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ON A FINITE DIFFERENCE SCHEME FOR BLOW UP SOLUTIONS FOR THE CHIPOT-WEISSLER EQUATION

by

Houda Hani and Moez Khenissi

Abstract. — In this paper, we are interested in the numerical analysis of blow up for the Chipot-Weissler equation $u_t = \Delta u + |u|^{p-1}u - |\nabla u|^q$ with Dirichlet boundary conditions in bounded domain when $p > 1$ and $1 \leq q \leq \frac{2p}{p+1}$.

To approximate the blow up solution, we construct a finite difference scheme and we prove that the numerical solution satisfies the same properties of the exact one and blows up in finite time.

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1. Introduction

In this paper, we study the numerical approximation of solutions which achieve blow up in finite time of the Chipot-Weissler equation

$$u_t = \Delta u + |u|^{p-1}u - |\nabla u|^q \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.1)$$

with Dirichlet boundary conditions

$$u(x, t) = 0, \quad x \in \partial\Omega \quad \text{and} \quad t > 0, \quad (1.2)$$

and initial data

$$u(x, 0) = u_0(x); \quad x \in \Omega, \quad (1.3)$$

Key words and phrases. — Nonlinear parabolic equation, Chipot-Weissler equation, finite time blow up, finite difference scheme, numerical solution, nonlinear gradient term.

where Ω is a regular bounded domain in $\mathbb{R}^d$, and p, q are fixed finite parameters.

This problem represents a model in population dynamics which is proposed by Souplet in [17], where (1.1)-(1.3) describes the evolution of the population density of a biological species, under the effect of certain natural mechanisms; $u(x, t)$ denote the spatial density of individuals located near a point $x \in \Omega$ at a time $t \geq 0$. The evolution of this density depends on three types of mechanisms: displacements, births and deaths. The reaction term represents the rate of births. If we suppose that the individuals can be destroyed by some predators during their displacements, then the dissipative gradient term represents the density of predators.

We are concentrated on solution u of (1.1)-(1.3) which blow up in the L^∞ norm in the following sense : There exists $T^* < \infty$, called the blow up time such that the solution u exists on $[0, T^*)$ and

$$\lim_{t \rightarrow T^*} \|u(t)\|_{L^\infty} = +\infty.$$

Numerous articles have been published concerning the problem of global existence or nonexistence of solutions to nonlinear parabolic equations. Problem (1.1)-(1.3), has been widely analyzed from a mathematical point of view, on the profile, blow up rates, asymptotic behaviours and self similar solutions (see for example: [16], [19], [21], [15] and [22]), but to our knowledge, there are no results concerning its numerical approximation.

Let us first have a look at the theoretical analysis of this problem. The quasilinear parabolic equation (1.1) was introduced in 1989 by Chipot and Weissler [4], in order to investigate the effect of a damping term on global existence or nonexistence of solutions. They proved that under appropriate conditions on q, p and d , there exists a suitable initial value u_0 so that the corresponding solution of (1.1)-(1.3) blows up in a finite time. More precisely:

Theorem 1.1. — [4] : Let $p > 1$, $1 \leq q \leq \frac{2p}{p+1}$ and $u_0 \in W^{3,s}(\Omega)$ for s sufficiently large, u_0 not identically zero.

Suppose in addition that:

1. $u_0 = 0$ on $\partial\Omega$.
2. $\Delta u_0 - |\nabla u_0|^q + |u_0|^p = 0$ on $\partial\Omega$.
3. $u_0 \geq 0$ in Ω .
4. $\Delta u_0 - |\nabla u_0|^q + u_0^p \geq 0$ in Ω .
5. $E(u_0) = \frac{1}{2} \|\nabla u_0\|_2^2 - \frac{1}{p+1} \|u_0\|_{p+1}^{p+1} \leq 0$.
6. If $q < \frac{2p}{p+1}$ then $\|u_0\|_{p+1}$ is sufficiently large.

7. If $q = \frac{2p}{p+1}$ then p is sufficiently large.

Then the corresponding solution of (1.1)-(1.3) blows up in finite time, in the L^∞ norm.

The obvious difficulty with this result is the existence of such initial data.

Souplet and Weissler have proved in [20] that finite time blow up occurs in $W^{1,s}$ norm (s sufficiently large), for large data whenever $p > q$, more precisely we have :

Theorem 1.2. — [20] : Assume $p > q$ and let $\psi \in W_0^{1,s}(\Omega)$ for s sufficiently large with $\psi \geq 0$, ($\psi \neq 0$).

1. There exists some $\lambda_0 = \lambda_0(\psi) > 0$ such that for all $\lambda > \lambda_0$, the solution of (1.1)-(1.3) with initial data $\phi = \lambda\psi$ blows up in finite time in $W^{1,s}$ norm.
2. There is some $C > 0$ such that

$$T^*(\lambda\psi) \leq \frac{C}{(\lambda \|\psi\|_\infty)^{p-1}}, \quad \lambda \rightarrow \infty.$$

The phenomenon of blow up in finite time for nonlinear parabolic equations has been extensively studied for last decades. Several papers contain numerous references on blow up results, see for example [6], [12], [7] and [10]. There has been many works in the past concerning numerical computation of solutions of nonlinear parabolic equation (see [9], [5], [13] and [3]) but without the gradient term. By studying various papers, we found that many interesting numerical problems for the Chipot-Weissler equation are left unsolved, and we would like to solve some of them in this and other forthcoming works. The results of this paper are used to study the properties of the numerical solution associated to (1.1)-(1.3) and hence we can assimilate the dissipatif role of the gradient term [8].

Although the details are explained in the subsequent sections, we outline here the main ideas of this study. Let us recall a result of Chen where he considered the Fujita equation $u_t = u_{xx} + u^{1+\alpha}$ and proved that some numerical solutions can blow up in finite time at more than one point [3]. The following questions may naturally arise:

1. What happens numerically, if we add a gradient term in the Fujita equation?
2. Which conditions on the reaction term and the gradient term to provide or prevent blow-up?
3. Can we compare the blowing-up rates of the equations with and without gradient term?

In this paper, we develop a numerical scheme in order to approximate the solutions of the nonlinear Chipot-Weissler equation in $\Omega = [-1, 1]$, and we show that the finite difference solution blow up in finite time if a certain condition is assumed. Next,

we study the numerical blow up set and the asymptotic behaviours of our numerical solution near the blow up point, we also try to give an approximation of the blowing up time.

Our paper is organized as follows: In the next section, we present some properties of the exact solution. In section 3, we construct a finite difference scheme and we prove that if the initial data is positive, monotone and symmetric then the numerical solution is also positive, monotone and symmetric. In section 4, we shall prove that the difference solution blows up in $x = 0$ the middle of the interval $[-1, 1]$. In section 5, we give some numerical results to illustrate our analysis. In the last section, we present some interesting questions which will be solved in the future study.

2. Properties of the exact solution

We consider the semilinear equation

$$u_t = u_{xx} + |u|^{p-1}u - |u_x|^q \text{ in }]0, T[\times]-1, 1[, \quad (2.1)$$

with initial data

$$u(0, x) = u_0(x) \text{ for } x \in [-1, 1] \quad (2.2)$$

and Dirichlet boundary conditions

$$u(t, -1) = u(t, 1) = 0 \text{ for } t \in [0, T]. \quad (2.3)$$

where $p > 1$, $1 \leq q \leq \frac{2p}{p+1}$ and $T < T^*$.

Here if $t \in [0, T]$ then $\|u(t, \cdot)\|_\infty := \max_x |u(t, x)| < \infty$.

For the sake of simplicity, we assume that the initial function u_0 satisfies the following conditions:

- (A1) u_0 is continuous, nonconstant and nonnegative in $[-1, 1]$.
- (A2) u_0 is spatially symmetric about $x = 0$.
- (A3) u_0 is strictly monotone increasing in $[-1, 0]$.
- (A4) u_0 is large in the sense that $\|u_0\|_\infty \gg 1$.
- (A5) $u_0(-1) = u_0(1) = 0$.

These properties will be preserved by our numerical scheme and make computations easier.

Under these conditions, it is known from [2] that the solution blows up only at the central point, that is

$$\exists T^* < +\infty \text{ such that } \lim_{t \rightarrow T^*} u(t, 0) = +\infty \text{ but } \lim_{t \rightarrow T^*} u(t, x) < \infty \text{ when } x \neq 0.$$

2.1. Regularity. — It is known from [1] that

Theorem 2.1. — [1] : For $\Omega =]-1, 1[$ and $r > \max(1, q-1)$, problem (2.1)-(2.3) generates a local semiflow on

$$W_0^{1,r}(\Omega)^+ := \{u \in W^{1,r}(\Omega); u \geq 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}$$

and for any $u_0 \in W_0^{1,r}(\Omega)^+$ there exists a unique maximal solution $u \in C([0, T^*), W_0^{1,r}(\Omega)^+)$ where T^* is the maximal existence time for u .

2.2. Positivity. — Chipot and Weissler have proved in [4] that

Theorem 2.2. — For s sufficiently large such that $s \geq 2q$ and $s \geq \frac{p}{1+p}$, if $u_0 \in W_0^{1,s}([-1, 1])$ with $u_0 \geq 0$ in $[-1, 1]$, then $u(t) \geq 0$ for all $t \in [0, T^*)$.

2.3. Symmetry. —

Theorem 2.3. — If the initial data satisfies (A2), then the exact solution u of (2.1)-(2.3) is symmetric, that is:

$$\forall x \in [-1, 1], \quad u(x, t) = u(-x, t) \quad \forall t \in [0, T^*).$$

Proof. — Let $u(x, t)$ be the solution of (2.1)-(2.3). We define, for all $t \geq 0$, the function

$$v(x, t) = \begin{cases} u(-x, t) & x \in [-1, 0] \\ u(x, t) & x \in [0, 1]. \end{cases}$$

We shall prove that v is a solution of (2.1)-(2.3) in $[-1, 1]$.

– In $[0, 1]$, we have $v(x, t) = u(x, t)$.

Then for all $x \in]0, 1[$ and $t \geq 0$, v satisfies,

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) - \frac{\partial^2 v}{\partial x^2}(x, t) - |v(x, t)|^p + \left| \frac{\partial v}{\partial x}(x, t) \right|^q = 0 \\ v(1, t) = 0. \end{cases} \quad (2.4)$$

Then v is a solution of (2.1) in $[0, 1[$.

– In $] -1, 0]$, we have $v(x, t) = u(-x, t)$.

Then for $x \in] -1, 0]$ and $t \geq 0$ we get

$$\begin{aligned} & \frac{\partial v}{\partial t}(x, t) - \frac{\partial^2 v}{\partial x^2}(x, t) - |v(x, t)|^p + \left| \frac{\partial v}{\partial x}(x, t) \right|^q \\ &= \frac{\partial u}{\partial t}(-x, t) - \frac{\partial^2 u}{\partial x^2}(-x, t) - |u(-x, t)|^p + \left| \frac{\partial u}{\partial x}(-x, t) \right|^q. \end{aligned}$$

Since $-x \in [0, 1[$ and $u(-x, t)$ is a solution in $[0, 1[$, we deduce then that

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) - \frac{\partial^2 v}{\partial x^2}(x, t) - |v(x, t)|^p + \left| \frac{\partial v}{\partial x}(x, t) \right|^q = 0 \\ v(-1, t) = 0. \end{cases} \quad (2.5)$$

Using (A2), we obtain

$$v(x, 0) = u_0 \quad \forall x \in [-1, 1]. \quad (2.6)$$

Finally, by (2.4), (2.5), (2.6) and by unicity of the solution of (2.1)-(2.3), we get $u = v$ in $[-1, 1]$.

This finishes the proof of the symmetry. □

2.4. Monotony. —

Theorem 2.4. — *If the initial data satisfies (A3), then the exact solution u of (2.1)-(2.3) is increasing in $[-1, 0[$, that is:*

$$\forall x \in [-1, 0[\text{ and } t \geq 0 \text{ we have } \frac{\partial u}{\partial x}(x, t) \geq 0.$$

Proof. — Let $u(x, t)$ be the solution of (2.1)-(2.3) in $[-1, 0[$, and $v(x, t) = \frac{\partial u}{\partial x}(x, t)$. For all $x \in [-1, 0[$ and $t > 0$, we have

$$\frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) - u^p(x, t) + \left| \frac{\partial u}{\partial x}(x, t) \right|^q = 0.$$

Then

$$\frac{\partial v}{\partial t}(x, t) - \frac{\partial^2 v}{\partial x^2}(x, t) - pu^{p-1}(x, t)v(x, t) + q \frac{\partial v}{\partial x} \text{sign}(v) |v|^{q-1}(x, t) = 0. \quad (2.7)$$

Let

$$v^- = \max(0, -v).$$

Multiplying (2.7) by $v^-(x, t)$, we get

$$\begin{aligned} & \frac{\partial v}{\partial t} v^- - \frac{\partial^2 v}{\partial x^2} v^- - pu^{p-1} v v^- - q \frac{\partial v}{\partial x} v^- |v|^{q-1} = 0 \\ \Rightarrow & -\frac{\partial v^-}{\partial t} v^- + \frac{\partial^2 v^-}{\partial x^2} v^- + pu^{p-1} (v^-)^2 + q \frac{\partial v^-}{\partial x} v^- |v^-|^{q-1} = 0. \end{aligned}$$

Using symmetry and integrating over $[-1, 1]$, we get

$$\begin{aligned} & \int_{-1}^1 \frac{\partial v^-}{\partial t} v^- dx - \int_{-1}^1 \frac{\partial^2 v^-}{\partial x^2} v^- dx - p \int_{-1}^1 u^{p-1} (v^-)^2 dx - \int_{-1}^1 \frac{\partial v^-}{\partial x} (v^-)^q dx = 0 \\ \Rightarrow & \frac{1}{2} \frac{d}{dt} \int_{-1}^1 (v^-)^2 dx - \left(\left[\frac{\partial v^-}{\partial x} v^- \right]_{-1}^1 - \int_{-1}^1 \left(\frac{\partial v^-}{\partial x} \right)^2 dx \right) - p \int_{-1}^1 u^{p-1} (v^-)^2 dx \\ & - \int_{-1}^1 \frac{\partial v^-}{\partial x} (v^-)^q dx = 0. \end{aligned}$$

But

$$\begin{aligned} \left[\frac{\partial v^-}{\partial x} v^- \right]_{-1}^1 &= \frac{\partial v^-}{\partial x}(1, t) v^-(1, t) - \frac{\partial v^-}{\partial x}(-1) v^-(-1, t) \\ &= 2v^-(1, t) \frac{\partial v^-}{\partial x}(1, t) \\ &= -2v^-(1, t) \frac{\partial v}{\partial x}(1, t). \end{aligned} \tag{2.8}$$

In the other hand, from (2.2), we have

$$u^p(1, t) = \frac{\partial u}{\partial t}(1, t) = 0,$$

then by (2.1)

$$\frac{\partial v}{\partial x}(1, t) = \frac{\partial^2 u}{\partial x^2}(1, t) = \left| \frac{\partial u}{\partial x}(1, t) \right| \geq 0,$$

and then we can deduce from (2.8) that

$$\left[\frac{\partial v^-}{\partial x} v^- \right]_{-1}^1 < 0.$$

So

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{-1}^1 (v^-)^2 dx + \int_{-1}^1 \left(\frac{\partial v^-}{\partial x} \right)^2 dx - p \int_{-1}^1 u^{p-1} (v^-)^2 dx - \int_{-1}^1 \frac{\partial v^-}{\partial x} (v^-)^q dx \\ &= \left[\frac{\partial v^-}{\partial x} v^- \right]_{-1}^1 \\ &\leq 0. \end{aligned}$$

Using the symmetry propriety, we get

$$\begin{aligned} & \frac{d}{dt} \int_{-1}^0 (v^-)^2 dx + 2 \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx - 2p \int_{-1}^0 u^{p-1} (v^-)^2 dx - 2 \int_{-1}^0 \frac{\partial v^-}{\partial x} (v^-)^q dx \leq 0 \\ \Rightarrow & \frac{1}{2} \frac{d}{dt} \int_{-1}^0 (v^-)^2 dx \leq - \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx + p \int_{-1}^0 u^{p-1} (v^-)^2 dx + \left| \int_{-1}^0 \frac{\partial v^-}{\partial x} (v^-)^q dx \right|. \end{aligned}$$

We refer to proposition 2.2 in [4], we can see that u and ∇u are bounded before blow up, then there exists $M, N > 0$ such that

$$\begin{aligned} |u^{p-1}(x, t)| &\leq M \text{ for all } x \in [-1, 1] \text{ and } 0 < t < T, \\ |\nabla u(x, t)| &\leq N \text{ for all } x \in [-1, 1] \text{ and } 0 < t < T. \end{aligned}$$

We use Young's inequality, then for all $\epsilon > 0$, there exists $C_\epsilon > 0$, such that

$$\left| \int_{-1}^0 \frac{\partial v^-}{\partial x} (v^-)^q dx \right| \leq \epsilon \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx + C_\epsilon \int_{-1}^0 (v^-)^{2q} dx.$$

Which implies that

$$\left| \int_{-1}^0 \frac{\partial v^-}{\partial x} (v^-)^q dx \right| \leq \epsilon \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx + C_\epsilon N^{2q-2} \int_{-1}^0 (v^-)^2 dx.$$

Then we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-1}^0 (v^-)^2 dx &\leq - \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx + M \int_{-1}^0 (v^-)^2 dx + \epsilon \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx \\ &\quad + C_\epsilon N^{2q-2} \int_{-1}^0 (v^-)^2 dx \\ &= (\epsilon - 1) \int_{-1}^0 \left(\frac{\partial v^-}{\partial x} \right)^2 dx + M_1 \int_{-1}^0 (v^-)^2 dx, \end{aligned}$$

where M_1 is a constant depending on N, M and C_ϵ .

For ϵ sufficiently small we get

$$\frac{d}{dt} \int_{-1}^0 (v^-)^2 dx \leq M_1 \int_{-1}^0 (v^-)^2 dx.$$

Integrations over $[0, t]$, for $t \leq T$, we get

$$\int_{-1}^0 (v^-)^2(x, t) dx \leq \int_{-1}^0 (v^-)^2(x, 0) dx + M \int_0^t \int_{-1}^0 (v^-)^2(x, s) dx ds.$$

From (A3), we have

$$v^-(x, 0) = \left(\frac{\partial u}{\partial x} \right)^-(x, 0) = \left(\frac{\partial u_0}{\partial x} \right)^-(x) = 0.$$

Then using Gronwall lemma, we deduce that $\forall x \in [-1, 0[, t \geq 0 \quad v^-(x, t) = 0$, which achieve the proof of Theorem 2.4. $\square$

3. Full discretization

We consider the semilinear parabolic equation

$$\begin{cases} u_t = u_{xx} + |u|^{p-1}u - |u_x|^q & \text{in }]0, T[\times]-1, 1[, \\ u(0, x) = u_0(x) & \text{for } x \in [-1, 1], \\ u(t, -1) = u(t, 1) = 0 & \text{for } t \in [0, T]. \end{cases} \quad (3.1)$$

In this section, we construct a finite difference scheme whose solution approximate the solution of (3.1) and satisfies the same properties proved above.

Throughout this paper, we use the following notations in the list below:

1. τ : size parameter for the variable time mesh τ_n .
2. h : positive parameter for which $\lambda := \frac{\tau}{h^2} < \frac{1}{16}$ kept fixed.
3. t_n : n -th time step on $t > 0$ determined as:

$$\begin{cases} t_0 = 0 \\ t_n = t_{n-1} + \tau_{n-1} = \sum_{k=0}^{n-1} \tau_k; \quad n \geq 1. \end{cases}$$

4. x_j : j -th net point on $[-1, 1]$ determined as:

$$\begin{cases} x_0 = -1 \\ x_j = x_{j-1} + h_n; \quad j \geq 1 \text{ and } n \geq 0 \\ x_{N_n+1} = 1. \end{cases}$$

5. u_j^n : approximate of $u(t_n, x_j)$.
6. τ_n : discrete time increment of n -th step determined by $\tau_n = \tau \min(1, \|u^n\|_\infty^{-p+1})$.
7. h_n : discrete space increment of n -th step determined by $h_n = \min(h, (2\|u^n\|_\infty^{-q+1})^{\frac{1}{2-q}})$.
8. $N_n = \frac{1}{h_n} - 1$.

Under the assumption that a spatial net point x_m coincides with the middle point $x = 0$ (we can easily achieve this by taking $N_n + 1 = 2m$), we will prove that

$$u_m^n = \max_{0 \leq j \leq N_n+1} |u_j^n| = \|u^n\|_\infty \quad \text{and} \quad u_{m-1}^n = \max_{j \neq m} |u_j^n|.$$

By using the notations above, our difference equation is introduced by: For $j = 1, \dots, N_n$ and $n \geq 0$:

$$\begin{cases} \frac{u_j^{n+1} - u_j^n}{\tau_n} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h_n^2} + (u_j^n)^p - \frac{1}{(2h_n)^q} |u_{j+1}^n - u_{j-1}^n|^{q-1} |u_{j+1}^{n+1} - u_{j-1}^{n+1}| \\ u_j^0 = u_0(x_j) \\ u_0^n = u_{N_n+1}^n = 0. \end{cases} \quad (3.2)$$

We denote by $U^n := (u_0^n, \dots, u_{N_n+1}^n)^t$.

Remark 3.1. — *The convergence of solution of (3.2) to the solution of (3.1) is proved in [8].*

Properties of the discrete solution

In this section, we prove that the difference solution has the same properties as the exact solution.

3.1. Positivity:— Let u be the classical solution of (3.1). By the maximum principal [4], we know that if the initial condition is nonnegative, then u is nonnegative too. We shall prove now the same result, as Theorem 2.2, for the discrete solution of (3.2).

Lemma 3.2. — *Suppose U^0 satisfies (A1)-(A5). Let U^n be the solution of (3.2), then we have $U^n \geq 0$ for all $n \geq 0$.*

Proof. — In view of assumption (A1), we see that $U^n \geq 0$ holds for $n = 0$. Supposing that it holds for some $n \geq 0$, we have to show that $U^{n+1} \geq 0$. For all $j = 1, \dots, N_n$, define

$$(u_j^{n+1})^- := \max_j(0, -u_j^{n+1}) \text{ and } (u_j^{n+1})^+ := \max_j(0, u_j^{n+1}).$$

If we multiply the equation of (3.2) by $(u_j^{n+1})^-$ we obtain

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\tau_n} (u_j^{n+1})^- &= \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h_n^2} (u_j^{n+1})^- + (u_j^n)^p (u_j^{n+1})^- \\ &\quad - \frac{1}{(2h_n)^q} |u_{j+1}^n - u_{j-1}^n|^{q-1} |u_{j+1}^{n+1} - u_{j-1}^{n+1}| (u_j^{n+1})^-. \end{aligned}$$

We use

$$u_j^{n+1} = (u_j^{n+1})^+ - (u_j^{n+1})^- \text{ and } (u_j^{n+1})^+ \cdot (u_j^{n+1})^- = 0.$$

Then we have

$$\begin{aligned} &- \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- - \frac{1}{h_n^2} (u_{j+1}^{n+1})^+ (u_j^{n+1})^- + \frac{1}{h_n} \frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} (u_j^{n+1})^- \\ &\quad - \frac{1}{h_n^2} (u_{j-1}^{n+1})^+ (u_j^{n+1})^- - \frac{1}{h_n} \frac{(u_j^{n+1})^- - (u_{j-1}^{n+1})^-}{h_n} (u_j^{n+1})^- + \frac{1}{2} \left| \frac{u_{j+1}^n - u_{j-1}^n}{2h_n} \right|^{q-1} \left| \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{h_n} \right| (u_j^{n+1})^- \\ &= \frac{1}{\tau_n} (u_j^n)^+ (u_j^{n+1})^- + (u_j^n)^p (u_j^{n+1})^- \\ &\geq 0 \end{aligned}$$

We multiply by (-1) and we sum for $j = 1, \dots, N_n$, we obtain

$$\begin{aligned} & \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{N_n} (u_{j+1}^{n+1})^+ (u_j^{n+1})^- - \frac{1}{h_n} \sum_{j=1}^{N_n} \frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} (u_j^{n+1})^- \\ & + \frac{1}{h_n^2} \sum_{j=1}^{N_n} (u_{j-1}^{n+1})^+ (u_j^{n+1})^- + \frac{1}{h_n} \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_{j-1}^{n+1})^-}{h_n} (u_j^{n+1})^- \\ & \leq \frac{1}{2} \sum_{j=1}^{N_n} \frac{1}{(2h)^q} |u_{j+1}^n - u_{j-1}^n|^{q-1} |u_{j+1}^{n+1} - u_{j-1}^{n+1}| (u_j^{n+1})^-. \end{aligned}$$

But

$$\sum_{j=1}^{N_n} (u_{j-1}^{n+1})^+ (u_j^{n+1})^- = \sum_{j=1}^{N_n} (u_j^{n+1})^+ (u_{j+1}^{n+1})^-,$$

and

$$\sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_{j-1}^{n+1})^-}{h_n} (u_j^{n+1})^- = \sum_{j=1}^{N_n} \frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} (u_{j+1}^{n+1})^- + \frac{((u_1^{n+1})^-)^2}{h_n}.$$

Then

$$\begin{aligned} & \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{N_n} (u_{j+1}^{n+1})^+ (u_j^{n+1})^- - \frac{1}{h_n} \sum_{j=1}^{N_n} \frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} (u_j^{n+1})^- \\ & + \frac{1}{h_n^2} \sum_{j=1}^{N_n} (u_{j-1}^{n+1})^+ (u_j^{n+1})^- + \frac{1}{h_n} \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_{j-1}^{n+1})^-}{h_n} (u_j^{n+1})^- \\ & = \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{N_n} ((u_{j+1}^{n+1})^+ (u_j^{n+1})^- + (u_j^{n+1})^+ (u_{j+1}^{n+1})^-) \\ & + \sum_{j=1}^{N_n} \left(\frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} \right)^2 + \left(\frac{(u_1^{n+1})^-}{h_n} \right)^2. \end{aligned}$$

We use $M := \|U^n\|_\infty < +\infty$ before blow-up, we can write that

$$\frac{|u_{j+1}^n - u_{j-1}^n|}{2h_n} \leq \frac{M}{h_n},$$

which implies that

$$\begin{aligned}
& \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{N_n} ((u_{j+1}^{n+1})^+ (u_j^{n+1})^- + (u_j^{n+1})^+ (u_{j+1}^{n+1})^-) \\
& + \sum_{j=1}^{N_n} \left(\frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} \right)^2 + \left(\frac{(u_1^{n+1})^-}{h_n} \right)^2 \\
& \leq \frac{1}{2} \sum_{j=1}^{N_n} \frac{1}{(2h)^q} |u_{j+1}^n - u_{j-1}^n|^{q-1} |u_{j+1}^{n+1} - u_{j-1}^{n+1}| (u_j^{n+1})^- \\
& \leq \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \frac{1}{h_n} \sum_{j=1}^{N_n} ((u_{j+1}^{n+1})^+ (u_j^{n+1})^- + (u_j^{n+1})^+ (u_{j+1}^{n+1})^-) + \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \frac{((u_1^{n+1})^-)^2}{h_n} \\
& + \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \sum_{j=1}^{N_n} \frac{|(u_{j+1}^{n+1})^- - (u_j^{n+1})^-|}{h_n} ((u_{j+1}^{n+1})^- + (u_j^{n+1})^-).
\end{aligned}$$

We define now the operator

$$(D(U^{n+1})^-)_j := \frac{(u_{j+1}^{n+1})^- - (u_j^{n+1})^-}{h_n} \text{ for } j = 1, \dots, N_n$$

and we denote by $\|X\| = \sum_{j=1}^{N_n} (X_j)^2$, then we have

$$\begin{aligned}
& \left(\frac{1}{h_n^2} - \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \frac{1}{h_n} \right) \sum_{j=1}^{N_n} ((u_{j+1}^{n+1})^+ (u_j^{n+1})^- + (u_j^{n+1})^+ (u_{j+1}^{n+1})^-) \\
& + \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \|D(U^{n+1})^-\|^2 \\
& + \left(\frac{1}{h_n^2} - \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \frac{1}{h_n} \right) ((u_1^{n+1})^-)^2 \\
& \leq \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \sum_{j=1}^{N_n} \frac{|(u_{j+1}^{n+1})^- - (u_j^{n+1})^-|}{h_n} ((u_{j+1}^{n+1})^- + (u_j^{n+1})^-).
\end{aligned}$$

Using the definition of h_n , we get

$$h_n = \min \left(h, (2 \|u^n\|_\infty^{-q+1})^{\frac{1}{2-q}} \right) \leq (2 \|u^n\|_\infty^{-q+1})^{\frac{1}{2-q}} = (2M^{-q+1})^{\frac{1}{2-q}},$$

which implies that

$$\frac{1}{h_n^2} - \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \frac{1}{h_n} > 0,$$

and we obtain

$$\begin{aligned} & \sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \|D(U^{n+1})^-\|^2 \\ & \leq \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \sum_{j=1}^{N_n} \frac{|(u_{j+1}^{n+1})^- - (u_j^{n+1})^-|}{h_n} ((u_{j+1}^{n+1})^- + (u_j^{n+1})^-). \end{aligned}$$

For the second term, we use Young's inequality to obtain that, for all $\epsilon > 0$

$$\begin{aligned} & \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \sum_{j=1}^{N_n} \frac{|(u_{j+1}^{n+1})^- - (u_j^{n+1})^-|}{h_n} ((u_{j+1}^{n+1})^- + (u_j^{n+1})^-) \\ & \leq \epsilon \sum_{j=1}^{N_n} \left(\frac{|(u_{j+1}^{n+1})^- - (u_j^{n+1})^-|}{h_n} \right)^2 + \frac{1}{4\epsilon} \left(\frac{M}{h_n} \right)^{2(q-1)} \sum_{j=1}^{N_n} ((u_{j+1}^{n+1})^- + (u_j^{n+1})^-)^2 \\ & \leq \epsilon \|D(U^{n+1})^-\|^2 + \frac{1}{4\epsilon} \left(\frac{M}{h_n} \right)^{2(q-1)} \sum_{j=1}^{N_n} ((u_{j+1}^{n+1})^- + (u_j^{n+1})^-)^2 \\ & \leq \epsilon \|D(U^{n+1})^-\|^2 + \frac{1}{\epsilon} \left(\frac{M}{h_n} \right)^{2(q-1)} \|(U^{n+1})^-\|^2. \end{aligned}$$

If we take $\epsilon = \frac{1}{2}$ then we obtain

$$\sum_{j=1}^{N_n} \frac{(u_j^{n+1})^- - (u_j^n)^-}{\tau_n} (u_j^{n+1})^- + \|D(U^{n+1})^-\|^2 \leq \frac{1}{2} \|D(U^{n+1})^-\|^2 + 2 \left(\frac{M}{h_n} \right)^{2(q-1)} \|(U^{n+1})^-\|^2,$$

and hence we get

$$\sum_{j=1}^{N_n} ((u_j^{n+1})^- - (u_j^n)^-) (u_j^{n+1})^- \leq 2\tau_n \left(\frac{M}{h_n} \right)^{2(q-1)} \|(U^{n+1})^-\|^2.$$

We use

$$(a - b)a = \frac{a^2}{2} - \frac{b^2}{2} + \frac{(a - b)^2}{2},$$

then we have

$$\frac{1}{2} \sum_{j=1}^{N_n} (u_j^{n+1})^{-2} - \frac{1}{2} \sum_{j=1}^{N_n} (u_j^n)^{-2} + \frac{1}{2} \sum_{j=1}^{N_n} ((u_j^{n+1})^- - (u_j^n)^-)^2 \leq 2\tau_n \left(\frac{M}{h_n} \right)^{2(q-1)} \|(U^{n+1})^-\|^2$$

which implies

$$\left(1 - 4\tau_n \left(\frac{M}{h_n} \right)^{2(q-1)} \right) \|(U^{n+1})^-\|^2 \leq \|(U^n)^-\|^2.$$

We use that $\lambda = \frac{\tau}{h^2} < \frac{1}{16}$, we can verify that

$$1 - 4\tau_n \left(\frac{M}{h_n} \right)^{2(q-1)} > 0$$

and finally we have

$$\|(U^{n+1})^-\|^2 \leq \frac{1}{1 - 4\tau_n \left(\frac{M}{h_n} \right)^{2(q-1)}} \|(U^n)^-\|^2.$$

By $U^n \geq 0$, we have $(U^n)^- = 0$, which implies that $(U^{n+1})^- = 0$, this gives $U^{n+1} \geq 0$. $\square$

3.2. Monotony:— The following result, analogue of Theorem 2.4, establishes monotony for the difference solution:

Lemma 3.3. — *Under the assumption (A1)-(A5), for U^n solution of (3.2) and $m = \frac{N_n + 1}{2}$ we have*

$$\begin{cases} 0 < u_j^n < u_{j+1}^n \text{ for } j = 1, \dots, m-1 \text{ and } n \geq 0 \\ 0 < u_{j+1}^n < u_j^n \text{ for } j = m, \dots, N_n \text{ and } n \geq 0 \end{cases}$$

Proof. — We will prove monotony by applying the similar argument as the nonnegativity to $v_j^n = u_{j+1}^n - u_j^n$ for $n \geq 0$ and $1 \leq j \leq m-1$. Let

$$V^n = (v_1^n, v_2^n, \dots, v_{m-1}^n)^t.$$

It is easy to see that for $j = 1, \dots, m-1$, v_j^n satisfies

$$\begin{aligned} \frac{v_j^{n+1} - v_j^n}{\tau_n} &= \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{h_n^2} + (u_{j+1}^n)^p - (u_j^n)^p \\ &- \frac{1}{(2h_n)^q} (|v_{j+1}^n - v_j^n|^{q-1} |v_{j+1}^{n+1} - v_j^{n+1}| - |v_j^n - v_{j-1}^n|^{q-1} |v_j^{n+1} - v_{j-1}^{n+1}|). \end{aligned} \quad (3.3)$$

In view of assumption (A3), we see that $V^n \geq 0$ holds for $n = 0$. Supposing that it holds for some $n \geq 0$, we have to show that $V^{n+1} \geq 0$. We use that

$$(u_{j+1}^n)^p - (u_j^n)^p > 0 \text{ and } \frac{1}{(2h_n)^q} (|v_j^n - v_{j-1}^n|^{q-1} |v_j^{n+1} - v_{j-1}^{n+1}|) > 0.$$

We multiply equation (3.4) by $(v_j^{n+1})^-$ and we sum for $j = 1, \dots, m-1$, we obtain

$$\begin{aligned}
& \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_j^n)^-}{\tau_n} (v_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{m-1} (v_{j+1}^{n+1})^+ (v_j^{n+1})^- \\
& - \frac{1}{h_n} \sum_{j=1}^{m-1} \frac{(v_{j+1}^{n+1})^- - (v_j^{n+1})^-}{h_n} (v_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{m-1} (v_{j-1}^{n+1})^+ (v_j^{n+1})^- \\
& + \frac{1}{h_n} \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_{j-1}^{n+1})^-}{h_n} (v_j^{n+1})^- - \sum_{j=1}^{m-1} \left| \frac{v_{j+1}^n - v_j^n}{2h_n} \right|^{q-1} \left| \frac{v_{j+1}^{n+1} - v_j^{n+1}}{2h_n} \right| (v_j^{n+1})^- \\
& = - \sum_{j=1}^{m-1} (v_j^n)^+ (v_j^{n+1})^- - \sum_{j=1}^{m-1} ((u_{j+1}^n)^p - (u_j^n)^p) (v_j^{n+1})^- \\
& - \sum_{j=1}^{m-1} \left| \frac{v_{j+1}^n - v_j^n}{2h_n} \right|^{q-1} \left| \frac{v_{j+1}^{n+1} - v_j^{n+1}}{2h_n} \right| (v_j^{n+1})^- \\
& \leq 0,
\end{aligned}$$

so that

$$\begin{aligned}
& \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_j^n)^-}{\tau_n} (v_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{m-1} (v_{j+1}^{n+1})^+ (v_j^{n+1})^- \\
& - \frac{1}{h_n} \sum_{j=1}^{m-1} \frac{(v_{j+1}^{n+1})^- - (v_j^{n+1})^-}{h_n} (v_j^{n+1})^- + \frac{1}{h_n^2} \sum_{j=1}^{m-1} (v_{j-1}^{n+1})^+ (v_j^{n+1})^- \\
& + \frac{1}{h_n} \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_{j-1}^{n+1})^-}{h_n} (v_j^{n+1})^- \\
& \leq \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \sum_{j=1}^{m-1} \left| \frac{v_{j+1}^{n+1} - v_j^{n+1}}{h_n} \right| (v_j^{n+1})^-.
\end{aligned}$$

We use the same calculations as the proof of Lemma 3.1, we see that

$$\begin{aligned}
& \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_j^n)^-}{\tau_n} (v_j^{n+1})^- + \sum_{j=1}^{m-1} \left(\frac{(v_{j+1}^{n+1})^- - (v_j^{n+1})^-}{h_n} \right)^2 \\
& \leq \frac{1}{2} \left(\frac{M}{h_n} \right)^{q-1} \sum_{j=1}^{m-1} \left| \frac{(v_{j+1}^{n+1})^- - (v_j^{n+1})^-}{h_n} \right| (v_j^{n+1})^- + \frac{((v_m^{n+1})^-)^2}{h_n}.
\end{aligned}$$

We denote by $\|X\|_* := \sum_{j=1}^{m-1} (X_j)^2$ and for all $\epsilon > 0$ we have

$$\begin{aligned} & \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_j^n)^-}{\tau_n} (v_j^{n+1})^- + \|D(V^{n+1})^-\|_*^2 \\ & \leq \epsilon \|D(V^{n+1})^-\|_*^2 + \frac{1}{4\epsilon} \left(\frac{M}{h_n}\right)^{2(q-1)} \|(V^{n+1})^-\|_*^2 + \frac{1}{h_n} \|(V^{n+1})^-\|_*^2. \end{aligned}$$

If we take $\epsilon = \frac{1}{2}$ we obtain

$$\begin{aligned} & \sum_{j=1}^{m-1} \frac{(v_j^{n+1})^- - (v_j^n)^-}{\tau_n} (v_j^{n+1})^- + \|D(V^{n+1})^-\|_*^2 \\ & \leq \frac{1}{2} \|D(V^{n+1})^-\|_*^2 + \frac{1}{2} \left(\frac{M}{h_n}\right)^{2(q-1)} \|(V^{n+1})^-\|_*^2 + \frac{1}{h_n} \|(V^{n+1})^-\|_*^2. \end{aligned}$$

Then we can deduce that

$$\sum_{j=1}^{m-1} ((v_j^{n+1})^- - (v_j^n)^-) (v_j^{n+1})^- \leq \tau_n \left(\frac{1}{h_n} + \frac{1}{2} \left(\frac{M}{h_n}\right)^{2(q-1)} \right) \|(V^{n+1})^-\|_*^2,$$

which implies that

$$\|(V^{n+1})^-\|_*^2 - \|(V^n)^-\|_*^2 + \frac{1}{2} \sum_{j=1}^{m-1} ((v_j^{n+1})^- - (v_j^n)^-) (v_j^{n+1})^- \leq \tau_n \left(\frac{1}{h_n} + \frac{1}{2} \left(\frac{M}{h_n}\right)^{2(q-1)} \right) \|(V^{n+1})^-\|_*^2.$$

And hence we get

$$\left(1 - 2\tau_n \left(\frac{1}{h_n} + \frac{1}{2} \left(\frac{M}{h_n}\right)^{2(q-1)} \right) \right) \|(V^{n+1})^-\|_*^2 \leq \|(V^n)^-\|_*^2.$$

We use that $\lambda = \frac{\tau}{h^2} < \frac{1}{16}$ we can verify that

$$1 - 2\tau_n \left(\frac{1}{h_n} + \frac{1}{2} \left(\frac{M}{h_n}\right)^{2(q-1)} \right) > 0.$$

And finally by $V^n \geq 0$, we can deduce that $(V^{n+1})^- = 0$, this gives $V^{n+1} \geq 0$.

We do the same thing to obtain that $v_j^n = u_j^n - u_{j+1}^n \geq 0$ for $n \geq 0$ and $m \leq j \leq N_n$. $\square$

3.3. Symmetry:— The last property of the difference solution is the symmetry, analogue of Theorem 2.4

Lemma 3.4. — *Under the assumption (A1)-(A5), for U^n solution of (3.2) we have*

$$u_{m-i}^n = u_{m+i}^n \text{ for all } i = 1, \dots, m-1 \text{ and } n \geq 0.$$

Proof. — For $n \geq 0$, let $\lambda_n := \frac{\tau_n}{h_n^2}$. For $j = 1, \dots, N_n$ the first equation of (3.2) can be rewritten as

$$-\lambda_n u_{j-1}^{n+1} + (1+2\lambda_n)u_j^{n+1} - \lambda_n u_{j+1}^{n+1} + \frac{\tau_n}{(2h_n)^q} |u_{j+1}^n - u_{j-1}^n|^{q-1} |u_{j+1}^{n+1} - u_{j-1}^{n+1}| = u_j^n + \tau_n (u_j^n)^p.$$

In view of the assumption (A2), we see that $u_{m-i}^0 = u_{m+i}^0$ for all $i = 1, \dots, m-1$. Supposing that it holds for some $n \geq 0$, then for $i = 1, \dots, m-1$,

$$u_{m-i}^n = u_{m+i}^n. \quad (3.4)$$

We have to show that $u_{m-i}^{n+1} = u_{m+i}^{n+1}$ for $i = 1, \dots, m-1$. Let $i \in \{1, \dots, m-1\}$, for $j = m-i$ we have

$$\begin{aligned} -\lambda_n u_{m-i-1}^{n+1} + (1+2\lambda_n)u_{m-i}^{n+1} - \lambda_n u_{m-i+1}^{n+1} \\ + \frac{\tau_n}{(2h_n)^q} |u_{m-i+1}^n - u_{m-i-1}^n|^{q-1} |u_{m-i+1}^{n+1} - u_{m-i-1}^{n+1}| = u_{m-i}^n + \tau_n (u_{m-i}^n)^p. \end{aligned} \quad (3.5)$$

Let

$$\alpha_{i,i}^n = \frac{\tau_n}{(2h_n)^q} |u_{m-i+1}^n - u_{m-i-1}^n|^{q-1}; \quad \alpha_{i,j}^n = 0 \text{ for } i \neq j \text{ and } \alpha^n = (\alpha_{i,j}^n)_{i,j}. \quad (3.6)$$

Then (3.6) can be rewritten as

$$A_1 U^{n+1} + \alpha^n |B_1 U^{n+1}| = V_1^n, \quad (3.7)$$

where

$$A_1 = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ with } A_{11} = \begin{pmatrix} -\lambda_n & 0 & \cdots & \cdots & \cdots & 0 \\ 1+2\lambda_n & \ddots & \ddots & & & \vdots \\ -\lambda_n & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda_n & 1+2\lambda_n & -\lambda_n \end{pmatrix}$$

$$B_1 = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ with } B_{11} = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ 1 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & -1 \end{pmatrix}$$

and $V_1^n = (v_{1j}^n)_j$ with

$$v_{1j}^n = u_j^n + \tau_n (u_j^n)^p \text{ if } 0 \leq j \leq m \text{ and } v_{1j}^n = 0 \text{ if } j > m.$$

We do the same thing for $i = 1, \dots, m-1$ and $j = m+i$, we obtain

$$\begin{aligned} -\lambda_n u_{m+i-1}^{n+1} &+ (1+2\lambda_n)u_{m+i}^{n+1} - \lambda_n u_{m+i+1}^{n+1} \\ &+ \frac{\tau_n}{(2h_n)^q} |u_{m+i+1}^n - u_{m+i-1}^n|^{q-1} |u_{m+i+1}^{n+1} - u_{m+i-1}^{n+1}| = u_{m+i}^n + \tau_n (u_{m+i}^n)^p. \end{aligned} \quad (3.8)$$

Using (3.4) and (3.6), then (3.9) can be rewritten as

$$A_2 U^{n+1} + \alpha^n |B_2 U^{n+1}| = V_2^n, \quad (3.9)$$

where

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix} \text{ with } A_{22} = \begin{pmatrix} -\lambda_n & 1+2\lambda_n & -\lambda_n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\lambda_n \\ \vdots & & & \ddots & \ddots & 1+2\lambda_n \\ 0 & \cdots & \cdots & \cdots & 0 & -\lambda_n \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix} \text{ with } B_{22} = \begin{pmatrix} -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 \end{pmatrix}$$

and $V_2^n = (v_{2j}^n)_j$ with

$$v_{2j}^n = u_j^n + \tau_n (u_j^n)^p \text{ if } m+1 \leq j \leq N_n \text{ and } v_{2j}^n = 0 \text{ if } j \leq m.$$

We sum (3.7) and (3.9) we obtain

$$A U^{n+1} + \alpha^n (|B_1 U^{n+1}| + |B_2 U^{n+1}|) = V^n,$$

$$\text{with } A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \text{ and } V^n = \begin{pmatrix} 0 \\ u_1^n + \tau_n(u_1^n)^p \\ \vdots \\ u_{m-1}^n + \tau_n(u_{m-1}^n)^p \\ u_m^n + \tau_n(u_m^n)^p \\ u_{m+1}^n + \tau_n(u_{m+1}^n)^p \\ \vdots \\ u_{N_n}^n + \tau_n(u_{N_n}^n)^p \\ 0 \end{pmatrix} := \begin{pmatrix} 0 \\ v_1^n \\ \vdots \\ v_{m-1}^n \\ v_m^n \\ v_{m+1}^n \\ \vdots \\ v_{N_n}^n \\ 0 \end{pmatrix}$$

We use $u_{m-i}^n = u_{m+i}^n$, we can see that V^n is symmetric, so we can write

$$V^n = (0 \ v_1^n \ \cdots \ v_{m-1}^n \ v_m^n \ v_{m-1}^n \ \cdots \ v_1^n \ 0)^t.$$

If we use monotony, we have

$$AU^{n+1} + \alpha^n(B_1U^{n+1} - B_2U^{n+1}) = V^n. \quad (3.10)$$

To ensure unicity of the solution, we should prove that $A + \alpha^n(B_1 - B_2)$ is invertible.

But it is easy to verify that for $X = (x_1 \ x_2 \ \cdots \ x_{N_n})^t$, we have

$$(A + \alpha^n(B_1 - B_2))X = 0 \Rightarrow X = 0.$$

Now, it is easy to see that if

$$U^{n+1} = (0 \ u_1^{n+1} \ \cdots \ u_{m-1}^{n+1} \ u_m^{n+1} \ u_{m+1}^{n+1} \ \cdots \ u_{N_n}^{n+1} \ 0)^t$$

is a solution of (3.10) then

$$W^{n+1} = (0 \ u_1^{n+1} \ \cdots \ u_{m-1}^{n+1} \ u_m^{n+1} \ u_{m-1}^{n+1} \ \cdots \ u_1^{n+1} \ 0)^t$$

is also a solution of (3.10). Then by uniqueness, we deduce that $U^{n+1} = W^{n+1}$ which achieve the proof of symmetry. $\square$

4. Blow up theorem

In this section we will prove that the solution of the numerical problem blows up for all $p > 1$ and $1 \leq q \leq \frac{2p}{p+1}$.

Theorem 4.1. — *We suppose that the initial data satisfies (A1) – (A5), then the solution of (3.2) blows up and we have*

$$\lim_{n \rightarrow +\infty} u_m^n = +\infty.$$

To prove the theorem, we need the next lemma:

Lemma 4.2. — *For a large initial data, we have $u_m^n \gg 1$ for all $n \geq 0$. Moreover, $u_m^{n+1} \geq u_m^n$ for all $n \geq 0$.*

Proof. — For $n = 0$, we have $u_m^0 \gg 1$ because of (A.3). Supposing that it holds for some $n \geq 0$, we have to show that $u_m^{n+1} \gg 1$. In the equation of (3.2), if we take $j = m$ and we use symmetry we obtain

$$(1 + 2\lambda_n)u_m^{n+1} = u_m^n + 2\lambda_n u_m^{n+1} + \tau_n (u_m^n)^p,$$

where $\lambda_n := \frac{\tau_n}{h_n^2}$, and then we have

$$u_m^{n+1} \geq \frac{1 + \tau_n (u_m^n)^{p-1}}{1 + 2\lambda_n} u_m^n. \quad (4.1)$$

Using the recurrence hypothesis we get

$$1 + \tau_n (u_m^n)^{p-1} = 1 + \tau \quad \text{and} \quad \lambda_n = \frac{\tau}{2^{\frac{2}{2-q}} (u_m^n)^{p-\frac{q}{2-q}}}.$$

Then

$$\begin{aligned} \frac{1 + \tau_n (u_m^n)^{p-1}}{1 + 2\lambda_n} &= \frac{1 + \tau}{1 + \tau 2^{1-\frac{2}{2-q}} (u_m^n)^{-p+\frac{q}{2-q}}} \\ &= \frac{1 + \tau}{1 + \tau 2^{\frac{-q}{2-q}} (u_m^n)^{\frac{-2p+q(p+1)}{2-q}}}. \end{aligned}$$

For $1 \leq q \leq \frac{2p}{p+1}$, we have

$$-r := \frac{-2p + q(p+1)}{2-q} \leq 0, \quad (4.2)$$

which implies that

$$u_m^{n+1} \geq \frac{1 + \tau}{1 + \tau (2^{\frac{-q}{2-q}} (u_m^n)^{-r})} u_m^n. \quad (4.3)$$

Now we have to show that

$$\frac{1 + \tau}{1 + \tau (2^{\frac{-q}{2-q}} (u_m^n)^{-r})} \geq 1.$$

Using $u_m^n \gg 1$ and $r \geq 0$, we have

$$2^{\frac{-q}{2-q}} (u_m^n)^{-r} = \frac{1}{2^{\frac{q}{2-q}} (u_m^n)^r} < 1.$$

Then

$$1 + \tau > 1 + \tau 2^{\frac{-q}{2-q}} (u_m^n)^{-r},$$

hence

$$\frac{1 + \tau}{1 + \tau (2^{\frac{-q}{2-q}} (u_m^n)^{-r})} > 1,$$

and then, by $u_m^n \gg 1$

$$u_m^{n+1} \geq \frac{1 + \tau}{1 + \tau(2^{\frac{-q}{2-q}}(u_m^n)^{-r})} u_m^n \gg 1. \quad (4.4)$$

Moreover, for all $n \geq 0$ we deduce from (4.3) and (4.4) that $u_m^{n+1} \geq u_m^n$ which proves Lemma 4.2. $\square$

Now we can prove theorem 4.1.

Proof. — Using Lemma 4.2, (4.2) and (4.3), we can write that:

$$\begin{aligned} u_m^{n+1} &\geq \frac{1 + \tau}{1 + \tau 2^{\frac{-q}{2-q}}(u_m^n)^{\frac{-2p+q(1+p)}{2-q}}} u_m^n \\ &\geq \frac{1 + \tau}{1 + \tau 2^{\frac{-q}{2-q}}(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}} u_m^n, \end{aligned}$$

which implies by iterations

$$u_m^n \geq \left(\frac{1 + \tau}{1 + \tau 2^{\frac{-q}{2-q}}(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}} \right)^n u_m^0.$$

For a large initial data, we have

$$\frac{1 + \tau}{1 + \tau 2^{\frac{-q}{2-q}}(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}} > 1,$$

this implies that $u_m^n \rightarrow +\infty$ as $n \rightarrow +\infty$, which achieve the proof of Theorem 4.1. $\square$

5. Numerical simulation

In this section, we present some numerical simulation that illustrate our results.

As it is shown in figure 1, we take $u_0(x) = 10^3 \sin(\frac{\pi}{2}(x+1))$, which satisfies the conditions (A1)-(A5). We take $p = 3$, $q = 1.3 < \frac{2p}{p+1}$. Figures 2, 3 and 4 show the evolution of the numerical solution for different iterations. One can see that, numerically, the solution blows up in $x = 0$ and the growth of the solution leads to the reduction of h_n and hence increasing the number of points of discretisation.

In Table 1, we present some results about the decreasing of h_n and the increasing of N_n (the number of points of discretisation of the interval $[-1, 1]$) in each iteration. Initially, simulation started with a discrete space step $h_n = 0.138$, a discrete time step $\tau_n = 10^{-4}$, a number of points of discretisation of the interval $[-1, 1]$, $N_n = 15$ and a maximum value $M = 10^3$, after 350 iterations, we have increasing of the maximum value which leads to the decreasing of the discrete space step and discrete time step. From figure 5, we observe the evolution of the maximum point (blow-up point) $x = 0$, it gives an idea about the blow up rates given in [18]. We show in

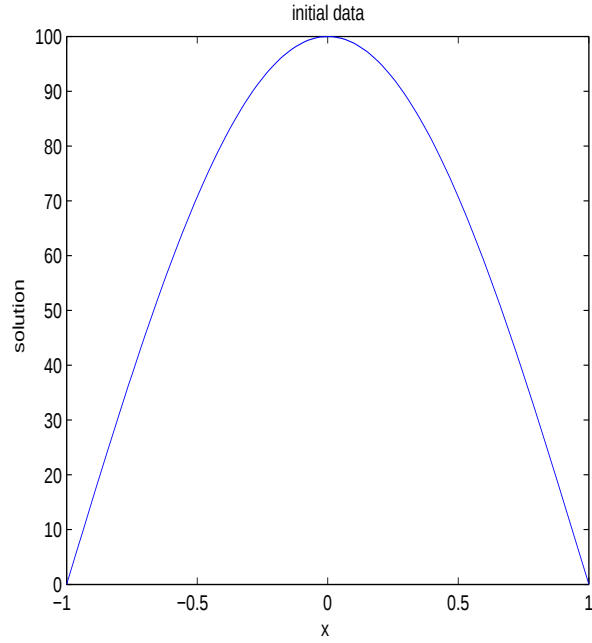


FIGURE 1. initial data: $u_0(x) = 1000 \sin(\frac{\pi}{2}(x+1))$

<i>Iteration</i>	1	110	145	200	260	280	300	350
M	10^3	$2.928.10^3$	$4.190.10^3$	$7.245.10^3$	$1.317.10^4$	$1.607.10^4$	$1.980.10^4$	$3.203.10^4$
τ_n	10^{-4}	10^{-6}	5.10^{-7}	2.10^{-9}	2.10^{-10}	10^{-10}	4.10^{-11}	10^{-11}
h_n	0.13	$0.87.10^{-1}$	$0.75.10^{-1}$	$0.59.10^{-1}$	$0.46.10^{-1}$	$0.42.10^{-1}$	$0.38.10^{-1}$	$0.31.10^{-1}$
N_n	15	23	27	33	41	47	51	63

TABLE 1. Reduction of h_n , τ_n and increasing of the number of points of discretisation.

figure 6, that the solution decays with a small initial data $u_0(x) = \sin(\frac{\pi}{2}(x+1))$, and hence blowing up can not occur, this was proved theoretically in [11]. We can see that the solution without gradient term shown in figure 8 blows up more rapidly than the solution of the Chipot-Weissler equation shown in figure 7, this proves the damping effect of the gradient term.

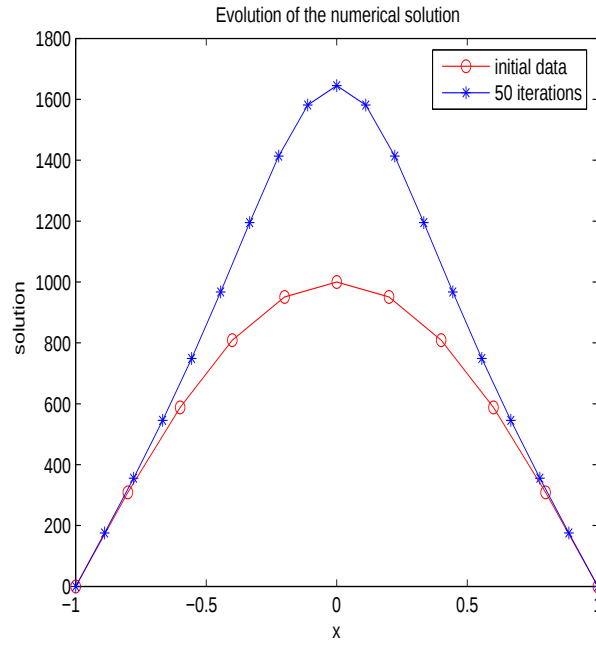


FIGURE 2. Evolution of the numerical solution in 50 iterations.

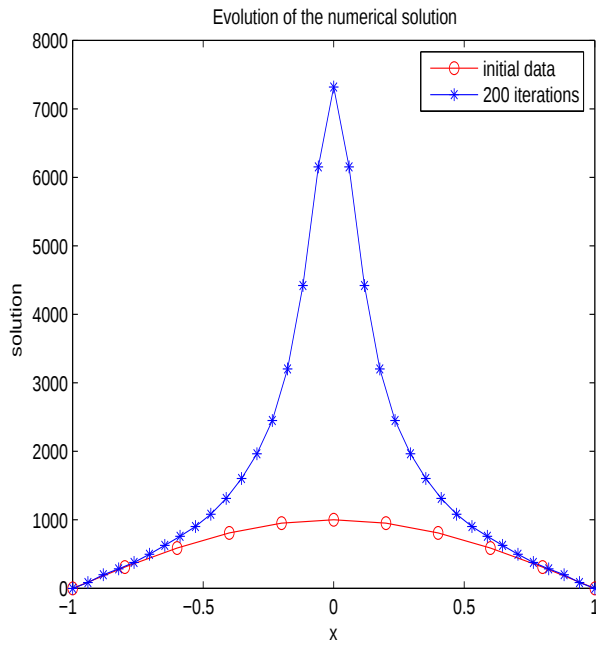


FIGURE 3. Evolution of the numerical solution in 200 iterations.

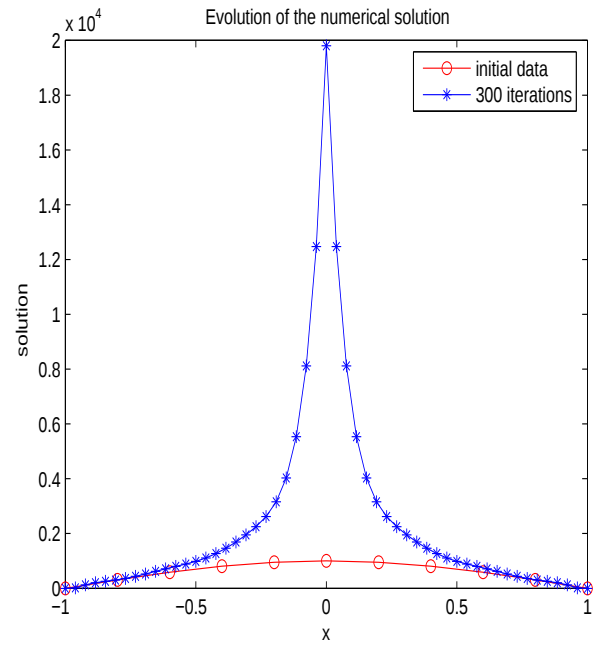


FIGURE 4. Evolution of the numerical solution in 300 iterations.

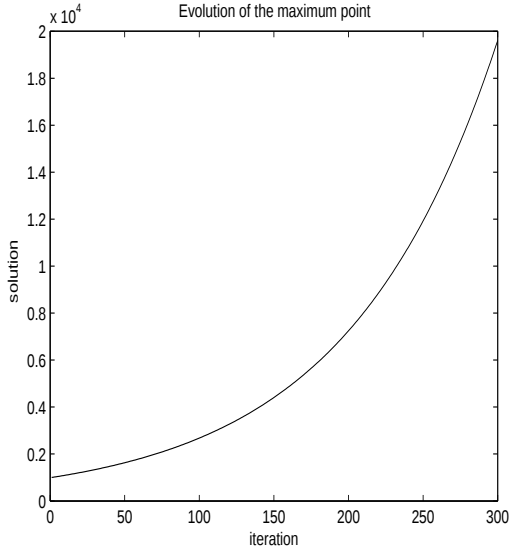


FIGURE 5. Evolution of the maximum point (blow-up point) $x = 0$.

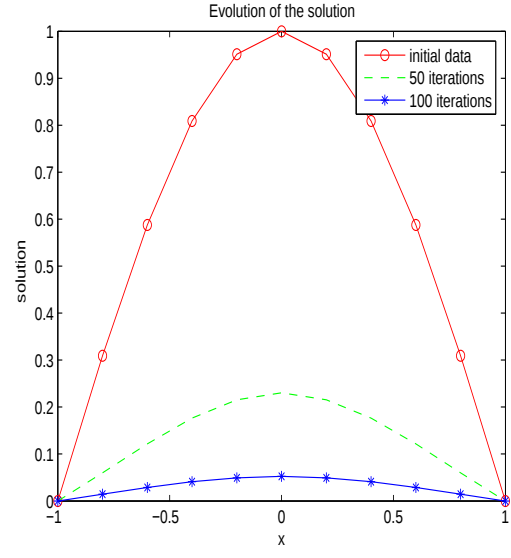


FIGURE 6. Global solution with a small initial data $u_0(x) = \sin(\frac{\pi}{2}(x+1))$.

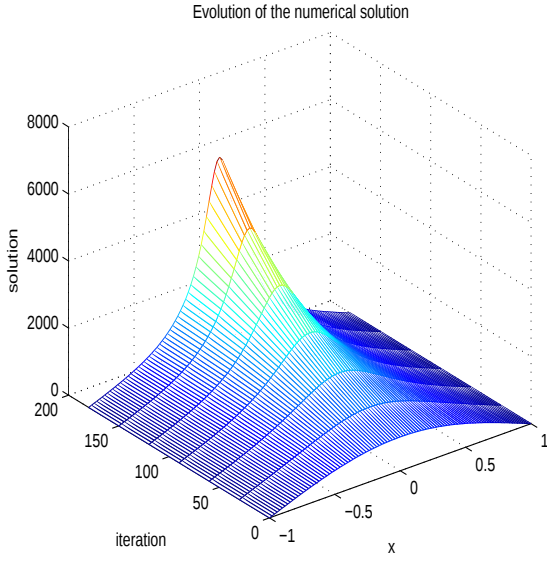


FIGURE 7. The shape of the numerical blow-up solution with gradient term in (t, x, u) -space

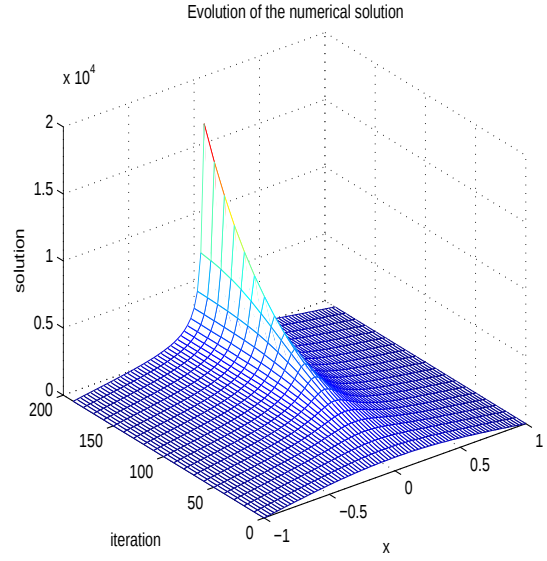


FIGURE 8. The shape of the numerical blow-up solution without gradient term in (t, x, u) -space

6. Concluding remarks

In this paper we have developed a numerical scheme in order to approximate the blow-up solution of the Chipot-Weissler equation, we have showed that we have blow up in $x_m = 0$.

Our goal in another work, is to use our scheme to study the competition between the gradient term which fights against blow up and the reaction term which may cause blow up in finite time as in the Fujita equation (without gradient term). In particular, we would like to answer some questions in the future study:

1. Can we determine the numerical blow-up set exactly?
2. What can we say about the asymptotic behaviours of the numerical solution near the blow-up set?
3. Can we give an approximation about the blow up time?
4. Let consider the equation $u_t = \Delta u + a|u|^p - b|\nabla u|^q$. What conditions should be satisfied by a and b to reproduce blowing-up phenomena?

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References

- [1] H. Amann, *parabolic evolution equations and nonlinear boundary conditions*. J. Diff. Equations 72 (1988), 201-269.
- [2] M. Cheblik, M. Fila and P. Quittner, *blowup of positive solutions of a semilinear parabolic equation with a gradient term*. Dyn. Contin. Discrete Impulsive Syst. Ser. A Math. Anal. 10 (2003),no. 4, 525-537.
- [3] Y. G. Chen, *asymptotic behaviours of blowing up solutions for finite difference analogue of $u_t = u_{xx} + u^{1+\alpha}$* . J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), 541-574.
- [4] M. Chipot and F. B. Weissler, *some blow up results for a nonlinear parabolic problem with a gradient term*. SIAM J. Math. Anal. 20(1989), 886-907.
- [5] C, H, Cho, *A finite difference scheme for blow up solutions for nonlinear wave equations*. Numer. Math. Theory Methods Appl. 3 (2010), 475-498.
- [6] A. Friedman, *blow up solutions of nonlinear parabolic equations*. W, M. Ni, L. A. Peletier, J. Serrin (Eds.), nonlinear diffusion equations and their equilibrium states, vol 1, Birkhaser Verlag, Basel, (1988), 301-318.
- [7] H. Fujita, *on the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$* . J. Fac. Sci. Univ. tokyo Sect. IA Math 13 (1966), 109-124.
- [8] H. Hani, *asymptotic behaviours of solutions for finite difference analogue of the Chipot-Weissler equation*. Preprint.
- [9] C. H, S. Hamada and H. Okamoto, *on the finite difference approximation for a parabolic blow up problem*. Japan J. Indust. Appl Math. 24 (2007), 131-160.

- [10] K. Hayakawa, *on nonexistence of global solutions of some semilinear parabolic equations*. Proc. Japan Acad. Ser. A Math 49 (1973), 503-525.
- [11] M. Hesaaraki, A.Moameni, *blow-up of positive solutions for a family of nonlinear parabolic equations in general domain in $\mathbb{R}^N$* . Michigan Math. J. 52 (2004), 375-389.
- [12] H. A. Levine, *the role of critical exponents in blow up theorems*. SIAM Rev 32 (1990), 262-288.
- [13] T. Nakagawa, *blowing up on a finite difference solution to $u_t = u_{xx} + u^2$* . Appl. Math. Optim. 2(1976), 337-350.
- [14] Pavol Quittner, *Blow-up for Semilinear Parabolic Equations with a Gradient Term*. Mathematical Methods in the Applied Sciences, Vol. 14, (1991), 413-417.
- [15] Pavol Quittner, *on global existence and stationary solutions for two classes of semilinear parabolic problems*. Comment. Math. Univ. Carolin. 34, 1 (1993), 105-124.
- [16] S. Snoussi, S. Tayachi and F. B. Weissler, *asymptotically self similar global solutions of a semilinear parabolic equation with a nonlinear gradient term*. Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 1291-1307.
- [17] Ph. Souplet, *finite time blow up for a nonlinear parabolic equation with a gradient term and applications*. Math. Methods Appl. sci, 19(1996), 1317-1333.
- [18] Ph. Souplet, *Recent results and open problems on parabolic equations with gradient nonlinearities*. Electronic Journal Of Differential Equations, vol 2001 (2001), 1-19.
- [19] Ph. Souplet and S. Tayachi, *blow up rates for nonlinear heat equations with gradient terms and for parabolic inequalities*. Colloquium Mathematicum, vol. 88 (2001), 135-154.
- [20] Ph. Souplet and F. B. Weissler, *self similar sub-solutions and blow up for nonlinear parabolic equations*. CJ. Math. Anal. Appl. 212 (1997), 60-74.
- [21] Ph. Souplet, S. Tayachi and F. B. Weissler, *exact self similar blow up of solutions of a semilinear parabolic equation with a nonlinear gradient term*. Indiana Univ. Math. J. 48 (1996), 655-682.
- [22] H. Zaag, and M.A. Ebde, *Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term*. Bol. Soc. Esp. Mat. Apl. SeMA 53 (2011), 5-21