Two floor building needing eight colors

Stéphane Bessy, Daniel Gonçalves, Jean-Sébastien Sereni

Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 19, no. 1, pp. 1–9 (2015)
DOI: 10.7155/jgaa.00344

Two-floor buildings need eight colors

S. Bessy [1] *D. Gonçalves* [1] *J.-S. Sereni* [2]

[1] LIRMM,
Université de Montpellier & CNRS
[2] Centre National de la Recherche Scientifique (LORIA),
Vandœuvre-lès-Nancy, France.

Abstract

Motivated by frequency assignment in office blocks, we study the chromatic number of the adjacency graph of a 3-dimensional parallelepiped arrangement. In the case each parallelepiped is within one floor, a direct application of the Four-Colour Theorem yields that the adjacency graph has chromatic number at most 8. We provide an example of such an arrangement needing exactly 8 colors. We also discuss bounds on the chromatic number of the adjacency graph of general arrangements of 3-dimensional parallelepipeds according to geometrical measures of the parallelepipeds (side length, total surface area or volume).

Submitted:	Reviewed:	Revised:	Accepted:	Final:
May 2014	August 2014	September 2014	December 2014	January 2015

Published:
January 2015

Article type:	Communicated by:
Regular paper	J.S.B. Mitchell

The second and third author's work were partially supported by the French *Agence Nationale de la Recherche* under references ANR-12-JS02-002-01 and ANR-10-JCJC-0204-01, respectively.
E-mail addresses: bessy@lirmm.fr (S. Bessy) goncalves@lirmm.fr (D. Gonçalves) sereni@kam.mff.cuni.cz (J.-S. Sereni)

1 Introduction

The Graph Colouring Problem for Office Blocks was raised by BAE Systems at the 53rd European Study Group with Industry in 2005 [1]. Consider an office complex with space rented by several independent organisations. It is likely that each organisation uses its own wireless network (WLAN) and asks for a safe utilisation of it. A practical way to deal with this issue is to use a so-called "stealthy wallpaper" in the walls and ceilings shared between different organisations, which would attenuate the relevant frequencies. Yet, the degree of screening produced is not sufficient if two distinct organisations have adjacent offices, that is, two offices in face-to-face contact on opposite sides of just one wall or floor-ceiling. In this case, the WLANs of the two organisations need to use two different channels (the reader is referred to the report by Allwright *et al.* [1] for the precise technical motivations).

This problem can be modeled as a graph coloring problem by building a *conflict graph* corresponding to the office complex: each organisation corresponds a vertex, and two vertices are adjacent if the corresponding territories share a wall, floor, or ceiling area. The goal is to assign a color (frequency) to each vertex (organisation) such that adjacent vertices are assigned distinct colors. In addition, not every graph may occur as the conflict graph of an existing office complex. However, the structure of such conflict graphs is not clear and various fundamental questions related to the problem at hands were asked. Arguably, one of the most natural questions concerns the existence of bounds on the chromatic number of such conflict graphs. More specifically, which additional constraints one should add to the model to ensure "good" upper bounds on the chromatic number of conflict graphs? These additional constraints should be meaningful regarding the practical problem, reflecting real-world situations as much as is possible. Indeed, as noted by Tietze [7], complete graphs of arbitrary sizes are conflict graphs, that is, for every integer n, there can be n organisations whose territories all are in face-to-face contact with each other. The reader is referred to the paper by Reed and Allwright [6] for a description of Tietze's construction. Besicovitch [4] and Tietze [8] proved that this is still the case if the territories are asked to be convex polyhedra.

An interesting condition is when the territories are required to be *rectangular parallelepipeds* (sometimes called *cuboids*), that is, a 3-dimensional solid figure bounded by six rectangles aligned with a fixed set of Cartesian axes. For convenience, we call a rectangular parallelepiped a *box*. When all territories are boxes, the *clique number* of any conflict graph, that is, the maximum size of a complete subgraph, is at most 4. However, Reed and Allwright [6] and also later Magnant and Martin [5] designed arrangements of boxes that yield conflict graphs requiring an arbitrarily high number of colors.

On the other hand, if the building is assumed to have floors (in the usual way) and each box is 1-*floor*, i.e. restricted to be within one floor, then the chromatic number is bounded by 8: on each floor, the obtained conflict graph is planar and hence can be colored using 4 colors [2, 3]. It is natural to ask whether this bound is tight. As noted during working sessions in Oxford (see the acknowledgments), it can be shown that up to 6 colors can be needed, by using an arrangement of boxes spanning three floors. Such a construction is shown in Figure 1.

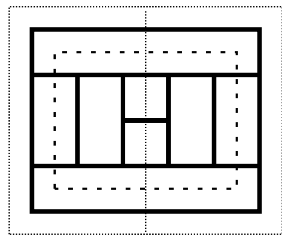

Figure 1: An arrangement of 1-floor boxes spanning three floors and requiring six colors. The solid, dotted, and dashed lines indicate the middle, top, and bottom floors, respectively.

The purpose of this note is to show that the upper bound is actually tight. More precisely, we shall build an arrangement of 1-floor boxes that spans two floors and yields a conflict graph requiring 8 colors. From now on, we shall identify a box arrangement with its conflict graph for convenience. In particular, we assign colors directly to the boxes and define the *chromatic number of an arrangement* as that of the associated conflict graph.

Theorem 1 *There exists an arrangement of* 1*-floor boxes spanning two floors with chromatic number* 8*.*

The boxes considered in Theorem 1 have one of their geometrical measures bounded: their heights are at most one floor. We also discuss bounds on the chromatic number of box arrangements with respect to some other geometrical measures: the side lengths, the surface area and the volume. More precisely, assuming that boxes have integer coordinates, we obtain the following.

Theorem 2 *We consider a box arrangement A with integer coordinates. We let ℓ, s and v be non-negative real numbers.*

(1) *If there exists one fixed dimension such that every box in A has length at most ℓ in this dimension, then A has chromatic number at most $4(\ell+1)$.*

(2) *If for each box, there is one dimension such that the length of the box in this dimension is at most ℓ, then A has chromatic number at most $12(\ell+1)$.*

(3) *If the total surface area of each box in A is at most s, then A has chromatic number at most $9\sqrt[3]{4s}+25$.*

(4) *If the volume of each box in A is at most v, then A has chromatic number at most $24\sqrt[4]{6v}+25$.*

In the next section, we give the proof of Theorem 1 and in the last section we indicate how to obtain the bounds given in Theorem 2.

2 Proof of Theorem 1

We shall construct an arrangement of 1-floor boxes that is not 7-colorable. To this end, we use the arrangement X of 1-floor boxes described in Figure 2 as a building brick. The arrangement X has three specific *regions*, $R_1(X)$, $R_2(X)$ and $R_3(X)$. We define X_i to be the subset of boxes of X intersecting the region $R_i(X)$, for each $i \in \{1, 2, 3\}$ (note that some boxes may belong to several subsets). We start by giving two straightforward but crucial properties of X with respect to proper colorings.

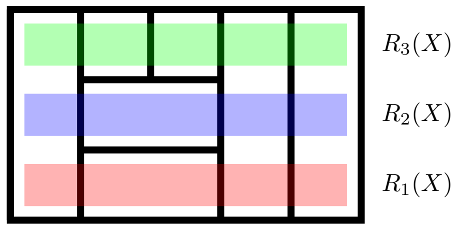

Figure 2: The gadget X with the regions $R_1(X)$, $R_2(X)$ and $R_3(X)$.

Assertion 3 *For every proper coloring c of X,*

(1) $|c(X)| \geqslant 4$*; and*

(2) $|c(X_2)| \leqslant 2$ *implies that* $|c(X_3)| \geqslant 4$*.*

The proof of this assertion does not need any insight, we thus omit it. (One can note, though, that the conflict graph of the arrangement X contains the 5-wheel as an induced subgraph.)

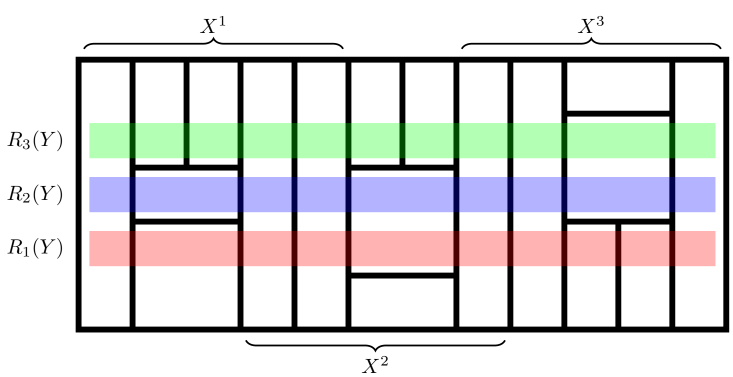

Figure 3: The gadget Y with the three regions $R_1(Y)$, $R_2(Y)$ and $R_3(Y)$.

Next, we obtain arrangement Y from three copies X^1, X^2 and X^3 of arrangement X. We define three regions $R_1(Y)$, $R_2(Y)$ and $R_3(Y)$ on Y as depicted in Figure 3. As previously, we define Y_i to be the subset of boxes of Y intersecting the region $R_i(Y)$, for each $i \in \{1,2,3\}$. We also define $X_i^j \coloneqq Y_i \cap X^j$ for $(i,j) \in \{1,2,3\}^2$.

Assertion 4 *If c is a proper coloring of Y, then $|c(Y_i)| \geqslant 4$ for some $i \in \{1,2,3\}$, that is, at least four colors are used in one of the three regions $R_1(Y)$, $R_2(Y)$ and $R_3(Y)$.*

Proof: Suppose on the contrary that there is a proper coloring c of Y with at most three colors on the boxes in each of Y_1, Y_2 and Y_3. For each $i \in \{1,2,3\}$, the restriction of c to X^i is a proper coloring of X^i, which we identify to c. We consider X^1. Since $X_3^1 \subset Y_3$ and $|c(Y_3)| \leqslant 3$, Assertion 3(2) implies that $\left|c(X_2^1)\right| \geqslant 3$. Consequently, $\left|c(X_2^1)\right| = 3$ and $c(X_2^1) = c(Y_2)$. The same argument applied to X^2 and to X^3 yields that $c(X_2^2) = c(Y_2) = c(X_2^3)$. Since $X_2^2 = X_1^2$, we infer that $c(X_1^1) \subseteq c(X_1^2) = c(Y_2)$. Similarly, since $X_2^3 = X_3^3$, we infer that $c(X_3^1) \subseteq c(X_3^3) = c(Y_2)$. Therefore, c is a proper coloring of X^1 using only 3 different colors, which contradicts Assertion 3(1). $\square$

To finish the construction, we need the following definition. Consider two copies Y^1 and Y^2 of Y. For $(i,j) \in \{1,2,3\}^2$, the regions $R_i(Y^1)$ and $R_j(Y^2)$ *fully overlap* if every box in $R_i(Y^1)$ is in face-to-face contact with every box in $R_j(Y^2)$. Observe that for every pair $(i,j) \in \{1,2,3\}$, there exists a 2-floor arrangement of Y^1 and Y^2 such that $R_i(Y^1)$ and $R_j(Y^2)$ fully overlap: it is obtained by rotating Y^2 ninety degrees, adequately scaling it (that is, stretching it horizontally) and placing it on top of Y^1.

We are now in a position to build the desired arrangement Z spanning two floors. To this end, we use several copies of Y. The first floor of Z is composed of seven parallel copies $Y^1, \ldots, Y^7$ of Y (drawn horizontally in Figure 4). The second floor of Z is composed of fifteen parallel copies of Y (drawn vertically in Figure 4): for each $j \in \{1,2,3\}$ and each $i \in \{2,\ldots,6\}$, a copy $Y(i,j)$ of Y is placed such that the region $R_1(Y(i,j))$ fully overlaps the regions $R_j(Y^1), \ldots, R_j(Y^{i-1})$, the region $R_2(Y(i,j))$ fully overlaps the region $R_j(Y^i)$, and finally the region $R_3(Y(i,j))$ fully overlaps the regions $R_j(Y^{i+1}), \ldots, R_j(Y^7)$.

Consider a proper coloring of Z. Assertion 4 ensures that each copy of Y in Z has a region in which at least four different colors are used. In particular, there exists $j \in \{1,2,3\}$ such that at least three regions among $R_j(Y^1), \ldots, R_j(Y^7)$ are colored using four colors. Let three of these regions be $R_j(Y^{i_1})$, $R_j(Y^{i_2})$ and $R_j(Y^{i_3})$ with $1 \leqslant i_1 < i_2 < i_3 \leqslant 7$. Now, consider the arrangement $Y(i_2,j)$. By Assertion 4, there exists $k \in \{1,2,3\}$ such that the region $R_k(Y(i_2,j))$ is also colored using at least four different colors. Consequently, as $R_k(Y(i_2,j))$ and $R_j(Y^{i_k})$ fully overlap, they are colored using at least eight different colors. This concludes the proof.

We note that the arrangement Z consists of 396 boxes. At the expanse of some technicalities, one can remove three boxes in Y (the two clearly useless

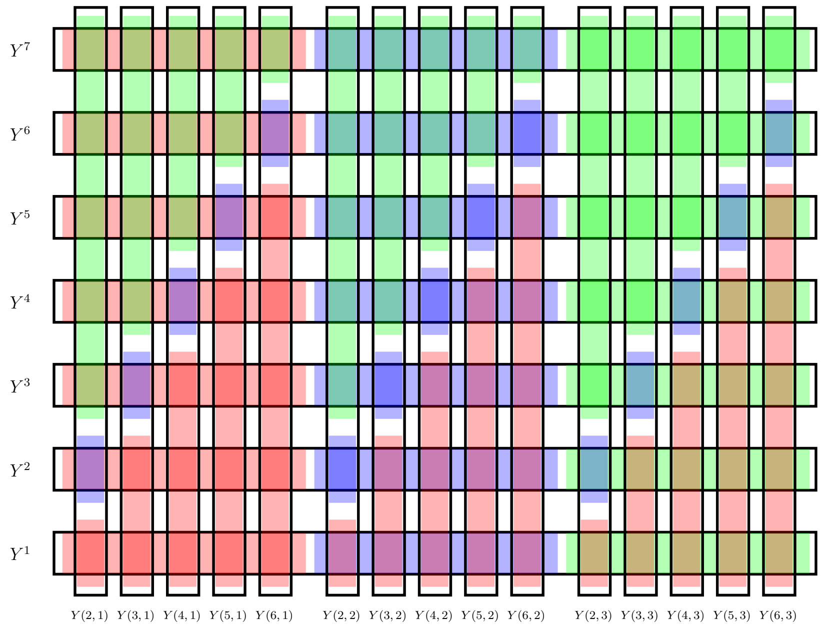

Figure 4: Schematic view of the arrangement Z.

boxes and also a non-end vertical one) and obtain, in a similar way, a two-floor arrangement Z that needs 8 colors and contains 330 boxes. We are not aware of a two-floor arrangement needing eight colors and containing less boxes.

3 Bounds with respect to geometrical measures

In this part, we provide bounds on the chromatic number of boxes arrangements provided that the boxes satisfy some geometrical constraints. Namely, we prove Theorem 2, which is recalled here for the reader's ease.

Theorem 2 *We consider a box arrangement A with integer coordinates. We let ℓ, s and v be non-negative real numbers.*

(1) *If there exists one fixed dimension such that every box in A has length at most ℓ in this dimension, then A has chromatic number at most $4(\ell+1)$.*

(2) *If for each box, there is one dimension such that the length of the box in this dimension is at most ℓ, then A has chromatic number at most $12(\ell+1)$.*

(3) *If the total surface area of each box in A is at most s, then A has chromatic number at most $9\sqrt[3]{4s}+25$.*

(4) *If the volume of each box in A is at most v, then A has chromatic number at most $24\sqrt[4]{6v}+25$.*

Proof:
(1) The conflict graph corresponding to an arrangement where the boxes have height at most ℓ can be vertex partitioned into $\ell+1$ planar graphs $P_0, \dots, P_\ell$. Indeed if the distance between the levels of two boxes is at least $\ell+1$, then these two boxes are not adjacent. So the planar graphs are obtained by assigning, for each x, all the boxes that have their floor at level x to be in the graph P_k where $k \coloneqq x \mod (\ell+1)$. Consequently, the whole conflict graph has chromatic number at most $4(\ell+1)$.

(2) The boxes can be partitioned into three sets according to the dimension in which the length is bounded. In other words, A is partitioned into U_1, U_2 and U_3 such that for each $i \in \{1, 2, 3\}$, all boxes in U_i have length at most ℓ in dimension i. Consequently, (1) ensures that each of U_1, U_2 and U_3 has chromatic number at most $4(\ell+1)$ and, therefore, A has chromatic number at most $3 \cdot 4(\ell+1) = 12(\ell+1)$.

(3) For each box, the minimum length taken over all three dimensions is $O(\sqrt{s})$, and thus (2) implies that the chromatic number of A is $O(\sqrt{s})$. However, one can be more precise. Let us fix a positive real number ℓ, to be made precise later. The set of boxes is partitioned as follows. Let U be the set of boxes with lengths in every dimension at least ℓ and let R be the set all remaining boxes, that is, $R \coloneqq A \setminus U$. By (2), the arrangement R has chromatic number at most 12ℓ. Now consider a box B in U with dimensions x, y and z, each being at least ℓ. We shall give an upper bound on the number of boxes of U that can be adjacent to B. The surface area of a face of a box in U is at least ℓ^2. So in U there are at most s/ℓ^2 boxes that have a totally adjacent face to B. Some boxes of U could also be adjacent to B without having a totally adjacent face to B. In this case, such a box is adjacent to an edge of B. For an edge of length w, there are at most $w/\ell + 2$ such boxes. So the number of boxes of U adjacent to B but having no face totally adjacent to a face of B is at most $4(x+y+z)/\ell + 24$. Since $\ell \leqslant \min\{x, y, z\}$, we deduce that $2\ell(x+y+z) \leqslant 2xy + 2yz + 2xz \leqslant s$. Hence the total number of boxes in U that are adjacent to B is at most $s/\ell^2 + 2s/\ell^2 + 24 = 3s/\ell^2 + 24$. Consequently, by degeneracy, U has chromatic number at most $3s/\ell^2 + 25$. Therefore, A has chromatic number at most $12\ell + 3s/\ell^2 + 25$. Thus setting $\ell \coloneqq \sqrt[3]{s/2}$ yields the upper bound $9\sqrt[3]{4 \cdot s} + 25$.

(4) Once again, for a fixed parameter ℓ to be made precise later, the set of boxes is partitioned into two parts: the part U, composed of all the boxes with lengths in every dimension at least ℓ and the part R, composed of all the remaining boxes. By (2), we know that R has chromatic number at most 12ℓ. Let B be a box in U with dimensions x, y and z. Since $\ell \leqslant \min\{x, y, z\}$, the volume v_B of B satisfies $6v \geqslant 6v_B = 6xyz \geqslant 2(\ell xy + \ell xz + \ell yz) = \ell s_B$, where s_B is the total surface area of B. So every box in U has total surface area at most $6v/\ell$ and thus (3) implies that U has chromatic number at most $9\sqrt[3]{4.6v/\ell} + 25$. Therefore, A has chromatic number at most $9\sqrt[3]{24v/\ell} + 12\ell + 25$. Thus setting ℓ to be $\sqrt[4]{3v/8}$ yields the upper bound $24\sqrt[4]{6v} + 25$. $\square$

In the previous theorem, we are mainly concerned with the order of magnitude of the functions of the different parameters. However, even in this context, we do not have any non trivial lower bound on the corresponding chromatic numbers.

Acknowledgments

The third author thanks Louigi Addario-Berry, Frédéric Havet, Ross Kang, Colin McDiarmid and Tobias Müller for stimulating discussions on the topic of this paper during a stay in Oxford, in 2005.

References

[1] D. Allwright, T. Gould, J. Gravesen, R. Leese, and H. Petersen. Graph colouring for office blocks. Technical report, 53rd European Study Group with Industry, 2005.

[2] K. Appel and W. Haken. Every planar map is four colorable. part I: Discharging. *Illinois J. Math.*, 21:429–490, 1977.

[3] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. part II: Reducibility. *Illinois J. Math.*, 21:491–567, 1977.

[4] A. S. Besicovitch. On Crum's problem. *J. London Math. Soc.*, 22:285–287, 1947. doi:10.1112/jlms/s1-22.4.285.

[5] C. Magnant and D. M. Martin. Coloring rectangular blocks in 3-space. *Discuss. Math. Graph Theory*, 31:161–170, 2011. doi:10.7151/dmgt.1535.

[6] B. Reed and D. Allwright. Painting the office. *Mathematics-in-Industry Case Studies Journal*, 1:1–8, 2008.

[7] H. Tietze. Über nachbargebiete im raume. *Monatshefte für Mathematik und Physik*, 16:211–216, 1905. doi:10.1007/BF01693778.

[8] H. Tietze. *Famous problems of mathematics. Solved and unsolved mathematical problems from antiquity to modern times.* B. K. Hofstadter and H. Komm, Graylock Press, 1965. Authorized translation from the second (1959) revised German edition.