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V. Sládek

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Completely Regularized Integral Representations and Integral Equations for Anisotropic Bodies with Initial Strains

The present paper is devoted to the boundary formulation for anisotropic bodies subjected to a distribution of initial strains. Concerning both interior and exterior problems, the results are featured by the complete regularization of all the derived expressions: the integral representation of the displacement gradients and the stresses as well as the ordinary or derivative integral equations. The formulation also includes the case of piecewise regular boundary with edges or corners. Furthermore, explicit expressions in terms of physical meaningful boundary quantities such as the stresses related to the normal vector and the tangent derivatives of the displacements are given, which is particularly useful for further numerical implementations.

Keywords: integral equation, regularization, anisotropy, initial strains

1. Introduction

The advanced computer codes aim to incorporate a wide variety of engineering problems including various physical effects and the possibility of treating geometrical complexity. Thus, theoretically elaborated formulations are required before their numerical implementation. This becomes extremely expressive in the case of Boundary Elements Methods (BEM), when the fundamental solution of the governing equation is needed and one intends to deal with singular integrals. Sometimes there is also a necessity to treat the domain integrals in addition to boundary ones in the BEM formulations.

In this paper, we shall deal with both the domain integrals and the singularities in the BEM formulation of boundary value problems for linear elastic anisotropic bodies with initial strains in three dimensions. The initial strains or stresses can arise during the manufacturing process, but from the point of view of the solution procedure one should also deal with the same domain integrals if the incremental procedure is applied to the solution of problems in which the elastoplastic behaviour of materials is to be taken into account.

An early integral formulation of three dimensional anisotropic elastostatic boundary value problems was given by VOGEL and RIZZO [1], who obtained also the integral expression for the fundamental solution. SLADEK and SLADEK [2] and BALAS et al. [3] presented the partially regularized integral formulation, with the strong singularity being the leading one even in the derivative Boundary Integral Equations (BIE). Recently, LE VAN [4] extended such a formulation to problems with a distribution of initial strains. The presence of initial strains gives rise to serious difficulties even in the isotropic case, if the integral representations of displacement gradients or stresses are derived. Owing to the singular behaviour of the kernel contained in the volume integral of initial strains in the integral representation of displacements, this integral cannot be differentiated behind the integral sign. By using the convective derivative concept (see e.g. KUPRADZE et al. [5]), MUKHERJEE [6], BUI [7], TELLES and BREBBIA [8], BONNET [9] succeeded in deriving the correct integral representation of the stress tensor (in the case of initial or plastic strains for isotropic materials) but involving a Cauchy Principal Value (CPV) integral together with a free (convective) term. Assuming the initial strains to be Hölder continuous and making use of the subtraction regularization technique [10], one can finally remove the CPV integrals [11], [12]. LE VAN [4] has proved the theorem according to which the integral representations of displacement gradients and stresses also in the anisotropic case are made free of the CPV, containing only weakly singular, i.e. convergent improper, integrals. Nevertheless, in that formulation there are certain boundary surface integrals which are to be taken in the CPV sense as the interior point approaches the boundary. This is due to the strong singularity of the corresponding integrands which give rise to large computational errors of the numerical evaluation of such integrals even if the source point is internal but very close to the boundary. Thus, in view of the numerical integration, one should deal with not only singular but also nearly singular integrals.

In this paper, we proceed further in the regularization. The nearly strongly singular integrals are removed from the integral representations of displacement gradients by regularizing them. Performing simple algebraic rearrangements, we arrive at the regularized derivative BIE. The derivative BIE can be used successfully in a unique BEM formulation of crack problems or in numerical computation of the tangent derivatives of displacements and the stress vector at the same boundary point. The formulation includes both the interior and exterior problems in three dimensions. The boundary surface is allowed to be piecewise regular with edges and corners. The stress vector is assumed to be continuous on finite sized patches of the boundary surface.

2. Integral representation of displacements and ordinary BIE (OBIE)

Consider a three dimensional body occupying an open region Ω with boundary S , and made of a homogeneous linear elastic anisotropic material. The components of the fourth order elastic tensor C are material constants satisfying the usual symmetries: $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$. The total deformation ε in Ω is composed of the sum of two symmetrical tensors: $\varepsilon = \varepsilon^e + \varepsilon^I$, where ε^I is the initial strain and ε^e the elastic part of the strain, in the meaning that it is related to the stress tensor by $\sigma(y) = C : \varepsilon^e(y) = C : (\varepsilon(y) - \varepsilon^I(y))$, i.e. $\sigma_{ij}(y) = C_{ijkl}\varepsilon_{kl}^e(y) = C_{ijkl}\varepsilon_{kl}(y)$.

This section gives the displacement field in the body in terms of boundary quantities, together with the corresponding OBIE. Since we are concerned with both interior and exterior problems, let us state the following conditions referred to as the regularity conditions which must be satisfied for an exterior problem.

Regularity conditions:

i) The radiation condition for unknown displacement and stress is assumed as:

$u(y) = o(1)$, $\sigma(y) = o(1/r)$, i.e., the stress vector $t(y, n_y) = \sigma(y) n_y = o(1/r)$ when $r = \|y - x\| \rightarrow \infty$, x being a fixed point.

ii) The radiation condition for the body forces and the initial strains is assumed as:

$f(y) = O(r^{-2-\delta})$, $\varepsilon^I(y) = O(r^{-1-\delta'})$ when $r \rightarrow \infty$, where δ, δ' are positive constants smaller than 1.

Note that condition (ii) is identically verified if the body forces f and the initial strains ε^I are confined to a finite region. The displacement vector at any point inside the body can be expressed in the integral form as follows (see e.g. BREBBIA [13], BALAS et al. [3]):

Theorem 1: *Provided the regularity conditions are fulfilled for an exterior problem, the displacement inside the body is given by*

$$u(x) = \int_S [E(x, y) t(y, n_y) - T^T(x, y, n_y) u(y)] d_y S + \int_{\Omega}^* [E(x, y) f(y) + \varepsilon^I(y) : D(x, y)] d_y V$$

for all $x \in \Omega \setminus S$,

(1)

where the asterisk denotes an improper integral. The stress vector $t(y, n_y)$ relates to the outward normal n_y on S . E denotes the fundamental displacement tensor for an anisotropic medium without any initial strains (cf. [1]). D is the third order tensor of the fundamental stress related to tensor E by $D_{ijk}(x, y) = C_{ijpq} \frac{\partial E_{pk}}{\partial y_q}(x, y)$. T is the Kupradze tensor defined by: $T_{ik}(x, y, n_y) = C_{ijpq} n_j(y) \frac{\partial E_{pk}}{\partial y_q}(x, y)$. Eventually, the double inner product $\varepsilon^I : D$ is a vector defined as $(\varepsilon^I : D)_k = \varepsilon_{mn}^I D_{mnk}$.

In equation (1), we have written $x \in \Omega \setminus S$ although Ω is defined as an open region, just to emphasize that x must lie outside the boundary S . For exterior problems, let $B(x, R)$ denote the ball centred at a given point x with radius R large enough for $B(x, R)$ to include the boundary S . Let S_R be the exterior surface of $B(x, R)$ and Ω_R be the bounded region within S and S_R , the second regularity condition ensures that

$$\lim_{R \rightarrow \infty} \int_{\Omega_R} E(x, y) f(y) d_y V \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_{\Omega_R} \varepsilon^I(y) : D(x, y) d_y V$$

are bounded.

Theorem 2:

i) *the regularity conditions are fulfilled for an exterior problem,*

ii) $u \in C^{0,\beta}(S)$, $0 < \beta \leq 1$,

then the regularized OBIE writes

$$\begin{aligned} e \cdot u(y_0) + \int_S^* \{T^T(y_0, y, n_y) [u(y) - u(y_0)] - E(y_0, y) t(y, n_y)\} d_y S \\ = \int_{\Omega}^* [E(y_0, y) f(y) + \varepsilon^I(y) : D(y_0, y)] d_y V \quad \text{for all } y_0 \in S, \end{aligned}$$
(2)

where e is a parameter for distinguishing between the interior ($e = 0$) and exterior ($e = 1$) problems.

Proof: By using the rigid body identity $\int_S T^T(x, y, n_y) d_y S = -(1 - e) I$ for all $\Omega \setminus S$, where I is the identity tensor, we get from equation (1):

$$\begin{aligned} u(x) - (1 - e) u(y_0) = \int_S \{E(x, y) t(y, n_y) - T^T(x, y, n_y) [u(y) - u(y_0)]\} d_y S \\ + \int_{\Omega}^* [E(x, y) f(y) + \varepsilon^I(y) : D(x, y)] d_y V \quad \text{for all } x \in \Omega \setminus S \text{ and all } y_0 \in S. \end{aligned}$$
(3)

According to hypothesis (ii) one can perform the limit $x \rightarrow y_0 \in S$ by putting $x = y_0$ in equation (3) and obtain the OBIE (2). $\square$

3. Displacement gradients and stresses on the boundary

In this section, we attempt to calculate the complete displacement gradient and stress tensors at any point y_0 of the boundary, assuming the boundary displacements and stresses related to the normal to be known. The following theorem provides an efficient way to achieve this requiring no integration but only simple algebraic computations.

Theorem 3: *At any point $y_0 \in S$, let (τ, ϱ, n_{y_0}) denote a local basis with vector n_{y_0} normal to S at y_0 and unit vectors τ and ϱ lying in the tangent plane. The displacement gradients and the stresses at y_0 are related to the tangent derivatives of the displacements $u_{p,\tau}(y_0)$, $u_{p,\varrho}(y_0)$, and the stress vector $t(y_0, n_{y_0})$ related to normal n_{y_0} by the following relations:*

$$\frac{\partial u_k}{\partial y_l}(y_0) = \alpha_{klm}(y_0) t_m(y_0, n_{y_0}) + \beta_{klp}(y_0) \frac{\partial u_p}{\partial \tau}(y_0) + \gamma_{klp}(y_0) \frac{\partial u_p}{\partial \varrho}(y_0) + \omega_{klpq}(y_0) \varepsilon_{pq}^I(y_0) \quad (4)$$

for all $y_0 \in S$ and all k, l ,

$$\sigma_{ij}(y_0) = \lambda_{ijm}(y_0) t_m(y_0, n_{y_0}) + \mu_{ijp}(y_0) \frac{\partial u_p}{\partial \tau}(y_0) + \nu_{ijp}(y_0) \frac{\partial u_p}{\partial \varrho}(y_0) + \psi_{ijpq}(y_0) \varepsilon_{pq}^I(y_0) \quad (5)$$

for all $y_0 \in S$ and all i, j ,

where

$$\begin{aligned} \alpha_{klm} &= A_{km}^{-1} n_l, & \lambda_{ijm} &= C_{ijkl} \alpha_{klm}, & A_{mp} &= C_{mnpq} n_n n_q, \\ \beta_{klp} &= \delta_{kp} \tau_l - B_{kpq} n_l \tau_q, & \mu_{ijp} &= C_{ijkl} \beta_{klp}, & B_{kpq} &= A_{km}^{-1} C_{mnpq} n_n, \\ \gamma_{klp} &= \delta_{kp} \varrho_l - B_{kpq} n_l \varrho_q, & \nu_{ijp} &= C_{ijkl} \gamma_{klp}, & & \\ \omega_{klpq} &= B_{kpq} n_l, & \psi_{ijpq} &= C_{ijkl} \omega_{klpq} - C_{ijpq}. \end{aligned}$$

Proof: Substituting the displacement gradients

$$\frac{\partial u_k}{\partial y_l}(y_0) = \frac{\partial u_k}{\partial n}(y_0) n_l(y_0) + \frac{\partial u_k}{\partial \tau}(y_0) \tau_l(y_0) + \frac{\partial u_k}{\partial \varrho}(y_0) \varrho_l(y_0) \quad (6)$$

into the constitutive law, we obtain

$$A_{mp} \frac{\partial u_p}{\partial n} = t_m + C_{mnpq} n_n \left(\varepsilon_{pq}^I - \frac{\partial u_p}{\partial \tau} \tau_q - \frac{\partial u_p}{\partial \varrho} \varrho_q \right).$$

Multiplying this relation by $\sum_m A_{km}^{-1}$, we get

$$\frac{\partial u_k}{\partial n} = A_{km}^{-1} t_m + B_{kpq} \left(\varepsilon_{pq}^I - \frac{\partial u_p}{\partial \tau} \tau_q - \frac{\partial u_p}{\partial \varrho} \varrho_q \right). \quad (7)$$

Inserting (7) into (6), one can eliminate the normal derivative of the displacements and obtain equation (4). Hence follow the stresses (5) by applying the constitutive law. $\square$

In the particular case of isotropy, the elastic tensor is $C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$, where λ, μ are the Lamé constants. Then the tensor A is symmetrical: $A_{mp} = \mu\delta_{mp} + (\lambda + \mu)n_m n_p$. Simple but tedious calculations lead to $\det A = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{lmn} A_{il} A_{jm} A_{kn} = \mu^2(\lambda + 2\mu)$. Consequently

$$A_{ij}^{-1} = \frac{1}{2 \det A} \varepsilon_{jpr} \varepsilon_{irs} A_{pr} A_{qs} = \frac{1}{\mu} \delta_{ij} - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} n_i n_j$$

and

$$B_{kpq} = \delta_{kp} n_q + \delta_{kq} n_p + \frac{\nu}{1 - \nu} \delta_{pq} n_k - \frac{1}{1 - \nu} n_k n_p n_q.$$

Whence the tensors for calculating the stresses on the boundary are

$$\begin{aligned} \lambda_{ijm} &= \sigma_{im} n_j + \delta_{jm} n_i - \frac{1}{1 - \nu} n_i n_j n_m + \frac{\nu}{1 - \nu} \delta_{ij} n_m, \\ \mu_{ijp} &= \mu(\delta_{ip} \tau_j + \delta_{jp} \tau_i) - \mu(n_i \tau_j + n_j \tau_i) n_p + 2\mu \frac{\nu}{1 - \nu} (\delta_{ij} - n_i n_j) \tau_p, \\ \nu_{ijp} &= \mu(\delta_{ip} \varrho_j + \delta_{jp} \varrho_i) - \mu(n_i \varrho_j + n_j \varrho_i) n_p + 2\mu \frac{\nu}{1 - \nu} (\delta_{ij} - n_i n_j) \varrho_p, \\ \psi_{ijpq} &= \mu(\delta_{ip} n_j n_q + \delta_{iq} n_j n_p + \delta_{jp} n_i n_q + \delta_{jq} n_i n_p - \delta_{ip} \delta_{jq} - \delta_{jp} \delta_{iq}) - 2\mu \frac{\nu}{1 - \nu} (\delta_{ij} - n_i n_j) (\delta_{pq} - n_p n_q). \end{aligned}$$

Eventually, the stress tensor on the boundary writes

$$\begin{aligned}
\sigma_{ij} = & t_i n_j + t_j n_i + \frac{(t \cdot n)}{1 - \nu} (\nu \delta_{ij} - n_i n_j) \\
& + \mu \left(\frac{\partial u_i}{\partial \tau} \tau_j + \frac{\partial u_j}{\partial \tau} \tau_i \right) - \mu (n_i \tau_j + n_j \tau_i) \frac{\partial u_p}{\partial \tau} n_p + 2\mu \frac{\nu}{1 - \nu} (\delta_{ij} - n_i n_j) \frac{\partial u_p}{\partial \tau} \tau_p \\
& + \mu \left(\frac{\partial u_i}{\partial Q} Q_j + \frac{\partial u_j}{\partial Q} Q_i \right) - \mu (n_i Q_j + n_j Q_i) \frac{\partial u_p}{\partial Q} n_p + 2\mu \frac{\nu}{1 - \nu} (\delta_{ij} - n_i n_j) \frac{\partial u_p}{\partial Q} Q_p \\
& + 2\mu (\varepsilon_{ip}^I n_p n_j + \varepsilon_{jp}^I n_p n_i - \varepsilon_{ij}^I) + 2\mu \frac{\nu}{1 - \nu} (\delta_{ij} - n_i n_j) (\varepsilon_{pq}^I n_p n_q - \varepsilon_{pp}^I).
\end{aligned}$$

Discarding the initial strains, we can verify that the above expression is the same as that given in BALAS et al. [3].

Although the tangent derivatives of displacements are available from the known displacements on S , such a procedure requires differentiation. Usually, the displacements are given at isolated nodal points and approximated over boundary elements. Then, the accuracy of tangent derivatives is often not satisfactory and the uniqueness across the element boundary requires C^1 continuous elements. Thus, equations (4) and (5) should not be combined with numerical differentiation for computing the tangent derivatives of displacements. For this purpose, one can use the tangent derivative boundary integral equations.

4. Integral representation of the displacement gradients

The displacement gradients and stresses at internal points are obtained by differentiating relation (1). In fact, the main difficulty lies in the presence of the volume integral of the initial strains. Since tensor $D(x, y)$ is singular as $\|y - x\|^{-2}$ for $x \in \Omega$, its derivatives with respect to x_l are not integrable over Ω , not even in the Cauchy principal value sense. This means that the volume integral in equation (1) containing ε^I cannot be differentiated behind the integral sign. By using the convective derivative formula and some basic properties established for tensors E and D in anisotropy, we obtain the following integral representation of the displacement gradients (see [4]).

Theorem 4: *Assuming that*

- i) *the regularity conditions are fulfilled for an exterior problem,*
- ii) *the boundary S is a Lyapunov surface: $S \in C^{1,\alpha}$, $0 < \alpha \leq 1$,*
- iii) *$u \in C^1(S)$,*
- iv) *$\varepsilon^I \in C^{0,\gamma}(\Omega)$, $0 < \gamma \leq 1$,*
i.e., there exists $C > 0$, so that $|\varepsilon_{ij}^I(y) - \varepsilon_{ij}^I(x)| \leq C\|y - x\|^\gamma$ for all $x, y \in \Omega$ and all i, j .

Then we have the following integral representation of the gradient of the displacements:

$$\begin{aligned}
\frac{\partial u_k}{\partial x_l}(x) = & \int_S \left\{ \frac{\partial E_{mk}}{\partial x_l}(x, y) t_m(y, n_y) - D_{mnk}(x, y) \mathcal{D}_{nl}(\partial y, n_y) u_m(y) \right\} d_y S \\
& + \int_\Omega^* \left\{ \frac{\partial E_{mk}}{\partial x_l}(x, y) f_m(y) + [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(x)] \frac{\partial D_{mnk}}{\partial x_l}(x, y) \right\} d_y V \\
& - \varepsilon_{mn}^I(x) \int_S D_{mnk}(x, y) n_l(y) d_y S - e A_{mnkl} \varepsilon_{mn}^I(x),
\end{aligned} \tag{8}$$

where $\mathcal{D}_{nl}(\partial y, n_y)$ is the tangential differential operator defined as

$$\mathcal{D}_{nl}(\partial y, n_y) = n_n(y) \frac{\partial}{\partial y_l} - n_l(y) \frac{\partial}{\partial y_n}.$$

The coefficient A_{mnkl} is defined as an integral over the sphere $\partial B(x, 1)$ centred at x with unit radius, $A_{mnkl} = \int_{\partial B(x, 1)} D_{mnk}(x, y) n_l(y) d_y S$, which does not actually depend on the point x and is merely a material constant.

In the particular case of an isotropic material, the closed form expression for $D_{mnk}(x, y)$ is known and the coefficients A_{mnkl} can be computed as follows:

$$\begin{aligned}
A_{mnkl} = & \int_{\partial B(x, 1)} D_{mnk}(x, y) n_l(y) d_y S \\
= & \frac{1}{8\pi(1 - \nu)} \int_0^\pi \int_0^{2\pi} \{ (1 - 2\nu) [\delta_{mn} r_{,k} r_{,l} - \delta_{nk} r_{,m} r_{,l} - \delta_{mk} r_{,n} r_{,l}] - 3r_{,k} r_{,l} r_{,m} r_{,n} \} \sin \theta d\theta d\varphi,
\end{aligned}$$

where ν is the Poisson ratio. Using

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} r_{,i} r_{,j} \sin \theta \, d\theta \, d\varphi = \frac{4\pi}{3} \delta_{ij}$$

and

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} r_{,k} r_{,l} r_{,m} r_{,n} \sin \theta \, d\theta \, d\varphi = \frac{4\pi}{15} (\delta_{kl} \delta_{mn} + \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}),$$

we obtain

$$A_{mnkl} = \frac{1}{15(1-\nu)} [(1-5\nu) \delta_{mn} \delta_{kl} - (4-5\nu) (\delta_{kn} \delta_{lm} + \delta_{km} \delta_{ln})].$$

Since all the kernels in the boundary integrals in (8) behave like r^{-2} as $r = \|y - x\| \rightarrow 0$, these integrals exist only in the Cauchy principal value sense, when $x \rightarrow y_0 \in S$. In order to obtain a fully regularized derivative BIE, equation (8) must be transformed into another one which contains regular integrands. Use will be made of the following lemma.

Lemma: *If $S \in C^{1,\alpha}$, $0 < \alpha \leq 1$, then one has the integral identity*

$$(1-e) \frac{\partial u_k}{\partial x_l}(y_0) = \int_S \left\{ \frac{\partial E_{mk}}{\partial x_l}(x, y) \sigma_{mn}(y_0) n_n(y) - D_{mnk}(x, y) \mathcal{D}_{nl}(\partial y, n_y) u_m(y_0) \right\} d_y S \\ - \int_S D_{mnk}(x, y) n_l(y) d_y S \cdot \varepsilon_{mn}^I(y_0) \quad \text{for all } x \in \Omega \setminus S, \text{ all } y_0 \in \Omega \cup S, \text{ and all } k, l, \quad (9)$$

where the normal n_y is outward to the domain Ω . Recall that tensors σ , ε^I , and vector u are related by the constitutive law: $\sigma = C : (\text{grad } u - \varepsilon^I)$.

Proof: Using the following results:

- the variable interchange property: $\frac{\partial E_{mk}}{\partial x_l}(x, y) = -\frac{\partial E_{mk}}{\partial y_l}(x, y)$
- the definition of operator $\mathcal{D}_{nl}$: $\mathcal{D}_{nl}(\partial y, n_y) u_m(y_0) = n_n(y) \frac{\partial u_m}{\partial y_l}(y_0) - n_l(y) \frac{\partial u_m}{\partial y_n}(y_0)$
- the rigid body identity: $\int_S D_{mnk}(x, y) n_n(y) d_y S = -(1-e) \delta_{mk}$

the right hand side of (9) can be rewritten as

$$\text{RHS of (9)} = (1-e) \frac{\partial u_k}{\partial y_l}(y_0) + \int_S D_{mnk}(x, y) n_l(y) d_y S \cdot \left(\frac{\partial u_m}{\partial y_n}(y_0) - \varepsilon_{mn}^I(y_0) \right) \\ - \int_S \frac{\partial E_{mk}}{\partial y_l}(x, y) \sigma_{mn}(y_0) n_n(y) d_y S.$$

Since

$$D_{mnk}(x, y) \cdot \left(\frac{\partial u_m}{\partial y_n}(y_0) - \varepsilon_{mn}^I(y_0) \right) = C_{mnpq} \frac{\partial E_{pk}}{\partial y_q}(x, y) \left(\frac{\partial u_m}{\partial y_n}(y_0) - \varepsilon_{mn}^I(y_0) \right) \\ = \frac{\partial E_{pk}}{\partial y_q}(x, y) \sigma_{pq}(y_0) = \frac{\partial E_{mk}}{\partial y_n}(x, y) \sigma_{mn}(y_0),$$

we get

$$\text{RHS of (9)} = (1-e) \frac{\partial u_k}{\partial y_l}(y_0) + \int_S \mathcal{D}_{ln}(\partial y, n_y) E_{mk}(x, y) d_y S \cdot \sigma_{mn}(y_0).$$

Since $S \in C^{1,\alpha}$, the Stokes theorem is valid and the last surface integral is zero. $\square$

For interior problems, another proof is possible by applying the integral representation (8) to the auxiliary state:

$$u_k(y) = u_k(y_0) + [u_{k,j}(y_0) - \varepsilon_{kj}^I(y_0)](y_j - y_{0j}) \quad (\text{linear displacement field})$$

with $f_i = 0$ and $\varepsilon_{ij}^I = 0$. For exterior problems, equation (8) is not directly applicable since the auxiliary displacement field does not satisfy the regularity condition $u(y) = o(1)$ at infinity. However, using equation (8) in a finite domain Ω_R with the boundary $S \cup S_R$, one can verify that the sum of the integrals over S_R tends to the finite value $\partial u_k / \partial y_l(y_0)$ as $R \rightarrow \infty$ and the lemma is proved.

The following theorem gives the regularized integral representations of the displacement gradients and the stresses.

Theorem 5: *If the hypotheses in Theorem 4 are fulfilled, then we have*

a) *the completely regularized integral representation of the displacement gradients*

$$\begin{aligned}
\frac{\partial u_k}{\partial x_l}(x) = & \int_S \frac{\partial E_{mk}}{\partial x_l}(x, y) [t_m(y, n_y) - t_m(y_0, n_{y_0})] d_y S \\
& - \int_S D_{mnk}(x, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \frac{\partial u_m}{\partial \tau}(y_0) \right] - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \frac{\partial u_m}{\partial \varrho}(y_0) \right] \right\} d_y S \\
& + \int_{\Omega}^* \left\{ \frac{\partial E_{mk}}{\partial x_l}(x, y) f_m(y) + [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(x)] \frac{\partial D_{mnk}}{\partial x_l}(x, y) \right\} d_y V \\
& + (1 - e) \frac{\partial u_k}{\partial x_l}(y_0) + L_{klms}(x, y_0) \frac{\partial u_m}{\partial x_s}(y_0) - M_{klmn}(x, y_0) \sigma_{mn}(y_0) \\
& - [\varepsilon_{mn}^I(x) - \varepsilon_{mn}^I(y_0)] N_{mnkl}(x) - e A_{mnkl} \varepsilon_{mn}^I(x) \\
& \text{for all } x \in \Omega \setminus S, \text{ all } y_0 \in \Omega \cup S, \text{ and all } k, l,
\end{aligned} \tag{10}$$

where $u_{m,s}(y_0)$ and $\sigma_{mn}(y_0)$ are given by equations (4) and (5), and

$$\begin{aligned}
L_{klms}(x, y_0) &= \int_S D_{mnk}(x, y) \varepsilon_{nlr} \{ \tau_r(y) [\tau_s(y) - \tau_s(y_0)] - \tau_r(y) [\varrho_s(y) - \varrho_s(y_0)] \} d_y S, \\
M_{klmn}(x, y_0) &= \int_S \frac{\partial E_{mk}}{\partial x_l}(y, x) [n_n(y) - n_n(y_0)] d_y S, \quad N_{mnkl}(x) = \int_S D_{mnk}(x, y) n_l(y) d_y S;
\end{aligned}$$

b) *the completely regularized integral representation of the stresses*

$$\begin{aligned}
\sigma_{ij}(x) = & \int_S D_{ijm}(y, x) [t_m(y, n_y) - t_m(y_0, n_{y_0})] d_y S \\
& - \int_S C_{ijkl} D_{mnk}(x, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \frac{\partial u_m}{\partial \tau}(y_0) \right] - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \frac{\partial u_m}{\partial \varrho}(y_0) \right] \right\} d_y S \\
& + \int_{\Omega}^* \{ D_{ijm}(y, x) f_m(y) + H_{ijmn}(x, y) [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(x)] \} d_y V \\
& + (1 - e) \sigma_{ij}(y_0) + P_{ijms}(x, y_0) \frac{\partial u_m}{\partial x_s}(y_0) - Q_{ijmn}(x, y_0) \sigma_{mn}(y_0) - R_{ijmn}(x) [\varepsilon_{mn}^I(x) - \varepsilon_{mn}^I(y_0)] \\
& - B_{ijmn}^e \varepsilon_{mn}^I(x) + (1 - e) C_{ijkl} \varepsilon_{kl}^I(y_0) \quad \text{for all } x \in \Omega \setminus S, \text{ all } y_0 \in \Omega \cup S, \text{ and all } i, j,
\end{aligned} \tag{11}$$

where $H_{ijmn}(x, y) = C_{ijkl} \frac{\partial D_{mnk}}{\partial x_l}(x, y)$, $P_{ijms}(x, y_0) = C_{ijkl} L_{klms}(x, y_0)$,

$$Q_{ijmn}(x, y_0) = C_{ijkl} M_{klmn}(x, y_0) = \int_S D_{ijm}(y, x) [n_n(y) - n_n(y_0)] d_y S,$$

$$R_{ijmn}(x) = C_{ijkl} N_{mnkl}(x), \quad B_{ijmn}^e = C_{ijmn} + e C_{ijkl} A_{mnkl}.$$

Proof: The differential operator $\mathcal{D}_{nl}$ can be rearranged as follows:

$$\mathcal{D}_{nl}(\partial y, n_y) = n_n(y) \frac{\partial}{\partial y_l} - n_l(y) \frac{\partial}{\partial y_n} = \varepsilon_{nlr} (\tau_s \varrho_r - \tau_r \varrho_s) \frac{\partial}{\partial y_s}.$$

This gives

$$\begin{aligned}
& \mathcal{D}_{nl}(\partial y, n_y) u_m(y) - \mathcal{D}_{nl}(\partial y, n_y) u_m(y_0) \\
&= \varepsilon_{nlr} (\tau_s \varrho_r - \tau_r \varrho_s) \left[\frac{\partial u_m}{\partial y_s}(y) - \frac{\partial u_m}{\partial y_s}(y_0) \right] \\
&= \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \frac{\partial u_m}{\partial \tau}(y_0) \right] - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \frac{\partial u_m}{\partial \varrho}(y_0) \right] \\
&\quad - \frac{\partial u_m}{\partial y_s}(y_0) \{ \varrho_r(y) [\tau_s(y) - \tau_s(y_0)] - \tau_r(y) [\varrho_s(y) - \varrho_s(y_0)] \}.
\end{aligned} \tag{12}$$

Subtracting (9) from (8) and taking (12) into account leads to equation (10). By using the constitutive law $\sigma = C : (\text{grad } u - \varepsilon^I)$ there readily follows equation (11). $\square$

If the interior point x is very close to the boundary surface S , it is appropriate to select y_0 as the nearest boundary point to x . Then, the nearly singularities of the integral kernel in surface integrals are cancelled out by the continuous behaviour of the density terms, because $\|x - y\|$ is minorized by $\|y - y_0\|$. Thus, the integral representations given by equations (10) and (11) can be efficiently used in near surface numerical calculations with eliminating the boundary layer effect [14].

In practice, a significant case to be considered is that of piecewise regular boundaries, i.e., when the surface S presenting edges or corners is made up of a finite number of smooth surfaces. By splitting up the boundary integrals and slightly modifying the previous proof, theorem 5 can be generalized to the following

Theorem 6: *Assuming that*

i) *the regularity conditions are fulfilled for an exterior problem;*

ii) $S = \sum_p S_p$, $S_p \in C^{1, \alpha_p}$, $0 < \alpha_p \leq 1$;

iii) $u \in C^1(S_p)$ for all p ;

iv) $\varepsilon^I \in C^{0, \gamma}(\Omega)$, $0 < \gamma \leq 1$,

then we have:

a) *the completely regularized integral representation of the displacement gradients*

$$\begin{aligned} \frac{\partial u_k}{\partial x_l}(x) = & \sum_p \int_{S_p} \frac{\partial E_{mk}}{\partial x_l}(x, y) [t_m(y, n_y) - \mathcal{L}_p t_m(y_0, n_{y_0})] d_y S \\ & - \sum_p \int_{S_p} D_{mnk}(x, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \mathcal{L}_p \frac{\partial u_m}{\partial \tau}(y_0) \right] - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \mathcal{L}_p \frac{\partial u_m}{\partial \varrho}(y_0) \right] \right\} d_y S \\ & + \int_{\Omega}^* \left\{ \frac{\partial E_{mk}}{\partial x_l}(x, y) f_m(y) + [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(x)] \frac{\partial D_{mnk}}{\partial x_l}(x, y) \right\} d_y V \\ & + (1 - e) \frac{\partial u_k}{\partial x_l}(y_0) + L_{klms}(x, y_0) \frac{\partial u_m}{\partial x_s}(y_0) - M_{klmn}(x, y_0) \sigma_{mn}(y_0) \\ & - [\varepsilon_{mn}^I(x) - \varepsilon_{mn}^I(y_0)] N_{mnkl}(x) - e A_{mnkl} \varepsilon_{mn}^I(x) \\ & \text{for all } x \in \Omega \setminus S, \text{ all } y_0 \in \Omega \cup S, \text{ and all } k, l, \end{aligned} \quad (13)$$

where the tensors L and M are accordingly modified:

$$L_{klms}(x, y_0) = \sum_p \int_{S_p} D_{mnk}(x, y) \varepsilon_{nlr} \{ \varrho_r(y) [\tau_s(y) - \mathcal{L}_p \tau_s(y_0)] - \tau_r(y) [\varrho_s(y) - \mathcal{L}_p \varrho_s(y_0)] \} d_y S,$$

$$M_{klmn}(x, y_0) = \sum_p \int_{S_p} \frac{\partial E_{mk}}{\partial x_l}(y, x) [n_n(y) - \mathcal{L}_p n_n(y_0)] d_y S;$$

b) *the completely regularized integral representation of the stresses*

$$\begin{aligned} \sigma_{ij}(x) = & \sum_p \int_{S_p} D_{ijm}(y, x) [t_m(y, n_y) - \mathcal{L}_p t_m(y_0, n_{y_0})] d_y S \\ & - \sum_p \int_{S_p} C_{ijkl} D_{mnk}(x, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \mathcal{L}_p \frac{\partial u_m}{\partial \tau}(y_0) \right] - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \mathcal{L}_p \frac{\partial u_m}{\partial \varrho}(y_0) \right] \right\} d_y S \\ & + \int_{\Omega}^* \{ D_{ijm}(y, x) f_m(y) + H_{ijmn}(x, y) [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(x)] \} d_y V \\ & + (1 - e) \sigma_{ij}(y_0) + P_{ijms}(x, y_0) \frac{\partial u_m}{\partial x_s}(y_0) - Q_{ijmn}(x, y_0) \sigma_{mn}(y_0) \\ & - R_{mnij}(x) [\varepsilon_{mn}^I(x) - \varepsilon_{mn}^I(y_0)] - B_{ijmn}^e \varepsilon_{mn}^I(x) + (1 - e) C_{ijkl} \varepsilon_{kl}^I(y_0) \\ & \text{for all } x \in \Omega \setminus S, \text{ all } y_0 \in \Omega \cup S, \text{ and all } i, j, \end{aligned} \quad (14)$$

where $\mathcal{S}_p$ denotes the selection operator defined by

$$\mathcal{S}_p g(y_0) = \begin{cases} \lim_{S_p \ni y \rightarrow y_0} g(y) & \text{if } y_0 \in S_p, \\ 0 & \text{if } y_0 \notin S_p. \end{cases}$$

If the boundary stress vector is prescribed discontinuously by boundary conditions on a smooth portion of the boundary, one should divide it into a finite number of smooth surface segments. The assumptions as well as the statements of Theorem 6 are satisfied also for such a case.

5. Completely regularized derivative boundary integral equations (DBIEs)

Equations (13) and (14) can be used successfully to obtain the derivative BIEs (DBIEs) for numerical calculation of the tangent derivatives of the displacements as shown by the following theorem.

Theorem 7: *Assuming that*

i) *the regularity conditions are fulfilled for an exterior problem;*

ii) $S = \sum_p S_p$, $S_p \in C^{1, \alpha_p}$, $0 < \alpha_p \leq 1$;

iii) $u \in C^{1, \beta_p}(S_p)$, $0 < \beta_p \leq 1$, *for all* p

iv) $\varepsilon^I \in C^{0, \gamma}(\Omega)$, $0 < \gamma \leq 1$

(these hypotheses imply that $t \in C^{0, \delta_p}(S_p)$, $0 < \delta_p \leq 1$, *for all* p), *then we have:*

1. *the completely regularized normal DBIE*

$$\begin{aligned} e \cdot t_i(y_0, n_{y_0}) - \sum_p \int_{S_p}^* D_{ijm}(y, y_0) [t_m(y, n_y) - \mathcal{S}_p t_m(y_0, n_{y_0})] d_y S \cdot n_j(y_0) \\ + \sum_p \int_{S_p}^* C_{ijkl} D_{mnk}(y_0, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \mathcal{S}_p \frac{\partial u_m}{\partial \tau}(y_0) \right] \right. \\ \left. - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \mathcal{S}_p \frac{\partial u_m}{\partial \varrho}(y_0) \right] \right\} d_y S \cdot n_j(y_0) \\ - P_{ijms}(y_0, y_0) \frac{\partial u_m}{\partial x_s}(y_0) n_j(y_0) + Q_{ijmn}(y_0, y_0) \sigma_{mn}(y_0) n_j(y_0) \\ = \int_{\Omega}^* \{ D_{ijm}(y, y_0) f_m(y) + H_{ijmn}(y_0, y) [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(y_0)] \} d_y V \cdot n_j(y_0) - e B_{ijmn} \varepsilon_{mn}^I(y_0) n_j(y_0) \\ \text{for all } y_0 \in S \text{ and all } i; \end{aligned} \quad (15)$$

2. *the completely regularized tangent DBIEs*

$$\begin{aligned} e \frac{\partial u_k}{\partial \tau}(y_0) + \sum_p \int_{S_p}^* D_{mnk}(y_0, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \mathcal{S}_p \frac{\partial u_m}{\partial \tau}(y_0) \right] \right. \\ \left. - \tau_r(y) \left[\frac{\partial u_m}{\partial \varrho}(y) - \mathcal{S}_p \frac{\partial u_m}{\partial \varrho}(y_0) \right] \right\} d_y S \cdot \tau_l(y_0) \\ + \left[M_{klmn}(y_0, y_0) \sigma_{mn}(y_0) - L_{klms}(y_0, y_0) \frac{\partial u_m}{\partial x_s}(y_0) \right] \tau_l(y_0) \\ = \sum_p \int_{S_p}^* \frac{\partial E_{mk}}{\partial \tau(y_0)}(y_0, y) [t_m(y, n_y) - \mathcal{S}_p t_m(y_0, n_{y_0})] d_y S \\ + \int_{\Omega}^* \left\{ \frac{\partial E_{mk}}{\partial \tau(y_0)}(y_0, y) f_m(y) + [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(y_0)] \frac{\partial D_{mnk}}{\partial \tau(y_0)}(y_0, y) \right\} d_y V - e A_{mnkl} \varepsilon_{mn}^I(y_0) \tau_l(y_0) \\ \text{for all } y_0 \in S \text{ and all } k \end{aligned} \quad (16)$$

and

$$\begin{aligned}
& e \frac{\partial u_k}{\partial Q}(y_0) + \sum_p \int_{S_p}^* D_{mnk}(y_0, y) \varepsilon_{nlr} \left\{ \varrho_r(y) \left[\frac{\partial u_m}{\partial \tau}(y) - \mathcal{L}_p \frac{\partial u_m}{\partial \tau}(y_0) \right] \right. \\
& \quad \left. - \tau_r(y) \left[\frac{\partial u_m}{\partial Q}(y) - \mathcal{L}_p \frac{\partial u_m}{\partial Q}(y_0) \right] \right\} d_y S \cdot \varrho_l(y_0) \\
& \quad + \left[M_{klmn}(y_0, y_0) \sigma_{mn}(y_0) - L_{klms}(y_0, y_0) \frac{\partial u_m}{\partial x_s}(y_0) \right] \varrho_l(y_0) \\
& = \sum_p \int_{S_p}^* \frac{\partial E_{mk}}{\partial Q(y_0)}(y_0, y) [t_m(y, n_y) - \mathcal{L}_p t_m(y_0, n_{y_0})] d_y S \\
& \quad + \int_{\Omega}^* \left\{ \frac{\partial E_{mk}}{\partial Q(y_0)}(y_0, y) f_m(y) + [\varepsilon_{mn}^I(y) - \varepsilon_{mn}^I(y_0)] \frac{\partial D_{mnk}}{\partial Q(y_0)}(y_0, y) \right\} d_y V - e A_{mnkl} \varepsilon_{mn}^I(y_0) \varrho_l(y_0) \\
& \quad \text{for all } y_0 \in S \text{ and all } k,
\end{aligned} \tag{17}$$

where $u_{m,s}(y_0)$ and $\sigma_{mn}(y_0)$ are given by equations (4) and (5), and $B_{ijmn} = C_{ijmn} + C_{ijkl} A_{mnkl}$, the asterisk denoting an improper integral applied only on that surface S_p which contains y_0 .

Proof: Let us contract equation (14) on the normal vector n_{y_0} and take the limit $x \rightarrow y_0 \in S$. The continuity requirements imposed on the boundary densities lead to the existence of the improper surface integrals as well as P_{ijms} and Q_{ijmn} in the ordinary sense. As for the integrals $N_{mnkl}(y_0)$ which exist in the Cauchy principal value sense, they do not contribute to equation (15) since they are multiplied by a vanishing term. Equations (16) and (17) are obtained in the same way. $\square$

All the improper integrals in equations (15)–(17) can be converted into integrals existing in an ordinary sense by using the local polar or spherical coordinates, respectively, for boundary and domain integrals. The normal derivative BIE (15), sometimes also called the traction BIE, is helpful in a unique formulation of crack problems (cf. [3]).

6. Conclusions

In this paper we have been concerned with the problem of anisotropic bodies with initial strains. The following results have been obtained:

The regularized integral representations of the displacement gradients and stresses have been given in equations (10) and (11), or (13) and (14). The nearly singular and/or singular integrals are eliminated.

For computation of quantities at boundary points, the straight algebraic expressions (4) and (5) have been given. In order to avoid numerical differentiation, one can compute the tangent derivatives of displacements by solving the system of the DBIE given by equations (16) and (17).

The OBIE (2) and the DBIEs (15)–(17) which are completely regularized, containing no Cauchy principal value integrals.

The formulation has been carried out for both interior and exterior problems. Also, it has included the case of bodies with a piecewise regular boundary by introducing a selector operator, equations (13)–(17), in order to unambiguously select the boundary quantities which could be nonuniquely defined on the boundaries of finite segments of the surface. Furthermore, these expressions have been explicitly written down in terms of physically meaningful boundary quantities such as the stresses related to the normal vector and the tangent derivatives of the displacements, which is particularly useful for further numerical implementations (cf. [15]).

References

- 1 VOGEL, S. M.; RIZZO, F. J.: An integral formulation of three dimensional anisotropic elastostatic boundary value problems. *J. Elasticity* **3** (1973) 3, 203–216.
- 2 SLADEK, J.; SLADEK, V.: Three dimensional crack analysis for an anisotropic body. *Appl. Math. Modelling* **6** (1982), 374–380.
- 3 BALAS, J.; SLADEK, J.; SLADEK, V.: Stress analysis by boundary equation methods. *Studies in Applied Mechanics* 23, Elsevier 1989.
- 4 LE VAN, A.: Boundary integro differential equation for anisotropic bodies with initial strains or stresses. *Appl. Math. Modelling* **21** (1997), 145–153.
- 5 KUPRADZE, V. D.; GEGELIA, T. G.; BASHELEISHVILI, M. O.; BURCHULADZE, T. V.: Three dimensional problems of the mathematical theory of elasticity and thermoelasticity. *North Holland Series in Applied Mathematics and Mechanics*, Amsterdam 1979.

- 6 MUKHERJEE, S.: Corrected boundary integral equations in planar thermoelastoplasticity. *Internat. J. Solids Struct.* **13** (1977), 331–335.
- 7 BUI, H. D.: Some remarks about the formulation of three dimensional thermoelastoplastic problems by integral equations. *Internat. J. Solids Struct.* **14** (1978), 935–939.
- 8 TELLES, J. C. F.; BREBBIA, C. A.: On the application of the boundary element method to plasticity. *Appl. Math. Modelling* **3** (1979), 446–470.
- 9 BONNET, M.: *Equations intégrales et éléments de frontière. Applications en mécanique des solides et des fluides.* Eyrolles CNRS, Paris 1995.
- 10 TANAKA, M.; SLADEK, V.; SLADEK, J.: Regularization techniques applied to boundary integral equation methods. *Appl. Mech. Reviews* **47** (1994) 10, 457–499.
- 11 DALLNER, R.; KUHN, G.: Efficient evaluation of volume integrals in the boundary element method. *Computat. Meth. Appl. Mech. Engin.* **109** (1993), 95–109.
- 12 SLADEK, J.; SLADEK, V.: Boundary element analysis for an interface crack between dissimilar elastoplastic materials. *Computat. Mech.* **16** (1995), 396–405.
- 13 BREBBIA, C. A.: *Boundary Element Method for engineers.* Pentech Press, Plymouth and London. Second revised and corrected edition 1984.
- 14 SLADEK, V.; SLADEK, J.; TANAKA, M.: Regularization of hypersingular and nearly singular integrals in the potential theory and elasticity. *Internat. J. Numer. Meth. Engin.* **36** (1993), 1609–1628.
- 15 SLADEK, V.; SLADEK, J.: Regularization of hypersingular integrals in BEM formulations using various kinds of continuous elements. *Engin. Anal. Bound. Elem.* **17** (1996), 5–18.

Addresses: Dres. Ing. habil. J. and V. SLÁDEK, Institute of Construction and Architecture, Slovak Academy of Sciences, SK 84220 Bratislava, Slovakia; Dr. A. LE VAN, Laboratoire de Mécanique des Structures et des Matériaux, Ecole Centrale de Nantes, BP 92101, F 44321 Nantes Cedex 3, France