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BLOCKS OF THE GROTHENDIECK RING
OF EQUIVARIANT BUNDLES ON A FINITE GROUP

by

CÉDRIC BONNAFÉ

Abstract. — If $G$ is a finite group, the Grothendieck group $K_G(G)$ of the category of $G$-equivariant $\mathbb{C}$-vector bundles on $G$ (for the action of $G$ on itself by conjugation) is endowed with a structure of (commutative) ring. If $K$ is a sufficiently large extension of $\mathbb{Q}_p$ and $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}_p$ in $K$, the $K$-algebra $K K_G(G) = K \otimes_\mathbb{Z} K_G(G)$ is split semisimple. The aim of this paper is to describe the $\mathcal{O}$-blocks of the $\mathcal{O}$-algebra $K K_G(G)$.

1. Notation, introduction

1.A. Groups. — We fix in this paper a finite group $G$, a prime number $p$ and a finite extension $K$ of the $p$-adic field $\mathbb{Q}_p$ such that $KH$ is split for all subgroups $H$ of $G$. We denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}_p$ in $K$, by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, by $k$ the residue field of $\mathcal{O}$ (i.e. $k = \mathcal{O}/\mathfrak{p}$) We denote by $\text{Irr}(KG)$ the set of irreducible characters of $G$ (over $K$).

A $p$-element (respectively $p'$-element) of $G$ is an element whose order is a power of $p$ (respectively prime to $p$). If $g \in G$, we denote by $g_p$ and $g'_p$ the unique elements of $G$ such that $g = g_p g'_p = g'_p g_p$, $g_p$ is a $p$-element and $g'_p$ is a $p'$-element. The set of $p$-elements (respectively $p'$-elements) of $G$ is denoted by $G_p$ (respectively $G_p'$).

If $X$ is a $G$-set (i.e. a set endowed with a left $G$-action), we denote by $[G \setminus X]$ a set of representatives of $G$-orbits in $X$. The reader can check that we will use formulas like

$$\sum_{x \in [G \setminus X]} f(x)$$

(or families like $(f(x))_{x \in [G \setminus X]}$) only whenever $f(x)$ does not depend on the choice of the representative $x$ in its $G$-orbit. If $X$ is a set-$G$ (i.e. a set endowed with a right $G$-action), we define similarly $[X/G]$ and will use it according to the same principles.

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1.B. Blocks. — A block idempotent of \( kG \) (respectively \( \theta G \)) is a primitive idempotent of the center \( Z(kG) \) (respectively \( Z(\theta G) \)) of \( \theta G \). We denote by \( \text{Blocks}(kG) \) (respectively \( \text{Blocks}(\theta G) \)) the set of block idempotents of \( kG \) (respectively \( \theta G \)). Reduction modulo \( \mathfrak{p} \) induces a bijection \( \text{Blocks}(\theta G) \leftrightarrow \text{Blocks}(kG), e \mapsto e \) (whose inverse is denoted by \( e \mapsto \bar{e} \)).

A \( p \)-block of \( G \) is a subset \( \mathcal{B} \) of \( \text{Irr}(G) \) such that \( \mathcal{B} = \text{Irr}(kGe) \), for some block idempotent \( e \) of \( \theta G \).

1.C. Fourier coefficients. — Let

\[ \text{IrrPairs}(G) = \{(g, \gamma) \mid g \in G \text{ and } \gamma \in \text{Irr}(kC_G(g))\} \]

and

\[ \text{BlPairs}_p(G) = \{ (s, e) \mid s \in G_{p'} \text{ and } e \in \text{Blocks}(\theta C_G(s)) \}. \]

The group \( G \) acts (on the left) on these two sets by conjugation. We set

\[ \mathcal{M}(G) = [G \setminus \text{IrrPairs}(G)] \quad \text{and} \quad \mathcal{M}^p(G) = [G \setminus \text{BlPairs}(G)]. \]

If \( (g, \gamma), (h, \eta) \in \text{IrrPairs}(G) \), we define, following Lusztig [Lu, 2.5(a)],

\[ \{(g, \gamma), (h, \eta)\} = \frac{1}{|C_G(g)| \cdot |C_G(h)|} \sum_{x \in G} \gamma(xhx^{-1})\eta(x^{-1}g^{-1}x). \]

Note that \( \{(g, \gamma), (h, \eta)\} \) depends only on the \( G \)-orbit of \( (g, \gamma) \) and on the \( G \)-orbit of \( (h, \eta) \).

1.D. Vector bundles. — Except from Proposition 2.3 below, all the definitions, all the results in this subsection can be found in [Lu, §2]. We denote by \( \mathfrak{B}un_G(G) \) the category of \( G \)-equivariant finite dimensional \( K \)-vector bundles on \( G \) (for the action of \( G \) by conjugation). Its Grothendieck group \( K_G(G) \) is endowed with a ring structure. For each \( (g, \gamma) \in \mathcal{M}(G) \), let \( V_{g, \gamma} \) be the isomorphism class (in \( K_G(G) \)) of the simple object in \( \mathfrak{B}un_G(G) \) associated with \( (g, \gamma) \), as in [Lu, §2.5] (it is denoted \( U_{g, \gamma} \) there). Then

\[ K_G(G) = \bigoplus_{(g, \gamma) \in \mathcal{M}(G)} \mathbb{Z} V_{g, \gamma}. \]

The \( K \)-algebra \( KK_G(G) = K \otimes_{\mathbb{Z}} K_G(G) \) is split semisimple and commutative. Its simple modules (which have dimension one) are also parametrized by \( \mathcal{M}(G) \): if \( (g, \gamma) \in \mathcal{M}(G) \), the \( K \)-linear map

\[ \Psi_{g, \gamma} : KK_G(G) \rightarrow K \]

defined by

\[ \Psi_{g, \gamma}(V_{h, \eta}) = \frac{|C_G(g)|}{\gamma(1)} \cdot \{(h^{-1}, \eta), (g, \gamma)\} \]
is a morphism of $K$-algebras and all morphisms of $K$-algebras $K\mathcal{K}_G(G)\to K$ are obtained in this way.

We define similarly block idempotents of $k\mathcal{K}_G(G)$ and $\mathcal{O}\mathcal{K}_G(G)$, as well as $p$-blocks of $\mathcal{M}(G)\leftrightarrow \text{Irr}(K\mathcal{K}_G(G))$.

1.E. Brauer maps. — Let $\Lambda$ denote one of the two rings $\mathcal{O}$ or $k$. If $g \in G$ (and if we set $s = g_{p'}$), we denote by $\text{Br}^\Lambda_g$ the $\Lambda$-linear map

$$\text{Br}^\Lambda_g : \Lambda C_G(s) \to \Lambda C_G(g)$$

such that

$$\text{Br}^\Lambda_g(h) = \begin{cases} h & \text{if } h \in C_G(g), \\ 0 & \text{if } h \notin C_G(g), \end{cases}$$

for all $h \in C_G(s)$. Recall [Is, Lemma 15.32] that

$$\text{Br}^k_g \text{ induces a morphism of algebras } Z(kC_G(s)) \to Z(kC_G(g)).$$

Therefore, if $e \in \text{Blocks}(\mathcal{O}C_G(s))$, then $\text{Br}^k_g(e)$ is an idempotent of $Z(kC_G(g))$ (possibly equal to zero) and we can write it a sum $\text{Br}^k_g(e) = e_1 + \cdots + e_n$, where $e_1, \ldots, e_n$ are pairwise distinct block idempotents of $kC_G(g)$. We then set

$$\beta_g^e(e) = \sum_{i=1}^n e_i.$$

It is an idempotent (possibly equal to zero, possibly non-primitive) of $Z(\mathcal{O}C_G(g))$.

1.F. The main result. — In order to state more easily our main result, it will be more convenient (though it is not strictly necessary) to fix a particular set of representatives of conjugacy classes of $G$.

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**Hypothesis and notation.** From now on, and until the end of this paper, we denote by:

- $[G/\sim]$ a set of representatives of conjugacy classes of elements of $G$ such that, for all $g \in [G/\sim]$, $g_{p'} \in [G_{p'}/\sim]$.

We also assume that, if $(g, \gamma) \in \mathcal{M}(G)$ or $(s, e) \in \mathcal{M}^p(G)$, then $g \in [G/\sim]$ and $s \in [G_{p'}/\sim]$.

If $(s, e) \in \mathcal{M}^p(G)$, we define $\mathcal{B}_G(s, e)$ to be the set of pairs $(g, \gamma) \in \mathcal{M}(G)$ such that:

1. $g_{p'} = s$.
2. $\gamma \in \text{Irr}(KC_G(g)\beta_g^e(e))$. 
Theorem 1.2. — The map \((s, e) \mapsto \mathcal{B}_G(s, e)\) induces a bijection between \(\mathcal{M}^p(G)\) to the set of \(p\)-blocks of \(\mathcal{M}(G)\).

2. Proof of Theorem 1.2

2.A. Central characters and congruences. — If \((g, \gamma) \in \text{IrrPairs}(G)\), we denote by \(\omega_{g, \gamma} : Z(KC_G(g)) \to K\) the central character associated with \(\gamma\) (if \(z \in Z(KC_G(g))\), then \(\omega_{g, \gamma}(z)\) is the scalar through which \(z\) acts on an irreducible \(KC_G(g)\)-module affording the character \(\gamma\)). It is a morphism of algebras: when restricted to \(Z(O_C G(g))\), it has values in \(O_C G(g)\).

If \(h \in C_G(g)\), we denote by \(\Sigma_g(h)\) conjugacy class of \(h\) in \(C_G(g)\) and we set

\[
\hat{\Sigma}_g(h) = \sum_{v \in \Sigma_g(h)} v \in Z(O_C G(g)).
\]

We have

\[
(2.1) \quad \omega_{g, \gamma}(\hat{\Sigma}_g(h)) = \frac{|\Sigma_g(h)| \cdot \gamma(h)}{\gamma(1)}.
\]

We also recall the following classical results:

**Proposition 2.2.** — If \(g \in G\) and \(\gamma, \gamma'\) are two irreducible characters of \(C_G(g)\), then \(\gamma\) and \(\gamma'\) lie in the same \(p\)-block of \(C_G(g)\) if and only if

\[
\omega_{g, \gamma}(\hat{\Sigma}_g(h)) \equiv \omega_{g, \gamma'}(\hat{\Sigma}_g(h)) \mod p
\]

for all \(h \in C_G(g)\).

**Proposition 2.3.** — Let \((g, \gamma)\) and \((g', \gamma')\) be two elements of \(\mathcal{M}(G)\). Then \((g, \gamma)\) and \((g', \gamma')\) belong to the same \(p\)-block of \(\mathcal{M}(G)\) if and only if

\[
\Psi_{g, \gamma}(V_{h, \eta}) \equiv \Psi_{g', \gamma'}(V_{h, \eta}) \mod p
\]

for all \((h, \eta) \in \mathcal{M}(G)\).

2.B. Around the Brauer map. — As Brauer maps are morphisms of algebras, we have

\[
\sum_{e \in \text{Blocks}(kC_G(g_p))} \text{Br}_G^p(e) = 1,
\]

and so

\[
(2.4) \quad \text{The family } \left( \mathcal{B}_G(g, e) \right)_{(g, e) \in \mathcal{M}^p(G)} \text{ is a partition of } \mathcal{M}(G).
\]
Now, let \((g, \gamma) \in \mathcal{M}(G)\) and let \(s = g_p\). If \(e \in \text{Blocks}(\mathcal{O}_{C_G}(s))\) is such that \(\gamma \in \text{Irr}(K C_G(g)\beta^g(e))\), and if \(\sigma \in \text{Irr}(K C_G(s)e)\), then [Is, Lemma 15.44]

\[
\omega_{s, \sigma}(z) \equiv \omega_{g, \gamma}(\text{Br}^g_e(z)) \mod p
\]

for all \(z \in \mathcal{Z}(\mathcal{O}_{C_G}(s))\).

2.C. Rearranging the formula for \(\Psi_{g, \gamma}^\ast\). — If \((g, \gamma), (h, \eta) \in \text{Irpairs}(g)\) then

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(g)\setminus G/C_G(h)]} \sum_{x h x^{-1} \in C_G(g)} \eta(x^{-1}g x) \omega_{g, \gamma}(\text{Br}^g_e(x h x^{-1})).
\]

Proof. — By definition,

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \frac{1}{\gamma(1) \cdot |C_G(h)|} \sum_{x \in [G/C_G(h)]} \sum_{x h x^{-1} \in C_G(g)} \eta(x^{-1}g x) \gamma(x h x^{-1}) = \frac{1}{\gamma(1)} \sum_{x \in [C_G(g)\setminus G/C_G(h)]} \sum_{x h x^{-1} \in C_G(g)} \eta(x^{-1}g x) \gamma(x h x^{-1}).
\]

Now, if \(x \in G\) is such that \(x h x^{-1} \in C_G(g)\) and if \(u \in C_G(g)\), then

\[
\eta((u x)^{-1} g (u x)) \gamma((u x) h (u x) x^{-1}) = \eta(x^{-1}g x) \gamma(x h x^{-1}).
\]

So we can gather the terms in the last sum according to their \(C_G(g)\)-orbit. We get

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(g)\setminus G/C_G(h)]} \sum_{x h x^{-1} \in C_G(g)} \eta(x^{-1}g x) \frac{|C_G(g)|}{|C_G(g) \cap x C_G(h) x^{-1}|} \frac{\gamma(x h x^{-1})}{\gamma(1)}.
\]

But, for \(x \in G\) such that \(x h x^{-1} \in C_G(g)\),

\[
\frac{|C_G(g)|}{|C_G(g) \cap x C_G(h) x^{-1}|} = |\Sigma_g(x h x^{-1})|,
\]

so the result follows from 2.1.

**Corollary 2.7.** — Let \(g \in [G/\sim]\) and let \(\gamma, \gamma' \in \text{Irr}(K C_G(g))\) lying in the same \(p\)-block of \(C_G(g)\). Then \((g, \gamma)\) and \((g, \gamma')\) lie in the same \(p\)-block of \(\mathcal{M}(G)\).

Proof. — This follows from 2.6 and Proposition 2.3.
2.D. $p'$-part. — Fix $(g, \gamma) \in \mathcal{M}(G)$. Then it follows from 2.6 that, for all $\chi \in \text{Irr}(KG)$,
\begin{equation}
(2.8) \quad \Psi_{g,\gamma}(V_{\chi}) = \chi(g).
\end{equation}

**Proposition 2.9.** — Let $(g, \gamma)$ and $(h, \eta)$ be two elements in $\mathcal{M}(G)$ which lie in the same $p$-block. Then $g_{p'} = h_{p'}$.

**Proof.** — By Proposition 2.3 and Equality 2.8, it follows from the hypothesis that $\chi(g) \equiv \chi(h) \mod p$
for all $\chi \in \text{Irr}(KG)$. Hence $g_{p'}$ and $h_{p'}$ are conjugate in $G$ (see [Bo, Proposition 2.14]), so they are equal according to our conventions explained in §1.F. 

**Proposition 2.10.** — Let $s \in G_{p'}$ and let $\sigma, \sigma' \in \text{Irr}(KC_G(s))$. Then $(s, \sigma)$ and $(s, \sigma')$ lie in the same $p$-block if and only if $\sigma$ and $\sigma'$ lie in the same $p$-block of $C_G(s)$.

**Proof.** — The if part has been proved in Corollary 2.7. Conversely, assume that $(s, \sigma)$ and $(s, \sigma')$ lie in the same $p$-block. Fix $h \in C_G(s)$. Then $s \in C_G(h)$. Let $\eta_{s,h} : C_G(h) \to K$ be the class function on $C_G(h)$ defined by:
\[
\eta_{s,h}(g) = \begin{cases} 
1 & \text{if } g_{p'} \text{ and } s \text{ are conjugate in } C_G(h), \\
0 & \text{otherwise}.
\end{cases}
\]

It follows from [Bo, Proposition 2.20] that $\eta_{s,h} \in \O(\text{Irr}(KC_G(h)))$. Therefore, by 2.6 and Proposition 2.3,
\[
(\#) \quad \sum_{s \in C_G(s) \setminus \{h\}} \sum_{x \in C_G(s) \setminus \{h\}} (\omega_{s,\sigma}(\hat{\Sigma}_g(xh^{-1})) - \omega_{s,\sigma'}(\hat{\Sigma}_g(xh^{-1}))) \equiv 0 \mod p.
\]

Now, let $x \in G$ be such that $xh^{-1} \in C_G(s)$. Since $x^{-1}sx$ is also a $p'$-element, $\eta_{s,h}(x^{-1}sx) = 1$ if and only if $s$ and $x^{-1}sx$ are conjugate in $C_G(h)$ that is, if and only if $x \in C_G(s)C_G(h)$. So it follows from (\#) that
\[
\omega_{s,\sigma}(\hat{\Sigma}_g(h)) \equiv \omega_{s,\sigma'}(\hat{\Sigma}_g(h)) \mod p
\]
for all $h \in C_G(s)$. This shows that $\sigma$ and $\sigma'$ lie in the same $p$-block of $C_G(s)$. 

2.E. Last step. — We shall prove here the last intermediate result:

**Proposition 2.11.** — Let \((s, e) \in \mathcal{M}^p(G)\) and let \((g, \gamma), (g', \gamma') \in \mathcal{B}_G(s, e)\). Then \((g, \gamma)\) and \((g', \gamma')\) are in the same \(p\)-block of \(\mathcal{M}(G)\).

**Proof.** — We fix \(\sigma \in \text{Irr}(KC_G(s)e)\). It is sufficient to show that \((g, \gamma)\) and \((s, \sigma)\) are in the same \(p\)-block of \(\mathcal{M}(G)\). For this, let \((h, \eta) \in \mathcal{M}(G)\). By Proposition 2.9, we have \(g_{\nu} = s\), so \(C_G(g) \subset C_G(s)\). So 2.6 can be rewritten:

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(s) \setminus G/C_G(h)]} \sum_{y \in [C_G(s) \setminus C_G(s) \mathcal{C}_G(h)/C_G(h)]} \eta(y^{-1}g y) \omega_{g, \gamma}(y h y^{-1}).
\]

Now, let \(x \in [C_G(s) \setminus G/C_G(h)]\) and \(y \in [C_G(g) \setminus C_G(s) x C_G(h) / C_G(h)]\) be such that \(y h y^{-1} \in C_G(g)\). Then \(y h y^{-1} \in C_G(s)\) and so \(x h x^{-1} \in C_G(s)\). Moreover \(y^{-1} s y\) is conjugate to \(x^{-1} s x\) in \(C_G(h)\). Finally, it is well-known (nd easy to prove) that \(\eta(y^{-1} h y) \equiv \eta(y^{-1} s y) \mod p\) (see for instance [Bo, Proposition 2.14]). Therefore:

\[
(\diamond) \quad \Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(s) \setminus G/C_G(h)]} \eta(x^{-1} s x) \omega_{g, \gamma} \left( \sum_{y \in [C_G(g) \setminus C_G(s) x C_G(h) / C_G(h)]} \hat{\psi}_g(y h y^{-1}) \right) \mod p.
\]

Now, let \(x \in [C_G(s) \setminus G/C_G(h)]\) be such that \(x h x^{-1} \in C_G(s)\). Then, by definition of the Brauer map,

\[
(\bigwedge) \quad \text{Br}_G^{\nu}(\hat{\psi}_s(x h x^{-1})) = \sum_{z \in [C_G(g) \setminus C_G(s) / C_G(x h x^{-1})]} \hat{\psi}_g((z x) h (z x)^{-1}).
\]

But \((z x) \in [C_G(g) \setminus C_G(s) / C_G(x h x^{-1})]\) is a set of representatives of double classes in \(C_G(g) \setminus C_G(s) x C_G(h) / C_G(h)\). So it follows from \((\diamond)\) and \((\bigwedge)\) that

\[
\Psi_{g, \eta}(V_{h, \eta}) = \sum_{x \in [C_G(s) \setminus G/C_G(h)]} \eta(x^{-1} s x) \omega_{g, \gamma}(\text{Br}_G^{\nu}(\hat{\psi}_s(x h x^{-1}))).
\]

Using now 2.5 and 2.6, we obtain that

\[
\Psi_{g, \eta}(V_{h, \eta}) \equiv \Psi_{s, \eta}(V_{h, \eta}) \mod p,
\]

as desired. \(\square\)

**Proof of Theorem 1.2.** — Let \((s, e)\) and \((s', e')\) be two elements of \(\mathcal{M}^p(G)\) such that \(\mathcal{B}_G(s, e)\) and \(\mathcal{B}_G(s', e')\) are contained in the same \(p\)-block of \(\mathcal{M}(G)\) (see Proposition 2.11). Let \(\sigma \in \text{Irr}(KC_G(s)e)\) and \(\sigma' \in \text{Irr}(KC_G(s')e')\).

Then \((s, \sigma)\) and \((s', \sigma')\) are in the same \(p\)-block, so it follows from Proposition 2.9 that \(s = s'\) and it follows from Proposition 2.10 that \(\gamma\) and \(\gamma'\) are in the same \(p\)-block of \(C_G(s)\), that is \(e = e'\). This completes the proof of Theorem 1.2. \(\square\)
References


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