Subexponentially increasing sums of partial quotients in continued fraction expansions

Lingmin Liao, Michal Rams

SUBEXPONENTIALLY INCREASING SUMS OF PARTIAL QUOTIENTS IN CONTINUED FRACTION EXPANSIONS

LINGMIN LIAO AND MICHAŁ RAMS

Abstract. We investigate from a multifractal analysis point of view the increasing rate of the sums of partial quotients $S_n(x) = \sum_{j=1}^n a_j(x)$, where $x = [a_1(x), a_2(x), \cdots]$ is the continued fraction expansion of an irrational $x \in (0,1)$. Precisely, for an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, one is interested in the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

Several cases are solved by Iommi and Jordan, Wu and Xu, and Xu. We attack the remaining subexponential case $\exp(n^\gamma)$, $\gamma \in [1/2, 1)$. We show that when $\gamma \in [1/2, 1)$, E_φ has Hausdorff dimension $1/2$. Thus, surprisingly, the dimension has a jump from 1 to $1/2$ at $\varphi(n) = \exp(n^{1/2})$. In a similar way, the distribution of the largest partial quotient is also studied.

1. Introduction

Each irrational number $x \in [0,1)$ admits a unique infinite continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}, \tag{1.1}$$

where the positive integers $a_n(x)$ are called the partial quotients of x. Usually, (1.1) is written as $x = [a_1, a_2, \cdots]$ for simplicity. The n-th finite truncation of (1.1): $p_n(x)/q_n(x) = [a_1, \cdots, a_n]$ is called the n-th convergent of x. The continued fraction expansions can be induced by the Gauss transformation $T : [0,1) \to [0,1)$ defined by

$$T(0) := 0, \text{ and } T(x) := \frac{1}{x} \pmod 1, \text{ for } x \in (0,1).$$

It is well known that $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integer part) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$.

For any $n \geq 1$, we denote by $S_n(x) = \sum_{j=1}^n a_j(x)$ the sum of the n first partial quotients. It was proved by Khintchine [5] in 1935 that $S_n(x)/(n \log n)$ converges in measure (Lebesgue measure) to the constant $1/\log 2$. In 1988, Philipp [7] showed that there is no reasonable normalizing sequence $\varphi(n)$

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such that a strong law of large numbers is satisfied, i.e., $S_n(x)/\varphi(n)$ will never converge to a positive constant almost surely.

From the point of view of multifractal analysis, one considers the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

where $\varphi : \mathbb{N} \to \mathbb{N}$ is an increasing function.

The case $\varphi(n) = \gamma n$ with $\gamma \in [1, \infty)$ was studied by Iommi and Jordan [3]. It is proved that with respect to γ, the Hausdorff dimension (denoted by $\dim_H$) of E_φ is analytic, increasing from 0 to 1, and tends to 1 when γ goes to infinity. In [9], Wu and Xu proved that if $\varphi(n) = n^\gamma$ with $\gamma \in (1, \infty)$ or $\varphi(n) = \exp(n^\gamma)$ with $\gamma \in (0, 1/2)$, then $\dim_H E_\varphi = 1$. Later, it was shown by Xu [10], that if $\varphi(n) = \exp(n)$ then $\dim_H E_\varphi = 1/2$ and if $\varphi(n) = \exp(\gamma^n)$ with $\gamma > 1$ then $\dim_H E_\varphi = 1/(\gamma + 1)$. The same proofs of [10] also imply that for $\varphi(n) = \exp(n^\gamma)$ with $\gamma \in (1, \infty)$ the Hausdorff dimension $\dim_H E_\varphi$ stays at 1/2. So, only the subexponentially increasing case: $\varphi(n) = \exp(n^\gamma), \gamma \in [1/2, 1)$ was left unknown. In this paper, we fill this gap.

Theorem 1.1. *Let $\varphi(n) = \exp(n^\gamma)$ with $\gamma \in [1/2, 1)$. Then*

$$\dim_H E_\varphi = \frac{1}{2}.$$

We also show that there exists a jump of the Hausdorff dimension of E_φ between $\varphi(n) = \exp(n^{1/2})$ and slightly slower growing functions, for example $\varphi(n) = \exp(\sqrt{n}(\log n)^{-1})$.

Theorem 1.2. *Let $\varphi(n) = \exp(\sqrt{n} \cdot \psi(n))$ be an increasing function with ψ being a $\mathcal{C}^1$ positive function on $\mathbb{R}_+$ satisfying*

$$\lim_{x\to\infty} \frac{\sup_{y\ge x} \psi(y)^2}{\psi(x)} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{x\psi'(x)}{\psi(x)} = 0. \tag{1.2}$$

Then

$$\dim_H E_\varphi = 1.$$

We remark that the assumption (1.2) on the function ψ says that ψ decreases to 0 slower than any polynomial. We also remark that when ψ is decreasing, then the first condition of (1.2) is automatically satisfied.

Theorems 1.1 and 1.2 show that, surprisingly, there is a jump of the Hausdorff dimensions from 1 to 1/2 in the class $\varphi(n) = \exp(n^\gamma)$ at $\gamma = 1/2$ and that this jump cannot be easily removed by considering another class of functions. See Figure 1 for an illustration of the jump of the Hausdorff dimension.

By the same method, we also prove some similar results on the distribution of the largest partial quotient in continued fraction expansions. For $x \in [0, 1) \setminus \mathbb{Q}$, define

$$T_n(x) := \max\{a_k(x) : \ 1 \le k \le n\}.$$

One is interested in the following lower limit:

$$T(x) := \liminf_{n\to\infty} \frac{T_n(x) \log\log n}{n}.$$

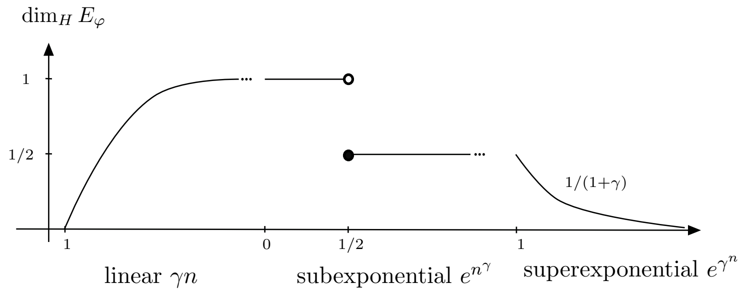

FIGURE 1. $\dim_H E_\varphi$ for different φ.

It was conjectured by Erdös that almost surely $T(x) = 1$. However, it was proved by Philipp [6] that for almost all x, one has $T(x) = 1/\log 2$. Recently, Wu and Xu [8] showed that

$$\forall \alpha \geq 0, \quad \dim_H \left\{ x \in [0,1) \setminus \mathbb{Q} : \lim_{n\to\infty} \frac{T_n(x) \log\log n}{n} = \alpha \right\} = 1.$$

They also proved that if the denominator n is replaced by any polynomial the same result holds. In this paper, we show the following theorem.

Theorem 1.3. *For all $\alpha > 0$,*

$$F(\gamma, \alpha) = \left\{ x \in [0,1) \setminus \mathbb{Q} : \lim_{n\to\infty} T_n(x)/\exp(n^\gamma) = \alpha \right\}$$

satisfies

$$\dim_H F(\gamma,\alpha) = \begin{cases} 1 & \text{if } \gamma \in (0, 1/2) \\ \frac{1}{2} & \text{if } \gamma \in (1/2, \infty). \end{cases}$$

We do not know what happens in the case $\gamma = 1/2$.

2. Preliminaries

For any $a_1, a_2, \cdots, a_n \in \mathbb{N}$, call

$$I_n(a_1, \cdots, a_n) := \{x \in [0,1) : a_1(x) = a_1, \cdots, a_n(x) = a_n\}$$

a *rank-n basic interval.* Denote by $I_n(x)$ the rank-n basic interval containing x. Write $|I|$ for the length of an interval I. The length of the basic interval $I_n(a_1, a_2, \cdots, a_n)$ satisfies

$$\prod_{k=1}^{n} (a_k + 1)^{-2} \leq \Big| I_n(a_1, \cdots, a_n) \Big| \leq \prod_{k=1}^{n} a_k^{-2}. \tag{2.1}$$

Let $A(m,n) := \left\{ (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n : \sum_{k=1}^n i_k = m \right\}$. Let $\zeta(\cdot)$ be the Riemann zeta function.

Lemma 2.1. *For any $s \in (1/2, 1)$, for all $n \geq 1$ and for all $m \geq n$, we have*

$$\sum_{(i_1,\ldots,i_n)\in A(m,n)} \prod_{k=1}^{n} i_k^{-2s} \leq \left(\frac{9}{2} \big(2 + \zeta(2s)\big) \right)^n m^{-2s}.$$

Proof. The proof goes by induction. First consider the case $n = 2$. For $m = 2$ the assertion holds, assume that $m > 2$. We will estimate the sum $\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s}$. For any $u \in [1, m/2]$ we have

$$\begin{aligned}
\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} &= 2\sum_{i=1}^{u-1} i^{-2s}(m-i)^{-2s} + \sum_{i=u}^{m-u} i^{-2s}(m-i)^{-2s} \\
&\le 2\Big(\sum_{i=1}^{u-1} i^{-2s}\Big)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s} \\
&\le 2\zeta(2s)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s}.
\end{aligned}$$

Take $u = \lfloor m/3 \rfloor$. Then one has

$$(m-2u+1)u^{-2s} = (m+1)u^{-2s} - 2u^{1-2s} \le (m+1)\lfloor \frac{m}{3} \rfloor^{-2s} - 2 \le 4.$$

Hence, the above sum is bounded from above by

$$(4 + 2\zeta(2s)) \cdot \big(\frac{2m}{3}\big)^{-2s} \le \frac{9}{2}(2 + \zeta(2s)) \cdot m^{-2s}.$$

Suppose now that the assertion holds for $n \in \{2, n_0\}$. Then for $n = n_0+1$, we have

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{n_0+1})\in\{1,\dots,m\}^{n_0+1},\ \sum i_k = m} \ \prod_{k=1}^{n_0+1} i_k^{-2s} \\
&= \sum_{i=1}^{m-1} i^{-2s} \sum_{(i_1,\dots,i_{n_0})\in\{1,\dots,m\}^{n_0},\ \sum i_k = m-i} \ \prod_{k=1}^{n_0} i_k^{-2s} \\
&\le \sum_{i=1}^{m-1} i^{-2s} \left(\frac{9}{2}(2+\zeta(2s))\right)^{n_0} (m-i)^{-2s} \\
&= \left(\frac{9}{2}(2+\zeta(2s))\right)^{n_0} \cdot \sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} \\
&\le \left(\frac{9}{2}(2+\zeta(2s))\right)^{n_0} \cdot \left(\frac{9}{2}(2+\zeta(2s))\right) m^{-2s} \\
&= \left(\frac{9}{2}(2+\zeta(2s))\right)^{n_0+1} m^{-2s}.
\end{aligned}$$

□

Let

$$A(\gamma, c_1, c_2, N) := \left\{ x \in (0,1) : c_1 < \frac{a_n(x)}{e^{n^\gamma}} < c_2,\ \forall n \ge N \right\}.$$

Denote by N_0 the smallest integer n such that $(c_2 - c_1) \cdot e^{n^\gamma} > 1$. Then the set $A(\gamma, c_1, c_2, N)$ is non-empty when $N \ge N_0$.

Lemma 2.2. *For any $\gamma > 0$, any $N \ge N_0$ and any $0 < c_1 < c_2$,*

$$\dim_H A(\gamma, c_1, c_2, N) = \frac{1}{2}.$$

Proof. This lemma is only a simple special case of [2, Lemma 3.2], but we will sketch the proof (based on [4]), needed for the next lemma. Without loss of generality, we suppose $N_0 = 1$ and let $N = 1$ (the proof for other N is almost identical).

Let $a_1, a_2, \ldots, a_n$ satisfy $c_1 < a_j e^{-j^\gamma} < c_2$ for all j. Those are exactly the possible sequences for which the basic interval $I_n(a_1, \ldots, a_n)$ has nonempty intersection with $A(\gamma, c_1, c_2, 1)$.

There are approximately

$$\prod_{j=1}^{n} (c_2 - c_1) e^{j^\gamma} \approx e^{\sum_1^n j^\gamma} \tag{2.2}$$

of such basic intervals, each of diameter

$$|I_n(a_1, \ldots, a_n)| \approx e^{-2\sum_1^n j^\gamma}, \tag{2.3}$$

(both estimations are up to a factor exponential in n). Hence, by using the intervals $\{I_n(a_1, \ldots, a_n)\}$ as a cover, we obtain

$$\dim_H A(\gamma, c_1, c_2, 1) \leq \frac{1}{2}.$$

To get the lower bound, we consider a probability measure μ uniformly distributed on $A(\gamma, c_1, c_2, 1)$, in the following sense: given $a_1, \ldots, a_{n-1}$, the probability of a_n taking any particular value between $c_1 e^{n^\gamma}$ and $c_2 e^{n^\gamma}$ is the same.

The basic intervals $I_n(a_1, \ldots, a_n)$ have, up to a factor c^n, the length $\exp(-2\sum_1^n j^\gamma)$ and the measure $\exp(-\sum_1^n j^\gamma)$. They are distributed in clusters: all $I_n(a_1, \ldots, a_n)$ contained in a single $I_n(a_1, \ldots, a_{n-1})$ form an interval of length $\exp(n^\gamma) \cdot \exp(-2\sum_1^n j^\gamma)$ (up to a factor c^n, with c being a constant), then there is a gap, then there is another cluster. Hence, for any $r \in (\exp(-2\sum_1^n j^\gamma),\ \exp(-2\sum_1^{n-1} j^\gamma))$ and any $x \in A(\gamma, c_1, c_2, 1)$ we can estimate the measure of $B(x, r)$:

$$\mu(B(x, r)) \approx \begin{cases} r \cdot e^{-\sum_1^n j^\gamma} & \text{if } r < e^{-2\sum_1^n j^\gamma + n^\gamma} \\ e^{-\sum_1^{n-1} j^\gamma} & \text{if } r > e^{-2\sum_1^n j^\gamma + n^\gamma} \end{cases}$$

(up to a factor c^n). The minimum of $\log \mu(B(x, r))/\log r$ is thus achieved for $r = e^{-2\sum_1^n j^\gamma + n^\gamma}$, and this minimum equals

$$\frac{-\sum_1^{n-1} j^\gamma}{-2\sum_1^n j^\gamma + n^\gamma} \approx \frac{-n^{\gamma+1}/(\gamma+1)}{-2n^{\gamma+1}/(\gamma+1) - n^\gamma} = \frac{1}{2} - O(1/n).$$

Hence, the lower local dimension of μ equals $1/2$ at each point of $A(\gamma, c_1, c_2, 1)$, which implies

$$\dim_H A(\gamma, c_1, c_2, 1) \geq \frac{1}{2}$$

by the Frostman Lemma (see [1, Principle 4.2]). □

Let now c_1 and c_2 not be constant but depend on n:

$$B(\gamma, c_1, c_2, N) = \left\{ x \in (0,1) : c_1(n) < \frac{a_n(x)}{e^{n^\gamma}} < c_2(n)\ \forall n \geq N \right\}.$$

A slight modification of the proof of Lemma 2.2 gives the following.

Lemma 2.3. *Fix $\gamma > 0$. Assume $0 < c_1(n) < c_2(n)$ for all n. Assume also that*
$$\lim_{n\to\infty} \frac{\log(c_2(n) - c_1(n))}{n^\gamma} = 0$$
and
$$\liminf_{n\to\infty} \frac{\log c_1(n)}{\log n} > -\infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log c_2(n)}{\log n} < +\infty.$$
Then there exists an integer N_1 such that $(c_2(n) - c_1(n)) \cdot e^{n^\gamma} > 1$ for all $n \geq N_1$, and for all $N \geq N_1$,
$$\dim_H B(\gamma, c_1, c_2, N) = 1/2.$$

Proof. We need only to replace the constants c_1 and c_2 by $c_1(n)$ and $c_2(n)$ in the proof of Lemma 2.2. Notice that by the assumptions of Lemma 2.3, the formula (2.2) holds up to a factor $\exp(\varepsilon \sum_1^n j^\gamma)$ for a sufficiently small $\varepsilon > 0$. While the formula (2.3) holds up to a factor $\exp(cn \log n)$ for some bounded c. All these factors are much smaller than the main term $\exp(\sum_1^n j^\gamma)$ which is of order $\exp(n^{1+\gamma})$. The rest of the proof is the same as that of Lemma 2.2. □

3. Proofs

Proof of Theorem 1.1. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be defined by $\varphi(n) = \exp(n^\gamma)$ with $\gamma > 0$. For this case, we will denote E_φ by E_γ.

Let us start from some easy observations, giving (among other things) a simple proof of $\dim_H E_\gamma = 1/2$ for $\gamma \geq 1$.

Consider first $\gamma \geq 1/2$. If $x \in E_\gamma$ then for any $\varepsilon > 0$ and for n large enough
$$(1 - \varepsilon)e^{n^\gamma} \leq S_n(x) \leq (1 + \varepsilon)e^{n^\gamma} \tag{3.1}$$
and
$$(1 - \varepsilon)e^{(n+1)^\gamma} \leq S_{n+1}(x) \leq (1 + \varepsilon)e^{(n+1)^\gamma}.$$
Hence
$$(1 - \varepsilon)e^{(n+1)^\gamma} - (1 + \varepsilon)e^{n^\gamma} \leq a_{n+1}(x) \leq (1 + \varepsilon)e^{(n+1)^\gamma} - (1 - \varepsilon)e^{n^\gamma}.$$
For $\gamma \geq 1$ this implies
$$E_\gamma \subset \bigcup_N A(\gamma, c_1, c_2, N)$$
for some constants c_1, c_2. By Lemma 2.2,
$$\dim_H E_\gamma \leq \frac{1}{2}, \quad \forall \gamma \geq 1.$$

Consider now any $\gamma > 0$. Set
$$c_1(n) = (e^{n^\gamma} - e^{(n-1)^\gamma})e^{-n^\gamma} \quad \text{and} \quad c_2(n) = \frac{n+1}{n} c_1(n).$$
For $\gamma \geq 1$, $c_1(n)$ and $c_2(n)$ are bounded from below. For $\gamma < 1$ and n large, we have
$$(e^{n^\gamma} - e^{(n-1)^\gamma})e^{-n^\gamma} \approx \gamma n^{\gamma - 1}.$$

Thus, in both cases the assumptions of Lemma 2.3 are satisfied. Checking $B(\gamma, c_1, c_2, N) \subset E_\gamma$, we deduce by Lemma 2.3 that

$$\dim_H E_\gamma \geq \frac{1}{2}, \quad \forall \gamma > 0.$$

Therefore, we have obtained $\dim_H E_\gamma = 1/2$ for $\gamma \geq 1$ and $\dim_H E_\gamma \geq 1/2$ for $\gamma > 0$. What is left to prove is that for $\gamma \in [1/2, 1)$ we have $\dim_H E_\gamma \leq 1/2$.

Let us first assume that $\gamma > 1/2$. Remember that if $x \in E_\gamma$, then for any $\varepsilon > 0$ and for n large enough we have (3.1). Take a subsequence $n_0 = 1$, and $n_k = k^{1/\gamma}$ $(k \geq 1)$. Then there exists an integer $N \geq 1$ such that for all $k \geq N$,

$$(1-\varepsilon)e^{n_k^\gamma} \leq S_{n_k}(x) \leq (1+\varepsilon)e^{n_k^\gamma},$$

and (as $\exp(n_k^\gamma) = e^k$)

$$(1-\varepsilon)e^k - (1+\varepsilon)e^{k-1} \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq (1+\varepsilon)e^k - (1-\varepsilon)e^{k-1}.$$

Thus

$$E_\gamma \subset \bigcup_N \bigcap_{k \geq N} A(\gamma, k, N),$$

with $A(\gamma, k, N)$ being the union of the intervals $\{I_{n_k}(a_1, a_2, \cdots, a_{n_k})\}$ such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \leq \ell \leq k,$$

where $D_\ell := [(1-\varepsilon)e^{n_\ell^\gamma} - (1+\varepsilon)e^{n_\ell^\gamma - 1}, (1+\varepsilon)e^{n_\ell^\gamma} - (1-\varepsilon)e^{n_\ell^\gamma - 1}]$.

Now, we are going to estimate the upper bound of the Hausdorff dimension of $E_\varphi^{(1)} = \bigcap_k A(\gamma, k, 1)$. For $E_\varphi^{(N)} = \bigcap_{k \geq N} A(\gamma, k, N)$ with $N \geq 2$ we have the same bound and the proofs are almost the same.

Observe that every set $A(\gamma, k, N)$ has a product structure: the conditions on a_i for $i \in (n_{\ell_1}, n_{\ell_1+1}]$ and for $i \in (n_{\ell_2}, n_{\ell_2+1}]$ are independent from each other. Hence, for any $s \in (1/2, 1)$ we can apply Lemma 2.1 together with the formula

$$|I_{n_k}|^s \leq \prod_{\ell=1}^k (a_{n_{\ell-1}+1} a_{n_{\ell-1}+2} \cdots a_{n_\ell})^{-2s}$$

to obtain

$$\sum_{I_{n_k} \subset A(\gamma,k,1)} |I_{n_k}|^s \leq \prod_{\ell=1}^k \sum_{m \in D_\ell} \left(\frac{9}{2}\big(2 + \zeta(2s)\big)\right)^{n_\ell - n_{\ell-1}} m^{-2s}.$$

Denote $r_1 := 2\varepsilon(1 - e^{-1})$ and $r_2 := (e - 1 - \varepsilon e - \varepsilon)/e$. Then we have $|D_\ell| \leq r_1 e^\ell$ and any $m \in D_\ell$ is not smaller than $r_2 e^\ell$. Thus we get

$$\sum_{I_{n_k} \subset A(\gamma,k,1)} |I_{n_k}|^s \leq \prod_{\ell=1}^k r_1 e^\ell \cdot \left(\frac{9}{2}\big(2 + \zeta(2s)\big)\right)^{\ell^{1/\gamma} - (\ell-1)^{1/\gamma}} \cdot r_2^{2s} e^{-2s\ell}. \tag{3.2}$$

We have $\ell^{1/\gamma} - (\ell - 1)^{1/\gamma} \approx \ell^{1/\gamma - 1}$. As $\gamma > 1/2$, we have $1/\gamma - 1 < 1$, and the main term in the above estimate is $e^{(1-2s)\ell}$. Thus for any $s > 1/2$, the product is uniformly bounded. Thus $\dim_H E_\varphi^{(1)} \leq 1/2$.

If $\gamma = 1/2$, we take $n_k = k^2/L^2$ with L being a constant and we repeat the same argument. Observe that now $\exp(n_k^\gamma) = e^{k/L}$. Then the same estimation will lead to

$$\sum_{I_{n_k} \subset A(\gamma,k,1)} |I_{n_k}|^s \leq \prod_{\ell=1}^{k} r_1 r_2^{2s} \cdot \left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{\frac{\ell^2-(\ell-1)^2}{L^2}} e^{(1-2s)\ell/L}. \tag{3.3}$$

The main term of the right side of the above inequality should be

$$\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{2\ell/L^2} \cdot e^{(1-2s)\ell/L}.$$

We solve the equation

$$\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{2/L^2} \cdot e^{(1-2s)/L} = 1,$$

which is equivalent to

$$\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right) = e^{\frac{2s-1}{2}L}. \tag{3.4}$$

Observe that the graphs of the two sides of (3.4) (as functions of the variable s) always have a unique intersection for some $s_L \in [1/2, 1]$, when L is large enough. These s_L are upper bounds for the Hausdorff dimension of $E_\varphi^{(1)}$. Notice that the intersecting point $s_L \to 1/2$ as $L \to \infty$ since the zeta function ζ has a pole at 1. Thus the dimension of $E_\varphi^{(1)}$ is not greater than 1/2.

So, in both cases, we have obtained $\dim_H E_\gamma \leq 1/2$. □

Sketch proof of Theorem 1.2. The proof goes like Section 4 of [9] with the following changes. We choose $\varepsilon_k = \psi(k)$. Let n_1 be such that $\varphi(n_1) \geq 1$ and define n_k as the smallest positive integer such that

$$\varphi(n_k) \geq (1+\varepsilon_{k-1})\varphi(n_{k-1}). \tag{3.5}$$

For a large enough integer M, set

$$\begin{aligned} E_M(\varphi) := \Big\{ & x \in [0,1) : a_{n_1}(x) = \lfloor (1+\varepsilon_1)\varphi(n_1) \rfloor + 1, \\ & a_{n_k}(x) = \lfloor (1+\varepsilon_k)\varphi(n_k) \rfloor - \lfloor (1+\varepsilon_{k-1})\varphi(n_{k-1}) \rfloor + 1 \text{ for all } k \geq 2, \\ & \text{and } 1 \leq a_i(x) \leq M \text{ for } i \neq n_k \text{ for any } k \geq 1 \Big\}. \end{aligned}$$

We can check that $E_M(\varphi) \subset E_\varphi$.

To prove $\dim_H E_\varphi = 1$, for any $\varepsilon > 0$, we construct a $(1/(1+\varepsilon))$-Lipschitz map from $E_M(\varphi)$ to E_M, the set of numbers with partial quotients less than some M in its continued fraction expansion. The theorem will be proved by letting $\varepsilon \to 0$ and $M \to \infty$.

Such a Lipschitz map can be constructed by send a point x in $E_M(\varphi)$ to a point $\tilde{x}$ by deleting all the partial quotients a_{n_k} in its continued fraction expansion. Define $r(n) := \min\{k : n_k \leq n\}$. The $(1/(1+\varepsilon))$-Lipschitz property will be assured if

$$\lim_{n\to\infty} \frac{r(n)}{n} = 0, \tag{3.6}$$

and

$$\lim_{n\to\infty} \frac{\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}})}{n} = 0. \tag{3.7}$$

In fact, by (1.2), we can check for any $\delta > 0$, $\psi(n) \le n^{\delta}$ for n large enough. Thus by definition of n_k, we can deduce that $r(n) \le n^{1/2+\delta}$. Hence (3.6) is satisfied.

Further, we have

$$\sum_{k=1}^{r(n)} \varepsilon_k \approx r(n)\psi(r(n)). \tag{3.8}$$

By (3.5)

$$\varphi(n) \ge \varphi(n_{r(n)}) \ge \prod_{k=1}^{r(n)-1} (1+\varepsilon_k)\varphi(n_1) \ge e^{\sum_{k=1}^{r(n)} \varepsilon_k/2}\varphi(n_1).$$

Thus (3.8) implies

$$r(n)\psi(r(n)) \ll \sqrt{n}\psi(n), \tag{3.9}$$

where $a_n \ll b_n$ means that a_n/b_n is bounded by some constant when $n \to \infty$.

On the other hand, by (2.1) and (3.5), we have

$$\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}}) \le r(n)\log(2\varphi(n)) + \sum_{k=1}^{r(n)} \varepsilon_k.$$

Hence (3.8) and (3.9) give

$$\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}}) \ll r(n)\sqrt{n}\psi(n) + r(n)\psi(r(n)) \ll \frac{n\psi^2(n)}{\psi(r(n))} + r(n).$$

Finally, (3.7) follows from the assumption (1.2) and the already proved formula (3.6). □

Proof of Theorem 1.3. For the case $\gamma < 1/2$, the set constructed in Section 4 of [9] (as a subset of the set of points for which $S_n(x) \approx e^{n^\gamma}$) satisfies also $T_n(x) \approx e^{n^\gamma}$ and has Hausdorff dimension one. We proceed to the case $\gamma > 1/2$.

The lower bound is a corollary of Lemma 2.3. Take $c_1(n) = \alpha(1-\frac{1}{n})$ and $c_2(n) = \alpha$. Let N_1 be the smallest integer n such that $\frac{\alpha}{n}e^{n^\gamma} > 1$. Then the conditions of Lemma 2.3 are satisfied, and for all points x such that $c_1(n)e^{n^\gamma} < a_n(x) < c_2(n)e^{n^\gamma}$, we have

$$T_n(x)/e^{n^\gamma} \ge c_1(n) = \alpha\left(1-\frac{1}{n}\right),$$

and

$$T_n(x)/e^{n^\gamma} = a_k/e^{n^\gamma} \le \alpha e^{k^\gamma}/e^{n^\gamma} \le \alpha,$$

where $k \le n$ is the position at which the sequence $a_1, \ldots, a_n$ achieves a maximum. Thus for all $x \in B(\gamma, c_1, c_2, N_1)$

$$\lim_{n\to\infty} T_n(x)/e^{n^\gamma} = \alpha.$$

Hence, $B(\gamma, c_1, c_2, N_1) \subset F(\gamma, \alpha)$ and the lower bound follows directly from Lemma 2.3.

The upper bound is a modification of that of Theorem 1.1. We consider the case $\alpha = 1$ only, since for other $\alpha > 0$, the proofs are similar.

Notice that for any $\varepsilon > 0$, if $x \in F(\gamma, 1)$, then for n large enough,

$$(1-\varepsilon)e^{n^\gamma} \le S_n(x) \le n(1+\varepsilon)e^{n^\gamma}.$$

Take a subsequence $n_k = k^{1/\gamma}(\log k)^{1/\gamma^2}$. Then

$$(1-\varepsilon)e^{k(\log k)^{1/\gamma}} \le S_{n_k}(x) \le k^{1/\gamma}(\log k)^{1/\gamma^2}(1+\varepsilon)e^{k(\log k)^{1/\gamma}},$$

and

$$u_k \le S_{n_k}(x) - S_{n_{k-1}}(x) \le v_k,$$

with

$$u_k := (1-\varepsilon)e^{k(\log k)^{1/\gamma}} - (k-1)^{1/\gamma}(\log(k-1))^{1/\gamma^2}(1+\varepsilon)e^{(k-1)(\log(k-1))^{1/\gamma}},$$

and

$$v_k := k^{1/\gamma}(\log k)^{1/\gamma^2}(1+\varepsilon)e^{k(\log k)^{1/\gamma}} - (1-\varepsilon)e^{(k-1)(\log(k-1))^{1/\gamma}}.$$

We remark that

$$u_k > \frac{1}{2}e^{k(\log k)^{1/\gamma}}, \quad v_k < \frac{3}{2}k^{1/\gamma}(\log k)^{1/\gamma^2}e^{k(\log k)^{1/\gamma}} \tag{3.10}$$

when k is large enough.

Observe that

$$F(\gamma, 1) \subset \bigcup_N B(\gamma, N),$$

with $B(\gamma, N)$ being the union of the intervals $\{I_{n_k}(a_1, a_2, \cdots, a_{n_k})\}_{k \ge N}$ such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \le \ell \le k,$$

where D_ℓ is the set of integers in the interval $[u_\ell, v_\ell]$.

As in the proof of Theorem 1.1, we need only study the set $B(\gamma, 1)$. For any $s \in (1/2, 1)$, since

$$|I_{n_k}|^s \le \prod_{\ell=1}^{k} (a_{n_{\ell-1}+1} a_{n_{\ell-1}+2} \cdots a_{n_\ell})^{-2s},$$

by Lemma 2.1,

$$\sum_{I_{n_k} \subset B(\gamma, N)} |I_{n_k}|^s \le \prod_{\ell=1}^{k} \sum_{m \in D_\ell} \left(\frac{9}{2}(2 + \zeta(2s))\right)^{n_\ell - n_{\ell-1}} m^{-2s}.$$

Note that by (3.10) the number of integers in D_ℓ satisfies

$$|D_\ell| \le v_\ell - u_\ell \le v_\ell < \frac{3}{2} \cdot \ell^{1/\gamma}(\log \ell)^{1/\gamma^2}.$$

By (3.10), we also have

$$m \ge u_\ell > \frac{1}{2}e^{\ell(\log \ell)^{1/\gamma}} \quad \text{for any } m \in D_\ell.$$

Similar to (3.2) and (3.3), we deduce that $\sum_{I_{n_k}\subset B(\gamma,N)} |I_{n_k}|^s$ is less than

$$\prod_{\ell=1}^{k} \frac{3}{2} \cdot \ell^{1/\gamma}(\log \ell)^{1/\gamma^2} e^{\ell(\log \ell)^{1/\gamma}} \left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_\ell - n_{\ell-1}} 2^{2s} e^{-2s\ell(\log \ell)^{1/\gamma}}.$$

Since $n_\ell - n_{\ell-1} \approx \ell^{1/\gamma - 1 + o(\varepsilon)}$ and $1/\gamma - 1 < 1$, the main term in the above estimation is $e^{(1-2s)\ell(\log \ell)^{1/\gamma}}$. Thus for any $s > 1/2$ the product is uniformly bounded and we have the Hausdorff dimension of $B(\gamma, 1)$ is not greater than $1/2$. Then we can conclude $\dim_H F(\gamma, 1) \leq 1/2$ and the proof is completed. □

4. Generalizations

In this section we consider after [4] certain infinite iterated function systems that are natural generalizations of the Gauss map. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to [0,1]$ be C^1 maps such that

(1) there exists $m \in \mathbb{N}$ and $0 < A < 1$ such that for all $(a_1, ..., a_m) \in \mathbb{N}^m$ and for all $x \in [0,1]$
$$0 < |(f_{a_1} \circ \cdots \circ f_{a_m})'(x)| \leq A < 1,$$
(2) for any $i, j \in \mathbb{N}$ $\; f_i((0,1)) \cap f_j((0,1)) = \emptyset$,
(3) there exists $d > 1$ such that for any $\varepsilon > 0$ there exist $C_1(\varepsilon), C_2(\varepsilon) > 0$ such that for $i \in \mathbb{N}$ there exist constants ξ_i, λ_i such that for all $x \in [0,1]$ $\; \xi_i \leq |f_i'(x)| \leq \lambda_i$ and
$$\frac{C_1}{i^{d+\varepsilon}} \leq \xi_i \leq \lambda_i \leq \frac{C_2}{i^{d-\varepsilon}}.$$

We will call such an iterated function system a *d-decaying system.* It will be further called *Gauss like* if

$$\bigcup_{i=1}^{\infty} f_i([0,1]) = [0,1)$$

and if for all $x \in [0,1]$ we have that $f_i(x) < f_j(x)$ implies $i < j$.

We have a natural projection $\Pi : \mathbb{N}^{\mathbb{N}} \to [0,1]$ defined by

$$\Pi(\underline{a}) = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1),$$

which gives for any point $x \in [0,1]$ its symbolic expansion $(a_1(x), a_2(x), \ldots)$. This expansion is not uniquely defined, but there are only countably many points with more than one symbolic expansions.

For a d-decaying Gauss like system we consider $S_n(x) = \sum_1^n a_i(x)$. Given an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ we denote

$$E_d(\varphi) = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

Theorem 4.1. *Let $\{f_i\}$ be a d-decaying Gauss like system. We have*

i) *if $\varphi(n) = e^{n^\gamma}$ with $\gamma < 1/d$,*
$$\dim_H E_d(\varphi) = 1,$$

ii) *if* $\varphi(n) = e^{n^\gamma}$ *with* $\gamma > 1/d$,

$$\dim_H E_d(\varphi) = \frac{1}{d},$$

iii) *if* $\varphi(n) = e^{\gamma^n}$ *with* $\gamma > 1$,

$$\dim_H E_d(\varphi) = \frac{1}{\gamma + d - 1}.$$

The proofs (both from Section 3 and from [9, 10]) go through without significant changes.

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Lingmin Liao, LAMA UMR 8050, CNRS, Université Paris-Est Créteil, 61 Avenue du Général de Gaulle, 94010 Créteil Cedex, France
E-mail address: lingmin.liao@u-pec.fr

Michał Rams, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail address: rams@impan.pl