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A new function space and applications

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Abstract

We define a new function space $B$, which contains in particular BMO, BV, and $W^{1/p,p}$, $1 < p < \infty$. We investigate its embedding into Lebesgue and Marcinkiewicz spaces. We present several inequalities involving $L^p$ norms of integer-valued functions in $B$. We introduce a significant closed subspace, $B_0$, of $B$, containing in particular VMO and $W^{1/p,p}$, $1 \leq p < \infty$. The above mentioned estimates imply in particular that integer-valued functions belonging to $B_0$ are necessarily constant. This framework provides a “common roof” to various, seemingly unrelated, statements asserting that integer-valued functions satisfying some kind of regularity condition must be constant.

1 Introduction

Let $\Omega$ be a connected domain in $\mathbb{R}^n$. Recall that if $f : \Omega \to \mathbb{Z}$ is a measurable function which satisfies one of the following regularity properties:

1. $f \in \text{VMO} (\Omega)$;
2. $f \in W^{1,1}(\Omega)$;
3. $f \in W^{1/p,p}(\Omega)$, with $1 < p < \infty$,

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then \( f \) is constant \([3, \text{Comment 2, pp. 223-224}], [2, \text{Theorem B.1}]\). The original motivation for this article was to provide a “common roof” to all these cases, and which yields in particular the following

**Theorem 1.** Assume that \( f : \Omega \to \mathbb{Z} \) is measurable and can be written as \( f = f_1 + f_2 + f_3 \), with \( f_1 \in \text{VMO}(\Omega; \mathbb{R}) \), \( f_2 \in W^{1,1}(\Omega; \mathbb{R}) \) and \( f_3 \in W^{1/p,p}(\Omega; \mathbb{R}) \) for some \( 1 < p < \infty \). Then \( f \) is constant.

The proof of Theorem 1 relies heavily on the introduction of a new space of functions, which might be of interest well beyond the scope of Theorem 1.

In what follows we denote by \( Q \) the unit cube \((0,1)^n\). For \( 0 < \varepsilon < 1 \), \( Q_\varepsilon(a) \) is the \( \varepsilon \)-cube centered at \( a \).

Given \( f \in L^1(Q; \mathbb{R}) \) and an \( \varepsilon \)-cube \( Q_\varepsilon \subset Q \), we set

\[
M(f, Q_\varepsilon) = \int_{Q_\varepsilon} |f - f_{Q_\varepsilon}|, \text{ where } f_{Q_\varepsilon} = \int_{Q_\varepsilon} f,
\]

and

\[
M^*(f, Q_\varepsilon) = \int_{Q_\varepsilon} \int_{Q_\varepsilon} |f(y) - f(z)| dy dz.
\]

Clearly, we have

\[
M(f, Q_\varepsilon) \leq M^*(f, Q_\varepsilon) \leq 2M(f, Q_\varepsilon).
\]

Note that if \( f = \mathbb{1}_A \), with \( A \subset Q \) measurable, then

\[
M(f, Q_\varepsilon) = M^*(f, Q_\varepsilon) = \frac{2|A \cap Q_\varepsilon|(|Q_\varepsilon| - |A \cap Q_\varepsilon|)}{|Q_\varepsilon|^2} \leq \frac{1}{2}.
\]

The following quantity plays an important role:

\[
[f]_\varepsilon = \sup_{\mathcal{F}} \left\{ \varepsilon^{n-1}\sum_{j \in J} M(f, Q_\varepsilon(a_j)) \right\}.
\]

Here, \( \mathcal{F} \) denotes a collection of mutually disjoint \( \varepsilon \)-cubes, \( \mathcal{F} = (Q_\varepsilon(a_j))_{j \in J} \), such that \#\( J \) = cardinality of \( J \leq 1/\varepsilon^{n-1} \) (instead of \#\( J \) we sometimes write \#\( \mathcal{F} \)) and the sup in (5) is taken over all such collections.

We then introduce the space

\[
B = \left\{ f \in L^1(Q; \mathbb{R}); \sup_{0 < \varepsilon < 1} [f]_\varepsilon < \infty \right\},
\]

and the corresponding norm (modulo constants)

\[
\|f\|_B = \sup_{0 < \varepsilon < 1} [f]_\varepsilon.
\]
The definition of $B$ is inspired by the celebrated BMO space of John–Nirenberg [4] equipped with the norm (modulo constants)

$$\|f\|_{\text{BMO}} := \sup_{0 < \varepsilon < 1} \sup_{a \in Q} \{M(f, Q_\varepsilon(a)); Q_\varepsilon(a) \subset Q\}. \quad (7)$$

Here are several examples of functions in $B$.

**Example 1.** $\text{BMO} \subset B$ with continuous injection. Indeed, using (7) we find that $\|f\|_B \leq \|f\|_{\text{BMO}}$.

When $n = 1$, we clearly have $B = \text{BMO}$; however, when $n \geq 2$, $B$ is strictly bigger than BMO (see e.g. Example 2 below).

**Example 2.** $\text{BV} \subset B$ with continuous injection. Indeed, by Poincaré’s inequality

$$\int_{Q_\varepsilon} |f - f_{Q_\varepsilon}| \leq \frac{c_n}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\nabla f|,$$

so that

$$\sum_{j \in J} M(f, Q_\varepsilon(a_j)) \leq \frac{c_n}{\varepsilon^{n-1}} \int_{\bigcup_{j \in J} Q_\varepsilon(a_j)} |\nabla f| \quad (8)$$

and

$$[f]_\varepsilon \leq c_n \int_Q |\nabla f|. \quad (9)$$

**Example 3.** $W^{1/p, p} \subset B$, $1 < p < \infty$, with continuous injection. Indeed, for every fixed $\alpha > 0$ we have

$$\int_{Q_\varepsilon} \int_{Q_\varepsilon} |f(y) - f(z)| dydz \leq n^{\alpha/2} \varepsilon^n \int_{Q_\varepsilon} \int_{Q_\varepsilon} \frac{|f(y) - f(z)|}{|y - z|^\alpha} dydz.$$

Choosing $\alpha = (n + 1)/p$ and applying Hölder’s inequality gives

$$M^*(f, Q_\varepsilon) \leq \frac{c_n}{\varepsilon^{(n-1)/p}} \left[ \int_{Q_\varepsilon} \int_{Q_\varepsilon} \frac{|f(y) - f(z)|}{|y - z|^{n+1}} dydz \right]^{1/p}, \quad \text{with } c_n = n^{(n+1)/2},$$

and since $\# J \leq 1/\varepsilon^{n-1}$ we obtain

$$\varepsilon^{n-1} \sum_{j \in J} M^*(f, Q_\varepsilon(a_j)) \leq c_n \left[ \sum_{j \in J} \int_{Q_\varepsilon(a_j)} \int_{Q_\varepsilon(a_j)} \frac{|f(y) - f(z)|}{|y - z|^{n+1}} dydz \right]^{1/p}. \quad (10)$$

Therefore

$$[f]_\varepsilon \leq c_n \|f\|_{W^{1/p, p}}.$$
An important quantity associated with $B$ is defined by

$$[f] = \lim_{\varepsilon \to 0} [f]_\varepsilon.$$  \hspace{1cm} (11)

The subspace

$$B_0 = \{ f \in B; [f] = 0 \}$$  \hspace{1cm} (12)

plays a key role in this article.

**Example 1’.** $\text{VMO} \subset B_0$.

This is clear, since VMO functions (see [5]) are characterized by

$$\lim_{\varepsilon \to 0} \sup_{a \in Q} \{ M(f, Q_\varepsilon(a)); Q_\varepsilon(a) \subset Q \} = 0.$$  

Moreover, $\text{VMO} = B_0$ when $n = 1$.

**Example 2’.** $W^{1,1} \subset B_0$.

This is clear from (8) and the fact that $|\bigcup_{j \in J} Q_\varepsilon(a_j)| \leq \varepsilon$.

**Example 3’.** $W^{1/p, p} \subset B_0$, $1 < p < \infty$.

This is an immediate consequence of (10) and the fact that $|\bigcup_{j \in J} Q_\varepsilon(a_j) \times Q_\varepsilon(a_j)| \leq \varepsilon^{n+1}$.

In particular we see that

$$\text{VMO} + W^{1,1} + W^{1/p, p} \subset B_0.$$  \hspace{1cm} (13)

\section{Some properties of $B$}

The main result of this section is

**Theorem 2.** Let $n \geq 2$. Then we have $B \subset L^{n/(n-1), w}$, and

$$\left\| f - \frac{1}{Q} \int_Q f \right\|_{L^{n/(n-1), w}} \leq C_n \|f\|_B, \forall f \in B.$$  \hspace{1cm} (14)

In Theorem 2, the Marcinkiewicz space $L^{n/(n-1), w}$ cannot be replaced by $L^{n/(n-1)}$, as a consequence of the next result.

**Proposition 3.** Let $n \geq 2$. There exists some $f \in B$ such that $f \not\in L^{n/(n-1)}$.

**Proof of Theorem 2.** We may assume that

$$\|f\|_B \leq 1 \text{ and } \int_Q f = 0.$$  \hspace{1cm} (15)

We also temporarily make the additional assumption that $f \in L^\infty$.  

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Under these assumptions, we will prove that
\[ \|f\|_{L^{n/(n-1),w}} \approx \sup_{t>0} t |\{|f| > t\}|^{(n-1)/n} \leq C_n. \tag{16} \]

For this purpose it suffices to consider, in (16), only \( t \geq 1 \).

We first note that, by (15), we have
\[ \hat{Q} |f| \leq 1. \tag{17} \]

In view of (17) we may consider, for \( t > 1 \), a Calderón-Zygmund decomposition at height \( t \), i.e., we consider families \( \mathcal{F}_j \) (with \( j \geq 1 \)) of mutually disjoint cubes \( Q_{2^{-j}} \subset Q \) of size \( 2^{-j} \) such that, if we set \( \mathcal{F} = \bigcup_{j \geq 1} \mathcal{F}_j \), then
\[ Q^* |f| \approx t \text{ for every } Q^* \in \mathcal{F}, \tag{18} \]
and
\[ |f| \leq t \text{ a.e. in } R := Q \setminus \bigcup_{Q \subset \mathcal{F} \in \mathcal{F}^* \mathcal{F}_j}. \tag{19} \]

We next decompose \( f = g + h \), with
\[ g = f_{\leq R} + \sum_{Q \subset \mathcal{F}} \left( \int_{Q} f \right) 1_{Q}, \]
\[ h = \sum_{j \geq 1} h_j, \text{ and } h_j = \sum_{Q \subset \mathcal{F}_j} \left( f - \int_{Q} f \right) 1_{Q}. \]

By (18) and (19), we have
\[ |g| \leq Ct \text{ and thus } |\{|f| > 2Ct\}| \subset |\{|h| > Ct\}|. \tag{20} \]

Using (20), we see that (16) amounts to the following:
\[ \sup_{t>1} t |\{|h| > Ct\}|^{(n-1)/n} \leq c. \tag{21} \]

We now proceed with the proof of (21). Since \( \|f\|_B = 1 \), for every family \( \mathcal{G} \subset \mathcal{F}_j \) such that
\[ \#\mathcal{G} \leq 1/(2^{-j})^{n-1} = 2^{j(n-1)}, \]
we have
\[ 2^{-j(n-1)} \sum_{Q \subset \mathcal{G} \mathcal{F}_j} \int_{Q} |f - \int_{Q} f| \leq 1. \]

By covering \( \mathcal{F}_j \) with mutually disjoint sets \( \mathcal{G} \) as above, we find that
\[ \sum_{Q, \subset \mathcal{F}_j} \int_{Q} |f - \int_{Q} f| \leq 2^{j(n-1)} + \#\mathcal{F}_j, \tag{22} \]
and thus
\[ \|h_j\|_{L^1} \leq 2^{-j} + 2^{-n_j} \# F_j. \]  
(23)

On the other hand, we have (using (18))
\[ 1 \geq \|f\|_{L^1} \geq \sum_{j \geq 1} \sum_{Q_j \in F_j} \int Q_j |f| = \sum_{j \geq 1} \sum_{Q_j \in F_j} 2^{-n_j} \int Q_j |f| \gtrsim \sum_{j \geq 1} 2^{-n_j} t \# F_j. \]  
(24)

From (23) and (24), we deduce that
\[ \sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \sum_{F_j \neq \emptyset} 2^{-j}. \]  
(25)

We next recall that
\[ \|f\|_{L^{n/(n-1),w}} = \sup_{A \subset Q} |A|^{-1/n} \int_A |f|. \]  
(26)

If \( F_j \neq \emptyset \) and \( Q_* \in F_j \), then (26) applied with \( A = Q_* \), combined with (18), implies that
\[ 2^{-j} \lesssim \left( \frac{\|f\|_{L^{n/(n-1),w}}}{t} \right)^{1/(n-1)}. \]  
(27)

By (25) and (27), we have
\[ \|h\|_{L^1} \leq \sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \left( \frac{\|f\|_{L^{n/(n-1),w}}}{t} \right)^{1/(n-1)}. \]  
(28)

In turn, (28) implies that (with \( C \) as in (21))
\[ |\{|h| > Ct\}| \lesssim \frac{\|h\|_{L^1}}{Ct} \lesssim \frac{1}{t^2} + \left( \frac{\|f\|_{L^{n/(n-1),w}}}{t^n} \right)^{1/(n-1)}, \]  
(29)

and thus
\[ t |\{|h| > Ct\}|^{(n-1)/n} \lesssim t^{(2-n)/n} + \|f\|_{L^{n/(n-1),w}}^{1/n} \lesssim 1 + \|f\|_{L^{n/(n-1),w}}^{1/n}. \]  
(30)

By taking, in (30), the supremum over \( t > 1 \), we find that
\[ \|f\|_{L^{n/(n-1),w}} \lesssim 1 + \|f\|_{L^{n/(n-1),w}}^{1/n}, \]  
and therefore \( \|f\|_{L^{n/(n-1),w}} \lesssim 1. \)

We complete the proof by removing the assumption that \( f \in L^\infty \). Let
\[ \Phi_N(s) = \begin{cases} 
  s, & \text{if } |s| \leq N \\
  N, & \text{if } s > N \\
  -N, & \text{if } s < -N 
\end{cases} \]
and set \( f_N := \Phi_N(f) \). By (3), we have \( \| f_N \|_B \leq 2 \| f \|_B \). In addition, \( f_N \) is bounded and thus satisfies (14), i.e.,

\[
\left\| f_N - \int_Q f_N \right\|_{L^{n/(n-1),w}} \leq 2C_n \| f \|_B. \tag{31}
\]

Using (26) and passing to the limit as \( N \to \infty \) in (31) yields (14) for every \( f \in B \).

\[\square\]

**Proof of Proposition 3.** Set

\[ \varphi(x) = (1 - |x|)^+, \forall x \in \mathbb{R}^n \]

and

\[ N_m = 2^{2^m}, \forall m \geq 1. \]

Consider a sequence of points \( (b_m)_{m \geq 1} \) such that the open balls \( B(b_m, 2/N_m) \) are contained in \( Q \) and mutually disjoint. (We may e.g. choose the points \( b_m \) on a line segment parallel to the \( x_1 \)-axis.) Set

\[ f_m(x) = N_m^{n-1} \varphi(N_m(x - b_m)), \forall m \geq 1 \tag{32} \]

and

\[ f(x) = \sum_{m \geq 1} f_m(x). \tag{33} \]

We will prove that \( f \in B \) and \( f \not\in L^{n/(n-1)} \).

Note that

\[ \text{supp} f_m = \overline{B}(b_m, 1/N_m), \]

and that the sets \( \text{supp} f_m, m \geq 1, \) are mutually disjoint.

Clearly,

\[ \| f_m \|_{L^1(Q)} = \frac{C}{N_m}, \forall m \geq 1, \tag{34} \]

and thus \( f \in L^1(Q) \); here and in what follows we denote by \( C \) a generic constant depending only on \( n \).

We have

\[ \| f_m \|_{L^{n/(n-1)}(Q)}^{n/(n-1)} = C, \forall m \geq 1, \]

so that \( f \not\in L^{n/(n-1)}(Q) \).

Given \( 0 < \varepsilon < 1 \) and integers \( M_1 = M_1(\varepsilon) \geq 1 \) and \( M_2 = M_2(\varepsilon) > M_1(\varepsilon) \) to be defined later, we write

\[ f = S_1 + S_2 + S_3, \tag{35} \]
with
\[ S_1 = \sum_{m \leq M_1} f_m, \quad S_2 = \sum_{M_1 < m \leq M_2} f_m, \quad S_3 = \sum_{m > M_2} f_m. \tag{36} \]

We now estimate separately \([S_1]_\epsilon, [S_2]_\epsilon \) and \([S_3]_\epsilon\).

**Estimate of \([S_1]_\epsilon\).** Here we use the fact that if \( h \in \text{Lip}(Q) \) then
\[ M(h, Q_\epsilon(a)) \leq \sqrt{n} \epsilon \|h\|_{\text{Lip}}, \tag{37} \]
and thus
\[ [h]_\epsilon \leq \sqrt{n} \epsilon \|h\|_{\text{Lip}}. \]
In particular,
\[ [f_m]_\epsilon \leq C \epsilon (N_m)^n. \tag{38} \]

Using (38) and the fact that
\[ \sum_{i=1}^{p} X^i \leq \frac{X^{p+1}}{X-1}, \forall X > 1, \]
we deduce that
\[ [S_1]_\epsilon \leq C \epsilon 2^n 2^{M_1}, \forall \epsilon \in (0,1). \tag{39} \]

**Estimate of \([S_2]_\epsilon\).** Applying (9) to \( f_m \) yields
\[ [f_m]_\epsilon \leq C, \forall m \geq 1, \forall \epsilon \in (0,1), \]
and in particular
\[ [S_2]_\epsilon \leq C(M_2 - M_1), \forall \epsilon \in (0,1). \tag{40} \]

**Estimate of \([S_3]_\epsilon\).** Clearly
\[ [h]_\epsilon \leq \frac{2}{\epsilon} \|h\|_{L^1(Q)}, \forall h \in L^1. \tag{41} \]

From (34) we deduce that
\[ [f_m]_\epsilon \leq \frac{C}{\epsilon N_m}. \tag{42} \]

Using (42) and the fact that
\[ \sum_{i=p}^{\infty} Y^i = \frac{Y^p}{1-Y}, \forall Y \in [0,1), \]
we see that
\[ [S_3]_\epsilon \leq \frac{C}{\epsilon 2^{2M_2}}. \] (43)

We now explain how to choose \( M_1(\epsilon) \) and \( M_2(\epsilon) \). Given \( 0 < \epsilon < 1 \), we denote by \( M_1 = M_1(\epsilon) \) the largest integer \( \ell \geq 1 \) such that
\[ 2^n 2^\ell \leq \frac{2^{2n}}{\epsilon}. \] (44)

Equivalently, we have
\[ 2^n 2^{M_1} \leq \frac{2^{2n}}{\epsilon} \] (45)
and
\[ 2^n 2^{M_1} > \frac{2^{2n}}{\epsilon}. \] (46)

Combining (39) and (45) yields
\[ [S_1]_\epsilon \leq C, \quad \forall \epsilon \in (0, 1). \] (47)

From (45) and (46) we obtain
\[ |M_1(\epsilon) - \log_2 \log_2 (1/\epsilon)| \leq C, \quad \forall \epsilon \in (0, 1/2). \] (48)

Next we denote by \( M_2 = M_2(\epsilon) \) the smallest integer \( \ell \geq 1 \) such that
\[ 2^n 2^\ell \geq \frac{4}{\epsilon}. \] (49)

(Note that \( M_2 > M_1 \) since \( 2^{2M_1} < 4/\epsilon \).)

Equivalently, we have
\[ 2^{2M_2} \geq \frac{4}{\epsilon} \] (49)
and
\[ 2^{2M_2-1} < \frac{4}{\epsilon}. \] (50)

Combining (43) and (49) yields
\[ [S_3]_\epsilon \leq C, \quad \forall \epsilon \in (0, 1). \] (51)

From (49) and (50) we obtain
\[ |M_2(\epsilon) - \log_2 \log_2 (1/\epsilon)| \leq C, \quad \forall \epsilon \in (0, 1/2). \] (52)
Therefore,

\[ |M_2(\varepsilon) - M_1(\varepsilon)| \leq C, \quad \forall \varepsilon \in (0, 1). \tag{53} \]

(Inequality (53) is deduced from (48) and (52) when \( \varepsilon \in (0, 1/2) \), and from (50) when \( \varepsilon \in [1/2, 1) \).

It follows from (40) and (53) that

\[ [S_2]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \tag{54} \]

Putting together (47), (51) and (54) we conclude that

\[ [f]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1), \]

and thus \( f \in B \).

\[ \square \]

3 Some properties of \( B_0 \) and \([f]\)

Our first result is

**Theorem 4.** Let \( f \) be a \( \mathbb{Z} \)-valued function on \( Q \) such that \( f \in B_0 \). Then \( f \) is constant.

Combining Theorem 4 with (13) we obtain Theorem 1.

When \( n = 1 \) we have \( B_0 = \text{VMO} \) and we may then invoke the fact that functions in VMO (\( Q; \mathbb{Z} \)) are constant (for any \( n \geq 1 \)); see [3, Comment 2, p. 223–224]. Therefore it suffices to prove Theorem 4 when \( n \geq 2 \). Next, we observe that it suffices to prove Theorem 4 when \( f = 1_A \) for some \( A \subset Q \). Indeed, let \( k \in \mathbb{Z} \) be such that \( |f^{-1}(k)| > 0 \). Set \( A = f^{-1}(k) \) and \( g = 1_A \). Clearly \( M^*(f, Q_\varepsilon) \geq M^*(g, Q_\varepsilon) \) for every \( \varepsilon \)-cube \( Q_\varepsilon \). Since \( f \in B_0 \), we deduce that \( g \in B_0 \). If Theorem 4 holds for \( g \), then \( g \equiv 1 \), and thus \( f \equiv k \).

Hence it remains to prove Theorem 4 when \( n \geq 2 \) and \( f = 1_A \). In this case we have the following quantitative improvement of Theorem 4.

**Theorem 5.** Let \( n \geq 2 \). There exists a constant \( C_n \) such that if \( f = 1_A \) with \( A \subset Q \) measurable, then

\[ \left\| f - \frac{1}{Q} \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C_n[f]. \tag{55} \]

**Remark 6.** A much more precise result (see [1]) asserts that there exist two constants \( 0 < \underline{c}_n \leq \overline{c}_n < \infty \) such that if \( f = 1_A \), then

\[ \underline{c}_n \min \left\{ 1, \int_Q |\nabla f| \right\} \leq [f] \leq \overline{c}_n \min \left\{ 1, \int_Q |\nabla f| \right\}, \tag{56} \]
with the convention that $\int_Q |\nabla f| = \infty$ if $f \not\in BV$.

Note that

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \int_Q |\nabla f|$$

by the Sobolev embedding, and that clearly

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq 2 \text{ when } f = 1_A.$$  

Therefore

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \min\{1, \int_Q |\nabla f|\} \leq C'[f] \text{ by (56).}$$

In fact, using a variant of the definition (5) involving $\varepsilon$-cubes of general orientation, one obtains a quantity $[f]_{\varepsilon}$ satisfying

$$[f]_{\varepsilon} \leq [f]_{\varepsilon}^* \leq C_1[f]_{\varepsilon} C_2$$

for some constants $C_1 > 1$, $C_2 > 1$ depending only on $n$ (see [1]). The main result in [1] asserts that if $f = 1_A$, then

$$\lim_{\varepsilon \to 0} [f]_{\varepsilon}^* = \frac{1}{2} \min\left\{1, \int_Q |\nabla f|\right\};$$

the ingredients of the proof of (59) are much more sophisticated than the arguments presented below. We acknowledge that it was Theorem 5 which prompted one of us to conjecture that (59) holds.

The main tool in the proof of Theorem 5 is

**Lemma 7.** Let $n \geq 2$. Let $U = \bigcup_{j \in J} Q_\varepsilon(a_j)$ be a union of $\varepsilon$-cubes. Then $Q \setminus U$ contains a connected set $S$ of measure $\geq 1 - \alpha_n (\#J)^{n/(n-1)} \varepsilon^n$, for some positive constant $\alpha_n$ depending only on $n$.

Here, the $\varepsilon$-cubes are not necessarily mutually disjoint, and we do not assume that these cubes are completely contained in $Q$.

**Remark 8.** The conclusion of Lemma 7 is optimal. Indeed, consider a ball $B \subset Q$ of (small) radius $R$. We may cover the sphere $\Sigma = \partial B$ by a union of $\varepsilon$-cubes as above with $\#J \varepsilon^{n-1} \approx R^{n-1}$. Then $|B| \approx R^n \approx (\#J)^{n/(n-1)} \varepsilon^n$.

Granted Lemma 7, we turn to the

**Proof of Theorem 5.** Let $f = 1_A$, with $A \subset Q$. Fix any $\lambda \in (0, 1/2)$, e.g. $\lambda = 1/4$.

In view of (58), we may assume that

$$0 \leq [f] < 2\lambda(1 - \lambda),$$

(60)
for otherwise the conclusion is clear with $C_n = \frac{1}{\lambda(1-\lambda)}$.

Note that, by (4),

$$M(f, Q_\varepsilon) = 2f_{Q_\varepsilon}(1 - f_{Q_\varepsilon}).$$

Therefore,

$$M(f, Q_\varepsilon) < 2\lambda(1 - \lambda) \implies \text{either } f_{Q_\varepsilon} < \lambda, \text{ or } f_{Q_\varepsilon} > 1 - \lambda. \quad (61)$$

With $\varepsilon$ small and $\tilde{Q} = (\varepsilon, 1-\varepsilon)^n$, consider a maximal family $J = J_\varepsilon$ of points $a \in \tilde{Q}$ such that the cubes $Q_\varepsilon(a)$ are mutually disjoint and satisfy

$$M(f, Q_\varepsilon(a)) \geq 2\lambda(1 - \lambda), \quad \forall a \in J. \quad (62)$$

Let $\nu > 0$ (to be chosen arbitrarily small later). We claim that for $\varepsilon$ sufficiently small (depending on $\nu$) we have

$$\#J \leq \frac{\delta}{\varepsilon^{n-1}} \quad \text{with } \delta = \frac{[f] + \nu}{2\lambda(1 - \lambda)}. \quad (63)$$

Indeed, we first see that, for $\varepsilon$ sufficiently small,

$$\#J \leq \frac{1}{\varepsilon^{n-1}}. \quad (64)$$

Otherwise, we may choose a subfamily $\tilde{J}$ such that $\#\tilde{J} = I(1/\varepsilon^{n-1})$, where $I(t)$ denotes the integer part of $t$. Then

$$[f] \geq \varepsilon^{n-1}(\#\tilde{J}) \geq \varepsilon^{n-1}\left(\frac{1}{\varepsilon^{n-1}} - 1\right)2\lambda(1 - \lambda),$$

which, by (60), is impossible for $\varepsilon$ small. From (64) and the definition of $[f]_\varepsilon$ we have

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#J) \geq \varepsilon^{n-1}(\frac{1}{\varepsilon^{n-1}} - 1)2\lambda(1 - \lambda),$$

which yields (63) for $\varepsilon$ sufficiently small.

Set $U := \cup_{a \in J}Q_{2\varepsilon}(a)$. By Lemma 7 and a scaling argument, $\tilde{Q} \setminus U$ contains a connected set $S = S_\varepsilon$ such that

$$|S_\varepsilon| \geq (1 - 2\varepsilon)^n - \alpha_n' \delta^{n/(n-1)}, \quad (65)$$

where $\alpha_n' = 2^n \alpha_n$.

We next note that (by the maximality of $J$) $U$ contains the set

$$V = V_\varepsilon := \{b \in \tilde{Q}; M(f, Q_\varepsilon(b)) \geq 2\lambda(1 - \lambda)\}, \quad (66)$$

and thus $S \subset \tilde{Q} \setminus V$.

We consider the continuous function

$$f_\varepsilon : \tilde{Q} \to \mathbb{R}, \quad f_\varepsilon(a) = f_{Q_\varepsilon(a)}.$$
By (61) and (66), in the set $\tilde{Q} \setminus V$ the function $f_\epsilon$ takes values into $[0, \lambda) \cup (1 - \lambda, 1]$. $S \subset \tilde{Q} \setminus V$ being connected, we find that either $f_\epsilon < \lambda$, or $f_\epsilon > 1 - \lambda$ in $S$.

We assume e.g. that $f_\epsilon < \lambda$ in $S_\epsilon$ along a sequence $\epsilon_m \to 0$. Clearly,

$$\int_{A \setminus \tilde{Q}} |1 - f_\epsilon| \to 0 \text{ as } \epsilon \to 0,$$

and thus

$$(1 - \lambda)|S_{\epsilon_m} \cap A| \leq \int_{S_{\epsilon_m} \cap A} (1 - f_{\epsilon_m}) \to 0 \text{ as } m \to \infty. \quad (67)$$

On the other hand, by (65) and (67) we have

$$|A| = |S_{\epsilon_m} \cap A| + |(\tilde{Q} \setminus S_{\epsilon_m}) \cap A| + |(Q \setminus \tilde{Q}) \cap A| \leq \alpha'_n \delta^n(n-1) + o(1) \text{ as } m \to \infty,$$

and thus $|A| \leq \alpha''_n \delta^n(n-1)$, so that

$$|A|^{(n-1)/n} \leq \alpha''_n \delta = \alpha''_n \frac{|f| + \nu}{2\lambda(1 - \lambda)} \quad \text{with } \alpha''_n = (\alpha'_n)^{(n-1)/n}.$$  

Since $\nu > 0$ can be chosen arbitrarily small, we deduce that

$$|A|^{(n-1)/n} \leq \frac{\alpha''_n |f|}{2\lambda(1 - \lambda)}. \quad (68)$$

Finally, we note that

$$\left\| f - \bar{f} \right\|_{L_{n(n-1)}} = \left( |A| (1 - |A|)^{n/(n-1)} + (1 - |A|) |A|^{n/(n-1)} \right)^{(n-1)/n}$$

$$\leq 2 \min \left\{ |A|^{(n-1)/n}, |A^c|^{(n-1)/n} \right\}. \quad (69)$$

Combining (68) and (69) yields (55).

For further use, let us note that the proof of Theorem 5 leads to the following result.

**Lemma 9.** Let $n \geq 2$ and $\lambda \in (0, 1/2)$. Let $A \subset Q$ be measurable and set $\mathbb{1}_A$. Assume that there exists a sequence $\epsilon_m \to 0$ and families

$$J_m \subset \tilde{Q}_m := (3\epsilon_m, 1 - 3\epsilon_m)^n$$

of points $a$ with the following property:

If $b \in \tilde{Q}_m \setminus \cup_{a \in J_m} Q_{2\epsilon_m}(a)$, then $M(f, Q_{\epsilon_m}(b)) < 2\lambda(1 - \lambda)$.

Let

$$\delta := \lim_{m \to \infty} \epsilon_m^{n-1} \# J_m.$$  

Then either $|A| \geq 1 - \tilde{c}_n \delta^{n/(n-1)}$, or $|A^c| \geq 1 - \tilde{c}_n \delta^{n/(n-1)}$.  

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Proof of Lemma 7. Recall a standard “relative” isoperimetric inequality. Let $B \subset Q$ satisfy $|B| \leq 1/2$. By (57) (applied with $f = \mathbb{1}_B$) and (69), we have

$$|B| \leq c_n \left( \int_Q |\nabla \mathbb{1}_B| \right)^{n/(n-1)} = c_n [P(B)]^{n/(n-1)},$$

(70)

where $P(B)$ represents the perimeter of $B$ relative to $Q$. When $B$ is a Lipschitz domain (which will be the case in what follows), $P(B)$ is the (surface) measure of $\partial B \cap Q$.

We now turn to the proof of the lemma. Set $\delta = (\#J) \varepsilon^{n-1}$. Let $(A_i)_{i \in I}$ be the connected components of the open set $Q \setminus \bigcup_{j \in J} \overline{Q}_\varepsilon(a_j)$.

Note that each $A_i$ is Lipschitz, and that

$$\bigcup_{i \in I} (\partial A_i \cap Q) \subset \bigcup_{j \in J} (\partial Q_\varepsilon(a_j) \cap Q).$$

(71)

Let

$$G_j := \{ x \in \partial Q_\varepsilon(a_j) \cap Q; x \text{ does not belong to the } (n-2) \text{ skeleton of } \partial Q_\varepsilon(a_j) \}.$$

Note that

$$[\bigcup_{i \in I} (\partial A_i \cap Q)] \setminus [\bigcup_{j \in J} G_j] \text{ has zero } \mathcal{H}^{n-1} \text{ measure.}$$
Since a point \( x \in G_j \) belongs to at most one \( \partial A_i \), we find, using (71), that
\[
\sum_{i \in I} P(A_i) \leq \sum_{j \in J} P(Q(e(a_j))) \leq c'_n \delta. \tag{72}
\]

We claim that if \( \delta < \delta_n \) (a positive number depending only on \( n \)), then there exists some \( i_0 \in I \) such that \( |A_{i_0}| > 1/2 \). Indeed, argue by contradiction and assume that \( |A_i| \leq 1/2, \forall i \in I \). By (70) and (72), we have
\[
1 - |U| = |Q \setminus U| = \sum_{i \in I} |A_i| \leq c_n \sum_{i \in I} [P(A_i)]^{n/(n-1)} \leq c_n \left[ \sum_{i \in I} P(A_i) \right]^{n/(n-1)} \leq c_n (c'_n \delta)^{n/(n-1)} = c''_n \delta^{n/(n-1)}. \tag{73}
\]

On the other hand
\[
|U| \leq (\#J) \varepsilon^n = \delta \varepsilon < \delta. \tag{74}
\]

Combining (73) and (74) we obtain
\[
1 \leq \delta + c''_n \delta^{n/(n-1)};
\]
this is impossible when \( \delta < \delta_n \), where \( \delta_n \) is the solution of
\[
1 = \delta_n + c''_n (\delta_n)^{n/(n-1)},
\]
and thus the claim is established when \( \delta < \delta_n \).

Set \( S = A_{i_0} \), which is clearly connected and contained in \( Q \setminus U \). Applying (70) to \( B = S^c \) we find (using (72))
\[
1 - |S| \leq c_n [P(S^c)]^{n/(n-1)} = c_n [P(S)]^{n/(n-1)} \leq c''_n \delta^{n/(n-1)},
\]
which is the desired conclusion when \( \delta < \delta_n \).

Finally, we observe that
\[
1 - \frac{1}{(\delta_n)^{n/(n-1)}} \delta^{n/(n-1)} \leq 0
\]
when \( \delta \geq \delta_n \) and therefore Lemma 7 holds with
\[
\alpha_n = \max \left\{ c''_n, \frac{1}{(\delta_n)^{n/(n-1)}} \right\}.
\]

\( \square \)
4 An extension of Theorem 5 to $\mathbb{Z}$-valued functions

Our main result in this section is

**Theorem 10.** Let $n \geq 2$. There exists a positive constant $c$ (independent of $n$) such that if $f$ is a $\mathbb{Z}$-valued function in $B$ and $[f] < c$, then $f \in L^{n/(n-1)}(Q)$ and

$$
\left\| f - \frac{1}{Q} \sum_{k} g_k \right\|_{L^{n/(n-1)}(Q)} \leq C_n[f],
$$

(75)

for some constant $C_n$ depending only on $n$.

Theorem 5 can be deduced from Theorem 10. Indeed, let $f = \mathbb{1}_A$. Then either $[f] \leq c$, and Theorem 10 applies, or $[f] > c$, and then

$$
\left\| f - \frac{1}{Q} \sum_{k} g_k \right\|_{L^{n/(n-1)}(Q)} \leq 2 \leq (2/c)[f].
$$

The smallness condition on $[f]$ in Theorem 10 is essential, as shown by the following improvement of Proposition 3.

**Proposition 11.** Let $n \geq 2$. There exists a $\mathbb{Z}$-valued function $f \in B$ such that $f \not\in L^{n/(n-1)}(Q)$.

**Proof of Theorem 10.** Step 1. Decomposition of $f$ as a sum of characteristic functions.

We temporarily assume that $f \geq 0$. Then $f$ is a sum of characteristic functions. Indeed, set

$$
A_k := \{ x \in Q; f(x) \geq k \}, \forall k \in \mathbb{N}^*,
$$

and let $g_k := \mathbb{1}_{A_k}$. Then we claim that

$$
f = \sum_{k > 0} g_k \tag{76}
$$

and

$$
|f(x) - f(y)| = \sum_{k > 0} |g_k(x) - g_k(y)|, \forall x, y \in Q. \tag{77}
$$

Indeed, on the one hand (76) follows from

$$
\sum_{k > 0} g_k(x) = \sum_{0 < k \leq f(x)} 1 = f(x).
$$
On the other hand, assuming e.g. that \( f(x) \geq f(y) \), we have \( g_k(x) = g_k(y) \) provided either \( k \leq f(y) \) or \( k > f(x) \), and thus
\[
\sum_{k>0} |g_k(x) - g_k(y)| = \sum_{f(y) < k \leq f(x)} |g_k(x) - g_k(y)| = \sum_{f(y) < k \leq f(x)} 1 = f(x) - f(y) = |f(x) - f(y)|;
\]
that is, (77) holds.

We next note that (77) implies
\[
M^*(f, Q_\ell) = \sum_{k>0} M^*(g_k, Q_\ell), \tag{78}
\]
and in particular
\[
M(g_k, Q_\ell) \leq M^*(f, Q_\ell), \forall k > 0. \tag{79}
\]

**Step 2.** Construction of maximal families of “bad” cubes.

Fix some \( \lambda \in (0, 1/2) \) and consider a sequence \( \varepsilon_m \to 0 \). Let \( \tilde{Q}^m := (3\varepsilon_m, 1-3\varepsilon_m)^n \).

Let \( J_m \) be a maximal family of points \( a \in \tilde{Q}^m \) such that the cubes \( Q_{\varepsilon_m}(a) \), \( a \in J_m \), are mutually disjoint and satisfy \( M^*(f, Q_{\varepsilon_m}(a)) \geq 2\lambda(1-\lambda) \).

By the maximality of \( J_m \) and by (79), we have
\[
M(g_k, Q_{\varepsilon_m}(b)) \leq M^*(f, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda), \forall b \in \tilde{Q}_m \setminus \cup_{a \in J_m} Q_{2\varepsilon_m}(a). \tag{80}
\]

We next associate to each \( k \) an appropriate subfamily extracted from \( J_m \).

More specifically, let
\[
J^k_m := \{a \in J_m; 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) \geq 2\lambda(1-\lambda)\}. \tag{81}
\]

We claim that
\[
M(g_k, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda), \forall b \in \tilde{Q}_m \setminus \cup_{a \in J^k_m} Q_{2\varepsilon_m}(a). \tag{82}
\]

Indeed, (80) implies that (82) holds for \( b \in \tilde{Q}_m \setminus \cup_{a \in J_m} Q_{2\varepsilon_m}(a) \).

It remains to establish (82) when \( b \in Q_{2\varepsilon_m}(a) \) for some \( a \in J_m \setminus J^k_m \). In this case, we have \( Q_{\varepsilon_m}(b) \subset Q_{3\varepsilon_m}(a) \) and thus
\[
M^*(g_k, Q_{\varepsilon_m}(b)) \leq 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) < 2\lambda(1-\lambda).
\]

This completes the proof of (82).

**Step 3.** A first estimate of \( \|f - \int_Q f\|_{L^n(n-1)} \).

By (69), (82), and Lemma 9, we have
\[
\left\| g_k - \int_Q g_k \right\|_{L^n(n-1)} \leq 2(\tilde{c}_n)^{(n-1)/n} \lim_{m \to \infty} (\varepsilon_m)^{n-1} \# J^k_m. \tag{83}
\]
Thus
\[
\sum_{k>0} \left\| g_k - \int_Q g_k \right\|_{L^n(n-1)} \leq 2(\tilde{c}_n)^{(n-1)/n} \lim_{m \to \infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J^k_m, \tag{84}
\]
and therefore
\[ \left\| f - \frac{1}{Q} \int_Q f \right\|_{L^{n/(n-1)}} \leq 2(\tilde{c}_n)^{(n-1)/n} \lim_{m \to \infty} (\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k. \] (85)

**Step 4.** A second estimate of \( \| f - \frac{1}{Q} \int_Q f \|_{L^{n/(n-1)}} \).

In this step, we assume that
\[ [f] < d := \lambda(1 - \lambda), \text{ with } \lambda \text{ chosen as in Step 2}. \] (86)

Under this assumption, we will prove that
\[ c'_n \lim_{m \to \infty} (\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k \leq [f] \text{ for some constant } c'_n > 0. \] (87)

Granted this estimate, we obtain (using (85)) that
\[ \left\| f - \frac{1}{Q} \int_Q f \right\|_{L^{n/(n-1)}} \leq \tilde{C}_n[f], \text{ with } \tilde{C}_n = 2(\tilde{c}_n)^{(n-1)/n}/c'_n. \] (88)

We now proceed to the proof of (87). We first note that (by (3)) we have
\[ M(f, Q_{\varepsilon_m}(a)) \geq \lambda(1 - \lambda), \forall a \in J_m. \] (89)

Repeating the proof of (64) (and using (86) and (89)), for large \( m \) we have
\[ \#J_m \leq 1/(\varepsilon_m)^{n-1}. \] (90)

We next rely on the following lemma, well-known to the experts, whose proof is omitted.

**Lemma 12.** Let \( \{Q_\varepsilon(a); a \in J\} \) be a family of mutually disjoint \( \varepsilon \)-cubes. Then there exists a constant \( N = N(n) \) such that

1. \( J = J^1 \cup J^2 \cup \ldots J^N \).
2. For every \( j \), the cubes \( Q_{3\varepsilon}(a), a \in J^j \), are mutually disjoint.
3. For every \( j \), we have \( \#J^j \leq \#J/3^{n-1} \).

By Lemma 12, for every family of mutually disjoint \( \varepsilon \)-cubes \( Q_\varepsilon(a), a \in J \subset (3\varepsilon, 1-3\varepsilon)^n \), such that \( \#J \leq 1/\varepsilon^{n-1} \), we have
\[ (3\varepsilon)^{n-1} \sum_{a \in J} M(h, Q_{3\varepsilon}(a)) \leq N[h]_{3\varepsilon}, \forall h : Q \to \mathbb{R}. \] (91)

In particular, for large \( m \) we have (using (90) and (91))
\[ (\varepsilon_m)^{n-1} \sum_{a \in J_m} M(f, Q_{3\varepsilon_m}(a)) \leq N/3^{n-1}[f]_{3\varepsilon_m}. \] (92)
Combining (92) with (3), we see that

\[(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(f, Q_{3\varepsilon_m}(a)) \leq 2N/3^{n-1} [f]_{3\varepsilon_m}, \tag{93}\]

We now use successively (93), (78) and (81) and obtain that

\[ [f]_{3\varepsilon_m} \geq 3^{n-1}/(2N)(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(f, Q_{3\varepsilon_m}(a)) = 3^{n-1}/(2N)(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(g_k, Q_{3\varepsilon_m}(a)) \geq 3^{n-1}/(2N)(\varepsilon_m)^{n-1} \sum_{k>0} \sum_{a \in J_m^k} M^*(g_k, Q_{3\varepsilon_m}(a)) \geq \lambda(1-\lambda)/(3^{n+1}N)(\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k = c'_n (\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k, \tag{94}\]

with \(c'_n := \lambda(1-\lambda)/(3^{n+1}N)\).

We derive (87) by letting \(m \to \infty\) in (94).

**Step 5.** We remove the assumption \(f \geq 0\).

We note that \(f = f^+ - f^-\), and that

\[ |f^\pm(x) - f^\pm(y)| \leq |f(x) - f(y)|, \quad \forall \ x, y \in Q. \tag{95}\]

By (3) and (95), we have

\[ M^*(f^\pm, Q) \leq M^*(f, Q) \leq 2M(f, Q), \]

and thus \([f^\pm] \leq 2[f]\). By the first part of the proof of this theorem, we have

\[ \left\| f^\pm - \int_Q f^\pm \right\|_{L^{n/(n-1)}} \leq \tilde{C}_n[f^\pm] \leq 2\tilde{C}_n[f], \tag{96}\]

provided \([f] < c := d/2\).

Finally, (96) implies that

\[ \left\| f - \int_Q f \right\|_{L^{n/(n-1)}} \leq C_n[f] \quad \text{provided \([f] < c\)}, \]

with \(C_n := 4\tilde{C}_n\).

The proof of Theorem 10 is complete. \(\square\)

**Proof of Proposition 11.** We use the same notation and the same strategy as in the proof of Proposition 3, with some minor modifications.

Set

\(g_m(x) = I(f_m(x)), \ \forall \ m \geq 1\), where \(I(t)\) denotes the integer part of \(t\),
and
\[ g(x) = \sum_{m \geq 1} g_m(x). \]

Clearly,
\[ \|g_m\|_{L^1(Q)} \leq \|f_m\|_{L^1(Q)} = \frac{C}{N_m} \tag{97} \]
(by (34)), so that \( g \in L^1(Q) \). On the other hand
\[ \|g_m\|_{L^{n/(n-1)}(Q)} \geq \|f_m - 1\|_{L^{n/(n-1)}(Q)} \geq \alpha > 0, \ \forall \ m \geq 1, \]
and thus \( g \not\in L^{n/(n-1)}(Q) \).

We will now prove that \( g \in B \). Write
\[ g = T_1 + T_2 + T_3, \]
with
\[ T_1 = \sum_{m \leq M_1} g_m, \ T_2 = \sum_{M_1 < m \leq M_2} g_m, \ T_3 = \sum_{m > M_2} g_m, \]
where \( M_1 = M_1(\varepsilon) \) and \( M_2 = M_2(\varepsilon) \) are defined exactly as in the proof of Proposition 3.

**Estimate of \( |T_1|_\varepsilon \).** Since \( g_m \not\in \text{Lip}(Q) \), we need to modify the argument. We claim that, for sufficiently small \( \varepsilon \) (depending only on \( n \)), given any cube \( Q_\varepsilon(a) \) there exists at most one integer \( m \leq M_1(\varepsilon) \) such that
\[ Q_\varepsilon(a) \cap (\text{supp} g_m) \neq \emptyset. \tag{98} \]
Indeed, if (98) holds, then
\[ Q_\varepsilon(a) \cap B(b_m, 1/N_m) \neq \emptyset, \]
and thus
\[ Q_\varepsilon(a) \subset B(b_m, 2/N_m) \tag{99} \]
provided
\[ \frac{1}{N_m} + \sqrt{n} \varepsilon \leq \frac{2}{N_m}, \ \forall \ m \leq M_1. \tag{100} \]
On the other hand, (45) implies that
\[ N_{M_1} \leq \frac{4}{\varepsilon^{1/n}}, \]
and thus \((100)\) holds when
\[
\varepsilon \leq \varepsilon_0 := \frac{1}{4n(n-1)n^2(n-1)}.
\]

We deduce the claim using \((99)\) and the fact that the balls \(B(b_m, 2/N_m)\) are mutually disjoint.

Therefore, for \(\varepsilon \leq \varepsilon_0\) we have
\[
M(T_1, Q_\varepsilon(a)) \leq \int_{Q_\varepsilon(a)} \int_{Q_\varepsilon(a)} |g_m(y) - g_m(z)| dydz
\tag{101}
\]
for some \(m \leq M_1(\varepsilon)\).

If \(y, z \in Q_\varepsilon(a)\), we have
\[
|f_m(y) - f_m(z)| \leq |y - z|\|f_m\|_{\text{Lip}} \leq (N_m)^n \sqrt{n} \varepsilon \leq C
\]
(by \((45)\)). Hence
\[
|g_m(y) - g_m(z)| \leq C, \tag{102}
\]
since
\[
|I(t) - I(s)| \leq |t - s| + 1, \quad \forall t, s.
\]

Combining \((101)\) and \((102)\) yields \(M(T_1, Q_\varepsilon(a)) \leq C\) and therefore
\[
[T_1]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{103}
\]

For \(\varepsilon \in [\varepsilon_0, 1)\), we use \((41)\) to assert that
\[
[T_1]_\varepsilon \leq \frac{2}{\varepsilon_0} \|T_1\|_{L^1(Q)} \leq \frac{2}{\varepsilon_0} \|g\|_{L^1(Q)}. \tag{104}
\]

Combining \((103)\) with \((104)\) we deduce that
\[
[T_1]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \tag{105}
\]

**Estimate of \([T_2]_\varepsilon\).** We claim that
\[
\int_Q |\nabla g_m| \leq C, \quad \forall m \geq 1, \tag{106}
\]
and this implies via \((9)\) that
\[
[g_m]_\varepsilon \leq C, \quad \forall m \geq 1, \quad \forall \varepsilon \in (0, 1),
\]
so that
\[
[T_2]_\varepsilon \leq C(M_2 - M_1) \leq C, \quad \forall \varepsilon \in (0, 1) \tag{107}
\]
In order to prove (106), note that

\[
\int_{\mathbb{R}^n} |\nabla g_m| = \sum_{k=1}^{(N_m)^n-1} \mathcal{H}^{n-1}([f_m = k]) = C \sum_{k=1}^{(N_m)^n-1} \left(1 - \frac{k}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}}
\]

\[
= C \sum_{\ell=1}^{(N_m)^n-1} \left(\frac{\ell}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}} \leq C.
\]

**Estimate of** \([T_3]_\varepsilon\). The technique for estimating \([S_3]_\varepsilon\) in the proof of Proposition 3 gives

\[
[T_3]_\varepsilon \leq C, \quad \forall \varepsilon \in (0,1).
\]

(108)

Combining (105), (107) and (108) yields \(g \in B\).

\[\square\]

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