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A NOTE ON THE RESONANCE COUNTING FUNCTION FOR SURFACES WITH CUSPS

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Abstract. We prove sharp upper bounds for the number of resonances in boxes of size 1 at high frequency for the Laplacian on finite volume surfaces with hyperbolic cusps. As a corollary, we obtain a Weyl asymptotic for the number of resonances in balls of size $T \to \infty$ with remainder $O(T^{3/2})$.

In this short note, we intend to prove sharp bounds on resonance-counting functions for the Laplacian on finite volume surfaces with hyperbolic cusps. Let $M$ be a complete non-compact surface, equipped with a Riemannian metric $g$. We assume that $(M, g)$ can be decomposed as the union of a compact manifold with boundary and a finite number of hyperbolic cusps, each one being isometric to

$$(a, +\infty)_y \times S^1_\theta$$

with metric $\frac{dy^2 + d\theta^2}{y^2}$ for some $a > 0$. The spectral properties of the Laplacian $\Delta_g$ were first studied by Selberg [Sel89] and Lax-Phillips [LP76] in constant negative curvature, and by Colin-de-Verdière [CdV83], Müller [Mül92], Parnovski [Par95] in the non-constant curvature setting.

On such surfaces, the resolvent $R(s) = (\Delta_g - s(1 - s))^{-1}$ of the Laplacian admits a meromorphic extension from $\{\Re s > 1/2\}$ to $\mathbb{C}$ as an operator mapping $L^2_{\text{comp}}$ to $L^2_{\text{loc}}$ and the natural discrete spectral set for $\Delta_g$ is the set of poles denoted by

$$\mathcal{R} \subset \{s \in \mathbb{C} \mid \Re s \leq 1/2\} \cup (1/2, 1].$$

The poles are called resonances and are counted with multiplicity $m(s)$ (the multiplicity $m(s)$ is defined below and corresponds, for all but finitely many resonances, to the rank of the residue of the resolvent at $s$). We shall recall in the next section how the set of resonances is built. To study their distribution in the complex plane, we define two counting functions:

1. $$N_R(T) := \sum_{s \in \mathcal{R}, |s - 1/2| \leq T} m(s)$$
2. $$N_R(T, \delta) := \sum_{s \in \mathcal{R}, |s - 1/2 - iT| \leq \delta} m(s).$$
The first result on the resonance counting function was proved by Selberg [Sel89, p. 25] for the special case of hyperbolic surfaces with finite volume: the following Weyl type asymptotic expansion holds as $T \to \infty$

$N_R(T) = \frac{\text{Vol}(M)}{2\pi} T^2 + C_0 T \log(T) + C_1 T + O\left(\frac{T}{\log(T)}\right)$

for some explicit constants $C_0, C_1$. In variable curvature, Müller gives a Weyl asymptotic [Müll92, Th. 1.3.a] of the form

$N_R(T) = \frac{\text{Vol}(M)}{2\pi} T^2 + o(T^2)$

and this was improved by Parnovski [Par95] who showed that for all $\epsilon > 0$

$N_R(T) = \frac{\text{Vol}(M)}{2\pi} T^2 + O(T^{3/2+\epsilon})$.

Parnovski’s proof relies on a Weyl type asymptotic expansion involving the scattering phase $S(T)$ (see next section for a precise definition):

$2\pi N_d(T) + S(T) = \frac{\text{Vol}(M)}{2} T^2 - 2kT \ln T + O(T)$,

where $k$ is the number of cusps, and $N_d$ is the counting function for the $L^2$ eigenvalues of $\Delta_g$ embedded in the continuous spectrum.

Using a Poisson formula proved by Müller [Müll92] and estimate (5), we are able to improve the results of Parnovski:

**Theorem 1.** For $T > 1$, and $0 \leq \delta \leq T/2$, the following estimates hold

$N_R(T, \delta) = O(T \delta + T)$,

$N_R(T) = \frac{\text{Vol}(M)}{2\pi} T^2 + O(T^{3/2})$.

In the first estimate with $\delta = 1$, the exponent in $T$ is sharp in general, as can be seen from Selberg’s result (3) which implies that there is $C > 0$ such that as $T \to \infty$

$N(T, 1) = CT + O\left(\frac{T}{\log T}\right)$

In $n$-dimensional Euclidan scattering, upper bounds $O(T^{n-1})$ on the number of resonances in boxes of fixed size at frequency $T$ were obtained by Petkov-Zworski [PZ99] using Breit-Wigner approximation and the scattering phase; our scheme of proof is inspired from their approach. Their result was extended to the case of non-compact perturbations of the Laplacian by Bony [Bon01]. In general, it is expected that the number of resonances in such boxes is controlled by the (fractal) dimension of the trapped set (see for example Zworski [Zwo99], Guillopé-Lin-Zworski [GLZ04],
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1. Preliminaries

We start by recalling well-known facts on scattering theory on surfaces with cusps, and we refer to the article of Müller [Mül92] for details. Let \((M, g)\) be a complete Riemannian surface that can be decomposed as follows:

\[ M = M_0 \cup Z_1 \cup \cdots \cup Z_k, \]

where \(M_0\) is a compact surface with smooth boundary, and \(Z_j\) are hyperbolic cusps

\[ Z_j \simeq (a_j, +\infty) \times S^1, \quad j = 1 \ldots k, \]

with \(a_j > 0\) and the metric on \(Z_j\) in coordinates \((y, \theta) \in (a_j, +\infty) \times S^1\) is

\[ ds^2 = \frac{dy^2 + d\theta^2}{y^2}. \]

Notice that the surface has finite volume when equipped with this metric.

The non-negative Laplacian \(\Delta\) acting on \(C_0^\infty(M)\) functions has a unique self-adjoint extension to \(L^2(M)\) and its spectrum consists of

1. Absolutely continuous spectrum \(\sigma_{ac} = [1/4, +\infty)\) with multiplicity \(k\) (the number of cusps).
2. Discrete spectrum \(\sigma_d = \{\lambda_0 = 0 < \lambda_1 \leq \cdots \leq \lambda_i \leq \ldots\}\), possibly finite, and which may contain embedded eigenvalues in the continuous spectrum. To \(\lambda \in \sigma_d\), we associate a family of orthogonal eigenfunctions that generate its eigenspace \((u^i_\lambda)_{i=1\ldots d_\lambda} \in L^2(M) \cap C^\infty(M)\).

The generalized eigenfunctions associated to the absolutely continuous spectrum are the Eisenstein functions, \((E_j(x, s))_{i=1\ldots k}\). Each \(E_j\) is a meromorphic family (in \(s\)) of smooth functions on \(M\). Its poles are contained in the open half-plane \(\{\Re s < 1/2\}\) or in \((1/2, 1]\). The Eisenstein functions are characterized by two properties:

1. \(\Delta_g E_j(., s) = s(1-s)E_j(., s)\)
2. In the cusp \(Z_i, i = 1 \ldots k\), the zeroth Fourier coefficient of \(E_j\) in the \(\theta\) variable equals \(\delta_{ij} y_i^s + \phi_{ij}(s)y_i^{1-s}\) where \(y_i\) denotes the \(y\) coordinate in the cusp \(Z_i\) and \(\phi_{ij}(s)\) is a meromorphic function of \(s\).
We can collect the scattering coefficients $\phi_{ij}$ in a meromorphic family of matrices, $\phi(s) = (\phi_{ij})_{ij}$ called scattering matrix. We denote its determinant by $\varphi(s) = \det \phi(s)$. Then the following identities hold

$$\phi(s)\phi(1-s) = Id, \quad \phi(s) = \phi(\overline{s}), \quad \phi(s)^* = \phi(\overline{s}).$$

The line $\Re s = 1/2$ corresponds to the continuous spectrum. On that line, $\phi(s)$ is unitary, $\varphi(s)$ has modulus 1. We also define the scattering phase

$$S(T) = -\int_0^T \frac{\varphi'(1/2 + it)}{\varphi(1/2 + it)} dt$$

The set of poles of $\varphi$, $\phi$ and $(E_j)_{j=1...k}$ is the same, we call them them scattering poles and we shall denote $\Lambda$ this set. It is contained in $\{\Re s < 1/2\} \cup (1/2, 1]$. The union of this set with the set of $s \in \mathbb{C}$ such that $s(1-s)$ is an $L^2$ eigenvalue, is called the resonance set, and denoted $\mathcal{R}$. Following [Müll92, pp.287], the multiplicities $m(s)$ are defined as :

1. If $\Re s \geq 1/2$, $s \neq 1/2$, $m(s)$ is the dimension of $\ker_{L^2}(\Delta_g - s(1-s))$.
2. If $\Re s < 1/2$, $m(s)$ is the dimension of $\ker_{L^2}(\Delta_g - s(1-s))$ minus the order of $\varphi$ at $s$.
3. $m(1/2)$ equals $(\text{Tr}(\phi(1/2)) + k)/2$ plus twice the dimension of $\ker_{L^2}(\Delta_g - 1/4)$.

For convenience, we define two counting functions for the discrete spectrum and the poles of $\varphi$:

$$N_d(T) := \sum_{|s_i - 1/2| \leq T} m(s_i),$$

$$N_\Lambda(T) := \sum_{s \in \Lambda, |s - 1/2| \leq T} m(s),$$

so that

$$N_\mathcal{R}(T) := \sum_{s \in \mathcal{R}, |s - 1/2| \leq T} m(s) = 2N_d(T) + N_\Lambda(T).$$

### 2. Main Observation

In this Section, we explain how to obtain estimate for $N_\mathcal{R}(T)$ in boxes at high frequency.

From the asymptotic expansion (5), we deduce that for $0 \leq \delta \leq T/2$,

$$2\pi(N_d(T+\delta) - N_d(T-\delta)) + S(T+\delta) - S(T-\delta) = 2\text{Vol}(M)T\delta - 4k\delta \ln T + O(T).$$

Next, we recall the Poisson formula for resonances proved by Müller [Müll92, Th. 3.32]

$$S'(T) = \log \frac{1}{q} + \sum_{\rho \in \Lambda} \frac{1 - 2\Re \rho}{(\Re \rho - 1/2)^2 + (\Im \rho - T)^2}.$$
where \( q \) is some constant (not necessarily \(< 1 \)). Let \( C > 1, 0 < \epsilon < 1 \) and

\[
\Omega_{T,\delta} := \{ s \in \mathbb{C}; \ |s - 1/2 - iT| \leq \delta/C \text{ and } 0 \leq 1/2 - \Re s \leq \epsilon \delta \}.
\]

Then, for \( s \in \Omega_{T,\delta} \),

\[
\int_{[T-\delta,T+\delta]} \frac{1 - 2\Re s}{(\Re s - 1/2)^2 + (t - \Im s)^2} dt = 2 \left[ \arctan \frac{t - \Im s}{1/2 - \Re s} \right]_{T-\delta}^{T+\delta}.
\]

The addition formula for arctan, with \( x, y > 0 \) and \( xy > 1 \) is given by

\[
\arctan x + \arctan y = \pi + \arctan \frac{x + y}{1 - xy}
\]

thus

\[
\int_{[T-\delta,T+\delta]} \frac{1 - 2\Re s}{(\Re s - 1/2)^2 + (t - \Im s)^2} dt = 2\pi - 2 \arctan \frac{2\delta(1/2 - \Re s)}{\delta^2 - |s - 1/2 - iT|^2} \geq 2\pi - 2 \arctan \tilde{C} \epsilon,
\]

where \( \tilde{C} \) is set to be \( 2/(1 - 1/C^2) \). For \( \epsilon \) small enough, this is bigger than, say \( \pi \).

Since all but a finite number of terms in (13) are positive, we have:

\[
S(T + \delta) - S(T - \delta) \geq O(\delta) + \sum_{\rho \in \Lambda \cap \Omega_{T,\delta}} \pi.
\]

Combining with (12), we deduce that

\[
N_d(T + \delta) - N_d(T - \delta) + \# \Lambda \cap \Omega_{T,\delta} = O(T\delta) + O(T) + O(\delta).
\]

This is the content of (6) in our main theorem.

3. Consequence

Now, we proceed to prove the second part of our theorem. We will follow the method of Müller [Müll92, pp. 282], which is a global and quantitative version of the argument used in the previous section. Integrating the Poisson formula over \([-T, T]\), we relate the scattering phase asymptotics to the poles of \( \phi \). Using the arctan addition formula, we are left with the sum of \( N_\Lambda(T) \) and an expression with arctan’s (equation (4.9) in [Müll92]):

\[
\frac{1}{2\pi} S(T) = \frac{1}{2} N_\Lambda(T) + \frac{1}{2\pi} \sum_{\rho \in \Lambda, \Re \rho < 1/2} \arctan \left[ \frac{1 - 2\Re \rho}{|\rho - 1/2|^2} T \left( 1 - \frac{T^2}{|\rho - 1/2|^2} \right)^{-1} \right] + O(T).
\]
The sum is then split between \{1\} the poles in \{|T - |\rho - 1/2| > T^{1/2}\}, and \{2\}, the others. Müller proved that the sum \{1\} is \(O(T^{3/2})\). The sum \{2\} can be bounded by

\[\frac{1}{4}(N_\Lambda(T + \sqrt{T}) - N_\Lambda(T - \sqrt{T})).\]

From [Mül92, Cor. 3.29], we also recall that

\[\sum_{\eta \in \Lambda, \eta \neq 1/2} m(\eta) \left| \frac{1 - 2\Re \eta}{|\eta - 1/2|^2} \right| < \infty.\]

Consider the set \(\tilde{\Lambda} = \{\eta \in \Lambda; (2\Re \eta - 1)^2 > 3\eta, |\eta| > 1\}\). On \(\tilde{\Lambda}\), we have that \(|\eta - 1/2|^{1/2} \leq 1 - 2\Re \eta\), thus

\[\sum_{\eta \in \tilde{\Lambda}, \eta \neq 1/2} m(\eta) \frac{1}{|\eta - 1/2|^{3/2}} < \infty.\]

If \(\tilde{n}(T)\) is the counting function for \(\tilde{\Lambda}\), we deduce that

\[\sum_{k=1}^{\infty} \tilde{n}(k) \left[ \frac{1}{k^{3/2}} - \frac{1}{(k + 1)^{3/2}} \right] < \infty.\]

Since \(\tilde{n}\) is non-decreasing, \(\tilde{n}(k) = o(k^{3/2})\). Now,

\[N_\Lambda(T - \sqrt{T}) - N_\Lambda(T + \sqrt{T}) \leq \tilde{n}(T) + N_\mathcal{R}(T, \sqrt{T}) + N(T, \sqrt{T}).\]

This concludes the proof.

References


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