Anisotropic damage modelling of biaxial behaviour and rupture of concrete structures

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Abstract: This paper deals with damage induced anisotropy modelling for concrete-like materials. A thermodynamics based constitutive relationship is presented coupling anisotropic damage and elasticity. The biaxial behaviour of such a model is analysed through the effects of yield surface modifications by the introduction of new equivalent strains.

INTRODUCTION

Regarding the ultimate behaviour of reinforced and prestressed concrete structures, the prevision of oriented micro-cracks openings is of main importance for the structural integrity. It is even a key issue for multi-physics analyses such as in diffusion problems. In that sense, the mesoscopic approach using Continuum Damage Mechanics at the Representative Element Volume scale is a relevant tool to deal with large scale structures if loading induced damage anisotropy is represented (Lemaitre & Desmorat 2005). It allows to represent the local loss of stiffness of the material and the strain localization zone representative of macroscopic cracks. Within the thermodynamics framework, the state damage variable may be a scalar or tensorial quantity. Dealing with induced anisotropy in cementitious materials and with non-symmetric tension/compression behaviour, the choice for a second order damage tensor (Cordebois & Sidoroff, 1982, Murakami, 1988) is a pragmatic approach with regard to robustness and numerical implementations. The 3D effects are not commonly taken into account with damage models. Plasticity is often required to gain some important features as of course permanent strains but also as confined features and responses.

A good approximation of the material response sustaining multiaxial states of stresses is required in Finite Elements, the states of stresses being naturally multiaxial at the structure Gauss points. Important works have then combined damage behaviour for tensile loadings and plasticity-like behaviour – eventually coupled with damage – for compressive or confined loadings (Gatuingt & Pijaudier-Cabot 2002). The drawback of such approaches is the complexity of the models and of their numerical implementation: one often has to deal with multi-surfaces plasticity-damage modelling, robustness is difficult to ensure and making the models nonlocal also becomes a difficult task. Damage models (with no plasticity) usually do not represent permanent strains, but they represent properly the monotonic softening response of materials, at least under low confinement conditions, conditions often encountered in structural design when tension and shear are the main cause of structural failure. Not modelling plasticity reduces the numbers of materials parameters introduced. It allows also to consider more naturally criterion surfaces in the strains space, choices computationally efficient.

The choice is made in the present work just to address such monotonic failure cases and to see if damage models, anisotropic but with a limited numbers of material parameters, may prove sufficient in this task. Concerning modelling, the tension/compression coupled to shear response usually needs an adequate expression for the equivalent stresses or strains used in the threshold or criterion function. Based on the works of Mazars (1984), Drucker-Prager (1952) and de Vree (1995), different strain based damage criteria are proposed next and extended to nonlocal framework. Their influence both at the material and structural levels is studied when ultimate behaviour and rupture occur. In a first part, the initial thermodynamics damage modelling is recalled allowing for the rigorous expression of constitutive equations. The main numerical features of the new modelling are mentioned in part 2. The third part of the paper allows for dealing with the effects of different expression for the equivalent strains and for the elasticity domains regarding the biaxial behaviour of concrete at the material scale. At last, a structural example is presented.

1. ANISOTROPIC MODELLING OF CONCRETE

The necessity to account for micro-cracks orientation in the description of the mechanical behaviour of concrete naturally leads to the use of continuum anisotropic damage framework (Chaboche, 1979, Cordebois & Sidoroff 1982, Ladevèze 1983, Chow & Wang 1987, Murakami 1988, Dragon & Halm, 1988, Papa & Talercio, 1996, Lemaitre & Desmorat 2005, Badel et al, 2007). To be computationally efficient, the expression of constitutive relationships has to be coupled with a proper numerical algorithm allowing dealing with analyses at the structural scale. From the numerical point of view, considering both a hydrostatic / deviatoric splitting and a damage threshold based on an equivalent strain allows to avoid the Gauss point iterative resolution of the constitutive equations and related evolution laws, even if implicitly discretized (Desmorat et al, 2007). The efficiency of such an approach for the use of anisotropic damage in reinforced and pre-stressed concrete has been pointed out treating large scale structures.

The mesh dependency induced by strain softening at the local level is avoided by adopting integral nonlocal type regularization (Pijaudier-Cabot & Bazant, 1987). The transition between an homogeneous state of cracking and a macro-crack propagation is solved by adopting a scalar critical damage value D_c close to unity for which the principal damages – i.e. the eigenvalues of the damage tensor - can no more increase in the direction where this critical value is reached.

1.1 Elasticity coupled with anisotropic damage

Modelling crack initiation and growth at the representative volume element scale within a macroscopic phenomenological framework needs the introduction of a thermodynamic variable. A choice has to be made concerning the damage kinematics, between scalar or tensorial representations. The easiest one consists in using a scalar damage variable, representing an isotropic state of concrete degradation (Mazars 1984). This approach allows for the expression of efficient material models dealing with robust stress integration algorithms. Large scale computation can be handled nevertheless without the information concerning the cracks orientation. If the description of the crack orientation is a major point, an anisotropic kinematics for the damage variable has to be introduced. Damage affecting directly the elasticity law, the most natural approach could use a fourth order tensor (Chaboche 1979, Leckie & Onat 1981). It is possible to express a new thermodynamic variable or to directly use the elasticity operator as a variable being able to degrade (Krajcinovic 1985, Simo & Ju 1987, Ju 1989, Govindjee *et al.* 1995, Meschke *et al.* 1998). In such a case, this framework encounters strong limitations due to the difficulty in proceeding to robust numerical integration within a finite element code and to deal with adequate identification for the large number of material parameters involved.

An alternative approach consists in using a symmetric orthotropic second order damage tensor as thermodynamic variable (Murakami & Ohno 1978, Cordebois & Sidoroff 1982, Chaw & Wang 1987, Murakami 1988). The resulting elasticity operator has also to be symmetric, depending on the strain, stress or energy used in the expression of effective stress (the real stress acting on the sane part of the material). Difficulty remains in dealing with effective stress, independent from material parameters (Lemaitre 2002, Lemaitre & Desmorat 2005), or to account for complete stiffness recovery when passing from tension to compression (Ladevèze 1983).

In the present study, only partial stress recovery is introduced. A single but tensorial damage variable is used in describing the non-symmetric behaviour of concrete in tension and in compression. According to the general expression (Ladevèze 1983, Lemaitre & Desmorat 2005) insuring continuity of the stress-strain path whatever the loadings, the thermodynamics potential takes the following form

$$\rho\psi^* = \frac{1+\nu}{2E} Tr\left[(\boldsymbol{1}-\boldsymbol{D})^{-1/2}\boldsymbol{\sigma}^D(\boldsymbol{1}-\boldsymbol{D})^{-1/2}\boldsymbol{\sigma}^D\right] + \frac{1-2\nu}{6E}\left[\frac{\langle Tr\boldsymbol{\sigma}\rangle_+^2}{1-Tr\boldsymbol{D}} + \langle Tr\boldsymbol{\sigma}\rangle_-^2\right] \tag{1}$$

where E and ν are the elasticity parameters, ρ the density, $\boldsymbol{\sigma}$ the Cauchy stress and $\boldsymbol{D}$ the damage tensor. The notation $(.)^D$ indicates the deviatoric part of a tensor and $(\boldsymbol{1}-\boldsymbol{D})^{-1/2}$ is gained from diagonal form of $(\boldsymbol{1}-\boldsymbol{D})$ as $(\boldsymbol{1}-\boldsymbol{D})_{diag} = \boldsymbol{P}^{-1}(\boldsymbol{1}-\boldsymbol{D})\boldsymbol{P}$ and $(\boldsymbol{1}-\boldsymbol{D})^{-1/2} = \boldsymbol{P}(\boldsymbol{1}-\boldsymbol{D})_{diag}^{-1/2}\boldsymbol{P}^{-1}$. The notation $\langle x\rangle_+ = \max(0,x)$ stands for the positive part of the scalar x and $\langle x\rangle_- = \min(0,x)$ stands for its negative part.

The state laws are obtained by derivation of the potential with respect to the thermodynamics variables. The elasticity law coupled with anisotropic damage reads:

$$\boldsymbol{\varepsilon} = \rho \frac{\partial \psi^*}{\partial \boldsymbol{\sigma}} = \frac{1+\nu}{E}\left[(\boldsymbol{1}-\boldsymbol{D})^{-1/2}\boldsymbol{\sigma}^D(\boldsymbol{1}-\boldsymbol{D})^{-1/2}\right]^D + \frac{1-2\nu}{3E}\left[\frac{\langle Tr\boldsymbol{\sigma}\rangle_+}{1-Tr\boldsymbol{D}} + \langle Tr\boldsymbol{\sigma}\rangle_-\right]\boldsymbol{1} \tag{2}$$

And the strain energy release rate density $\boldsymbol{Y}$ (the thermodynamics force associated with $\boldsymbol{D}$) is: $\boldsymbol{Y} = \rho \dfrac{\partial \psi^*}{\partial \boldsymbol{D}}$.

The decoupling between the deviatoric member of the stress-strain relation and its volumic part induces only a partial stiffness recovery sufficient for monotonic applications. In compression, damage does not affect the hydrostatic response with a bulk modulus $K = E/3(1-2\nu)$. In tension the damaged bulk modulus is $\tilde{K} = (1 - Tr\boldsymbol{D})K$.

Previous elasticity law rewritten $\tilde{\boldsymbol{\sigma}} = \mathbf{E} : \boldsymbol{\varepsilon}$ with $\mathbf{E}$ the undamaged Hooke tensor defines analytically the relationship between the Cauchy stress and the effective stress $\tilde{\boldsymbol{\sigma}}$,

$$\tilde{\boldsymbol{\sigma}} = \left[(\boldsymbol{1}-\boldsymbol{D})^{-1/2}\boldsymbol{\sigma}^D(\boldsymbol{1}-\boldsymbol{D})^{-1/2}\right]^D + \frac{1}{3}\left[\frac{\langle Tr\boldsymbol{\sigma}\rangle_+}{1-Tr\boldsymbol{D}} - \langle -Tr\boldsymbol{\sigma}\rangle_+\right]\boldsymbol{1} \tag{3}$$

so that the elasticity laws simply sums up as :

$$\boldsymbol{\varepsilon} = \boldsymbol{E}^{-1} : \tilde{\boldsymbol{\sigma}} = \frac{1+\nu}{E}\tilde{\boldsymbol{\sigma}} - \frac{\nu}{E} Tr\tilde{\boldsymbol{\sigma}}\boldsymbol{1} \tag{4}$$

1.2 Damage threshold and evolution laws

Damage evolution is linked to the violation or not of a criterion. Depending on the materials, the damage criterion may be expressed, similarly to plasticity, thanks to the stresses (Ortiz 1985, Warnke 1975, Voyiadjis & Abu-Lebdeh 1994), using the strains (Mazars 1984, Herrmann & Kestin 1988, Ramtani 1990, de Vree *et al.* 1995, Geers *et al.* 2000) or using energy quantities like the damage energy release rate (Marigo 1981, Laborderie *et al.* 1990). Most of the finite elements codes use displacements based interpolation functions. The most efficient way to express constitutive equations arguing for explicit numerical integration is to make use of a damage threshold based on the strains. For brittle materials like concrete, Mazars's criterion (1984) defining an equivalent strain is used for the anisotropic evolution of damage. The equivalent strain $\hat{\varepsilon}$ is constructed selecting the positive principal strain ε_I :

$$\hat{\varepsilon} = \sqrt{\langle \varepsilon \rangle_+ : \langle \varepsilon \rangle_+} = \sqrt{\sum \langle \varepsilon_I \rangle_+^2} \tag{5}$$

The damage threshold takes the simple form:

$$f = \hat{\varepsilon} - \kappa(tr\boldsymbol{D}) \leq 0 \tag{6}$$

$\kappa(tr\boldsymbol{D})$ is the consolidation function in the strains space, depending on the trace of the damage tensor. The initial value defines the damage threshold $\kappa_0 = \kappa(0)$.

The advantages are multiple. On the one hand, the use of equivalent quantities based on strains will make easier the explicit derivation of the subsequent numerical scheme for stress integration. On the other hand, dealing only with one scalar equivalent quantity based on strains, the classical drawbacks of spurious mesh dependency when using softening constitutive equations (Bazant 1976) are simply numerically avoided by implementing the law within the framework of nonlocal media through the use of a nonlocal weight function ψ (Pijaudier-Cabot & Bazant 1987). It only needs the identification of a new parameter l_c, which is the internal length of the nonlocal medium. This length can be linked to the size of the heterogeneities of the material or to the resulting scale effects when dealing with structural analyses. A equivalent strain $\hat{\varepsilon}$ is the made nonlocal as

$$\hat{\varepsilon}^{nl}(x) = \frac{1}{V_r(x)} \int_V \hat{\varepsilon}(s)\psi(s-x)dv \text{ with } V_r = \int_V \psi(s-x)dv \text{ and } \psi(s-x) = \exp(-2\|s-x\|^2 / l_c^2) \tag{7}$$

Only one damage tensorial variable is used to represent the micro-cracks pattern in concrete. The non-symmetric tension/compression response is obtained thanks to the anisotropic feature of damage. The damage evolution in one direction is guided by the level of extension in this direction. Then the damage can be considered proportional to the positive part of the strain tensor $\langle \boldsymbol{\varepsilon} \rangle_+$ or to $\langle \boldsymbol{\varepsilon} \rangle_+^2$. The power 2 is used next as $Tr\dot{\boldsymbol{D}} \propto Tr\langle \boldsymbol{\varepsilon} \rangle_+^2$ simplifies in $Tr\dot{\boldsymbol{D}} \propto \hat{\varepsilon}^2$.

Within the thermodynamics framework, the corresponding choice for the non associated potential is:

$$F = \boldsymbol{Y} : \langle \boldsymbol{\varepsilon} \rangle_+^2 \tag{8}$$

The damage evolution law is obtained by derivation with regard to the strain energy release rate density:

$$\dot{\boldsymbol{D}} = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{Y}} = \dot{\lambda}\langle \boldsymbol{\varepsilon} \rangle_+^2 \tag{9}$$

The damage multiplier $\dot{\lambda}$ is determined from the consistency condition $f = 0$ and $\dot{f} = 0$.

The identification of the behaviour of different concretes is performed by the definition of the consolidation function $\kappa(tr\boldsymbol{D})$. For concrete, a simple expression has been found allowing to fit the concrete responses in tension and in compression, introducing only two material parameters *a* and *A* in addition to the Young's modulus *E*, the Poisson's ratio ν and the damage threshold κ_0.

$$\kappa(Tr\boldsymbol{D}) = a.\tan\left[\frac{Tr\boldsymbol{D}}{aA} + \arctan\left(\frac{\kappa_0}{a}\right)\right] \tag{10}$$

1.3 Uniaxial responses

The responses of the model, subject to uniaxial states of stresses are presented in the next figures. The material parameters used in the analysis are E = 42 GPa, ν = 0.2, κ_0 = 5 10^{-5}, A = 5 10^3, a = 2.93 10^{-4}.

One can observe that with only one thermodynamic variable (the damage tensor), the model is able to correctly describe the non symmetric uniaxial behaviour of concrete. Due to the decoupling between the hydrostatic part and the volumic part of the model, the volumic strain of the response is not affected in compression. Even if dilatancy is not represented, this is an improvement compared to the loss of bulk modulus obtained with isotropic damage models.

2. NUMERICAL IMPLEMENTATION

2.1 Euler backward scheme for numerical stresses and damage computations

One major advantage of this model is its easy numerical implementation within any finite element code and its corresponding robustness regarding computations at the structural scale. In fact, any evolution equation can be analytically (explicitly) calculated, even when using an implicit discretisation scheme. No internal iteration, very CPU time consuming, are needed at Gauss point level. The principal steps for stress computation are recall hereafter.

Knowing all the variables and stress at time t_n as well as the final state of strain at time t_{n+1}, $\boldsymbol{\varepsilon}_{n+1}$, the stress-damage algorithm aims at computing the internal variable (the damage tensor $\boldsymbol{D}_{n+1}$) and the stresses $\boldsymbol{\sigma}_{n+1}$ at time t_{n+1}.

1. Calculate the threshold function trial : $f_{trial} = \varepsilon_{eq,n+1} - \kappa\left(tr\boldsymbol{D}_n\right)$ with any appropriate equivalent strain (Mazars's one $\varepsilon_{eq,n+1} = \hat{\varepsilon}_{n+1} = \sqrt{\langle \boldsymbol{\varepsilon}_{n+1} \rangle_+ : \langle \boldsymbol{\varepsilon}_{n+1} \rangle_+}$ or others..)

If the elasticity criterion is not violated, i.e. $f < 0$, the damage does not evolve ($\boldsymbol{D}_{n+1} = \boldsymbol{D}_n$) the stress increment is obtained thanks to a reversible elastic change of state (steps 6 and 7). On the contrary, the internal variable has to be corrected, according to the nonlinear constitutive equations and the consolidation function.

2. Calculate the non local equivalent strain if regularisation procedure is adopted (eq. 7).

3. Discretize the damage evolution as : $\Delta\boldsymbol{D} = \boldsymbol{D}_{n+1} - \boldsymbol{D}_n = \Delta\lambda\langle\boldsymbol{\varepsilon}_{n+1}\rangle_+^2$

4. Taking the trace of the previous expression (making $tr\langle\boldsymbol{\varepsilon}_{n+1}\rangle_+^2 = \hat{\varepsilon}_{n+1}^2$ appears), the damage multiplier increment can be explicitly computed

$$\Delta\lambda = \frac{Tr\boldsymbol{D}_{n+1} - Tr\boldsymbol{D}_n}{\hat{\varepsilon}_{n+1}^2}, \text{ with } Tr\,\boldsymbol{D}_{n+1} = \kappa^{-1}\left(\varepsilon_{eq,n+1}\right)$$

5. Actualize the damage tensor: $\boldsymbol{D}_{n+1} = \boldsymbol{D}_n + \Delta\lambda\langle\boldsymbol{\varepsilon}_{n+1}\rangle_+^2$

6. Calculate the effective stresses: $\tilde{\boldsymbol{\sigma}}_{n+1} = \boldsymbol{E} : \boldsymbol{\varepsilon}_{n+1}$

7. Determine the Cauchy stresses by inverting eq (3).

$$\begin{aligned}\boldsymbol{\sigma}_{n+1} = (\boldsymbol{1} - \boldsymbol{D}_{n+1})^{1/2}\,\tilde{\boldsymbol{\sigma}}_{n+1}(\boldsymbol{1} - \boldsymbol{D}_{n+1})^{1/2} - \frac{(\boldsymbol{1} - \boldsymbol{D}_{n+1}) : \tilde{\boldsymbol{\sigma}}_{n+1}}{3 - Tr\boldsymbol{D}_{n+1}}(\boldsymbol{1} - \boldsymbol{D}_{n+1}) \\ + \frac{1}{3}\left[(1 - Tr\boldsymbol{D}_{n+1})\langle Tr\tilde{\boldsymbol{\sigma}}_{n+1}\rangle_+ + \langle Tr\tilde{\boldsymbol{\sigma}}_{n+1}\rangle_-\right]\boldsymbol{1}\end{aligned}$$

2.2 Rupture control procedure

Describing rupture of structural elements needs to deal with damage constitutive equations reaching the ultimate state of damage (eigenvalues close to unity). For isotropic damage, respecting the Clausius-Duhem inequality, it is quite clear that the maximum value of the damage variable should not exceed one. Within an anisotropic framework with partial stiffness recovery, such a criterion is no more so simple. The tensorial damage acts by its individual values on the deviatoric part of the behaviour and it acts by its trace on the volumic part. It is obvious to consider two different treatments for the damage evolution law, depending on which part of the behaviour is treated. Such a treatment simply just ensures the property of a positive effective damaged elasticity tensor.

A critical value D_c for damage is introduced, allowing defining the numerical transition between a micro-cracked medium range and the occurrence of a macroscopic crack, for which continuum damage mechanics looses sense and consistency, benefiting to nonlinear fracture mechanics or extended finite element formulation. Both handle strong discontinuities in the material (Belytschko & Black 1999, Jirasek 2000). In first approximations for concrete, D_c is taken equal to 0.99.

Concerning the hydrostatic part of the constitutive equations, the limitations at high state of damage appears clearly when observing the reversible process equations (from eq (2)).

$$Tr\varepsilon = \frac{\langle Tr\sigma \rangle_+}{3K(1 - Tr\boldsymbol{D})} + \frac{\langle Tr\sigma \rangle_-}{3K} \tag{11}$$

To keep positive the damaged bulk modulus, it is necessary to only limit in tensile loadings the evolution of the trace of $\boldsymbol{D}$ to D_c. In such a way that, when $tr\boldsymbol{D}$ reaches D_c, the bulk modulus takes the critical value of $\widetilde{K} = (1 - D_c)K$ when $Tr\varepsilon > 0$.

For the deviatoric case, to ensure that the damaged elasticity operator remains positive definite, one only has to impose that the eigenvalues of the second order damage tensor are bounded by 1, or by D_c from a numerical point of view (Lemaitre *et al.* 2000, Badel 2001). Under these conditions, the general evolution equations of the previous section have to be adapted. If the maximum eigenvalue of damage D_I reaches its critical value in the direction n_I, damage growth in that direction is stopped, defining a first plane of fixed crack in the solid. Damage only goes on growing in the remaining (n_{II}, n_{III}) directions. The damage evolution law is kept unchanged by only considering the terms of the strain tensor in the (n_{II}, n_{III}) plane, conserving $D_I = D_c$ along n_I. The corresponding (projected) evolution law formally reads:

$$\dot{\boldsymbol{D}} = \dot{\lambda}\, \Pi_{n_I} \langle \boldsymbol{\varepsilon} \rangle_+ \qquad (12)$$

with Π_{n_I} the projection operator. If the loading continues, a second direction for which damage reaches its critical value is detected, defining in the same way, orthogonally to the two first directions, the third one, and so the eigen base of the damage operator for the cracked medium. Only three families of cracks are introduced at the final stage and the fully broken behaviour is an elastic one (with different bulk modulus in tension and in compression),

$$\boldsymbol{\sigma} = 2G(1 - D_c)\boldsymbol{\varepsilon}^D + K\left[(1 - D_c)\langle Tr\boldsymbol{\varepsilon}\rangle_+ + \langle Tr\boldsymbol{\varepsilon}\rangle_-\right] \qquad (13)$$

with $G = E/2(1+\nu)$, the shear modulus.

3. BIAXIAL BEHAVIOUR AND EQUIVALENT STRAIN

The biaxial behaviour of brittle materials like concrete has to be handled with care, due to the complexity of the degradation modes and crack pattern (Kupfer *et al.* 1973). The suitable representation of biaxial rupture needs to possess an adequate elasticity limit as well as evolution equations able to deal with confinement effects (Mazars 1984, Ramtani 1990). This section is devoted to evaluate the biaxial response of the previous model using different equivalent strains. Although being within an anisotropic framework, such an analysis can easily be performed due to the high level of modularity of the constitutive model based on the use on equivalent strains.

3.1 Biaxial responses for an isotropic model and for the anisotropic model

The constitutive models based on continuum damage mechanics, isotropic or anisotropic, exhibit a high level of brittleness in biaxial compression. If the state of stress at rupture is plotted in the $(\sigma_{11}, \sigma_{22})$ plane, one can observe a decrease of the load bearing capacity in biaxial compression (with regards to the uniaxial response in compression, σ_{rupt}) while the experimental results show an increase of the stress at rupture of about 20 % (Kupfer & Gerstle 1973, van Mier 1984). In figure 2a, one shows the elasticity limit, using the Mazars equivalent strain as well as the rupture envelop for the isotropic (Mazars 1984) damage model. Kupfer (1973) and van Mier (1984) experimental data are drawn.

In figure 2b, the brittleness increase is even more clearly emphasized. The effects of damage in the non convexity of the rupture limit are even more pronounced for the anisotropic model, by localizing the crack in a unique plane, orthogonal to the loading plane. The lack of concavity in biaxial compression of the initial elasticity domain (flat end with Mazars's criterions) is the major source for this bad representation of biaxial responses. A possible remedy for compressive loadings is of course the consideration of both plasticity and damage mechanisms but the corresponding models becomes quite complex (Meschke & Lackner 1998, Ragueneau *et al.* 2000, Jason *et al.* 2006) A second remedy, much simpler in terms of numerical efforts, consists in keeping the elasticity coupled to damage but in increasing the elasticity non-symmetry by adding new invariants in the expression of the equivalent strain.

3.2 Modification of the equivalent strain

Different solutions may be adopted to improve the biaxial responses of the models subject to complex state of stress. By only changing the definition and expression of the elasticity domain, the numerical integration is kept unchanged as well as the nonlocal formulation allowing proceeding to structural case study without mesh dependency. Different formulations can be adopted for concrete, based on the original expression of the Mazars equivalent strain $\hat{\varepsilon} = \sqrt{\langle \varepsilon \rangle_+ : \langle \varepsilon \rangle_+}$ completed by terms function of the strain invariants : $I_1 = I_1(\varepsilon) = Tr(\varepsilon)$ and $J_2 = J_2(\varepsilon) = \frac{1}{2}\left[\varepsilon : \varepsilon - \frac{1}{3} I_1^2\right]$.

- Mazars-Drucker-Prager : Adding of the trace of the strain tensor. A complementary material parameter *k* has to be identified. Although it greatly improves the elasticity domain in bi-compression, the main disadvantage of this expression is to modify in the same way the tensile response making necessary a complete reidentification of all the material parameters.

$$\varepsilon_{eq} = \hat{\varepsilon} + kI_1 \tag{14}$$

- Modified Mazars-Drucker-Prager: Adding of the trace of the negative part of the strain tensor, without modifying the response in tension. This non-convex surface may generate, depending on the material parameters choice, instability of the mechanical response.

$$\varepsilon_{eq} = \hat{\varepsilon} + k\left\langle I_1 \right\rangle_- \tag{15}$$

- Mazars-Mises-Drucker-Prager : : Adding of the trace of the strain tensor and of the second invariant of the deviatoric part of the strain tensor. This equivalent strain defines a convex elasticity domain making easier the constitutive equation parameters identification.

$$\varepsilon_{eq} = \hat{\varepsilon} + kI_1 + \frac{1}{\sqrt{2}}\sqrt{J_2} \tag{16}$$

- De Vree criterion: a non-centered von Mises criterion in the strain space (de Vree *et al.* 1995). The ellipsoïdal shape, far from the previous one allows for a direct control of the non-symmetric behaviour of concrete through the introduction of the material parameter *k*.

$$\varepsilon_{eq} = \frac{k-1}{2k(1-2\nu)} I_1 + \frac{1}{2k}\sqrt{\frac{(k-1)^2}{(1-2\nu)^2} I_1^2 + \frac{12k}{(1+\nu)^2} J_2} \qquad (17)$$

Such a difference in the biaxial behaviour is more relevant when looking at the shear response of the different models. The isotropic Mazars model in shear is compared to the anisotropic one using the previous different expressions for the equivalent strain and elasticity surface. The shear behaviour of concrete, due to aggregate interlocks, roughness and frictional sliding along the crack surfaces should be much more ductile than the response in pure tension. This point is illustrated in figure 5, especially for the de Vree surface.

4. NOORU-MOHAMED'S STRUCTURAL CASE-STUDY

The classical experiment of Nooru-Hohamed (1992) is used to analyse, at the structural level, the effects of the changes in the expression of the equivalent strain. Due to the interaction between tension and shear following a non proportional path, this test is relevant for comparison in our case.

The specimen geometry and the experimental testing set up are shown in Fig. 6. It is a symmetric 200 mm · 200 mm mortar square with two notches, 30 mm long and 5 mm thick. The case study is here carried out for a maximum shear load FMax = 10 kN exhibiting mixed mode fracure. The 3D Finite Element discretization of the specimen is made by the use of four node tetrahedron elements with one integration point. In order to perform the computations in 3D at reasonable cost, a FE mesh with a 5 mm width is used when the real width of the specimen is 50 mm. The mesh, the boundary conditions as well as the loading specifications are presented in the figure 6.

The model parameters used for the simulation are those of section 2 for concrete: E = 42000 MPa, $\nu = 0.2$, $\kappa_0 = 5.10^{-5}$, $A = 5.10^{3}$, $a = 2.9310^{-4}$.

In order to avoid any spurious mesh dependency due to strain localisation, nonlocal computations have been performed on a medium mesh. The characteristic length is set to 2 mm for all the analyse. The convergence of the computations has been numerically shown using three types of mesh in a previous work (Desmorat *et al.* 2007). Two types of computations are performed with the two models giving the most different responses in bi-compression and in shear. The first one uses the original anisotropic model, i.e. with the damage threshold based on Mazars equivalent strain. The second one uses the de Vree equivalent strain. The results are given in the following using the medium mesh. The comparisons between the experimental crack pattern and the anisotropic D_{11} and D_{22} damage fields are plotted in the figure 7. Whatever the criterion, the damage fields obtained are quite close so that only the results obtained with the original Mazars equivalent strain are plotted in figure 7.

One can observe that the crack path is well suited by the numerical computations with the non-symmetric boundary conditions of figure 6. The difference between the two criteria is emphasized in figure 8, for which the results obtained with the two different equivalent strains are plotted in the tensile load – vertical displacement diagram. The ductility as well as the peak displacement is better simulated using the de Vree equivalent strain. Although it allows for a better response modelling in the biaxial regime, the de Vree criterion overestimates the shear response. This point may explain the overload registered in the numerical computations in figure 8.

CONCLUSIONS

Dealing with multiaxial behaviour of brittle materials like concrete needs to account for micro-cracks orientations. A 3D robust model introducing damage induced anisotropy is presented in this paper, as well as its stresses computation numerical scheme. Important feature, even if Euler backward scheme is used no iterations are needed at the Gauss point level. The model only needs the consideration of a single thermodynamics tensorial variable and the identification of 6 material parameters (2 for elasticity, one damage threshold, 2 for damage growth and a new parameter k for both the shear and the biaxial response) to handle the non-symmetric response of concrete in tension and in compression. Benchmarking the modelling responses under biaxial compression state of stress emphasizes the lack in ductility encountered by models based on continuum damage mechanics, whatever the damage kinematics, scalar or tensorial. The modifications of the damage threshold and the definition of new equivalent strains improve the multiaxial response. The elasticity domain, defined thanks to the different equivalent strains and the high level of modularity of the model here exposed allows for 'parametric' analyses of different yield surfaces. The equivalent strain, by the addition of strain tensor invariants, is modified to account for a better asymmetry in the elastic response of the material in tension and in bi-compression. Equivalent strain, based on a modified non-centered von Mises criterion proves its efficiency to deal with the bi-compression state of stresses. When applied to a structural case study of a concrete sample subject to both shear and tension, this equivalent strain, coupled to the anisotropic model predicts an overestimation of the specimen load bearing capacity, due to the local shear ductility increase.

Some intermediate solutions should be retained in the future to improve the numerical responses of continuous anisotropic damage based models. For example, the permanent strains, as well as the confinement effects induced by dilatancy help the model to catch the ductility increase in confined cases without penalising the shear response. But in any case, the definition of an equivalent strain is the key-feature for 3D modelling.

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LIST OF FIGURES

(a) (b)

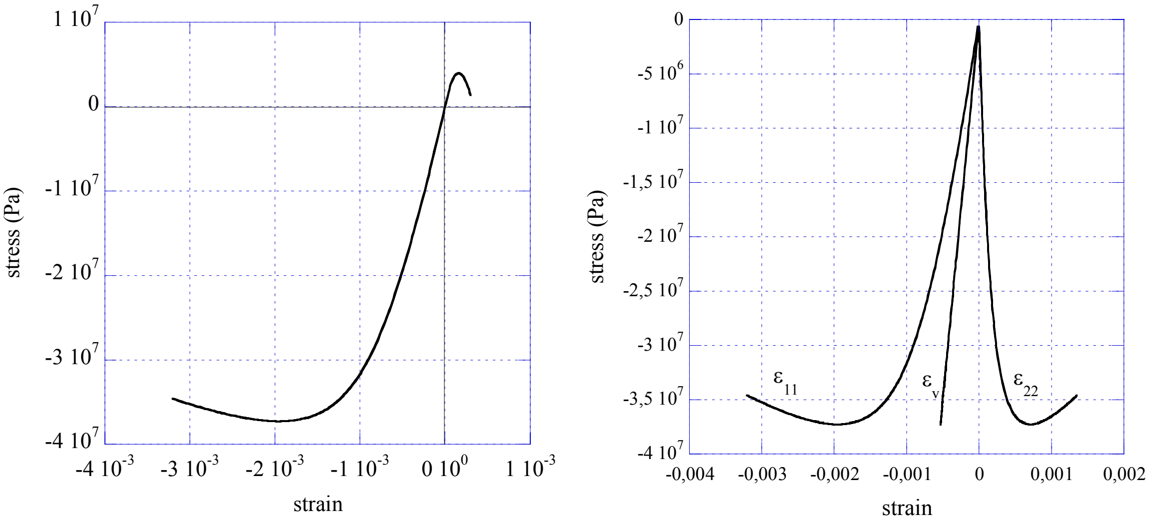

Fig 1 : Uniaxial response of the anisotropic damage model. (a) Tension/compression non symmetric feature and (b) volumic behaviour in compression ($\varepsilon_V = Tr\boldsymbol{\varepsilon}$).

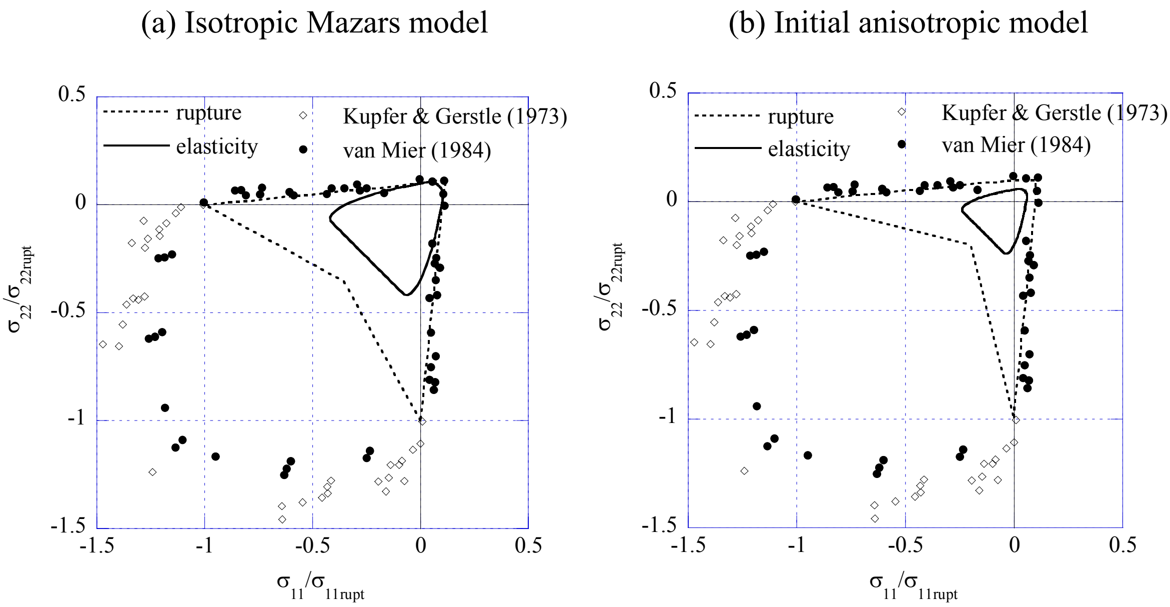

Fig 2. Elasticity domain and ultimate state at rupture for (a) the isotropic model (Mazars 1984) and (b) the initial anisotropic model using the Mazars equivalent strain.

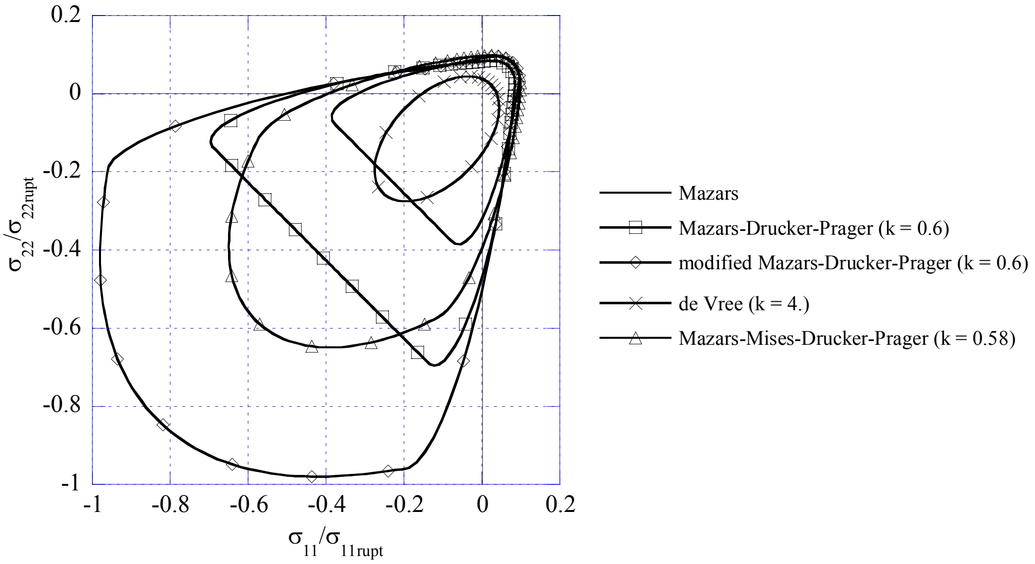

Fig 3. Elasticity domains comparisons for the different equivalent strains. The *k* parameter has been adjusted to obtain for each model the same responses in tension and in compression as in figure 1.

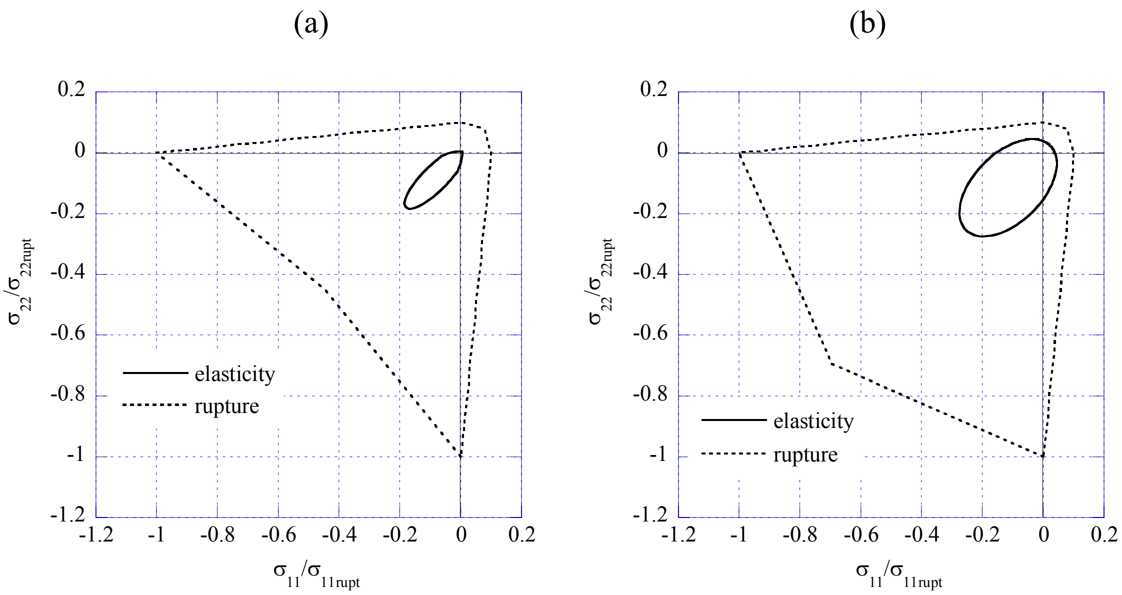

Fig 4. Elasticity domain and ultimate state at rupture for (a) the anisotropic damage model using Mazars-Mises-Drucker-Prager equivalent strain and (b) the de Vree Criterion.

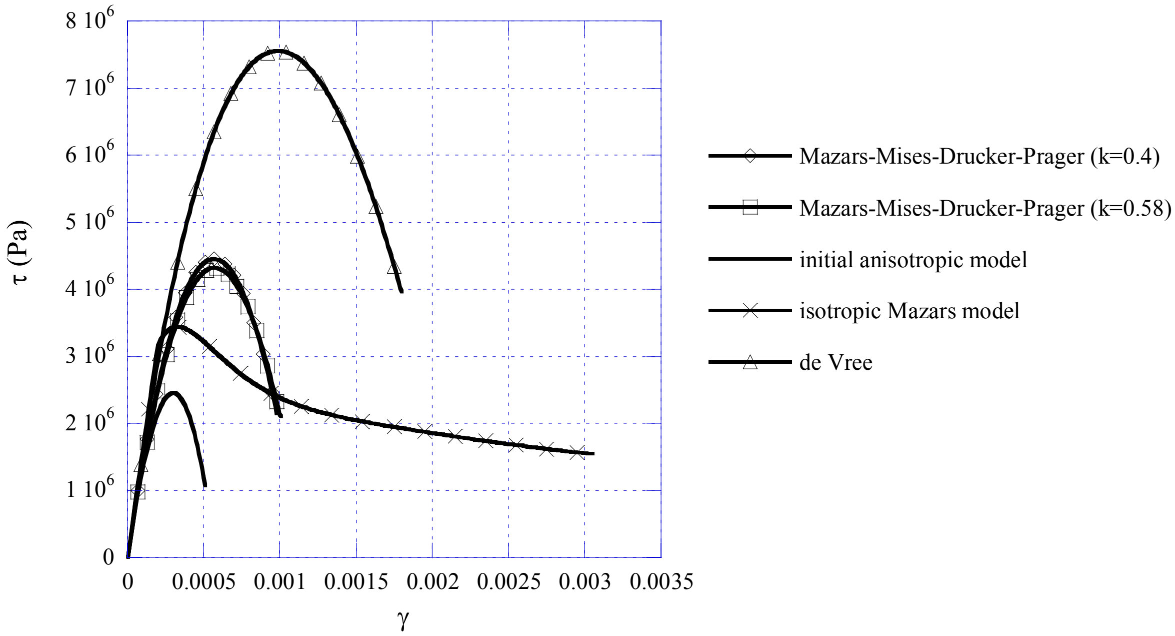

Fig 5. Shear response for the isotropic and the anisotropic damage models using the different expressions for the equivalent strains.

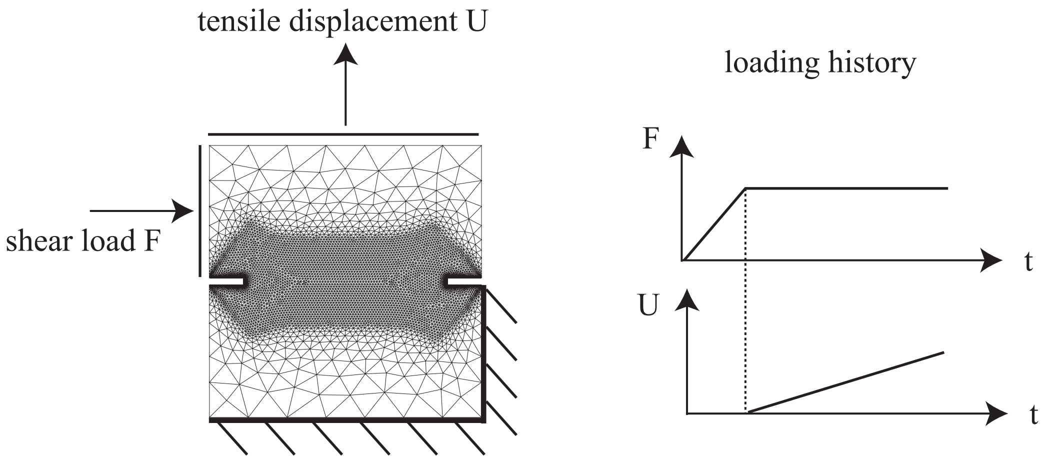

Fig 6. Nooru-Mohamed test. Mesh, boundary conditions and loading history

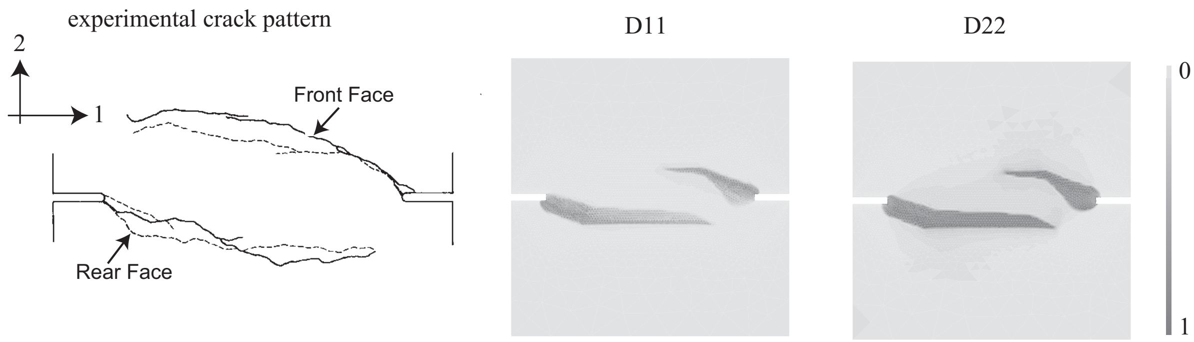

Fig 7. Nooru-Mohamed test. Experimental crack pattern. Damage fields in the direction 1 (horizontal) and 2 (vertical).

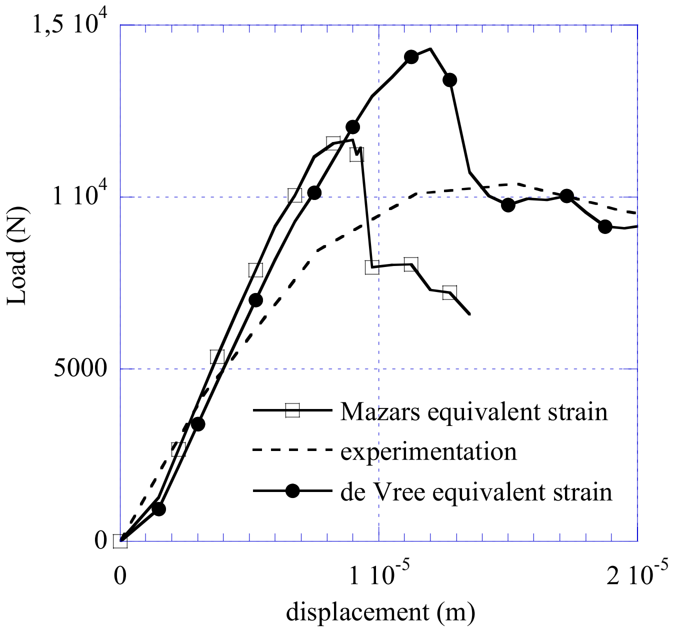

Fig 8. Nooru-Mohamed test. Comparisons of two types of damage threshold during the loading.