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Abstract. We show that strongly positively recurrent Markov shifts (including shifts of finite type) are classified up to Borel conjugacy by their entropy, period and their numbers of periodic points.

1. Introduction

Theorem 1.1 below is one of the results in the "full sets" paper of Hochman [10]. In the statement, 'Markov shift' means countable state Markov shift. The free part of a Borel system is the subsystem obtained by restriction to the nonperiodic points, and a full subset is an invariant subset of measure one for every invariant Borel probability measure. Two Borel systems are *almost-Borel isomorphic* if they are Borel isomorphic after restriction to full subsets of their free parts. Detailed definitions for the Introduction are given in the next section.

Theorem 1.1. [10] *Two mixing Markov shifts are almost-Borel isomorphic if and only if (1) they have equal entropy and (2) one has a measure of maximum entropy if and only the other does.*

An important observation [10] in this setting is that two Borel systems that embed each into the other are Borel isomorphic, by a Borel variant of Cantor-Bernstein Theorem (a.k.a. the measurable Schröder-Bernstein Theorem). Consequently Theorem 1.1 was an immediate corollary of the following embedding theorem.

Theorem 1.2. [10] *Suppose (Y, T) is a mixing Markov shift and (X, S) is a Borel system such that $h(S, \mu) < h(T)$ for every ergodic invariant Borel probability μ on X. Then there is an almost-Borel embedding of (X, S) into (Y, T).*

This theorem easily leads to a decisive almost-Borel classification of Markov shifts, and has implications for other systems [10, 2].

The study of Borel dynamics, adopting weakly wandering sets as the relevant notion of negligible sets, was initated by Shelah and Weiss [15, 16, 17]. Here that notion of isomorphism preserves additionally the infinite and quasi-invariant measures (and again it is natural to restrict to free parts). Whether there is a theorem for Borel dynamics like Theorem 1.2 is a difficult open problem, discussed in [10]. Our purpose in this paper is to show that a generalization of Theorem 1.1 to this richer category holds in at least one meaningful case.

Theorem 1.3. *The free parts of mixing SPR Markov shifts are Borel isomorphic if and only if they have equal entropy.*

We note that Hochman [10] has asked if those free parts are in fact topologically conjugate, at least in the case of subshifts of finite type.

As in the almost-Borel case, Theorem 1.3 is an immediate corollary of an embedding result, stated next.

Theorem 1.4. *Suppose (Y, T) is a mixing SPR Markov shift and (X, S) is a Markov shift such that $h(X) = h(Y)$ and X has a unique irreducible component of full entropy and this component is a mixing SPR Markov shift. Then there is a Borel embedding of (X, S) into (Y, T).*

The proof is independent of Hochman's result and techniques. Roughly speaking, Hochman builds almost-Borel embeddings from the bottom up with a uniform version of the Krieger Generator Theorem [12]. In our much more special situation, we can build Borel embeddings with the following offshoot of the Krieger Embedding Theorem.

Theorem 1.5. *Suppose (Y, T) is a mixing Markov shift and (X, S) is a Markov shift such that $h(X) < h(Y)$. Then there is a Borel embedding of the free part of (X, S) into (Y, T).*

Theorem 1.5, though not completely trivial, is completely unsurprising. (The question of when a Markov shift embeds *continuously* into a mixing Markov shift is much harder [5, 6].) The novel feature in the proof of Theorem 1.4 is the use of a "top-down" embedding given by the almost isomorphism theorem of [3] to reduce the problem to embeddings of lower entropy systems.

At the end of the paper we state the Borel classification of the free parts of irreducible SPR Markov shifts, which follows from the mixing case.

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2. Definitions and background

A **Borel system** $(X, \mathcal{X}, T)$ is a standard Borel space[1] $(X, \mathcal{X})$ together with a Borel automorphism[2] $T : X \to X$. We often abbreviate $(X, \mathcal{X}, T)$ to (X, T) or X or T if it does not create confusion. A **Borel factor map** is a homomorphism of Borel systems: a (not necessarily onto) Borel measurable map intertwining the actions. An isomorphism or conjugacy of Borel systems

[1] $\mathcal{X}$ is a σ-algebra of subsets of X such that there is distance on X which turns it into a complete separable space whose collection of Borel subsets is $\mathcal{X}$.

[2] A bijection such that $T^{-1}\mathcal{X} := \{T^{-1}E : E \in \mathcal{X}\} = T\mathcal{X} = \mathcal{X}$.

is a bijective Borel factor map; an embedding of Borel systems is an injective Borel factor map. By an easy exercise in descriptive set theory (see [16, p.399]), there is a Borel conjugacy of two systems if and only if there is a Borel conjugacy between their free parts and for each n the cardinalities of their sets of periodic orbits of size n is the same.

Given a Borel system (X, T), we use $\mathbb{P}(X) \supset \mathbb{P}_{\text{erg}}(X) \supset \mathbb{P}'_{\text{erg}}(X)$ respectively to denote the sets of all measures[3], all ergodic measures, and all ergodic nonatomic measures. Recall from [16] that a set W is **wandering** if it is Borel and if $\bigcup_{k\in\mathbb{Z}} T^k W$ is a disjoint union (which we denote $\bigsqcup_{k\in\mathbb{Z}} T^k W$). A set is **weakly wandering** if it is a Borel subset of a countable union of wandering sets. Such a set has measure zero for all quasi-invariant measures [15, 16], not only for measures in $\mathbb{P}(X)$. To avoid any mystery, we record a simple remark.

Remark 2.1. Suppose (X, S) and (Y, T) are Borel systems and each contains an uncountable Borel set which is wandering. Then the systems are Borel isomorphic if and only if they are Borel isomorphic modulo wandering sets.

The basis of the remark is the following. Any weakly wandering set is contained in the orbit of a wandering set. Under the assumption, such wandering sets in X and Y can be enlarged to uncountable Borel subsets of the ambient Polish space. Any two such sets are Borel isomorphic.

A **Markov shift** (X, S) is a topological system $\Sigma(G)$ defined by the action of the left shift $\sigma : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ on the set $\Sigma(G)$ of paths on some oriented graph G with countably (possibly finitely) many vertices and edges. We will use the edge shift (rather than the vertex shift) presentation. The domain X is the set of $x = (x_n)_{n\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}}$ (where $\mathcal{E}$ is the set of oriented edges) such that for all n, the terminal vertex of x_n equals the initial vertex of x_{n+1}. The (Polish) topology on X is the relative topology of the product of the discrete topologies. When G is finite, $\Sigma(G)$ is a shift of finite type (SFT). $\Sigma(G)$ is **irreducible** if G contains a unique strongly connected component, i.e., a maximal set of the vertices such that for any pair, there is a loop containing both. An arbitrary Markov shift is the disjoint union of a wandering set and countably many disjoint irreducible Markov shifts. An irreducible Markov shift is mixing if and only if the g.c.d. of the periods of its periodic points is 1.

The Borel entropy of a system (X, S) is the supremum of the Kolmogorov-Sinai entropies $h(S, \mu)$, $\mu \in \mathbb{P}(X)$. Markov shifts of positive entropy contain uncountable wandering sets; so, by the Remark 2.1, for simplicity we can neglect weakly wandering sets in both statements and proofs. An irreducible Markov shift (X, S) (more generally, an irreducible component) has at most one measure of maximum (necessarily finite) entropy [7]; if this measure μ exists, then (S, μ) is measure-preservingly isomorphic to the product of a

[3]Unless specified otherwise, the word measure will denote an invariant Borel probability.

finite entropy Bernoulli shift and a finite cyclic rotation (see [2] for comment and references).

An irreducible Markov shift Σ is **strongly positively recurrent** (or **stably positive recurrent** or just **SPR**) if it admits a measure μ of maximal entropy which is *exponentially recurrent*: for every non-empty open subset $U \subset \Sigma$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\Big(\Sigma \setminus \bigcup_{k=0}^{n-1} \sigma^{-k} U\Big) < 0 \ .$$

We refer to [3, 8, 9] for more on SPR shifts. In the language of [8, 9], the SPR Markov shifts are the positively recurrent symbolic Markov chains defined by stably recurrent matrices (further developed in [9] as the fundamental class of "stably positive" matrices). The SPR Markov shifts are a natural subclass preserving some of the significant properties of finite state shifts [3, Sec.2].

3. Embedding a Markov shift with smaller entropy

In this section we will prove Theorem 1.5. First we recall and adapt some standard finite-state symbolic dynamics (for more detail on this, see [1] or [14]).

Lemma 3.1. *Suppose $\epsilon > 0$ and X is a mixing Markov shift with entropy $h(X) > 0$. Then X contains infinitely many mixing SFTs S_n, pairwise disjoint, such that $h(S_n) > h(X) - \epsilon$ for all n.*

Proof. X contains an SFT S with entropy greater than $h(X) - \epsilon$ [7]; S is easily enlarged to a mixing SFT S' in X. The complement of a given proper subshift of S' contains a mixing SFT with entropy arbitrarily close to $h(S')$ [4, Lemma 26.17]. Thus one can construct the required family inductively. □

Definition 3.2. For a system (X, S), $|P^o_n(X)|$ denotes the cardinality of the set of points in S-orbits of length n.

Theorem 3.3 (Krieger Embedding Theorem [13])**.** *Let X be a subshift on a finite alphabet and Y a mixing SFT such that $h(X) < h(Y)$ and $|P^o_n(X)| \le |P^o_n(Y)|$ for all n. Then there is a continuous embedding of X into Y.*

Proposition 3.4. [1, Lemma 2.1 and p.546] *Suppose X is a mixing SFT and M is a positive integer. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be distinct finite orbits in X. Let W_i be the set of points whose positive iterates are positively asymptotic to $\mathcal{O}_i$, and let $W = \cup_i W_i$. Then there exist a mixing SFT Z and a continuous surjection $p : Z \to X$ such that:*

1. *$|p^{-1}(x)| = 1$ for all x outside W*
2. *The preimage of $\mathcal{O}_i$ is an orbit $\widetilde{\mathcal{O}_i}$ of length $M|\mathcal{O}_i|$.*
3. *$p^{-1}(W_i)$ is the set of points positively asymptotic to $\widetilde{\mathcal{O}_i}$.*

Corollary 3.5. *Let X and Y be SFTs such that $h(X) < h(Y)$ and Y is mixing. Then there is a continuous embedding of $X \setminus X_0$ into Y where X_0 is the union of a weakly wandering set and a finite set of periodic points.*

Proof. We have that $\lim_n(\,|P_n^o(Y)| - |P_n^o(X)|\,) = \infty$. Thus we may choose M to build Z as in Proposition 3.4 such that Z, by Theorem 3.3, embeds into Y. The map $Z \to X$ is a Borel isomorphism on the complement of a set X_0 of points positively asymptotic to finitely many periodic points. □

To reduce Theorem 1.5 to this corollary, we use reductions stated as three lemmas. A **loop system** is a Markov shift defined by a **loop graph**: a graph made of simple loops which are based at a common vertex and otherwise do not intersect. Given a power series $f = \sum_{n=1}^{\infty} f_n z^n$ with coefficients in $\mathbb{Z}_+$, we let Σ_f denote the loop system with exactly f_n simple loops of length n in the loop graph. If $h(\Sigma_f) = \log\lambda < \infty$, then

(1) $0 < f(1/\lambda) \leq 1$,
(2) $\alpha < \lambda \implies f(1/\alpha) = \infty$ and
(3) $f(1/\lambda) = 1$ if Σ_f has a measure of maximum entropy (i.e. is positive recurrent).

For more on loop systems and Markov shifts, see [3, 9, 11] and their references.

Lemma 3.6. *Any Markov shift X is Borel isomorphic to a Borel system*

$$W \;\sqcup\; \bigsqcup_{n\in\mathbb{N}} \Sigma(L_n)$$

where W is weakly wandering and for each n, L_n is a loop graph.

Lemma 3.7. *Let Σ be a loop system and $h > h(\Sigma)$. Then there is a SFT S with $h(S) < h$ such that Σ has a continuous embedding into S.*

Before proving the lemmas, we deduce the lower-entropy embedding theorem from them.

Proof of Theorem 1.5. According to Remark 2.1 and Lemma 3.6, we may assume that X is a disjoint union of loop systems $\Sigma(L_n)$. Let $h = (h(Y) + h(X))/2 > h(X)$. By Lemma 3.7, each loop system $\Sigma(L_n)$ can be (continuously) embedded into some SFT W_n with entropy less than h. Let $\epsilon = h(Y) - h > 0$. By Lemma 3.1 (with $\epsilon = (h(Y) - h))/2)$), there are pairwise disjoint mixing SFTs Y_n in Y' with $h(Y_n) > h$. Finally, Corollary 3.5 shows that each W_n (apart from finitely many periodic points) can be Borel embedded into $Y_n \subset Y$. Altogether, apart from a countable set of periodic points, X has been Borel embedded into Y. □

We now prove the lemmas.

Proof of Lemma 3.6. Let G be some graph presenting X. For convenience, we identify its vertices with $1, 2, \dots$. Observe that each $W_n^\epsilon := \{x \in X : x_0 = n \text{ and } \forall i > 0\; x_{\epsilon i} \neq n\}$ ($n \in \mathbb{N}^*, \epsilon \in \{-1, +1\}$) is wandering. Consider the loop graphs L_n defined by the first return loops of G at vertex n which avoid the vertices $k < n$.

For each $x \in X$, let $N := \inf\{n \geq 1 : \exists a_k, b_k \to \infty \; x_{-a_k} = x_{b_k} = n\}$ and consider the following three cases.

(1) $N = \infty$. Then there exists $\epsilon \in \{-1, +1\}$ such that $x \in \sigma^{-j} W^{\epsilon}_{x_0}$, where $j := \epsilon \sup\{\epsilon i \in \mathbb{Z} : x_i = x_0\} \in \mathbb{Z}$.
(2) $N < \infty$ and $\{x_m : m \in \mathbb{Z}\} \cap [1, N) \neq \varnothing$. Then there exist $k \in [1, N)$ and $\epsilon \in \{-1, +1\}$ such that $j := \epsilon \sup\{\epsilon i \in \mathbb{Z} : x_i = k\} \in \mathbb{Z}$, so $x \in \sigma^{-j} W^{\epsilon}_k$.
(3) Otherwise, $x \in \Sigma(L_N)$.

To conclude, observe that $\bigcup_{k \in \mathbb{N}^*, j \in \mathbb{Z}, \epsilon \in \{-1,+1\}} \sigma^{-j} W^{\epsilon}_k$ is a weakly wandering set. □

Proof of Lemma 3.7. Let $\Sigma = \Sigma_f$, a loop system described by a power series $f = \sum_{n=1}^{\infty} f_n z^n$. If f is a polynomial, then Σ_f is itself an SFT. From now on, we assume f to have infinitely many non-zero terms.

We are going to build the SFT as a finite loop system Σ_p, with a polynomial p obtained by truncating the power series f and then adding some monomials to ensure enough space for the embedding while keeping the entropy $< h$.

Let $\beta \in (h(\Sigma), h)$. Given a positive integer N, let $f^{(N)}$ denote the truncation of f to the polynomial $f_1 z + f_2 z^2 + \cdots + f_N z^N$. As $f(e^{-h(\Sigma)}) \leq 1$ and $h(\Sigma) < \beta$ we have $f_n < e^{n\beta}$ for all $n \geq 1$. Let $g^{<N>}$ denote the polynomial $g_{N+1} z^{N+1} + g_{N+2} z^{N+2} + \cdots + g_{2N} z^{2N}$, where $g_n = \lceil e^{n\beta} \rceil$ (the integer ceiling). Then

$$\begin{aligned} |g^{<N>}(z)| &\leq \left[(e^{(N+1)\beta} + 1) + \cdots + (e^{2N\beta} + 1)|z|^{N-1} \right] |z|^{N+1} \\ &= e^{(N+1)\beta} |z|^{N+1} \left[\frac{1 - (e^{\beta}|z|)^N}{1 - e^{\beta}|z|} \right] + |z|^{N+1} \left[\frac{1 - |z|^N}{1 - |z|} \right]. \end{aligned}$$

As $\beta > 0$, we see that $\lim_{N \to \infty} g^{<N>}(z) = 0$ uniformly for $|z|$ fixed, smaller than $e^{-\beta}$.

Recall that $f(r) < 1$ for $r < e^{-h(\Sigma)}$. Also if $r > 0$ and $|z| = r$ and $f^{(N)}(r) < f(r) < 1$, then $|1 - f^{(N)}(z)| \geq 1 - f^{(N)}(r) > 1 - f(r) > 0$. Fix some $\gamma \in (\beta, h)$ and then N sufficiently large that the following hold:

(1) $|2g^{<N>}(z)| < 1 - f(e^{-\gamma}) < 1 - f^{(N)}(e^{-\gamma}) \leq |1 - f^{(N)}(z)|$;
(2) both $1 - f^{(N)}(z)$ and $1 - f^{(N)}(z) - 2g^{<N>}(z)$ are non-zero.

It follows from Rouché's Theorem that $1 - f^{(N)}$ and $1 - f^{(N)} - 2g^{<N>}$ have the same number of zeros inside the circle $|z| = e^{-\gamma}$, i.e. no zeros. Thus, setting $p := f^{(N)} + 2g^{<N>}$, we get $h(\sigma_p) < \gamma < h$.

Now, set $k = g^{<N>}$ and split p as $p = (f^{(N)} + g^{<N>}) + g^{<N>} =: h + k$ and let $q := h(1 + k + k^2 + \cdots)$. σ_q is the loop system defined from σ_{h+k} by replacing the loops from k by all the loops made by concatenating a copy of a loop from h with an arbitrary positive number of copies of loops from

k (see [3, Lemma 5.1] for detail). It follows that σ_q can be identified to the subset of σ_p obtained by removing a copy of σ_k with the points asymptotic to it. Hence, there is a continuous embedding of σ_q into σ_p.

Note that for $n \leq N$ we have $f_n = p_n = q_n$. Also, for $n > N$, $f_n < e^{n\beta} \leq (1 + k + k^2 + \cdots)_n \leq q_n$. This yields an embedding $\sigma_f \to \sigma_q$ and concludes the proof. □

4. The SPR case

We now give the proof of Theorem 1.4. Let X' be the mixing SPR component of X with $h(X) = h(Y)$. Equal entropy mixing SPR Markov shifts are *almost isomorphic* as defined and proved in [3]. Consequently there will be a word w and a subsystem Σ^w of X' (consisting of the points which see w infinitely often in the past and in the future) such that there is a continuous embedding ψ_0 from $X_0 = \Sigma^w$ onto a subsystem Y_0 of Y and $\epsilon > 0$ such that the complements $X' \setminus X_0$ and $Y \setminus Y_0$ have Borel entropy less than $h(Y) - \epsilon$.

The Borel subsystem $X \setminus X_0$ is (after passing to a higher block presentation) the union of a Markov shift X_1 (the subsystem of X avoiding the word w) and a weakly wandering set W (defined by the occurence of w, with a failure of infinite recurrence in the past or future). By Remark 2.1, we can forget about W. We cannot expect X_1 to have entropy less than $h(Y \setminus Y_0)$, and therefore we cannot apply Theorem 1.5 to embed X_1 into a subsystem of $Y \setminus Y_0$. Instead, we will push X_1 into the image of X_0, and adjust the definition on X_0 to keep injectivity.

For L large enough,

$$\Sigma^{w,L} := \{x \in \Sigma : \forall n \in \mathbb{Z}\ \exists k \in \{0, \dots, L\}\ x_{n+k} \dots x_{n+k+|w|-1} = w\}$$

is a mixing Markov subshift with $h(\Sigma^{w,L}) > h(X_1)$. We apply Lemma 3.1 to get pairwise disjoint mixing SFTs $Y_1, Y_2, \dots$ in $\Sigma^{w,L}$ satisfying $h(Y_i) > h(X_1)$ for all $i \in \mathbb{N}$.

Let C denote the complement in X_1 of the periodic points. Theorem 1.5 gives Borel embeddings $\gamma_i : C \to Y_i$. Let $Z_i := \gamma_i(C) \subset Y_i$ and let ϕ_i be the conjugacy $\gamma_{i+1} \circ \gamma_i^{-1} : Z_i \to Z_{i+1}$. We define $\psi : X_0 \cup C \to \Sigma'$ by

$$\begin{aligned} \psi : x &\mapsto \gamma_1(x) \in Z_1 && \text{if } x \in C \\ &\mapsto \phi_i(\psi_0(x)) \in Z_{i+1} && \text{if } \psi_0(x) \in Z_i \\ &\mapsto \psi_0(x) && \text{otherwise .} \end{aligned}$$

This ψ is a Borel embedding. This finishes the proof of Theorem 1.4. □

Lastly we record the obvious corollary of Theorem 1.3.

Theorem 4.1. *The free parts of two irreducible SPR Markov shifts are Borel isomorphic if and only if they have the same entropy and period.*

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Department of Mathematics - University of Maryland
E-mail address: mmb@math.umd.edu

Laboratoire de Mathématiques d'Orsay (CNRS, UMR 8628) - Université Paris-Sud
E-mail address: jerome.buzzi@math.u-psud.fr

Instituto de Matemáticas - Universidad Nacional Autónoma de México
E-mail address: rgomez@math.unam.mx