The incompressible Navier-Stokes equations on non-compact manifolds

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Abstract. We shall prove dispersive and smoothing estimates for Bochner type laplacians on some non-compact Riemannian manifolds with negative Ricci curvature, in particular on hyperbolic spaces.

These estimates will be used to prove Fujita-Kato type theorems for the incompressible Navier-Stokes equations.

We shall also discuss the uniqueness of Leray weak solutions in the two dimensional case.

1. Introduction

This work deals with the equations describing the motion of an incompressible fluid with viscosity in a non-compact space M, more precisely, we shall study the incompressible Navier-Stokes system on a non-compact Riemannian manifold. Let us first recall some classical results for the incompressible Navier-Stokes equations in the flat case $\mathbb{R}^n$. In this framework, the unknowns are the velocity $u : \mathbb{R}_t^+ \times \mathbb{R}_x^n \to \mathbb{R}_x^n$ of the fluid, a time dependent divergence free vector field on $\mathbb{R}^n$ and its pressure $p : \mathbb{R}_t^+ \times \mathbb{R}_x^n \to \mathbb{R}$. The incompressible Navier-Stokes system takes the following form

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \operatorname{grad} p = \nu \Delta u, \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

The velocity is divergence free because of the incompressibility assumption and ν (which is the inverse of the Reynolds number when the system is written in non-dimensional coordinates) is positive since the fluid is viscous. Moreover, in cartesian coordinates the definitions of the operators arising in the previous system are: for all $j \in \{1, \dots, n\}$

$$((u \cdot \nabla)u)^j = (\nabla_u u)^j = u^i \partial_i u^j, \quad div\, u = \partial_i u^i, \quad (\operatorname{grad} p)^j = \partial_j p, \quad (\Delta u)^j = \partial_{ii} u^j.$$

where we sum over i. We notice that, in cartesian coordinates, the vectorial laplacian is made by the usual (scalar) Laplacian acting on each component of vector fields $u = (u^1, \dots, u^n)$.

We add to the system (1) an initial condition on the velocity $u_{|t=0} = u_0$, with the initial data u_0 divergence free (div $u_0 = 0$). The notion of C^2 solution (i.e. classical solution) is not efficient here. It has been pointed out by C. Ossen (see [45] and [46])

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that another concept of solution must be used. There are many notions of solutions that are appropriate for this system. The most famous ones are the Leray weak solutions that are based on the energy dissipation (see [37]) and the Kato type solutions that are based on the scaling of the equation (see [30]). One way of studying the initial value problem (NSE) is via the weak solutions introduced by Leray. Indeed, Leray and Hopf showed the existence of a global weak solution of the Navier-Stokes equations corresponding to initial data in $L^2(\mathbb{R}^n)$ (see [37], [28]). Lemarié extended this construction and obtained the existence of uniformly locally square integrable weak solutions. Questions about the uniqueness and regularity of these solutions are completely clear only when $n = 2$. In particular, in dimension 2 the energy inequality is verified and Leray weak solutions are unique for any initial data $u_0 \in L^2(\mathbb{R}^2)$ and global propagation of higher regularity holds. When $n \geq 3$, these questions have not been answered yet; the case of dimension 3 is one of millenium problems. But important contributions in understanding partial regularity and conditional uniqueness of weak solutions should be mentioned (see e.g. [8], [39], [40], [21], [19]). Because of the uniqueness problem with the weak solution in dimension $n \geq 3$, another approach was introduced by Kato and Fujita (1961) studying stronger solutions (or mild solutions) (see [20]). To define them, they use the Hodge decomposition in $\mathbb{R}^n$, i.e. for every L^2 vector field, one has the unique orthogonal decomposition

$$u = v + \operatorname{grad} q, \quad \operatorname{div} v = 0, \quad v = \mathbb{P}u, \tag{2}$$

where $\mathbb{P}$ is the Leray projector on divergence free vector fields. Formally if u solves the Navier-Stokes Cauchy problem, then applying the projector $\mathbb{P}$ to the equation, the pressure term dissapear and one gets the following Cauchy problem for this semi-linear parabolic system

$$\begin{cases} \partial_t u - \Delta u = -\mathbb{P}(u \cdot \nabla u), \\ u_{|t=0} = u_0. \end{cases} \tag{3}$$

By using the heat semi-group and the Duhamel formula, the PDE is reformulated as a fixed point problem in a suitable Banach space X_T

$$u(t, \cdot) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}(u \cdot \nabla u)\, ds.$$

Then the strong solutions are solutions of this fixed point problem. This approach of Kato allows us to get the well-posedness of the Cauchy problem to the Navier-Stokes equations locally in time and globally for small initial data in various subcritical or critical spaces. The critical spaces are the natural ones to solve the equation by the fixed point method since they are invariant under the scaling of the equation: if u is the solution, then $u^\lambda = \lambda u(\lambda^2 t, \lambda x)$ is also a solution. The main results for the critical spaces are the following: $\dot{H}^{-1+\frac{n}{2}} \subset L^n \subset \dot{B}^{-1+\frac{n}{p}}_{p,\infty} \subset BMO^{-1}$, obtained by [20], [24], [30], [57], [9], [49], [33], [5]. The largest critical spaces is $\dot{B}^{-1}_{\infty,\infty}$, but the Cauchy problem is showed to be ill-posed by [7]. Moreover, there are global well-posedness results for some classes of large data in all the above spaces, that uses the structure of the non-linear term (see for examples [36], [1], [11]).

In this paper, we shall mainly be interested in the Kato approach in the case of the Navier-Stokes equations on non-compact riemannian manifolds. The plan of the paper is as follows. In the next section, we give a more precise description of the manifolds that we shall consider and we recall some definitions and properties of Riemannian geometry and functional analysis on these non-compact manifolds. In section 3, we recall the natural way to write the Navier-Stokes equation on a non-flat manifold that was pointed out by [17], [54]. Note that the issue is that we need a Laplacian acting on vector fields and that there is no canonical object of this type on a manifold (there are many possibilities such as the Hodge Laplacian, the Bochner Laplacian). We shall also explain a good way to write the system under the form (3) on our manifolds. Note that we cannot use directly the decomposition (2) that does not hold in general on a manifold when non-trivial L^2 harmonic 1-forms are present. This phenomenon is at the origin of the non-uniqueness phenomenon on the hyperbolic plane pointed out in [13], [32] and produce non-unique $\mathcal{C}^\infty$ solutions. In section 4, we prove dispersive and smoothing estimates for the heat and Stokes equations associated to the Bochner Laplacian. The negative curvature yields better large time decay than in the Euclidian case. Our set of estimates for the Stokes problem is more complete when the Ricci curvature of the manifold is constant (thus in particular on the hyperbolic spaces and also on more general symmetric spaces of non-compact type). This comes from the fact that in this case the study of the Stokes problem can be reduced to the study of the vectorial heat equation. These are the crucial estimates needed in order to get Fujita Kato type theorems. In section 5, we prove well-posedness results for the Navier-Stokes equations in an L^n framework. Finally in section 6, we discuss how by eliminating the pressure from the Navier-Stokes system our approach can be used to recover the uniqueness of Leray weak solutions on two-dimensional non-compact manifolds. This gives another approach to the recent result [14].

2. Baby Geometry ♣

We shall recall in this section the main objects of Riemannian geometry and their properties that we need. For more details, we refer to Riemannian geometry textbooks [23],[29] for example.

2.1. **Connections.** We consider (M, g) a Riemannian manifold. We shall denote by ∇ the Levi-Civita connection:

$$\begin{aligned}\Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM)\\ (X, Y) &\mapsto \nabla_X Y\end{aligned}$$

where we denote by $\Gamma(TM)$ the set of vector fields on M. The crucial property of this connection is its compatibility with the metric: for any vector fields X, Y, Z, we have

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{4}$$

For $X \in \Gamma(TM)$, we can extend ∇_X to arbitrary (p, q) tensors by requiring that

i) $\nabla_X(c(S)) = c(\nabla_X S)$ for any contraction c,
ii) $\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T$

with the convention that for f a function $\nabla_X f = X \cdot f$. In particular, we get that for $S \in \Gamma(\bigotimes^p(TM)\bigotimes^q(T^*M))$

$$(\nabla_X S)(X_1, \cdots X_q) = \nabla_X\big(S(X_1, \cdots X_q)\big) - S(\nabla_X X_1, \cdots, X_q) - \cdots - S(X_1, \cdots, \nabla_X X_q).$$

We define the covariant derivatives ∇ on tensor field $S \in \Gamma(\bigotimes^p(TM)\bigotimes^q(T^*M))$ by

$$\nabla S(X, X_1, \cdots X_q) = (\nabla_X S)(X_1, \cdots X_q),$$

thus $\nabla S \in \Gamma(\bigotimes^p(TM)\bigotimes^{q+1} T^*M))$.

2.2. **Curvatures.** We shall use the following classical definitions for the various curvature tensors. The curvature tensor is defined by

$$R(X,Y)Z = -\nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) + \nabla_{[X,Y]}Z, \quad \forall X,\, Y,\, Z \in \Gamma(TM). \tag{5}$$

The Riemann curvature tensor is given by

$$\mathrm{Riem}(X,Y,Z,T) = g(R(X,Y)Z,T), \quad \forall X,\, Y,\, Z,\, T \in \Gamma(TM) \tag{6}$$

and the Ricci curvature tensor is defined by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} \mathrm{Riem}(X, e_i, Y, e_i), \tag{7}$$

for an orthonormal basis $(e_1, \cdots e_n)$. The notion of sectional curvature will be also used. For every $(X,Y) \in (T_xM)^2$, we define the sectional curvature of the plane (X,Y) as

$$\kappa(X,Y) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

2.3. **Metric on tensors.** Let us recall the musical applications: for a 1-form ω, we associate the vector field $\omega^\sharp$ defined by

$$g(\omega^\sharp, Y) = \omega(Y), \quad \forall Y \in \Gamma(TM)$$

and for a vector field X, we associate the 1-form $X^\flat$ defined by

$$X^\flat(Y) = g(X,Y), \quad \forall Y \in \Gamma(TM).$$

The Riemmanian gradient of a function is then defined as

$$\mathrm{grad}\, p = (dp)^\sharp.$$

More generally, for tensors $T \in \Gamma(\otimes^p TM \otimes^q T^*M)$, we have

$$\begin{aligned} T^\sharp &= C_1^2(g^{-1} \otimes T) \in \Gamma(\otimes^{p+1} TM \otimes^{q-1} T^*M), \\ T^\flat &= C_2^1(g \otimes T) \in \Gamma(\otimes^{p-1} TM \otimes^{q+1} T^*M), \\ \mathrm{div}\, T &= C_1^1 \nabla T \in \Gamma(\otimes^{p-1} TM \otimes^{q} T^*M) \end{aligned}$$

where C_j^i stands for the contraction of the i and j indices for tensors.

We can define a metric on 1-forms by setting

$$g(\omega, \eta) := g(\omega^\sharp, \eta^\sharp), \quad \forall \omega,\, \eta \in \Gamma(T^*M).$$

We can then extend the definition to general tensors fields in $\Gamma(\otimes^p TM \otimes^q T^*M)$, by setting

$$g := (\otimes^p g) \otimes (\otimes^q g).$$

In local coordinates $(x^1, \ldots, x^n)$, for $T, S \in \Gamma(\otimes^p TM \otimes^q T^*M)$, i.e.

$$T = T^{i_1 \ldots i_p}_{j_1 \ldots j_q}\, \partial_{x^{i_1}} \otimes \cdots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q},$$

$$S = S^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q}\, \partial_{x^{i'_1}} \otimes \cdots \otimes \partial_{x^{i'_p}} \otimes dx^{j'_1} \otimes \cdots \otimes dx^{j'_q},$$

this yields the expression

$$g(T, S) = g_{i_1 i'_1} \cdots g_{i_p i'_p} g^{j_1 j'_1} \cdots g^{j_q j'_q} T^{i_1 \ldots i_p}_{j_1 \ldots j_q} S^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q}.$$

We shall also use for tensors the notation

$$|T|^2 = g(T, T). \tag{8}$$

We define the Sobolev norms of tensors $T \in \Gamma(\otimes^p TM \otimes^q T^*M)$ by

$$\|T\|_{W^{m,p}} = \Big(\sum_{0 \leq k \leq m} \int_M g(\nabla^k T, \nabla^k T)^{\frac{p}{2}}\, dvol \Big)^{\frac{1}{p}}, \quad W^{m,2} = H^m.$$

2.4. **Normal coordinates.** To compute intrinsic objects in local coordinates, it will be very often convenient to use normal coordinates. More precisely, we shall use that in the vicinity of any point m_0, there exists a coordinate system $(x^1, \cdots, x^n)$ such that at the point m_0 the coordinates of the Riemannian metric and the Christoffel coefficients verify

$$g_{ij}(m_0) = \delta_{ij}, \qquad \Gamma^k_{ij}(m_0) = 0. \tag{9}$$

2.5. **Some useful geometric formulas.**

Lemma 2.1 (Kato inequality)**.** *For any vector field u*

$$|\nabla |u|| \leq |\nabla u|. \tag{10}$$

Proof. We prove the inequality at each point m by using a normal coordinate system centered at m. Let us set $e_i = \partial/\partial x^i = \partial_i$, then $(e_1, \cdots, e_n)$ is an orthonormal basis at m. By using (9) and by Cauchy-Schwartz inequality we have on the one hand

$$|\nabla(|u|^2)|^2_{/m} = \sum_i (\partial_i |u|^2)^2_{/m} = 4 \sum_i g(\nabla_{e_i} u, u)^2_{/m} \leq 4 |\nabla u|^2_{/m} |u|^2_{/m},$$

on the other hand

$$\nabla(|u|^2)|^2_{/m} = 4 |(\nabla(|u|)|u|)|^2_{/m},$$

so we obtain

$$|(\nabla(|u|)|u|)|^2_{/m} \leq |\nabla u|^2_{/m} |u|^2_{/m}$$

which yields the result. □

Let us denote by Δ_g the Laplace Beltrami operator and by $\overrightarrow{\Delta}$ the Bochner Laplacian,

$$\overrightarrow{\Delta}u = -\nabla^*\nabla u = \mathrm{Tr}_g(\nabla^2 u) \tag{11}$$

where ∇^* is the formal adjoint of ∇ for the L^2 scalar product and

$$\mathrm{Tr}_g(\nabla^2 u) = g^{ij}\nabla^2 u(e_i, e_j)$$

in local coordinates.

Lemma 2.2. *For any vector field u, we have Bochner's identity*

$$\frac{1}{2}\Delta_g\big(g(u,u)\big) = g(\overrightarrow{\Delta}u, u) + g(\nabla u, \nabla u). \tag{12}$$

Note that in the right hand side the scalar product $g(\nabla u, \nabla u)$ is the scalar product on $(1,1)$ tensors defined above.

Proof. To prove the formula, we shall compute each term in the formula in normal coordinates at m for any point m. Let us set $e_i = \partial/\partial x^i = \partial_i$, then $(e_1, \cdots, e_n)$ is an orthonormal basis at m. By using the properties (9) of the normal coordinates at m, we have

$$(\overrightarrow{\Delta}u)_{/m} = (\nabla^2_{e_i,e_i}u)_{/m} = \big(\nabla_{e_i}\big(\nabla_{e_i}u\big)\big)_{/m}$$

Therefore, we obtain by using (4)

$$g(\overrightarrow{\Delta}u, u)_{/m} = \Big(\partial_i\big(g(\nabla_{e_i}u, u)\big) - g(\nabla_{e_i}u, \nabla_{e_i}u)\Big)_{/m}$$

and hence

$$\Big(g(\overrightarrow{\Delta}u, u) + g(\nabla u, \nabla u)\Big)_{/m} = \big(\partial_i\big(g(\nabla_{e_i}u, u)\big)\big)_{/m}$$

by using again (9). To conclude, we observe by using again (4) that

$$g(\nabla_{e_i}u, u) = \frac{1}{2}\partial_i\big(g(u,u)\big)$$

and then that

$$\big(\partial_i\big(g(\nabla_{e_i}u, u)\big)\big)_{/m} = \frac{1}{2}\big(\partial_i^2(g(u,u)\big)_{/m} = \frac{1}{2}\Delta_g\big(g(u,u)\big)_{/m}.$$

□

2.6. **Functional Analysis on non-compact manifolds.** In all this paper, we shall consider smooth, complete, non-compact, simply connected Riemannian manifolds M of dimension $n \geq 2$ that verify the following assumptions

- **(H1)** $|R| + |\nabla R| + |\nabla^2 R| \leq K$;
- **(H2)** $-\frac{1}{c_0}g \leq \mathrm{Ric} \leq -c_0 g$, for some $c_0 > 0$;
- **(H3)** $\kappa \leq 0$;
- **(H4)** $\inf_{x\in M} r_x > 0$;

where R is the curvature tensor, Ric is the Ricci curvature tensor, κ is the sectional curvature and r_x stands for the injectivity radius for the exponential map at x.

Remark 2.3. *This set of assumptions have several important consequences, which will be crucial in the following.*

(1) $C_c^\infty(M)$ *is dense in* $H^1(M)$, *(see* [26]*);*
(2) *From Varopoulos* [56], *(see also* [26]*) the Sobolev inequalities are verified. In particular*
$$\eta_n \|f\|^2_{L^{2^*}(M)} \leq \|\nabla f\|^2_{L^2(M)}, \quad \forall f \in H^1(M)$$
is verified for some $\eta_n > 0$, *where* $2^* = 2n/(n-2)$ *if* $n \geq 3$ *and* 2^* *is arbitrary in* $(2, +\infty)$ *when* $n = 2$.
(3) *In dimension* $n = 2$, *we also have the continuous embedding* $W^{1,1}(M) \subset L^2(M)$, *therefore there exists* $C > 0$ *such that*
$$\|f\|^2_{L^2(M)} \leq C \left(\|\nabla f\|_{L^1(M)} + \|f\|_{L^1(M)} \right).$$
By using this inequality with $f = |g|^2$ *and by the Cauchy-Schwarz inequality we obtain the following Gagliardo-Nirenberg inequality*
$$\|g\|^4_{L^4(M)} \leq C \left(\|\nabla g\|^2_{L^2(M)} \|g\|^2_{L^2(M)} + \|g\|^4_{L^2(M)} \right).$$
(4) *From Setti* [51], *(see also* [42]*) the Poincaré inequality*
$$\delta_n \|f\|^2_{L^2(M)} \leq \|\nabla f\|^2_{L^2(M)}, \quad \forall f \in H^1(M)$$
is verified for $\delta_n \geq [c_0 - (n-1)(n-2)\kappa^*]/4 > 0$, *with* $\kappa^* = \sup_M \kappa$.

Remark 2.4. *An important example of non-compact Riemannian manifolds for which our hypothesis* **(H1-4)** *hold true are the well-known real hyperbolic spaces* $M = \mathbb{H}^n(\mathbb{R})$, $n \geq 2$, *defined as follows*
$$\mathbb{H}^n = \left\{ \Omega = (\tau, x) \in \mathbb{R}^{n+1},\, \Omega = (cosh\, r,\, \omega\, sinh\, r),\, r \geq 0,\, \omega \in \mathbb{S}^{n-1} \right\},$$
the metric g *being*
$$g = dr^2 + (sinh\, r)^2 d\omega^2$$
with $d\omega^2$ *the canonical metric on the sphere* $\mathbb{S}^{n-1}$.

The Ricci curvature tensor is constant, $\mathrm{Ric} = \kappa(n-1)g$ *with* κ *the sectional curvature given by* $\kappa = -1$. *In fact, the curvature tensor is*
$$R(X, Y)Z = \kappa R_0(X, Y)Z, \quad \forall X, Y, Z \in \Gamma(T\mathbb{H}^n)$$
where $R_0(X, Y)Z := g(X, Z)Y - g(Y, Z)X$ *(which also implies that* $\nabla R = 0$*) and thus the Riemann tensor is*
$$\mathrm{Riem}(X, Y, Z, T) = kg(R_0(X, Y)Z, T) = -\left[g(X, Z)g(Y, T) - g(Y, Z)g(X, T) \right].$$

Assumptions **(H1-4)** *are also verified by several other classical examples in geometry, like some Damek-Ricci spaces and all symmetric spaces of non-compact type (see* [15], [55], [27], [18]*).*

3. The Navier-Stokes equations on manifolds

The Navier-Stokes equations on a Riemannian manifold (M, g) takes the form

$$\partial_t u + \nabla_u u + \operatorname{grad} p = \nu L u, \quad \operatorname{div} u = 0,$$

where the diffusive part is defined by the operator L. The unknowns (u, p) are such that the velocity $u(t, \cdot) \in \Gamma(TM)$ is a vector field on M and the pressure $p(t, \cdot)$ is a real-valued function. For the left hand side of the equation all terms have a natural definition. Indeed, $\nabla_u u \in \Gamma(TM)$ stands for the covariant derivative of u along of u and $\operatorname{grad} p$ is the Riemannian gradient of the pressure. Note that since u is divergence free, we have also the following identity

$$\nabla_u u = \operatorname{div}(u \otimes u). \tag{13}$$

To define the vectorial Laplacian L, we have to make a choice since there is no canonical definition of a Laplacian on vector fields on Riemannian manifolds: there are at least two candidates for the role of Laplace operator, i.e. the Bochner and Hodge Laplacians. Following [17], [54] (see also [50], [43]), the correct formulation is obtained by introducing the stress tensor. Let us recall that on $\mathbb{R}^n$, if $\operatorname{div} u = 0$, we have

$$Lu = \operatorname{div}\big(\nabla u + \nabla u^t\big) = \Delta u.$$

The natural generalization on M is to take

$$Lu = \operatorname{div}\big(\nabla u + \nabla u^t\big)^\sharp \in \Gamma(TM).$$

Since u is divergence free, we can express L in the following way:

$$Lu = \overrightarrow{\Delta} u + r(u),$$

where r is the Ricci operator which is related to the Ricci curvature tensor by

$$r(u) = \big(\operatorname{Ric}(u, \cdot)\big)^\sharp \in \Gamma(TM).$$

By using the Weitzenbock formula on $1-$forms

$$\Delta_H u^\flat = \nabla^* \nabla u^\flat + \operatorname{Ric}(u, \cdot), \tag{14}$$

where $\Delta_H = d^* d + d d^*$ is the Hodge Laplacian on 1-forms, we can also relate L to the Hodge Laplacian:

$$Lu = \Big(- \Delta_H u^\flat + 2 \operatorname{Ric}(u, \cdot)\Big)^\sharp.$$

Let us consider the Cauchy problem for the incompressible Navier-Stokes equation on M (assume $\nu = 1$)

$$\begin{cases} \partial_t u + \nabla_u u + \operatorname{grad} p = \overrightarrow{\Delta} u + r(u), \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0, \quad u_0 \in \Gamma(TM). \end{cases} \tag{15}$$

In view of its own structure and by **(H2)**, we can deduce that the smooth solution of (15) satisfies the following energy inequality

$$\|u(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla u(s)\|_{L^2}^2 + c_0 \|u(s)\|_{L^2}^2\right) ds \le \|u_0\|_{L^2}^2. \tag{16}$$

Indeed, multiplying u in (15) and then integrating on M by part combining with the Bochner identity (2.2), we have (16). According to (16), it is natural to construct weak solution that verify the energy inequality. Nevertheless we expect at least the same difficulty as in the Euclidean case (at least in dimension greater than 3) and hence it is also natural to study Kato type solutions. In both cases, one has to be careful when eliminating the pressure. Indeed, in the Euclidean case for smooth solutions it is well known that the pressure term p can be eliminated via Leray-Hopf projections and that we can view Navier-Stokes system (1) as an evolution equation of u alone. On a Riemannian manifold M some problems may occur since the Kodaira-Hodge decomposition of L^2 1-forms on complete manifolds is under the form

$$L^2\big(\Gamma(T^*M)\big) = \overline{\operatorname{Image} d} \oplus \overline{\operatorname{Image} d^*} \oplus \mathcal{H}^1(M)$$

where $\mathcal{H}^1(M)$ is the space of L^2 harmonic 1-forms (see [34]). It may happen that there are non-trivial L^2 harmonic 1-forms which are responsable for non-uniqueness (even in dimension two, [13], [32] on the hyperbolic space $\mathbb{H}^2$). We shall make the following choice for the pressure in order to eliminate this non-uniqueness phenomenon. We first note that if (u, p) is a smooth solution of the Navier-Stokes equation (15), then by taking the divergence of the first equation in (15) and by noticing that $\operatorname{div} u = 0$, we obtain that

$$\Delta_g p + \operatorname{div}[\nabla_u u] - 2\operatorname{div}(ru) = 0. \tag{17}$$

We used the consequence of Weitzenbock formula (14) that $\operatorname{div}(\overrightarrow{\Delta} u) = \operatorname{div}(ru)$ if $\operatorname{div} u = 0$. In order to determine the pressure, we shall always choose the solution in L^p of this elliptic equation since $\Delta_g : W^{2,p} \to L^p$ is an isomorphism thanks to the assumptions **(H1-H4)** (see [41], [52] for $2 \le p < \infty$). It follows that

$$\operatorname{grad} p = \operatorname{grad}(-\Delta_g)^{-1} \operatorname{div}[\nabla_u u] - 2 \operatorname{grad}(-\Delta_g)^{-1}\operatorname{div}(ru).$$

We shall discuss why this choice is appropriate to get uniqueness results (in relation with the counterexamples of [13], [32]) in section 6. It will be convenient to use the notation

$$\mathbb{P} = I + \operatorname{grad}(-\Delta_g)^{-1}\operatorname{div}.$$

Solving p from (17) and inserting it into (15), we get

$$\begin{cases} \partial_t u - \overrightarrow{\Delta} u - r(u) + 2\operatorname{grad}(-\Delta_g)^{-1}\operatorname{div}(ru) = -\mathbb{P}[\nabla_u u], \\ \operatorname{div} u_0 = 0, \\ u_{|t=0} = u_0, \quad u_0 \in \Gamma(TM) \end{cases} \tag{18}$$

From (18), we see that the Navier-Stokes system on M belongs to a class of non-linear parabolic equations on vector fields.

We remark that when the Ricci tensor Ric is a negative constant scalar multiple of the metric and $\operatorname{div} u = 0$, we have $\operatorname{div}(ru) = 0$ and the linear non-local term disappear. In particular this occurs on the hyperbolic spaces $M = \mathbb{H}^n(\mathbb{R})$ (see Remark 2.4).

In order to use the fixed point method, we need to prove dispersive and smoothing estimates for the semi-group associated to the linear part of the Cauchy problem (18).

4. Dispersive and smoothing estimates

4.1. **The case of vectorial heat equations.** We study the Cauchy problem for the heat equation associated to the Bochner Laplacian on vector fields:

$$\left\{\begin{array}{l}\partial_t u = \overrightarrow{\Delta} u + r(u),\\ u_{|t=0} = u_0, \quad u_0 \in \Gamma(TM).\end{array}\right. \tag{19}$$

We shall prove dispersive and smoothing estimates for the semi-group associated to this vectorial heat equation (19) on M of dimension $n \geq 2$ satisfying our assumptions **(H1-4).**

These kind of estimates are related to the behaviour of the heat kernel which is well studied in the literature for various types of manifolds for both the Laplace-Beltrami and the Hodge Laplacian (see for example [38], [2], [3], [25], [41], [47], [56], [10], [12], [4], [48], [6] and others)

The main results of this section are:

Theorem 4.1 (Dispersive estimates)**.** *Assuming* **(H1-4)**, *the solution of* (19) *satisfies the following dispersive estimates*

$$\|e^{t(\overrightarrow{\Delta}+r)}u_0\|_{L^q} \leq c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)}\, e^{-t(\gamma_{p,q}+c_0)}\, \|u_0\|_{L^p}, \tag{20}$$

for every p, q such that $1 \leq p \leq q \leq +\infty$, with $\gamma_{p,q} = \frac{\delta_n}{2}\left[\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{8}{q}\left(1-\frac{1}{p}\right)\right]$, $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$ *and for all* $u_0 \in L^p(\Gamma(TM))$.

Theorem 4.2 (Smoothing estimates)**.** *Assuming* **(H1-4)**, *the solution of* (19) *satisfies the following smoothing estimates*

$$\|\nabla u(t)\|_{L^p} \leq C \max\left(\frac{1}{\sqrt{t}}, 1\right) e^{-t\left(c_0+\frac{4\delta_n}{p}\left(1-\frac{1}{p}\right)\right)}\|u_0\|_{L^p} \tag{21}$$

for every $1 < p < +\infty$ *and for all* $u_0 \in L^p(\Gamma(TM))$.

Under the same assumptions as in Theorem 4.2, we can deduce more general smoothing estimates $L^p \to W^{1,q}$.

Corollary 4.3. *Assuming* **(H1-4)**, *for every p, q such that $1 < p \leq q < +\infty$, we obtain for all times $t > 0$*

$$\|\nabla e^{t(\overrightarrow{\Delta}+r)}u_0\|_{L^q} \leq\ c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{n}\right)}\, e^{-t\left(c_0+\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)}\|u_0\|_{L^p}, \tag{22}$$

with $\gamma_{p,q} = \frac{\delta_n}{2}\left[\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{8}{q}\left(1-\frac{1}{p}\right)\right]$, $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$ *and for all* $u_0 \in L^p(\Gamma(TM))$. *Moreover, under the same assumption, we have*

$$\|e^{t(\overrightarrow{\Delta}+r)}\nabla^* T_0\|_{L^q} \leq\ c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{n}\right)}\, e^{-t\left(c_0+\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)}\|T_0\|_{L^p}, \tag{23}$$

with $\gamma_{p,q} = \frac{\delta_n}{2}\left[\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{8}{q}\left(1-\frac{1}{p}\right)\right]$, $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$ *and for all tensors* $T_0 \in L^p(\Gamma(TM \otimes T^*M))$

Proof. It is sufficient to use the semi-group property combined with the smoothing (21) and dispersive estimates (20)

$$e^{t(\overrightarrow{\Delta}+r)} = e^{\frac{t}{2}(\overrightarrow{\Delta}+r)}\left(e^{\frac{t}{2}(\overrightarrow{\Delta}+r)}\right) : L^p \to L^q \to W^{1,q}.$$

The second estimate follows by duality. Note that r is symmetric for the metric g by definition. □

Remark 4.4. *Note that, since*

$$\nabla^* T = -\operatorname{div}(T^\sharp), \quad \forall T \in \Gamma(TM \otimes T^*M)$$

from (23) *we also get the following smoothing estimate*

$$\|e^{t(\overrightarrow{\Delta}+r)} \operatorname{div} T_0^\sharp\|_{L^q} \leq c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{n}\right)} e^{-t\left(c_0+\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)} \|T_0^\sharp\|_{L^p}, \tag{24}$$

with $\gamma_{p,q} = \frac{\delta_n}{2}\left[\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{8}{q}\left(1-\frac{1}{p}\right)\right]$, $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$ *and for all tensors* $T_0^\sharp \in L^p(\Gamma(TM \otimes TM))$.

4.2. **Proof of Theorem 4.1.** We shall split the proof of Theorem 4.1 in several steps. We shall first start with a comparison lemma that allows to reduce the proof of estimates for the vectorial Laplacian heat equation to estimates for the Laplace Beltrami heat equation.

Lemma 4.5. *For any* $u_0 \in C_b^\infty(\Gamma(TM))$, *we have the pointwise estimate*

$$|e^{t(\overrightarrow{\Delta}+r)} u_0|_{/x} \leq e^{t(\Delta_g - c_0)}|u_0|_{/x}, \quad \forall x \in M$$

Proof. Let $u(t,x) = \left(e^{t(\overrightarrow{\Delta}+r)}u_0\right)(x)$ be the solution of the vectorial heat equation (19). We note that $|u| = g(u,u)^{\frac{1}{2}}$ solves the following scalar heat equation

$$\partial_t|u| - \Delta_g|u| = \frac{1}{|u|}\left(|\nabla|u||^2 - |\nabla u|^2\right) + g\left(r(u), \frac{u}{|u|}\right).$$

Indeed, we have the following consequence of the Bochner identity (12)

$$g\left(\overrightarrow{\Delta}u, \frac{u}{|u|}\right) = \frac{1}{2}\frac{\Delta_g|u|^2}{|u|} - \frac{|\nabla u|^2}{|u|} = \Delta_g|u| + \frac{|\nabla|u||^2}{|u|} - \frac{|\nabla u|^2}{|u|}.$$

By the Kato inequality (10), we have that

$$|\nabla|u||^2 - |\nabla u|^2 \leq 0$$

and thanks to **(H2)**, we also get that

$$g\left(r(u), \frac{u}{|u|}\right) \leq -c_0|u|,$$

therefore, we finally obtain that

$$\partial_t|u| - \Delta_g|u| + c_0|u| \leq 0$$

and the estimate follows from the maximum principle. □

As a consequence $L^p \to L^q$ estimates for $(\Delta_g - c_0)$ will imply $L^p \to L^q$ estimates for $(\overrightarrow{\Delta} + r)$. Therefore, we shall first establish the dispersive estimates for the heat equation associated to the Laplace Beltrami.

Proposition 4.6. *($L^p \to L^p$ estimates)*
For every $p \in [1, +\infty]$, we have for some c_0, $\delta_n > 0$, the following estimate

$$\|e^{t(\Delta_g - c_0)} f_0\|_{L^p(M)} \leq e^{-t\left(\frac{4\delta_n(p-1)}{p^2} + c_0\right)} \|f_0\|_{L^p(M)}.$$

Proof. Let us set $f(t,x) = e^{t(\Delta_g - c_0)} f_0$, then f is a solution of

$$\partial_t f - (\Delta_g - c_0) f = 0. \tag{25}$$

By multiplying the equation by $|f|^{p-2} f$ and by integrating on the manifold, we find

$$\frac{d}{dt}\|f\|^p_{L^p} + 4\,\frac{p-1}{p}\|\nabla(|f|^{\frac{p}{2}})\|^2_{L^2} + c_0 p \|f\|^p_{L^p} \leq 0. \tag{26}$$

By using the Poincaré inequality in (4) Remark 2.3, there is some $\delta_n > 0$ such that

$$\delta_n \|h\|^2_{L^2} \leq \|\nabla h\|^2_{L^2}$$

with $h = |f|^{\frac{p}{2}}$, we obtain that

$$\frac{d}{dt}\|f\|^p_{L^p} + \left(4\frac{\delta_n(p-1)}{p} + c_0 p\right)\|f\|^p_{L^p} \leq 0,$$

so by a Gronwall type inequality we can conclude. $\square$

Proposition 4.7 ($L^1 \to L^\infty$ estimates)**.** *For every $p \in [1, +\infty]$, we have the dispersive estimate*

$$\|e^{t(\Delta_g - c_0)} f_0\|_{L^\infty(M)} \leq c_n(t) e^{-t(\frac{\delta_n}{2} + c_0)} \|f_0\|_{L^1(M)}, \text{ with } c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right).$$

We shall give a proof suitable for any manifold that satisfies our assumptions **(H1-4)**. For non-compact manifolds that enjoy a nice Fourier analysis like the hyperbolic spaces, Damek-Ricci spaces or symmetric spaces of non-compact type, such results can be obtained directly from heat kernel estimates (see [2], [3] and others).

Proof. We need to distinguish the $n \geq 3$ and $n = 2$ cases due to the fact that the Sobolev embedding of $H^1(M)$ in $L^{2^*}(M)$ is critical in dimension 2.

- We begin with the proof of the case of dimension bigger than 3, which is more direct. We use a classical argument (see for example [58]) to prove dispersive estimates for the heat equation in euclidean cases by using suitable energy estimates and Sobolev embeddings. Here we can use the Poincaré inequality in our argument to improve the large time decay. We first use (26) with $p = 2$

$$\frac{d}{dt}\|f(t)\|^2_{L^2(M)} + 2\|\nabla f(t)\|^2_{L^2(M)} + 2c_0\|f(t)\|^2_{L^2(M)} \leq 0$$

and by combining it with Sobolev-Poincaré inequalities in (2) and (4) Remark 2.3, we have

$$\frac{d}{dt}\|f(t)\|^2_{L^2(M)} + \eta_n\|f(t)\|^2_{L^{2^*}(M)} + (\delta_n + 2c_0)\|f(t)\|^2_{L^2(M)} \leq 0.$$

Since by interpolation and the decay of the L^1 norm ((26) with $p = 1$), we have

$$\|f(t)\|_{L^2(M)} \leq \|f(t)\|^{\alpha}_{L^1(M)}\|f(t)\|^{1-\alpha}_{L^{2^*}(M)} \leq C^{\alpha}\|f(t)\|^{1-\alpha}_{L^{2^*}(M)}$$

with $\frac{1}{2} = \alpha + \frac{(1-\alpha)}{2^*}$ that is to say $\alpha = \frac{2}{n+2}$ and $C = \|f(0)\|_{L^1(M)}$, we obtain

$$\frac{d}{dt}\|f(t)\|^2_{L^2(M)} + \frac{\eta_n}{C^{\frac{2\alpha}{1-\alpha}}}\|f(t)\|^{\frac{2}{1-\alpha}}_{L^{2^*}(M)} + (\delta_n + 2c_0)\|f(t)\|^2_{L^2(M)} \leq 0.$$

Next by setting $y(t) = \|f(t)\|^2_{L^2(M)}$, we find the following differential inequality

$$y'(t) + (\delta_n + 2c_0)y(t) \leq -\frac{\eta_n}{C^{\frac{2\alpha}{1-\alpha}}}y(t)^{\frac{1}{1-\alpha}}.$$

Then

$$z(t) = y(t)e^{(\delta_n+2c_0)t}$$

solves

$$z'(t) \leq -\frac{\eta_n}{C^{\frac{2\alpha}{1-\alpha}}}z(t)^{\frac{1}{1-\alpha}}e^{-(\delta_n+2c_0)\frac{\alpha}{1-\alpha}t}$$

and hence by integrating, we obtain

$$z(t) \leq \left[\frac{C^{\frac{2\alpha}{1-\alpha}}}{\eta_n}(\delta_n + 2c_0)\right]^{\frac{1-\alpha}{\alpha}} \left(1 - e^{-(\delta_n+2c_0)t\frac{\alpha}{1-\alpha}}\right)^{\frac{\alpha-1}{\alpha}},$$

this yields

$$y(t) \leq \left[\frac{C^{\frac{2\alpha}{1-\alpha}}}{\eta_n}(\delta_n + 2c_0)\right]^{\frac{1-\alpha}{\alpha}} \left(e^{(\delta_n+2c_0)t\frac{\alpha}{1-\alpha}} - 1\right)^{\frac{\alpha-1}{\alpha}}.$$

We have thus proved that

$$\|f(t)\|_{L^2(M)} \leq c_n(t)^{\frac{1}{2}}e^{-t(\frac{\delta_n}{2}+c_0)}\|f_0\|_{L^1(M)}, \text{ with } c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right).$$

Therefore $e^{\frac{t}{2}(\Delta_g - c_0)} : L^1(M) \to L^2(M)$ with norm less than $c_n(\frac{t}{2})^{\frac{1}{2}}e^{-\frac{t}{2}(\frac{\delta_n}{2}+c_0)}$. By duality $e^{\frac{t}{2}(\Delta_g - c_0)} : L^2(M) \to L^\infty(M)$ with norm less than $c_n(\frac{t}{2})^{\frac{1}{2}}e^{-\frac{t}{2}(\frac{\delta_n}{2}+c_0)}$. Finally, since $e^{t(\Delta_g - c_0)} = e^{\frac{t}{2}(\Delta_g - c_0)}e^{\frac{t}{2}(\Delta_g - c_0)} : L^1(M) \to L^\infty(M)$, with norm less than $c_n(\frac{t}{2})e^{-t(\frac{\delta_n}{2}+c_0)}$ and we get the desired dispersive estimate for $n \geq 3$.

- In dimension $n = 2$, we shall first prove the $L^2(M) \to L^\infty(M)$ estimate by using the Nash iteration method (see [44]). To do so, we use (26) with $p = 4$

$$\frac{d}{dt}\|f(t)\|^4_{L^4(M)} + 3\|\nabla(|f(t)|^2)\|^2_{L^2(M)} + 4c_0\|f(t)\|^4_{L^4(M)} \leq 0$$

By multiplying by t the last inequality and by using the Gagliardo-Nirenberg and Poincaré inequalities (see (3) and (4) in Remark 2.3) we have

$$\frac{d}{dt}\left(t\|f(t)\|^4_{L^4(M)}\right) \leq C\|\nabla f(t)\|^2_{L^2(M)}\|f(t)\|^2_{L^2(M)}. \tag{27}$$

Since by using (26) with $p=2$, we have

$$\|f(t)\|^2_{L^2(M)} \leq \|f(0)\|^2_{L^2(M)} \quad \text{and} \quad \int_0^t \|\nabla f(\tau)\|^2_{L^2(M)} d\tau \leq \|f(0)\|^2_{L^2(M)},$$

we obtain by integrating (27) on $[0,t]$ the following estimate

$$\|f(t)\|^4_{L^4(M)} \leq \frac{C}{t}\|f(0)\|^4_{L^2(M)}. \tag{28}$$

We have also for $t > s > 0$

$$\|f(t)\|_{L^4(M)} \leq \left(\frac{C}{(t-s)}\right)^{\frac{1}{4}} \|f(s)\|_{L^2(M)}.$$

Since $|f|^{2^{k-1}}$ is a non-negative sub-solution of the heat equation (25), we can use (28) with f replaced by $|f|^{2^{k-1}}$. This yields

$$\|f(t)\|_{L^{2^{k+1}}(M)} \leq \left(\frac{C}{(t-s)^{\frac{1}{4}}}\right)^{\frac{1}{2^{k-1}}} \|f(s)\|_{L^{2^k}(M)}.$$

For every $t > 0$, let us set $t_k = t\left(1 - \frac{1}{k+1}\right)$; we deduce from the last inequality that

$$\|f(t_{k+1})\|_{L^{2^{k+1}}(M)} \leq \left(\frac{C(k+2)^{\frac{1}{2}}}{t^{\frac{1}{4}}}\right)^{\frac{1}{2^{k-1}}} \|f(t_k)\|_{L^{2^k}(M)}.$$

By induction we find

$$\|f(t_{k+1})\|_{L^{2^{k+1}}(M)} \leq \frac{C^{\left(\frac{1}{2^{k-1}}+\frac{1}{2^{k-2}}+\cdots+1\right)}\left[(k+2)^{\frac{1}{2^k}}(k+1)^{\frac{1}{2^{k-1}}}\cdots 2\right]}{t^{\left(\frac{1}{2^{k+1}}+\frac{1}{2^k}+\cdots+\frac{1}{4}\right)}}\|f(t_1)\|_{L^2(M)}.$$

Since, when $k \to \infty$, we have that

$$t^{\left(\frac{1}{2^{k+1}}+\frac{1}{2^k}+\cdots+\frac{1}{4}\right)} \to t^{\frac{1}{2}},$$

that $C^{\left(\frac{1}{2^{k-1}}+\frac{1}{2^{k-2}}+\cdots+1\right)}$ and the product $\left[(k+2)^{\frac{1}{2^k}}(k+1)^{\frac{1}{2^{k-1}}}\cdots 2\right]$ are bounded, we get

$$\|f(t)\|_{L^\infty(M)} \leq \frac{C}{t^{\frac{1}{2}}}\|f(0)\|_{L^2(M)}. \tag{29}$$

As expected on a non-compact manifold with negative curvature, we can improve the decay in the last estimate for large times.
Actually, for $t > 1$ by the semigroup property we can write

$$e^{t(\Delta_g - c_0)} = e^{\frac{t-1}{2}(\Delta_g - c_0)}\, e^{(\Delta_g - c_0)}\, e^{\frac{t-1}{2}(\Delta_g - c_0)}.$$

Thanks to Proposition 4.6 we have that $e^{\frac{t-1}{2}(\Delta_g - c_0)} : L^2(M) \to L^2(M)$ is bounded with norm less than $e^{-\frac{t-1}{2}(\delta_2 + c_0)}$ and that $e^{\frac{t-1}{2}(\Delta_g - c_0)} : L^\infty(M) \to L^\infty(M)$ is bounded with norm less than $e^{-c_0 \frac{t-1}{2}}$. Moreover, by (29) with $t = 1$, we have also that $e^{(\Delta_g - c_0)} : L^2(M) \to L^\infty(M)$ is bounded. Thus for $t > 1$ we obtain

$$\|f(t)\|_{L^\infty(M)} \leq Ce^{-t(\frac{\delta_2}{2} + c_0)} \|f(0)\|_{L^2(M)}. \tag{30}$$

From (29) and (30) we get

$$\|f(t)\|_{L^\infty(M)} \leq c_2(t)^{\frac{1}{2}} e^{-t(\frac{\delta_2}{2} + c_0)} \|f_0\|_{L^2(M)}, \text{ with } c_2(t) = C_2 \max\left(\frac{1}{t}, 1\right).$$

As before, by a duality and composition argument we deduce the claimed dispersive estimate in dimension 2

$$\|f(t)\|_{L^\infty(M)} \leq c_2(t) e^{-t(\frac{\delta_2}{2} + c_0)} \|f(0)\|_{L^1(M)}.$$

□

End of the proof of Theorem 4.1. Finally we prove $L^p \to L^q$ dispersive estimates for the Bochner heat equation. Thanks to Lemma 4.5, it suffices to prove the corresponding estimates for the Laplace Beltrami semi-group. To do so, we shall use many interpolation arguments. First, we can use Proposition 4.6 for $p = 1$ and Proposition 4.7 to obtain the following estimate

$$\| e^{t(\Delta_g - c_0)} \|_{L^1 \to L^r} \leq Ce^{-c_0 t} e^{-\frac{t\delta_n}{2}(1 - \frac{1}{r})} c_n(t)^{(1 - \frac{1}{r})} \quad \forall\, r \in [1, +\infty]$$

and by duality we deduce that

$$\| e^{t(\Delta_g - c_0)} \|_{L^p \to L^\infty} \leq Ce^{-c_0 t} e^{-\frac{t\delta_n}{2p}} c_n(t)^{\frac{1}{p}} \quad \forall\, p \in [1, +\infty]\,.$$

Next, by interpolating the last estimate and the $L^p \to L^p$ estimate in Proposition 4.6 for $p \in [1, \infty]$, we conclude the proof obtaining $L^p \to L^q$ estimates for $1 \leq p \leq q \leq \infty$ with the norm $c_n(t)^{\left(\frac{1}{p} - \frac{1}{q}\right)} e^{-c_0 t} e^{-t\,\gamma_{p,q}}$, where $\gamma_{p,q} = \frac{\delta_n}{2}\left[\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{8}{q}\left(1 - \frac{1}{p}\right)\right]$ and $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$.

4.3. **Proof of Theorem 4.2.** We shall split the proof into several Lemmas.

Lemma 4.8. *Assuming* **(H1-4)**, *we have the following estimate for* $p \geq 2$

$$\|\nabla e^{t(\overrightarrow{\Delta} - I)} u_0\|_{L^p} \leq C \max\left(\frac{1}{\sqrt{t}}, 1\right) \|u_0\|_{L^p}, \quad \forall\, t > 0. \tag{31}$$

Proof. The following proof generalizes to the vectorial Laplacian some arguments yielding log Sobolev inequalities for the Laplace Beltrami operator on Riemannian manifolds (see for example [6]). Let us consider $P_t = e^{t\Delta_g}$ and $Q_t = e^{t(\overrightarrow{\Delta} - I)}$. We shall prove the crucial pointwise estimate

$$|\nabla Q_t u_0|^2 \leq \frac{1}{d(t)}\left(P_t(|u_0|^2) - |Q_t u_0|^2\right) - |Q_t u_0|^2, \quad \frac{1}{d(t)} = \frac{\alpha_1}{(e^{2\alpha_1 t} - 1)}, \tag{32}$$

with $\alpha_1 = -\left(\max(c_0, \frac{1}{c_0}) + 2Kn + 2Kn^{\frac{3}{2}} + 2\right) < 0$.
Since

$$\frac{1}{d(t)} = \frac{\alpha_1}{(e^{2\alpha_1 t} - 1)} \leq C_1 \begin{cases} \frac{1}{t} & \text{if } 0 < t \leq 1, \\ \alpha_1 & \text{if } t \geq 1, \end{cases}$$

we obtain

$$|\nabla Q_t u_0|^2 \leq C_1 \max\left(\frac{1}{t}, \alpha_1\right) P_t(|u_0|^2)$$

which, by integrating on M and by using that $P_t : L^{\frac{p}{2}} \to L^{\frac{p}{2}}$ is bounded for $p \geq 2$ proved in Proposition 4.6, implies

$$\begin{aligned} \|\nabla Q_t u_0\|_{L^p} &\leq C_n \max\left(\frac{1}{\sqrt{t}}, 1\right) \|P_t(|u_0|^2)\|^{\frac{1}{2}}_{L^{\frac{p}{2}}} \\ &\leq C_n \max\left(\frac{1}{\sqrt{t}}, 1\right) e^{-\frac{4t\delta_n (p-2)}{p^2}} \|u_0\|_{L^p} \end{aligned} \tag{33}$$

which yields the proof of the Lemma.
It remains to prove (32). We note that by using the following properties

$$\frac{d}{ds} P_s = \Delta_g P_s = P_s \Delta_g \text{ and } \frac{d}{ds} Q_{t-s} = -(\overrightarrow{\Delta} - I) Q_{t-s}, \tag{34}$$

we can write

$$\begin{aligned} P_t\left(|u_0|^2\right) - |Q_t\, u_0|^2 &= \int_0^t \frac{d}{ds}\left(P_s\left(|Q_{t-s}\, u_0|^2\right)\right) ds \\ &= \int_0^t P_s \left[\left(\Delta_g |Q_{t-s}\, u_0|^2\right) - 2g\left(\overrightarrow{\Delta} Q_{t-s} u_0, Q_{t-s}\, u_0\right) + 2|Q_{t-s} u_0|^2\right] ds. \end{aligned}$$

From Bochner's identity (12) in Lemma 2.2 for the vector field $u = Q_{t-s}\, u_0$, we obtain

$$P_t\left(|u_0|^2\right) - |Q_t u_0|^2 = 2\int_0^t P_s\left(|\nabla Q_{t-s} u_0|^2 + |Q_{t-s} u_0|^2\right) ds := 2\int_0^t e^{2\alpha s} \psi(s)\, ds.$$

where

$$\psi(s) = e^{-2\alpha s} P_s\left(|\nabla Q_{t-s} u_0|^2 + |Q_{t-s} u_0|^2\right) \geq 0.$$

If we can choose the parameter $\alpha \in \mathbb{R}$ such that ψ is nondecreasing, we obtain

$$P_t\left(|u_0|^2\right) - |Q_t\, u_0|^2 \geq 2\,\psi(0) \int_0^t e^{2\alpha s}\, ds = d(t)\ \left(|\nabla Q_t\, u_0|^2 + |Q_{t-s} u_0|^2\right)$$

and the conclusion follows. Finally we have to prove that: there exists $\alpha \in \mathbb{R}$ such that $\psi'(s) \geq 0$. With explicit computations and by using again the semigroup properties (34), we write for all $m \in M$

$$\begin{aligned} \psi'(s)_{/m} = e^{-2\alpha s} P_s \Big[&-2\alpha\left(|\nabla Q_{t-s} u_0|^2 + |Q_{t-s} u_0|^2\right) + \Delta_g\left(|\nabla Q_{t-s} u_0|^2 + |Q_{t-s} u_0|^2\right) \\ &-2g\left(\nabla(\overrightarrow{\Delta} - I) Q_{t-s} u_0, \nabla Q_{t-s}\, u_0\right) + g\left((\overrightarrow{\Delta} - I) Q_{t-s} u_0, Q_{t-s}\, u_0\right)\Big]_{/m}, \end{aligned}$$

by using Bochner's identity (12) again, we can simplify the last expression obtaining

$$\psi'(s)_{/m} = e^{-2\alpha s}\, P_s\,[B(m)]$$

where

$$B(m) = \Big[\left(\Delta_g|\nabla Q_{t-s}\,u_0|^2\right) - 2g\left(\nabla\overrightarrow{\Delta}Q_{t-s}u_0, \nabla Q_{t-s}\,u_0\right)$$
$$-(2\alpha+1)|Q_{t-s}\,u_0|^2 - 2\alpha|\nabla Q_{t-s}\,u_0|^2\Big]_{/m}.$$

By the maximum principle it is sufficient to prove that $B(m) \geq 0, \forall\, m \in M$. We compute $B(m)$ for each $m \in M$ by using normal geodesic coordinates at m. Let us set $e_i = \partial/\partial x^i = \partial_i$, then $(e_1, \cdots, e_n)$ is an orthonormal basis at m. By using the properties (9) of the normal coordinates at m and by the connection property (4), we can write the first term of $B(m)$ as follows

$$\begin{aligned}
&\left(\Delta_g|\nabla Q_{t-s}\,u_0|^2\right)_{/m} = \partial_k^2\left(g^{ij}g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)\right)_{/m}\\
&= \left(\partial_k^2 g^{ij}\right)_{/m} g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m} + g^{ij}_{/m}\partial_k^2\left(g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)\right)_{/m}\\
&= \left(\partial_k^2 g^{ij}\right)_{/m} g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m} + 2g^{ij}_{/m} g\left(\nabla_{e_k}\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_k}\nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m}\\
&+ g^{ij}_{/m} g\left(\nabla_{e_k}\nabla_{e_k}\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m} + g^{ij}_{/m} g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_k}\nabla_{e_k}\nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m}.
\end{aligned}$$

Thus, by using the expression (11) of the Bochner Laplacian and the norm on tensors (8) in normal coordinates at m, we can write

$$\begin{aligned}
\left(\Delta_g|\nabla Q_{t-s}\,u_0|^2\right)_{/m} &= \left(\partial_k^2 g^{ij}\right)_{/m} g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m}\\
&\quad + 2|\nabla^2 Q_{t-s}u_0|^2_{/m} + 2g\left(\overrightarrow{\Delta}\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_i}(Q_{t-s}\,u_0)\right)_{/m}.
\end{aligned}$$

Therefore

$$\begin{aligned}
B(m) &= \left(\partial_k^2 g^{ij}\right)_{/m} g\left(\nabla_{e_i}(Q_{t-s}\,u_0), \nabla_{e_j}(Q_{t-s}\,u_0)\right)_{/m} + 2|\nabla^2 Q_{t-s}u_0|^2_{/m}\\
&\quad + 2g\left(\overrightarrow{\Delta}\nabla_{e_i}(Q_{t-s}\,u_0) - \nabla_{e_i}\overrightarrow{\Delta}(Q_{t-s}\,u_0), \nabla_{e_i}(Q_{t-s}\,u_0)\right)_{/m}\\
&\quad - (2\alpha+1)|Q_{t-s}\,u_0|^2_{/m} - 2\alpha|\nabla Q_{t-s}\,u_0|^2_{/m}.
\end{aligned}$$

Now, let us compute $\left(\overrightarrow{\Delta}\nabla_{e_i}(Q_{t-s}\,u_0) - \nabla_{e_i}\overrightarrow{\Delta}(Q_{t-s}\,u_0)\right)$ at m. By using (11) and (5), we have

$$\begin{aligned}
&\nabla_{e_i}\overrightarrow{\Delta}(Q_{t-s}\,u_0)_{/m} = \left(\nabla_{e_i}(\nabla_{e_k}\nabla_{e_k}(Q_{t-s}\,u_0) - \nabla_{\nabla_{e_k}e_k}(Q_{t-s}\,u_0))\right)_{/m}\\
&= \left[\nabla_{e_k}\nabla_{e_i}(\nabla_{e_k}(Q_{t-s}\,u_0)) - R(e_i,e_k)\nabla_{e_k}(Q_{t-s}\,u_0) + \nabla_{[e_i,e_k]}\nabla_{e_k}(Q_{t-s}\,u_0) - \nabla_{e_i}\nabla_{\nabla_{e_k}e_k}(Q_{t-s}\,u_0)\right]_{/m}
\end{aligned}$$

We note that $[e_i, e_k] = 0$ for every point p in a vicinity of m and since $\nabla_{\nabla_{e_k} e_k} = 0$ at m this yields

$$\nabla_{e_i} \nabla_{\nabla_{e_k} e_k} (Q_{t-s}\, u_0)_{/m}$$
$$= \Big[\nabla_{\nabla_{e_k} e_k} (\nabla_{e_i} Q_{t-s}\, u_0) - R(e_i, \nabla_{\nabla_{e_k} e_k}) Q_{t-s}\, u_0 + \nabla_{[e_i, \nabla_{\nabla_{e_k} e_k}]} (Q_{t-s}\, u_0) \Big]_{/m} = \nabla_{[e_i, \nabla_{\nabla_{e_k} e_k}]} (Q_{t-s}\, u_0)_{/m}$$

and applying again (5), we obtain

$$\nabla_{e_i} \overrightarrow{\Delta} (Q_{t-s}\, u_0)_{/m} =$$
$$\Big[\nabla_{e_k} \nabla_{e_k} \nabla_{e_i} (Q_{t-s}\, u_0) - \nabla_{e_k} \left(R(e_i, e_k) Q_{t-s}\, u_0 \right) - R(e_i, e_k) \nabla_{e_k} (Q_{t-s}\, u_0) - \nabla_{[e_i, \nabla_{\nabla_{e_k} e_k}]} (Q_{t-s}\, u_0) \Big]_{/m}$$
$$= \overrightarrow{\Delta} \nabla_{e_i} (Q_{t-s}\, u_0)_{/m} - [(\nabla R)(e_k, e_i, e_k)](Q_{t-s}\, u_0)_{/m}$$
$$- 2R(e_i, e_k) \nabla_{e_k} (Q_{t-s}\, u_0)_{/m} - \nabla_{[e_i, \nabla_{\nabla_{e_k} e_k}]} (Q_{t-s}\, u_0)_{/m}.$$

By using the Cristoffel symbols, we have

$$[e_i, \nabla_{\nabla_{e_k} e_k}]_{/m} = \left[\left(\partial_i \Gamma^l_{kk} \right) e_l \right]_{/m}.$$

Thus

$$B(m) = \left(\partial^2_k g^{ij} \right)_{/m} g \left(\nabla_{e_i} (Q_{t-s}\, u_0), \nabla_{e_j} (Q_{t-s}\, u_0) \right)_{/m} + 2 |\nabla^2 Q_{t-s} u_0|^2_{/m}$$
$$+ 2g \left([(\nabla R)(e_k, e_i, e_k)](Q_{t-s}\, u_0), \nabla_{e_i} (Q_{t-s}\, u_0) \right)_{/m} + 4g \left(R(e_i, e_k) \nabla_{e_k} (Q_{t-s}\, u_0), \nabla_{e_i} (Q_{t-s}\, u_0) \right)_{/m}$$
$$+ 2 \left(\partial_i \Gamma^l_{kk} \right)_{/m} g \left(\nabla_{e_l} (Q_{t-s}\, u_0), \nabla_{e_i} (Q_{t-s}\, u_0) \right)_{/m} - (2\alpha + 1) |Q_{t-s}\, u_0|^2_{/m} - 2\alpha |\nabla Q_{t-s}\, u_0|^2_{/m}.$$

We can rewrite the last expression with the curvature tensors. Indeed by (6), we have

$$4g \left(R(e_i, e_k) \nabla_{e_k} (Q_{t-s}\, u_0), \nabla_{e_i} (Q_{t-s}\, u_0) \right)_{/m} = 4 \,\mathrm{Riem}(e_i, e_k, \nabla_{e_k} (Q_{t-s} u_0), \nabla_{e_i} (Q_{t-s} u_0))_{/m}$$

By [16] (p.15) and (7)

$$\sum_{k=1}^n \left(\partial^2_k g^{ij} \right)_{/m} = \frac{2}{3} \sum_{k=1}^n \mathrm{Riem}(e_i, e_k, e_j, e_k)_{/m} = \frac{2}{3} \,\mathrm{Ric}(e_i, e_j)_{/m}.$$

By using

$$\Gamma^k_{ij} = \frac{1}{2} \left(g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \right)$$

and again by [16], we also deduce

$$\sum_{k=1}^n \left(\partial_i \Gamma^l_{kk} \right)_{/m} = \sum_{k=1}^n \left(-\frac{1}{3} \,\mathrm{Riem}(e_k, e_i, e_l, e_k) + \frac{1}{6} \,\mathrm{Riem}(e_k, e_l, e_k, e_i) + \frac{1}{6} \,\mathrm{Riem}(e_k, e_i, e_k, e_l) \right)_{/m},$$

by the symmetry properties of Riemann tensor and (6), we obtain

$$\sum_{k=1}^n \left(\partial_i \Gamma^l_{kk} \right)_{/m} = \frac{2}{3} \,\mathrm{Ric}(e_i, e_l)_{/m}.$$

Thus we have

$$\begin{aligned}B(m) = 2\,\mathrm{Ric}(e_i,e_j)_{/m}\, g(\nabla_{e_i}(Q_{t-s}u_0), \nabla_{e_j}(Q_{t-s}u_0))_{/m} + 2|\nabla^2 Q_{t-s}u_0|^2_{/m} + \\ 4\,\mathrm{Riem}(e_i,e_k,\nabla_{e_k}(Q_{t-s}u_0),\nabla_{e_i}(Q_{t-s}u_0))_{/m} - 2\alpha|\nabla Q_{t-s}\,u_0|^2_{/m} + \\ 2g\left([(\nabla R)(e_k,e_i,e_k)](Q_{t-s}\,u_0), \nabla_{e_i}(Q_{t-s}\,u_0)\right)_{/m} - (2\alpha+1)|Q_{t-s}\,u_0|^2_{/m}.\end{aligned}$$

By hypothesis **(H1-2)** we have

$$B(m) \geq |\nabla Q_{t-s}u_0|^2_{/m}\left(-2\alpha - 2\max\left(c_0, \frac{1}{c_0}\right) - 4Kn - 2Kn^{\frac{3}{2}}\right) + |Q_{t-s}u_0|^2_{/m}\left(-2\alpha - 1 - 2Kn^{\frac{3}{2}}\right).$$

We can see that $B(m)$ is positive for $\alpha \leq \alpha_1$. In particular, by choosing

$$\alpha_1 = -\left(\max\left(c_0, \frac{1}{c_0}\right) + 2Kn + 2Kn^{\frac{3}{2}} + 2\right),$$

we end the proof of the Lemma. □

Lemma 4.9. *Assuming* **(H1-4)**, *we have the following estimate for* $1 < p < +\infty$

$$\|\nabla e^{t(\overrightarrow{\Delta}-I)}u_0\|_{L^p} \leq C\,\max\left(\frac{1}{\sqrt{t}}, 1\right)\|u_0\|_{L^p}, \quad \forall\ t > 0. \tag{35}$$

Proof. We have only to prove the estimate (35) for $1 < p \leq 2$. We can use that, for vector fields $u \in \Gamma(TM)$,

$$\|\nabla u\|_{L^p} \sim \|(-\overrightarrow{\Delta})^{\frac{1}{2}}u\|_{L^p}, \quad 1 < p < +\infty,$$

which is essentially the L^p boundedness of the Riesz transform $\nabla(-\overrightarrow{\Delta})^{-\frac{1}{2}}$ (see [41], [52]). Thus $(-\overrightarrow{\Delta})^{\frac{1}{2}}\,e^{t(\overrightarrow{\Delta}-I)} : L^p \to L^p$ satisfies (31) for $2 \leq p < +\infty$ and by duality $\left((-\overrightarrow{\Delta})^{\frac{1}{2}}e^{t(\overrightarrow{\Delta}-I)}\right)^* = e^{t(\overrightarrow{\Delta}-I)}(-\overrightarrow{\Delta})^{\frac{1}{2}} = (-\overrightarrow{\Delta})^{\frac{1}{2}}e^{t(\overrightarrow{\Delta}-I)} : L^{p'} \to L^{p'}$, we obtain (35) for $1 < p' \leq 2$. □

We shall now establish short time $L^p \to L^p$ estimates for the operator $\nabla e^{t(\overrightarrow{\Delta}+r)}$.

Lemma 4.10. *Assuming* **(H1-4)**, *we have the following estimate for* $1 < p < +\infty$

$$\|\nabla e^{t(\overrightarrow{\Delta}+r)}u_0\|_{L^p} \leq C\,\max\left(\frac{1}{\sqrt{t}}, 1\right)\|u_0\|_{L^p}, \quad \forall\ 0 < t \leq 1. \tag{36}$$

Proof. Since

$$\nabla e^{t\overrightarrow{\Delta}} = e^t\,\nabla e^{t(\overrightarrow{\Delta}-I)},$$

we deduce from Lemma 4.9 that

$$\|\nabla e^{t\overrightarrow{\Delta}}u_0\|_{L^p} \leq C\,\max\left(\frac{1}{\sqrt{t}}, 1\right)\|u_0\|_{L^p} \tag{37}$$

for $0 < t \leq 1$ and $1 < p < +\infty$. By using the Duhamel formula, we have

$$\nabla e^{t(\overrightarrow{\Delta}+r)}u_0 = \nabla e^{t\overrightarrow{\Delta}}u_0 + \int_0^t \nabla e^{(t-s)\overrightarrow{\Delta}}\left(r\left(e^{s(\overrightarrow{\Delta}+r)}u_0\right)\right)ds,$$

thus we obtain

$$\|\nabla e^{t(\overrightarrow{\Delta}+r)}u_0\|_{L^p} \le \|\nabla e^{t\overrightarrow{\Delta}}u_0\|_{L^p} + \int_0^t \|\nabla e^{(t-s)\overrightarrow{\Delta}}\left(r\left(e^{s(\overrightarrow{\Delta}+r)}u_0\right)\right)\|_{L^p}ds$$

hence

$$\le C\,\max\left(\frac{1}{\sqrt{t}},1\right)\|u_0\|_{L^p} + C\int_0^t \max\left(\frac{1}{\sqrt{t-s}},1\right)\|r\left(e^{s(\overrightarrow{\Delta}+r)}u_0\right)\|_{L^p}ds.$$

By hypothesis **(H2)** and (20), we have

$$\|r\left(e^{s(\overrightarrow{\Delta}+r)}u_0\right)\|_{L^p} \le C\max\left(c_0,\frac{1}{c_0}\right)\|e^{t(\overrightarrow{\Delta}+r)}u_0\|_{L^p} \le C\max\left(c_0,\frac{1}{c_0}\right)\,e^{-t(\gamma_{p,q}+c_0)}\,\|u_0\|_{L^p}.$$

Thus we obtain

$$\|\nabla e^{t(\overrightarrow{\Delta}+r)}u_0\|_{L^p} \le C\,\max\left(\frac{1}{\sqrt{t}},1\right)\|u_0\|_{L^p}$$

for short time $0 < t \le 1$ and $1 < p < +\infty$. □

By previous $L^p \to L^p$ estimates (36) for the operator $\nabla e^{t(\overrightarrow{\Delta}+r)}$ and by duality argument, the following estimates are true for $1 < p < +\infty$ and short time $0 < t \le 1$

$$\|e^{t(\overrightarrow{\Delta}+r)}\nabla^* T_0\|_{L^p} \le C\,\max\left(\frac{1}{\sqrt{t}},1\right)\|T_0\|_{L^p}, \tag{38}$$

for tensors $T_0 \in L^p(\Gamma(TM \otimes T^*M))$. To get large time estimates, we use the semi-group property combined with last estimates at $t = 1$ and $L^p \to L^p$ dispersive estimates (20)

$$e^{t(\overrightarrow{\Delta}+r)}\nabla^* = e^{(t-1)(\overrightarrow{\Delta}+r)}\left(e^{(\overrightarrow{\Delta}+r)}\nabla^*\right) : L^p \to L^p \to L^p.$$

This yields for $1 < p < +\infty$ and $t \ge 1$

$$\|e^{t(\overrightarrow{\Delta}+r)}\nabla^* T_0\|_{L^p} \le C\,e^{-t\left(c_0+\frac{4\delta_n}{p}\left(1-\frac{1}{p}\right)\right)}\|T_0\|_{L^p}, \tag{39}$$

for all $T_0 \in L^p(\Gamma(TM \otimes T^*M))$. From (38) and (39) we deduce for all time $t > 0$ and $1 < p < +\infty$

$$\|e^{t(\overrightarrow{\Delta}+r)}\nabla^* T_0\|_{L^p} \le C\,\max\left(\frac{1}{\sqrt{t}},1\right)\,e^{-t\left(c_0+\frac{4\delta_n}{p}\left(1-\frac{1}{p}\right)\right)}\|T_0\|_{L^p}. \tag{40}$$

for all $T_0 \in L^p(\Gamma(TM \otimes T^*M))$. Finally, by duality again we finish the proof of Theorem 4.2 obtaining smoothing estimates (21) for vector fields.

4.4. **The case of the Stokes equations.** In this section, we shall consider the following Stokes type linear equations

$$\begin{cases} \partial_t u - \overrightarrow{\Delta} u - r(u) + 2\,\mathrm{grad}(-\Delta_g)^{-1}\,\mathrm{div}(ru) = 0, \\ \mathrm{div}\,u_0 = 0, \\ u_{|t=0} = u_0, \quad u_0 \in \Gamma(TM). \end{cases} \tag{41}$$

It will be convenient to use the linear operator

$$Bu = -2\,\mathrm{grad}(-\Delta_g)^{-1}\,\mathrm{div}(ru).$$

Note that thanks to the boundedness of the Riesz transform on a manifold that satisfies the assumptions **(H1-4)** (see again [41]), B is bounded as a linear operator $L^p \to L^p$ for every p, $1 < p < +\infty$. First we shall prove the following dispersive estimates for small time:

Proposition 4.11. *Assuming* **(H1-4)**, *the solution of* (41) *satisfies the following dispersive estimates: for every p, q such that $1 < p \le q < +\infty$, there exists $C > 0$ such that*

$$\|u(t)\|_{L^q} \le C c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L^p} \quad \forall\, 0 < t \le 2,\ \forall\, u_0 \in L^p(\Gamma(TM)), \tag{42}$$

with $c_n(t) = \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$.

Proof. We begin the proof with the case $p = q$. By using the Duhamel formula we can write

$$u(t) = e^{t(\overrightarrow{\Delta}+r)}u_0 + \int_0^t e^{(t-\tau)(\overrightarrow{\Delta}+r)} Bu(\tau)\, d\tau.$$

Thanks to (20) in Theorem 4.1 with $p = q$, we have

$$\|u(t)\|_{L^q} \le C\|u_0\|_{L^q} + \int_0^t C\|u(\tau)\|_{L^q} d\tau$$

and hence from the Gronwall inequality, we find

$$\|u(t)\|_{L^q} \le Ce^{Ct}\|u_0\|_{L^q}, \quad \forall t \ge 0. \tag{43}$$

Note that the large time behavior is not good and will be improved later.
Next, thanks to (20) in Theorem 4.1, we obtain

$$\begin{aligned}
\|u(t)\|_{L^q} \le{}& c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-t(\gamma_{p,q}+c_0)} \|u_0\|_{L^p} + \\
& \int_0^{t/2} \left[c_n(t-\tau)\right]^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-(t-\tau)(\gamma_{p,q}+c_0)} \|u(\tau)\|_{L^p} d\tau + C\int_{t/2}^t \|u(\tau)\|_{L^q} d\tau \\
\le{}& Cc_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L^p} + Cc_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \int_0^{t/2} \|u(\tau)\|_{L^p} d\tau + C\int_{t/2}^t \|u(\tau)\|_{L^q} d\tau
\end{aligned}$$

by using the estimate (43), we deduce the following estimate for $0 < t \le 2$

$$\|u(t)\|_{L^q} \le Cc_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L^p} + C\int_{t/2}^t \|u(\tau)\|_{L^q} d\tau$$

and hence by setting $y(t) = c_n(t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\|u(t)\|_{L^q}$, we get

$$y(t) \le C\|u_0\|_{L^p} + \frac{C}{c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)}} \int_{\frac{t}{2}}^t c_n(\tau)^{\left(\frac{1}{p}-\frac{1}{q}\right)} y(\tau)\, d\tau \le C\|u_0\|_{L^p} + C\int_0^t y(\tau)\, d\tau$$

and by the Gronwall inequality we can conclude. ☐

Theorem 4.12. *Assuming* **(H1-4)**, *there exist* $\beta \geq c_0 > 0$ *such that the solution of the Cauchy problem* (41) *satisfies the following estimates*

$$\|u(t)\|_{L^p} \leq Ce^{-\beta t}\left(\|u_0\|_{L^p} + \|u_0\|_{L^2}\right), \quad \forall\, t > 0, \tag{44}$$

for every p *such that* $2 \leq p < +\infty$, *for all* $u_0 \in L^p(\Gamma(TM)) \cap L^2(\Gamma(TM))$ *and some* $C > 0$.

Proof. In the case $p = 2$, the above estimate is a direct consequence of the energy estimate for the Stokes equations (41). Indeed, multiplying u in (41) and then integrating on M by part combining with the Bochner identity (2.2), we have the following energy estimate

$$\|u(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla u(s)\|_{L^2}^2 + c_0\,\|u(s)\|_{L^2}^2\right) ds \leq \|u_0\|_{L^2}^2. \tag{45}$$

For $p > 2$, the $L^p \to L^p$ type estimate that we used previously does not yield a good result for large times due to the additional term Bu that does not vanish. In the following, we shall use an argument that relies on the $L^2 \to L^p$ dispersive estimate. This is the reason for which we also need the initial data to be in L^2. By dispersive estimates (42), we have

$$\|u(t)\|_{L^p} \leq c_n(t)^{\left(\frac{1}{2}-\frac{1}{p}\right)} e^{-\beta t}\|u_0\|_{L^2}, \quad \forall 0 < t \leq 2,\ p \geq 2. \tag{46}$$

We use the semi-group property combined with the last estimate at $t = 1$ and the $L^2 \to L^2$ estimates (44) for $t > 1$, then

$$e^{t(\overrightarrow{\Delta}+r-B)} = e^{(\overrightarrow{\Delta}+r-B)}\left(e^{(t-1)(\overrightarrow{\Delta}+r-B)}\right) : L^2 \to L^2 \to L^p$$

is bounded and we have

$$\|u(t)\|_{L^p} \leq Ce^{-\beta t}\|u_0\|_{L^2}, \quad \forall t \geq 2 \text{ and } p \geq 2. \tag{47}$$

Combining the last estimate and (42) with $p = q$, we deduce

$$\|u(t)\|_{L^p} \leq Ce^{-\beta t}\left(\|u_0\|_{L^p} + \|u_0\|_{L^2}\right), \quad \forall t > 0 \text{ and } p \geq 2. \tag{48}$$

$\square$

Corollary 4.13. *Assuming* **(H1-4)**, *there exist* $\beta \geq c_0 > 0$ *such that the solution of the Cauchy problem* (41) *satisfies the following dispersive estimates*

$$\|u(t)\|_{L^q} \leq Cc_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-\beta t}\left(\|u_0\|_{L^p} + \|u_0\|_{L^2}\right), \quad \forall\, t > 0, \tag{49}$$

for every p, q *such that* $2 \leq p \leq q < +\infty$, *for all* $u_0 \in L^p(\Gamma(TM)) \cap L^2(\Gamma(TM))$ *and some* $C > 0$ *and*

$$\|u(t)\|_{L^q} \leq Cc_n(t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-\beta t}\,\|u_0\|_{L^p}, \quad \forall\, t > 0, \tag{50}$$

for every p, q *such that* $1 < p \leq 2 \leq q < +\infty$, *for all* $u_0 \in L^p(\Gamma(TM))$ *and some* $C > 0$, *with* $c_n(t) = \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$.

Proof. By using the semi-group property combined with dispersive estimates (42) at $t = 1$ and the $L^2 \cap L^p \to L^p$ estimates (44) for $t > 1$

$$e^{t(\overrightarrow{\Delta}+r-B)} = e^{(\overrightarrow{\Delta}+r-B)}\left(e^{(t-1)(\overrightarrow{\Delta}+r-B)}\right) : L^2 \cap L^p \to L^p \to L^q$$

is bounded and we have that

$$\|u(t)\|_{L^q} \leq Ce^{-\beta t}\left(\|u_0\|_{L^p} + \|u_0\|_{L^2}\right) \tag{51}$$

for $t \geq 2$ and for every p, q such that $2 \leq p \leq q < +\infty$. Combining the last estimate and (42) in Proposition 4.11, we deduce (49).

To prove (50), we note that (49) yields a $L^2 \to L^p$ estimate valid for all positive times and for $p \geq 2$. By duality, we deduce the $L^{p'} \to L^2$ estimate and we finally get (50) by using the semigroup property $e^{t(\overrightarrow{\Delta}+r-B)} = e^{\frac{t}{2}(\overrightarrow{\Delta}+r-B)}e^{\frac{t}{2}(\overrightarrow{\Delta}+r-B)}$.

□

Proposition 4.14. *Assuming* **(H1-4)**, *the solution of* (41) *satisfies the following smoothing estimates*

$$\|\nabla u(t)\|_{L^p} \leq C\frac{1}{\sqrt{t}}\|u_0\|_{L^p} \quad \forall\ 0 < t \leq 2, \tag{52}$$

for every $1 < p \leq q < +\infty$ *and for all* $u_0 \in L^p(\Gamma(TM))$.

Proof. By using the Duhamel formula we can write

$$\nabla u(t) = \nabla e^{t(\overrightarrow{\Delta}+r)}u_0 + \int_0^t \nabla e^{(t-\tau)(\overrightarrow{\Delta}+r)}Bu(\tau)\,d\tau.$$

Thanks to (21) in Theorem 4.2, we have

$$\|\nabla u(t)\|_{L^p} \leq C\frac{1}{\sqrt{t}}\|u_0\|_{L^p} + \int_0^t C\frac{1}{\sqrt{t-\tau}}\|u(\tau)\|_{L^p}d\tau \tag{53}$$

and by using (42) we find

$$\|\nabla u(t)\|_{L^p} \leq C\frac{1}{\sqrt{t}}\|u_0\|_{L^p}, \quad \forall 0 < t \leq 2.$$

□

Theorem 4.15. *Assuming* **(H1-4)**, *there exist* $\beta \geq c_0 > 0$ *such that the solution of the Cauchy problem* (41) *satisfies the following estimates*

$$\|\nabla u(t)\|_{L^q} \leq C\,c_n(t)^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{n}\right)}e^{-\beta t}\left(\|u_0\|_{L^p} + \|u_0\|_{L^2}\right), \quad \forall\ t > 0, \tag{54}$$

for every p, q *such that* $2 \leq p \leq q < +\infty$, *for all* $u_0 \in L^p(\Gamma(TM)) \cap L^2(\Gamma(TM))$ *and some* $C > 0$ *and*

$$\|\nabla u(t)\|_{L^q} \leq Cc_n(t)^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{n}\right)}e^{-\beta t}\,\|u_0\|_{L^p}, \quad \forall\ t > 0, \tag{55}$$

for every p, q *such that* $1 < p \leq 2 \leq q < +\infty$, *for all* $u_0 \in L^p(\Gamma(TM))$ *and some* $C > 0$, *with* $c_n(t) = \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$.

Proof. We use the semi-group property combined with smoothing estimates (52) at $t = 1$ and previous estimates (49) for $t > 1$, then

$$e^{t(\overrightarrow{\Delta}+r-B)} = e^{(\overrightarrow{\Delta}+r-B)}\left(e^{(t-1)(\overrightarrow{\Delta}+r-B)}\right) : L^2 \cap L^p \to L^q \to W^{1,q}$$

is bounded and we have

$$\|\nabla u(t)\|_{L^q} \leq C c_n (t-1)^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-\beta(t-1)} \left(\|u_0\|_{L^p} + \|u_0\|_{L^2}\right), \quad \forall t \geq 2 \text{ and } 2 \leq p \leq q < +\infty. \tag{56}$$

By using again the semi-group property combined with smoothing estimates (52) and dispersive estimates (42) for short time $0 < t \leq 2$, we have that

$$e^{t(\overrightarrow{\Delta}+r-B)} = e^{\frac{t}{2}(\overrightarrow{\Delta}+r-B)}\left(e^{\frac{t}{2}(\overrightarrow{\Delta}+r-B)}\right) : L^p \to L^q \to W^{1,q}$$

is bounded and we obtain

$$\|\nabla u(t)\|_{L^q} \leq C c_n (t)^{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{n}\right)} \|u_0\|_{L^p}, \quad \forall\, 0 < t \leq 2 \text{ and } 1 < p \leq q < +\infty.$$

Combining the last estimate and (56), we can conclude the proof of (54).

To prove (55), we use the same splitting, the only difference is that for the large time estimate we use (50) in place of (49). □

5. Fujita-Kato theorems on manifolds

5.1. **Strong solutions for Navier-Stokes on Einstein manifolds with negative curvature.** We will first restrict our attention to the case of non-compact Riemannian manifolds M for which the Ricci tensor Ric is a negative constant scalar multiple r of the metric. By using (13), in this case is more convenient to rewrite the nonlinear Cauchy problem for $u(t, \cdot) \in \Gamma(TM)$ in the following way

$$\begin{cases} \partial_t u - \left(\overrightarrow{\Delta} u + ru\right) = -\mathbb{P}(\operatorname{div}(u \otimes u)), \quad \mathbb{P}v = v + \operatorname{grad}(-\Delta_g)^{-1} \operatorname{div} v, \\ u_{|t=0} = u_0, \ \operatorname{div} u_0 = 0. \end{cases} \tag{57}$$

We recall the definition of well-posedness:

Definition 5.1. *the Cauchy problem is locally well-posed on a Banach space X if for any bounded subset B of X, there exists $T > 0$ and a Banach space X_T continuously contained into $\mathcal{C}([0, T], X)$ such that:*

- i) *for any Cauchy data $u_0(x) \in B$, (57) has a unique solution $u(t, x) \in X_T$;*
- ii) *the flow map $u_0 \in B \to u(t, x) \in X_T$ is continuous.*

We say that the problem is globally well-posed if these properties hold for $T = +\infty$.

Theorem 5.2. *(* **Well-posedness on** *M) For every $u_0 \in L^n(\Gamma(TM))$, with $\operatorname{div} u_0 = 0$, there exists $T > 0$ and a unique solution u of the incompressible Navier-Stokes equations such that $u \in \mathcal{C}([0, T], L^n(\Gamma(TM))) \cap X_T$.*
Moreover, there exists $\delta > 0$ such that if $\|u_0\|_{L^n(\Gamma(TM))} \leq \delta$ then the above solutions are global in time.
In dimension 2, the solutions are global for large data.

The Banach space X_T will be defined below.

Proof. We have to solve the fixed point problem

$$u(t) = e^{t(\overrightarrow{\Delta}+r)}u_0 - \int_0^t e^{(t-\tau)(\overrightarrow{\Delta}+r)}\mathbb{P}(\operatorname{div}(u \otimes u))(\tau)\, d\tau = u_1 + B(u,u)(t).$$

We use the following classical variant of the Banach fixed point Theorem :

Lemma 5.3. *Consider X a Banach space and B a bilinear operator such that*

$$\forall u,\, v \in X, \quad \|B(u,v)\|_X \leq \gamma \|u\|_X\, \|v\|_X,$$

then, for every $u_1 \in X$, such that $4\gamma\|u_1\|_X < 1$, the sequence defined by

$$u_{n+1} = u_1 + B(u_n,u_n), \quad u_0 = 0$$

converges to the unique solution of

$$u = u_1 + B(u,u)$$

such that $2\gamma\|u\|_X < 1$.

We notice that this is the Kato's scheme, which consist in finding a family of spaces $(X_T)_{T>0}$ such that the bilinear operator B maps $X_T \times X_T$ into X_T continuously. This will produce automatically local or global well-posedeness result. Then to prove the continuity of $B(u,u)$ on X_T, we use this Lemma with

$$X_T = \left\{ u \in L^\infty_{loc}([0,T], L^q(\Gamma(TM)) |\, c_n(t)^{-(\frac{1}{n}-\frac{1}{q})} e^{\beta t} \|u(t)\|_{L^q} \in L^\infty(0,T) \right\}$$

for some $n < q < +\infty$ and β adapted to the large time decay rate of our dispersive estimates. We recall that $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$. We notice that the $\overrightarrow{\Delta} + r$ has the property of commuting with the projection $\mathbb{P}$ as long as M has no boundary. Using this fact, we write thanks to the L^p boundedness of the Riesz transform (see [41]) and our smoothing estimates (24)

$$\begin{aligned}\|B(u,u)(t)\|_{L^q} &\leq C \int_0^t c_n(t-\tau)^{\frac{1}{n}+\frac{1}{q}} e^{-\beta(t-v)} \|u \otimes u(\tau)\|_{L^{\frac{q}{2}}}\, ds \\ &\leq C \int_0^t c_n(t-\tau)^{\frac{1}{n}+\frac{1}{q}} e^{-\beta(t-\tau)} \left(e^{-\beta\tau} c_n(\tau)^{\frac{1}{n}-\frac{1}{q}}\right)^2 d\tau \|u\|^2_{X_T}.\end{aligned}$$

Consequently, we obtain for any $q > n$

$$\|B(u,u)\|_{X_T} \leq C\|u\|^2_{X_T}.$$

By using the Lemma 5.3, we get a solution in X_T if $4C\|u_1\|_{X_T} < 1$. If u_0 is small in $L^n(\Gamma(TM)$, this is true with $T = +\infty$. If u_0 is not small, we use as usual that $\lim_{T\to 0}\|u_1\|_{X_T} = 0$ to get the local (in time) well-posedness result in X_T. By classical argument we finally get $u \in \mathcal{C}([0,T], L^n)$.

In dimension $n = 2$, we can prove that the above solutions are global (in time) with any initial data $u_0 \in L^2$. Though the energy estimate gives an unconditional control of the L^2 norm, this is not sufficient to obtain global existence since in the above fixed point argument the existence time T does not depend only on the L^2 norm of u_0. This is due to the fact that L^2 is the critical space in dimension two. Nevertheless, we can overcome this problem with the following classical argument. We first note that at time T the above solution is such that $u(T) \in H^1$ due to the smoothing effect. By an easy fixed point Theorem, we can then continue this solution in H^1 on $[T, T_1]$ with an existence time that only depends on the norm of the initial data in H^1. Consequently, we can obtain global existence if we derive an a priori bound on the H^1 norm of u. Thanks to the boundedness of the Riesz transform that gives the estimate

$$\|\nabla u\|_{L^2} \lesssim \|du^\flat\|_{L^2} + \|u\|_{L^2}$$

it actually suffices to get an estimate on the L^2 norm of $du^\flat$. To get this a priori estimate, we can observe that in terms of differential forms, the Navier-Stokes equation can be written as

$$\partial_t u^\flat + L_u u^\flat + \frac{1}{2} d|v|^2 + dp = \delta_H u^\flat + 2ru^\flat$$

where L_u is the Lie derivative. This yields for $\eta = du^\flat$ the equation

$$\partial_t \eta + L_u \eta = \Delta_H \eta + 2r\eta$$

and hence by identifying η with a scalar function ω, we obtain

$$\partial_t \omega + d\omega(v) = \Delta_g \omega + 2r\omega.$$

Since v is divergence free, we deduce from this equation that

$$\|\omega(t)\|_{L^2} \leq \|\omega(s)\|_{L^2}, \quad t \geq s$$

and the result follows.

□

5.2. **Strong solutions for Navier-Stokes on more general non-compact manifolds.** In this section, we shall study the well-posedness on suitable Banach spaces of the following non-linear Cauchy problem on more general non-compact Riemannian manifolds M satisfying our assumptions **(H1-4)** :

$$\begin{cases} \partial_t u - \overrightarrow{\Delta} u - r(u) - Bu = -\mathbb{P}\left[\nabla_u u\right], & \mathbb{P}v = v + \operatorname{grad}(-\Delta_g)^{-1} \operatorname{div} v, \\ \operatorname{div} u_0 = 0, \\ u_{|t=0} = u_0, \quad u_0 \in \Gamma(TM), \end{cases} \tag{58}$$

where $Bu = -2\operatorname{grad}(-\Delta_g)^{-1}\operatorname{div}(ru)$ and as remarked before B and $\mathbb{P}$ are bounded as linear operators $L^p \to L^p$ for every p, $1 < p < +\infty$ (see again [41]). We notice that the $(\overrightarrow{\Delta} + r - B)$ has not the property of commuting with the operator $\mathbb{P}$ on M, we thus have to modify the functional space where we use the fixed point argument.

Theorem 5.4. *(* **Well-posedness on** *M) For every $u_0 \in L^n(\Gamma(TM)) \cap L^2(\Gamma(TM))$, with $\operatorname{div} u_0 = 0$, there exists $T > 0$ and a unique solution u of the incompressible Navier-Stokes equations (58) such that $u \in \mathcal{C}([0,T], L^n(\Gamma(TM)) \cap L^2(\Gamma(TM)) \cap X_T$.*
Moreover, there exists $\delta > 0$ such that if $\|u_0\|_{L^n(\Gamma(TM))} + \|u_0\|_{L^2(\Gamma(TM))} \leq \delta$ then the above solutions are global in time.
In dimension 2, the solutions are global for large data.

Proof. We have to solve the fixed point problem

$$u(t) = e^{t(\overrightarrow{\Delta}+r-B)}u_0 - \int_0^t e^{(t-\tau)(\overrightarrow{\Delta}+r-B)}\mathbb{P}(\nabla_u u)(\tau)\,d\tau = u_1 + B(u,u)(t).$$

We use again the previous Lemma 5.3 with the following functional space X_T:

$$X_T = \Big\{u \in L^\infty_{loc}([0,T], L^q(\Gamma(TM))),\ \nabla u \in L^\infty_{loc}([0,T], L^{\tilde{q}}(\Gamma(TM)) \cap L^\infty_{loc}([0,T], L^s(\Gamma(TM)) | \\ e^{\beta t}c_n(t)^{-(\frac{1}{n}-\frac{1}{q})}\|u(t)\|_{L^q} + e^{\beta t}c_n(t)^{-(\frac{2}{n}-\frac{1}{\tilde{q}})}\|\nabla u(t)\|_{L^{\tilde{q}}} + e^{\beta t}c_n(t)^{-(\frac{2}{n}-\frac{1}{s})}\|\nabla u(t)\|_{L^s} \in L^\infty(0,T)\Big\}$$

for some suitable $n < q, \widetilde{q}, s < +\infty$ and β adapted to the large time decay rate of our dispersive estimates. We recall that $c_n(t) = C_n \max\left(\frac{1}{t^{\frac{n}{2}}}, 1\right)$. Thanks to the L^p boundedness of the Riesz transform (see [41]) and our dispersive estimates (49), we have

$$\|B(u,u)(t)\|_{L^q} \le C\int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{q}}e^{-\beta(t-\tau)}\left(\|\nabla_u u(\tau)\|_{L^r} + \|\nabla_u u(\tau)\|_{L^2}\right) d\tau \tag{59}$$

for $q \ge r \ge 2$ and by using the Hölder inequality and the definition of our X_T norm, we obtain

$$\le C\int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{q}}e^{-\beta(t-\tau)}\left(\|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^s} + \|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^{\widetilde{q}}}\right) d\tau$$
$$\le C\int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{q}}e^{-\beta(t-\tau)}e^{-2\beta\tau}c_n(\tau)^{\frac{1}{n}-\frac{1}{q}}\left(c_n(\tau)^{\frac{2}{n}-\frac{1}{s}} + c_n(\tau)^{\frac{2}{n}-\frac{1}{\widetilde{q}}}\right) d\tau\|u(\tau)\|^2_{X_T}$$

with

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{s}, \quad \frac{1}{2} = \frac{1}{q} + \frac{1}{\widetilde{q}}. \tag{60}$$

Consequently, this yields

$$\sup_{t>0}\left[c_n(t)^{-(\frac{1}{n}-\frac{1}{q})}e^{\beta t}\|B(u,u)(t)\|_{L^q}\right] \le C\|u\|^2_{X_T},$$

since

$$\sup_{t>0}\left[c_n(t)^{-(\frac{1}{n}-\frac{1}{q})}\int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{q}}e^{-\beta\tau}c_n(\tau)^{\frac{1}{n}-\frac{1}{q}}\left(c_n(\tau)^{\frac{2}{n}-\frac{1}{s}} + c_n(\tau)^{\frac{2}{n}-\frac{1}{\widetilde{q}}}\right) d\tau\right] < +\infty$$

if

$$q \ge n, \quad q \ge r, \quad 2 \le r < n, \quad s > \frac{n}{2}, \quad \frac{1}{r} \le \frac{1}{s} + \frac{1}{n}. \tag{61}$$

In similar way by using the smoothing estimates (54), we get

$$\|\nabla B(u,u)(t)\|_{L^s} \le C\int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{s}+\frac{1}{n}}e^{-\beta(t-\tau)}\left(\|\nabla_u u(\tau)\|_{L^r} + \|\nabla_u u(\tau)\|_{L^2}\right) d\tau \tag{62}$$

for $s \geq r \geq 2$ and again by using the Hölder inequality and the definition of our X_T norm, we obtain

$$\leq C \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{s}+\frac{1}{n}} e^{-\beta(t-\tau)} \left(\|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^s} + \|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^{\widetilde{q}}}\right) d\tau$$
$$\leq C \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{s}+\frac{1}{n}} e^{-\beta(t-\tau)} e^{-2\beta\tau} c_n(\tau)^{\frac{1}{n}-\frac{1}{q}} \left(c_n(\tau)^{\frac{2}{n}-\frac{1}{s}} + c_n(\tau)^{\frac{2}{n}-\frac{1}{\widetilde{q}}}\right) d\tau \|u(\tau)\|^2_{X_T}.$$

Then

$$\sup_{t>0}\left[c_n(t)^{-(\frac{2}{n}-\frac{1}{s})} e^{\beta t} \|\nabla B(u,u)(t)\|_{L^s}\right] \leq C\|u\|^2_{X_T},$$

since

$$\sup_{t>0}\left[c_n(t)^{-(\frac{2}{n}-\frac{1}{s})} \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{s}+\frac{1}{n}} e^{-\beta\tau} c_n(\tau)^{\frac{1}{n}-\frac{1}{q}} \left(c_n(\tau)^{\frac{2}{n}-\frac{1}{s}} + c_n(\tau)^{\frac{2}{n}-\frac{1}{\widetilde{q}}}\right) d\tau\right] < +\infty$$

if

$$s \geq n, \quad s \geq r, \quad 2 \leq r < n, \quad \frac{1}{r} \leq \frac{1}{s} + \frac{1}{n}. \tag{63}$$

Finally, in the same way by using again the smoothing estimates (54), we have

$$\|\nabla B(u,u)(t)\|_{L^{\widetilde{q}}} \leq C \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{\widetilde{q}}+\frac{1}{n}} e^{-\beta(t-\tau)} \left(\|\nabla_u u(\tau)\|_{L^r} + \|\nabla_u u(\tau)\|_{L^2}\right) d\tau$$

for $\widetilde{q} \geq r \geq 2$ and again by using the Hölder inequality and the definition of our X_T norm, we obtain

$$\leq C \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{\widetilde{q}}+\frac{1}{n}} e^{-\beta(t-\tau)} \left(\|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^s} + \|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^{\widetilde{q}}}\right) d\tau$$
$$\leq C \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{\widetilde{q}}+\frac{1}{n}} e^{-\beta(t-\tau)} e^{-2\beta\tau} c_n(\tau)^{\frac{1}{n}-\frac{1}{q}} \left(c_n(\tau)^{\frac{2}{n}-\frac{1}{s}} + c_n(\tau)^{\frac{2}{n}-\frac{1}{\widetilde{q}}}\right) d\tau \|u(\tau)\|^2_{X_T}.$$

Thus

$$\sup_{t>0}\left[c_n(t)^{-(\frac{2}{n}-\frac{1}{\widetilde{q}})} e^{\beta t} \|\nabla B(u,u)(t)\|_{L^{\widetilde{q}}}\right] \leq C\|u\|^2_{X_T},$$

since

$$\sup_{t>0}\left[c_n(t)^{-(\frac{2}{n}-\frac{1}{\widetilde{q}})} \int_0^t c_n(t-\tau)^{\frac{1}{r}-\frac{1}{\widetilde{q}}+\frac{1}{n}} e^{-\beta\tau} c_n(\tau)^{\frac{1}{n}-\frac{1}{q}} \left(c_n(\tau)^{\frac{2}{n}-\frac{1}{s}} + c_n(\tau)^{\frac{2}{n}-\frac{1}{\widetilde{q}}}\right) d\tau\right] < +\infty$$

if

$$\widetilde{q} \geq n, \quad \widetilde{q} \geq r, \quad 2 \leq r < n, \quad \frac{1}{r} \leq \frac{1}{\widetilde{q}} + \frac{1}{n}. \tag{64}$$

Consequently, by choosing $q, \widetilde{q}, s$ such that the conditions (60), (61), (63), (64) are verified, we obtain

$$\|B(u,u)\|_{X_T} \leq C\|u\|^2_{X_T}.$$

By using the Lemma 5.3, we get a solution in X_T if $4C\|u_1\|_{X_T} < 1$. If u_0 is small in $L^n(\Gamma(TM)) \cap L^2(\Gamma(TM))$, this is true with $T = +\infty$. If u_0 is not small, we use as usual that $\lim_{T\to 0}\|u_1\|_{X_T}=0$ to get the local (in time) well-posedness result in X_T. By classical arguments, we finally get $u\in\mathcal{C}([0,T],L^n\cap L^2)$.

To handle the two-dimensional case, we can proceed in the same way with X_T defined as

$$X_T=\Big\{u\in L^\infty_{loc}([0,T],L^q(\Gamma(TM)),\ \nabla u\in L^\infty_{loc}([0,T],L^{\tilde q}(\Gamma(TM))\cap L^\infty_{loc}([0,T],L^s(\Gamma(TM))| \\ e^{\beta t}c_n(t)^{-(\frac{1}{n}-\frac{1}{q})}\|u(t)\|_{L^q}+e^{\beta t}c_n(t)^{-(\frac{2}{n}-\frac{1}{s})}\|\nabla u(t)\|_{L^s}\in L^\infty(0,T)\Big\}$$

with $q>2$ and $s>2$. To estimate $B(u,u)$ in X_T the only differences with the previous computations is that when using the dispersive and smoothing estimates to get (59) and (62), we take $r<2$ and thus we apply (50), (55) in place of (49), (54).

To get the global well-posedness, we use the same argument as in the end of the proof of Theorem 5.2.

□

6. Remarks on the uniqueness of weak solutions

As shown in [13], [32], for two-dimensional manifolds, one has to be careful with the definition of Leray type solutions in order to get uniqueness. Indeed, it was proven that for the hyperbolic space $\mathbb{H}^2$, due to the presence of non-trivial bounded harmonic forms there exists infinitely many weak solutions $u\in L^\infty_T L^2\cap L^2_T H^1$ that satisfy the energy inequality:

$$\|u(t)\|^2_{L^2}+\int_0^t\|\nabla u(s)\|^2_{L^2}\,ds\le\|u(0)\|^2_{L^2}\tag{65}$$

for almost every $t\ge 0$. A way to recover the uniqueness, by carefully selecting the weak solution was recently proposed in [14]. We shall propose another way to recover the uniqueness in terms of the regularity of the pressure for two-dimensional manifolds that satisfy **(H1-4)**. Let us recall that the pressure is the solution of the elliptic equation

$$\Delta_g p=-\operatorname{div}(\operatorname{div}(u\otimes u)-2ru).\tag{66}$$

If u has the regularity of a Leray solution, $u\in L^\infty_T L^2\cap L^2_T H^1$, then $\operatorname{div}(ru)\in L^2_T L^2$ and $u\otimes u\in L^2_T L^2$. Indeed, we have that

$$\|u\otimes u\|_{L^2}\le\||u|\|^2_{L^4}\le C\big(\|\nabla u\|_{L^2}\|u\|_{L^2}+\|u\|^2_{L^2}\big)$$

thanks to the Gagliardo-Nirenberg inequality in Remark 2.3 (3). This yields

$$\|u\otimes u\|_{L^2_T L^2}\le C\big(\|u\|_{L^\infty_T L^2}\|\nabla u\|_{L^2_T L^2}+T^2\|u\|^2_{L^\infty_T L^2}\big).$$

Consequently $\operatorname{div}(\operatorname{div}(u\otimes u))\in L^2_T H^{-2}$. Since $\Delta_g : L^2\to H^{-2}$ is an isomorphism by using [41], there exists a unique solution p of (66) such that $p\in L^2_T L^2$. This motivates the following definition of Leray weak solutions:

Definition 6.1. *For every divergence free $u_0\in L^2\Gamma(TM)$, we shall say that $u\in L^\infty_T L^2\cap L^2_T H^1$ is a Leray weak solution of the Navier-Stokes equation with initial data u_0 if for* *every* $\phi \in \mathcal{C}^1_c(\overline{\mathbb{R}}_+ \times M, TM)$*, we have*

$$\int_{\mathbb{R}_+\times M} \Big(g(u,\partial_t\phi) + g(u\otimes u, \nabla\phi) + p \operatorname{div}\phi - g(\nabla u, \nabla\phi) + g(ru,u)\Big) dV_g dt \\ + \int_M g(u_0, \phi(0,\cdot))dV_g = 0 \tag{67}$$

with $p \in L^2_T L^2$ *the unique solution of the elliptic equation* (66).

We claim that

Theorem 6.2. *Assume that* M *is a two-dimensional complete simply connected non-compact manifold that satisfy* **(H1-4)***. Then, for every divergence free* $u_0 \in L^2\Gamma(TM)$*, there exists a unique weak Leray solution.*

Proof. There are many classical ways to prove the existence. Note that the strong solutions that we have constructed in section 5.1 are actually weak solutions, therefore, we shall focus on the uniqueness.

To prove the uniqueness, we shall first prove that our definition of weak solution contains that they satisfy the energy inequality (and even the energy equality). We first notice that if u is a weak Leray solution, then u is a solution of

$$\partial_t u = \Delta u + ru - \operatorname{div} u\otimes u - \nabla p \tag{68}$$

in the distribution sense and that $u_{|t=0} = u_0$ in the weak sense. Note that the right hand side belongs to $L^2_T H^{-1}$ and therefore $\partial_t u \in L^2_T H^{-1}$. We thus obtain that

$$\|u(t)\|^2_{L^2} - \|u(s)\|^2_{L^2} = 2\int_s^t \langle \partial_t u, u\rangle \, d\tau \tag{69}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket $H^1 - H^{-1}$. In particular, we obtain that $u \in \mathcal{C}L^2$.

Next, thanks to our assumptions **(H1-4)**, we have that $\mathcal{C}^1_c(\overline{\mathbb{R}}_+ \times M)$ is dense in $X = \mathcal{C}_{loc}([0,+\infty[, L^2) \cap L^2_{loc}(\mathbb{R}_+, H^1) \cap H^1_{loc}(\mathbb{R}_+, L^2)$. Moreover, all the bilinear terms that appear in the definition (67) are continuous on $X\times X$, and the trilinear term

$$B(u,v,\phi) = \int_{\mathbb{R}\times M} \Big(g(u\otimes v, \nabla\phi) - \Delta_g^{-1}\big(\operatorname{div}\operatorname{div}(u\otimes v)\big)\operatorname{div}\phi\Big)$$

is continuous on $X\times X\times X$ as a consequence of the Gagliardo-Nirenberg inequality since

$$\|u\otimes v\|_{L^2_T L^2} \le C(T)\big(\|u\|_{L^\infty_T L^2}\|u\|_{L^2_T H^1}\big)^{\frac12}\big(\|v\|_{L^\infty_T L^2}\|v\|_{L^2_T H^1}\big)^{\frac12}.$$

This yields that in our definition of weak solution we can take $\phi \in X$. In addition, since $u \in X$, we can use the definition for $\phi = (\rho_\varepsilon * 1_{[0,T]})u$, for every $T>0$ fixed. By using (69), we obtain by taking ε to zero that

$$\frac12\|u(T)\|^2_{L^2} + \int_0^T\int_M \big(|\nabla u|^2 - g(ru,u) - g(u\otimes u, \nabla u) - p \operatorname{div} u\big)\, dV_g\, dt = \frac12\|u(0)\|^2_{L^2}.$$

Since u solves (68) in the distribution sense, we obtain by taking the divergence that

$$\partial_t \operatorname{div} u - \Delta \operatorname{div} u = 0.$$

This proves that $\operatorname{div} u \in L^2_T L^2$ is a solution of the heat equation with zero initial data. Consequently, u stays divergence free for all times. Thus, we obtain

$$\int_M g(u \otimes u, \nabla u) = -\int_M g(\nabla_u u, u) = \frac{1}{2}\int_M |u|^2 \operatorname{div} u \, dV_g = 0.$$

Consequently, we have proven that

$$\frac{1}{2}\|u(T)\|^2_{L^2} + \int_0^T \int_M \big(|\nabla u|^2 - g(ru, u)\big)\, dV_g \, dt = \frac{1}{2}\|u(0)\|^2_{L^2}$$

which is the energy equality.

Now, consider u, v two Leray weak solutions. We can take $\phi = (\rho_\varepsilon * 1_{[0,T]})(u - v)$ and let ε to zero to get that

$$\begin{aligned}(u(T), u(T) - v(T))_{L^2} &- \int_0^T \langle u, \partial_t(u - v)\rangle dt \\ &+ \int_0^T \int_M \Big(g(\nabla u, \nabla(u - v)) - g(ru, u - v)) - g(u \otimes u, \nabla(u - v))\Big) dV_g dt = 0\end{aligned}$$

and that

$$\begin{aligned}(v(T), u(T) - v(T))_{L^2} &- \int_0^T \langle v, \partial_t(u - v\rangle)dt \\ &+ \int_0^T \int_M \Big(g(\nabla v, \nabla(u - v)) - g(rv, u - v)) - g(v \otimes v, \nabla(u - v))\Big) dV_g dt = 0\end{aligned}$$

since as already observed u and v are divergence free. Next, we can subtract the two identities to obtain

$$\begin{aligned}\frac{1}{2}\|u(T) - v(T)\|^2_{L^2} + \int_0^T \big(\|\nabla(u - v)\|^2_{L^2} + c_0\|u - v\|^2_{L^2}\big)\, dt &\leq \frac{1}{2}\|u_0 - v_0\|^2_{L^2} \\ &+ \int_0^T \int_M |g(\nabla_u u, u - v) - g(\nabla_v v, u - v)|\, dV_g dt.\end{aligned} \tag{70}$$

By using that

$$g(\nabla_u u, (u - v) - g(\nabla_v v, u - v) = g(\nabla_u(u - v), (u - v)) + g(\nabla_u v - \nabla_v v, u - v),$$

we obtain

$$\begin{aligned}\frac{1}{2}\|u(T) - v(T)\|^2_{L^2} + \int_0^T \big(\|\nabla(u - v)\|^2_{L^2} + c_0\|u - v\|^2_{L^2}\big)\, dt &\leq \frac{1}{2}\|u_0 - v_0\|^2_{L^2} \\ &+ \int_0^T \int_M |g(\nabla_u v - \nabla_v v, (u - v))|\, dV_g dt.\end{aligned} \tag{71}$$

To conclude, we can use again the Gagliardo-Nirenberg inequality which yields

$$\begin{aligned}\int_0^T\int_M |g(\nabla_u v-\nabla_v v,(u-v))|\,dV_g dt &\leq \int_0^T\int_M |\nabla v||u-v|^2\,dV_g dt\\ &\leq \int_0^T \|\nabla v\|_{L^2}\big(\|\nabla(u-v)\|_{L^2}+\|u-v\|_{L^2}\big)\|u-v\|_{L^2}dt.\end{aligned}$$

By using the Young inequality, we obtain from (70)

$$\frac{1}{2}\|u(T)-v(T)\|_{L^2}^2 \leq \frac{1}{2}\|u_0-v_0\|_{L^2}^2 + C\int_0^T \|\nabla v\|_{L^2}^2\|u-v\|_{L^2}^2 dt, \quad \forall T\geq 0$$

and hence from the Gronwall inequality, we have

$$\|u(T)-v(T)\|_{L^2}^2 \leq \|u_0-v_0\|_{L^2}^2 e^{\int_0^T C\|\nabla v\|_{L^2}^2 dt}, \quad \forall T\geq 0.$$

In particular, if $u_0=v_0$, we obtain that $u(T)=v(T)$ for all positive times.

□

As a final remark, we can analyze how the counterexample given in [32] in the case of $\mathbb{H}^2$ is excluded by our definition of Leray weak solution. The velocity field u was chosen under the form

$$u=f(t)(d\Phi)^\sharp$$

where Φ is an harmonic function such that $d\Phi\in L^2$ and f is an arbitrary function of time. In order, to ensure that v is a solution, the pressure p is chosen as

$$p=(2f(t)-f'(t))\Phi-\frac{1}{2}f(t)^2|d\Phi|^2.$$

The restriction given by the energy inequality is not sufficient to ensure that the time profile f is completely determined. In this construction $d\Phi\in L^2$ but Φ itself does not belong to L^2, consequently, if we require that $p\in L^2_T L^2$, then we necessarily have $f'=2f$ and thus $f(t)=f(0)e^{2t}$. This determines completely u from its initial value.

Let us finally note that the definition of weak solutions in [14] leads to the same selection of the velocity in the analysis of this counterexample.

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