP_1(h)/h}$, $P_1(h) = -h^2 \Delta + V_1$, $I = ]a, b[$ with $a > 0$, $\chi \in C^\infty_0(I)$. Then the estimate

$$\|\langle x \rangle^{-s} \chi(P_1(h))U(t, h)\langle x \rangle^{-s}\| \leq C_s(t)^{-\epsilon}, \quad t \in \mathbb{R},$$

holds for some $s, \epsilon > 0$ and uniformly in $h \in ]0, h_0[$ if and only if every energy $E$ in supp $\chi$ is non-trapping. If the latter is satisfied, (3) holds with $\epsilon = s$ and for any $s > 0$; see [2].

For non-selfadjoint operators, one can no longer use compactly supported cut-off. The main difficulty to prove a result like (1) is the semiclassical analysis near the threshold zero for dissipative Schrödinger operators. A closely related problem is a global limiting absorption principle on the whole real axis from the the upper half-complex plane and a nice resolvent estimate in $h > 0$. For energies away from zero and under the condition (2), this was recently obtained by J. Royer (see [1]). The question is open near the threshold zero.

References
