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Jean-François Chassagneux, Adrien Richou

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Jean-François Chassagneux *
Department of Mathematics, Imperial College London
180 Queen's Gate, London, SW7 2AZ, United Kingdom.
j.chassagneux@imperial.ac.uk

Adrien Richou
Univ. Bordeaux 1, IMB, UMR 5251, F-33400 Talence, France.
INRIA, Équipe ALEA, F-33400 Talence, France.
adrien.richou@math.univ-bordeaux1.fr

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Abstract

This article deals with the numerical approximation of Markovian backward stochastic differential equations (BSDEs) with generators of quadratic growth with respect to z and bounded terminal conditions. We first study a slight modification of the classical dynamic programming equation arising from the time-discretization of BSDEs. By using a linearization argument and BMO martingales tools, we obtain a comparison theorem, a priori estimates and stability results for the solution of this scheme. Then we provide a control on the time-discretization error of order $\frac{1}{2} - \varepsilon$ for all $\varepsilon > 0$. In the last part, we give a fully implementable algorithm for quadratic BSDEs based on quantization and illustrate our convergence results with numerical examples.

Key words: Backward stochastic differential equations; generator of quadratic growth; time-discretization, numerical approximation.

MSC Classification (2000): 60H10, 65C30

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1 Introduction

In this paper, we are interested in the numerical approximation of solutions to a special class of backward stochastic differential equations (BSDEs for short in the sequel). Let us recall that solving a BSDE consists in finding an adapted couple (Y, Z) satisfying the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leqslant t \leqslant T,$$

where W is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ the Brownian filtration. In their seminal paper [38], Pardoux and Peng prove the existence of a unique solution (Y, Z) to this equation for a given square integrable terminal condition ξ and a Lipschitz random driver f. Many extensions to this Lipschitz setting have been considered. In particular, the class of BSDE, with generators of quadratic growth with respect to the variable z, has received a lot of attention in recent years. These equations arise, by example, in the context of utility optimization problems with exponential utility functions, or alternatively in questions related to risk minimization for the entropic risk measure (see e.g. [41, 27, 36] among many other references). Existence and uniqueness of solution for such BSDEs has been first proved by Kobylanski [34]. Since then, many authors worked on this question. When the terminal condition is bounded, we refer to [34, 35, 42, 7], and, in the unbounded case, we refer to [8, 3, 9, 20, 19].

We will focus here on the numerical approximation of the so-called 'quadratic BSDE' in a Markovian setting namely

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{1.1}$$

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{1.2}$$

Throughout this paper, we assume that the functions $b : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are K-Lipschitz continuous functions and the function g is a bounded K-Lipschitz continuous function, for a positive constant K. We also assume that the function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d} \to \mathbb{R}$ is a K-Lipschitz continuous function with respect to x and y i.e.

$$|f(x_1, y_1, z) - f(x_2, y_2, z)| \leqslant K(|x_1 - x_2| + |y_1 - y_2|)$$

for all $y_1, y_2 \in \mathbb{R}$, $x_1, x_2 \in \mathbb{R}^d$ and $z \in \mathbb{R}^{1 \times d}$, and a L-locally Lipschitz continuous function with respect to z: for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^{1\times d}$,

$$\left|f(x,y,z) - f(x,y,z')\right| \leqslant L(1 + |z| + \left|z'\right|) \left|z - z'\right|,$$

where L is a positive constant. Moreover f is bounded with respect to x: for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{1\times d}$,

$$|f(x,y,z)| \leqslant L(1 + |y| + |z|^2).$$

Let us notice that all convergence results obtained in this paper do not need extra assumptions on b, σ, f and g. Especially, we emphasize that no uniform ellipticity condition is necessary on σ.

1.1 Known results on the approximation of quadratic BSDEs

The design of efficient algorithms to solve BSDEs in any reasonable dimension has been intensively studied since the first work of Chevance [15], see e.g. [43, 5, 23, 11, 10] and the references therein. In all these articles, the driver f of the BSDE is a Lipschitz function with respect to z and this assumption plays a key role in the proofs.

Up to now, there have been few results on the time-discretization and numerical simulation of quadratic BSDEs. We review now all the techniques that allow to compute the solution of quadratic BSDEs, to the best of our knowledge. None of them provide a suitable complete answer to the approximation of the BSDE (1.2).

First of all, when the generator has a specific form (roughly speaking the generator is a sum of a purely quadratic term $z \mapsto C|z|^2$ and a Lipschitz function) it is possible to solve almost explicitly the quadratic BSDE by using an exponential transformation method, also called Cole-Hopf transform (see e.g. [29]).

It is also possible to solve some specific quadratic Markovian BSDEs by solving a fully coupled forward backward system, i.e. when Y and Z appear also in the coefficients of (1.1). This is the method used by Delarue and Menozzi in [17, 18] where they solved in particular the deterministic KPZ equation. But approximation results for fully coupled forward backward systems need strong assumptions on the regularity of coefficients and a uniform ellipticity assumption for σ. Moreover, their implementation is not straightforward (due to the coupling).

In some cases, one can also rely on 'classical' schemes for Lipschitz BSDEs in order to numerically solve quadratic BSDEs. Indeed, when the terminal condition g is a bounded Lipschitz-continuous function and σ is bounded then it is known that Z is bounded by a constant M (see e.g. Theorem 3.6 in [40]). Since the generator f is assumed to be locally Lipschitz with respect to z, one only needs to replace the generator f by a new generator $\tilde{f}_M(.,.,.) = f(.,.,\varphi_M(.))$ where φ_M is the projection on the centered Euclidean ball of radius M. Then, one can easily show that these two BSDEs with generators f and $\tilde{f}_M$ have the same solution. It is then possible to solve the second BSDE with Lipschitz driver $\tilde{f}_M$ to retrieve the solution to the quadratic BSDE. Let us remark that some exponential terms appear in the constant M which lead to a new generator with possibly huge Lipschitz constant with respect to z and may cause numerical difficulties, see [4].

In the general case, Z may be unbounded. Nevertheless, when g is a bounded Lipschitz function and σ is Lipschitz but not necessarily bounded the following non-uniform bound holds true

$$|Z_t| \leqslant C(1 + |X_t|), \quad \text{for all } t \leqslant T, \tag{1.3}$$

see e.g. Theorem 3.6 in [40].

Now, replacing the generator f with the Lipschitz generator $\tilde{f}_M$ we obtain a solution (Y^M, Z^M) which is different from (Y, Z). But it is possible to estimate the error between the two using the estimate on Z. The error is bounded by $\frac{C_p}{M^p}$ for every $p > 1$, see [28, 40]. Once again, since the new generator $\tilde{f}_M$ is Lipschitz, we can easily apply classical numerical approximation schemes for Lipschitz BSDEs. Problems occur when one tries to obtain a rate of convergence for this technique. The classical (squared) error estimate for the discrete-time approximation of Lipschitz BSDEs is $\frac{C}{n}$ with n the number of time steps, but the constant C depends strongly on the Lipschitz constant of $\tilde{f}_M$ with respect to z and so it depends on M, see e.g. [43, 5]. In fact, one obtains an upper bound for the time-discretization error (squared) of order $Ce^{CM^2}n^{-1}$, the exponential term resulting from the use of Gronwall's lemma. Finally, an upper bound of the global error (squared) equals to

$$\frac{C_p}{M^p} + \frac{Ce^{CM^2}}{n}.$$

When M increases, n^{-1} will have to be small exponentially fast. The resulting rate of convergence turns out to be bad: setting $M = (\log n)^{1/2}$ the global error bound becomes $C_p(\log n)^{-p}$ which is not satisfactory.

To circumvent the above difficulties, one can impose a specific growth assumption on σ, leading to exponential moment control on X, in order to retrieve a better bound for the error between (Y, Z) and (Y^M, Z^M). In this case, the global error becomes satisfactory, see Theorem 5.9 in [40]. Reasonable convergence rate can also be retrieved for unbounded locally Lipschitz-continuous terminal conditions, using estimates in the spirit of (1.3), but in the very restrictive case of constant σ, see Theorem 5.7 in [40]. Note that dealing with an unbounded terminal condition is already a challenge for the theoretical study of (1.2).

In this paper, we focus on *Lipschitz-continuous bounded terminal condition* and *unbounded Lipschitz-continuous σ*. This covers the case of models with great practical interest as geometric Brownian motion (Black-Scholes model). Using a similar truncation procedure as the one described above, we are able to obtain a bound on the time discretization error which does not depend on M. The global (squared) error bound is shown to be almost the classical one, that is to say $\frac{C_\epsilon}{n^{1-\epsilon}}$, for all $\epsilon > 0$.

Let us conclude this review with the case of non-Lipschitz bounded terminal condition. In this case - even in the Lipschitz setting for the generator - new difficulties arise in the simulation of BSDEs, see e.g. [24]. In the quadratic case, when σ is bounded, it is possible to use estimates of the form

$$|Z_t| \leqslant \frac{C}{\sqrt{T-t}}, \quad \text{or} \quad |Z_t| \leqslant \frac{C}{(T-t)^{(1-\alpha)/2}}$$

if the terminal condition is α-Hölder, see [16, 39]. Thanks to these estimates one can replace the generator f by a Lipschitz generator such that the Lipschitz constant with respect to z depends on time and blows up near the time T. The time discretization problem is addressed in [39] and the approximation of discretized BSDEs thanks to least-squares regression is tackled in the paper [25]. In these two papers the time-discretization grid is not uniform taking into account the estimates on Z. In particular, there are more points near the terminal time T than near the initial time. We think that it would be very interesting to try to extend our results and techniques in the case of irregular terminal conditions.

1.2 Main results of the paper

We now present in more depth our main results. As already mentioned, to tackle the problem of the numerical approximation of (1.2), we introduce a Lipschitz-continuous approximation of the driver f: $f_N(.,.,.) = f(.,.,\varphi_N(.))$ and φ_N is the projection on the centered Euclidean ball of radius ρN with $\rho > 0$ chosen such that f_N is N-Lipschitz-continuous with respect to z.

Given a grid $\pi = \{0 = t_0 < t_1 < ... < t_n = T\}$ of the time interval $[0, T]$, we define

$h_i = t_{i+1} - t_i$ the time-step between times t_i and t_{i+1}, and $h := \max_i h_i$ assuming that

$$hn \leqslant C \text{ and there exists } \theta \geqslant 1 \text{ such that } h_i n^\theta \geqslant C > 0\ ,\ 0 \leqslant i < n\,. \tag{1.4}$$

Here and in the sequel, C is a positive constant, which may change from line to line but which does not depend on n. We denote it C_p if it depends on an extra parameter p.

Definition 1.1. *We denote $(Y_i^\pi, Z_i^\pi)_{0\leqslant i\leqslant n}$ the solution of the BTZ[1]-scheme satisfying*
(i) the terminal condition is $(Y_n^\pi, Z_n^\pi) = (g(X_n^\pi), 0)$,
(ii) for $i < n$, the transition from step $i+1$ to step i is given by

$$\begin{cases} Y_i^\pi = \mathbb{E}_{t_i}\left[Y_{i+1}^\pi + h_i f_N(X_i^\pi, Y_i^\pi, Z_i^\pi)\right] \\ Z_i^\pi = \mathbb{E}_{t_i}\left[Y_{i+1}^\pi H_i^R\right], \end{cases} \tag{1.5}$$

where $\mathbb{E}_t[\cdot]$ stands for $\mathbb{E}[\cdot|\mathcal{F}_t]$, $0 \leqslant t \leqslant T$.
The discrete-time process $(X_i^\pi)_{0\leqslant i\leqslant n}$ is an approximation of $(X_t)_{t\in[0,T]}$ and we choose to work here with the Euler scheme:

$$\begin{cases} X_0^\pi &= x\ , \\ X_{i+1}^\pi &= X_i^\pi + b(X_i^\pi)h_i + \sigma(X_i^\pi)(W_{t_{i+1}} - W_{t_i}), \quad 0 \leqslant i < n\ . \end{cases}$$

The coefficients $(H_i^R)_{0\leqslant i<n}$ are some $\mathbb{R}^{1\times d}$ independent random vectors defined, given $R > 0$, by

$$(H_i^R)^\ell = \frac{-R}{\sqrt{h_i}} \vee \frac{W^\ell_{t_{i+1}} - W^\ell_{t_i}}{h_i} \wedge \frac{R}{\sqrt{h_i}}, \qquad 1 \leqslant \ell \leqslant d. \tag{1.6}$$

We observe that $(H_i^R)_{0\leqslant i<n}$ satisfies

$$\mathbb{E}_{t_i}[H_i^R] = 0,\ h_i\mathbb{E}_{t_i}\left[(H_i^R)^\top H_i^R\right] = h_i\mathbb{E}\left[(H_i^R)^\top H_i^R\right] = c_i I_{d\times d} \text{ and } \frac{\lambda}{d} \leqslant c_i \leqslant \frac{\Lambda}{d}, \tag{1.7}$$

where λ, Λ are positive constants that do not depend on R, for R big enough. Moreover, it is well known (see e.g. [33]) that, under the Lipschitz continuity assumption on b and σ,

$$\mathbb{E}\left[\sup_{0\leqslant i\leqslant n} |X_i^\pi|^{2p}\right] \leqslant C_p \quad \text{and} \quad \max_{0\leqslant i\leqslant n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |X_t - X_i^\pi|^{2p}\right] \leqslant C_p h^p\ , \quad p \geqslant 1\,. \tag{1.8}$$

Combining (1.7), (1.8) and the Lipschitz continuity property of f_N, an easy induction argument proves that (Y^π, Z^π) are square integrable and thus conditional expectations involved at each step of the algorithm are well defined. Moreover, assuming $Kh < 1$ allows for the implicit definition of Y_i^π, $i < n$.

The first main result of the paper is the following theorem.

[1] Bouchard-Touzi-Zhang, the first authors to consider this scheme, see [43, 5].

Theorem 1.1. *Setting, for some $\alpha \in (0, 1/2)$,*

$$N = n^{\alpha} \quad \textit{and} \quad R = \log(n), \tag{1.9}$$

we have, for all $\eta > 0$,

$$\mathbb{E}\left[\sup_{0 \leqslant i \leqslant n} |Y_{t_i} - Y_i^{\pi}|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s - Z_i^{\pi}|^2 \, ds\right] \leqslant C_{\alpha,\eta} h^{1-\eta}.$$

The choice of N and R as specific functions of n will be made clear in the following. The truncation procedure guarantees the stability of the scheme. Letting these constants grow with n guarantees the convergence of the scheme. Obviously, a good balance between the two has to be found.
To obtain this theorem, we first prove stability results for the scheme given in Definition 1.1. This is a priori not straightforward because the Lipschitz constant explodes. In order to do this, we use a linearization argument leading to a comparison theorem and relying on BMO martingales tools. We then study carefully the truncation error induced by the time-discretization. In particular, we have to revisit Zhang's path regularity result.

One has to observe that the above scheme is still a theoretical one since it assumes a perfect computation of the conditional expectations. In practice, these conditional expectations have to be estimated. Many methods can be used and Theorem 1.1 is a key step toward a complete convergence analysis.

In this paper, we chose to compute the conditional expectation using a Markovian quantization procedure which is now quite well known. We refer to [26, 37] for general results about quantization and [2] for application to American options pricing and to [17] for application to coupled forward-backward SDEs. We present in Section 4 a fully implementable numerical scheme and prove the following upper bound for the convergence error:

$$|Y_0 - \hat{Y}_0^{\pi}| \leqslant C_{\alpha,\eta} h^{\frac{1}{2}-\eta}, \quad \text{for all } \eta > 0 \ ,$$

with $(\hat{Y}^{\pi}, \hat{Z}^{\pi})$ the solution of the scheme (1.1) where conditional expectations are replaced by implementable approximations. See Corollary 4.1 for a suitable choice of parameters.

The rest of this paper is organised as follows. In Section 2, we introduce the linearization tool for discrete schemes and we obtain some very useful estimates on (Y^{π}, Z^{π}) together with some stability results. Section 3 is devoted to the convergence analysis of the time discretization for quadratic BSDEs. In the last section, we give a fully implementable scheme, we study its convergence error and we provide some numerical illustrations.

2 Preliminary results

First of all, let us recall that under the assumptions on the generator f and the terminal condition g given in the previous section, existence and uniqueness result holds for (1.1)-(1.2). Moreover, the solution is known to have the following properties, see e.g. [34, 6, 1].

Proposition 2.1. *The FBSDE* (1.1)-(1.2) *has a unique solution* $(X, Y, Z) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times \mathcal{M}^2$. *Moreover, the martingale* $(\int_0^t Z_s dW_s)_{t\in[0,T]}$ *belongs to the space of BMO martingales. The* $\mathcal{S}^\infty$ *norm of* Y *and the BMO norm of* $(\int_0^t Z_s dW_s)_{t\in[0,T]}$ *are bounded by a constant that depends only on* T, $|g|_\infty$, *and the constant that appears in the growth assumption on the generator* f.

BMO martingales theory plays a key role for a priori estimates needed in our study. For details about the theory we refer the reader to [32]. We now recall the definition of a BMO martingale and introduce some notations. Let $(M_t)_{0\leqslant t\leqslant T}$ be a martingale for the filtration $(\mathcal{G}_t)_{0\leqslant t\leqslant T}$. We say that M is a BMO martingale if it is a square integrable martingale such that

$$\|M\|^2_{BMO(\mathcal{G})} := \sup_\tau \mathbb{E}\left[|M_T - M_{\tau^-}|^2 \,\middle|\mathcal{G}_\tau\right] < +\infty$$

where the supremum is taken over all stopping times $\tau \in [0, T]$.

2.1 Lipschitz approximation

We first recall a key result concerning the Lipschitz approximation of quadratic BSDEs. We introduce $(Y_t^N, Z_t^N)_{t\in[0,T]}$ the solution of the following BSDE

$$Y_t^N = g(X_T) + \int_t^T f_N(X_s, Y_s^N, Z_s^N)ds - \int_t^T Z_s^N dW_s \tag{2.1}$$

recalling that $f_N(.,.,.) = f(.,.,\varphi_N(.))$ and φ_N is the projection on the centered Euclidean ball of radius ρN with $\rho > 0$ chosen such that f_N is N-Lipschitz with respect to z.

Remark 2.1. The results of Proposition 2.1 hold true for processes (X, Y^N, Z^N). Importantly the $\mathcal{S}^\infty$ norm of Y^N and the BMO norm of $\left(\int_0^t Z_s^N dW_s\right)_{t\in[0,T]}$ are bounded by a constant that does not depend on N.

Theorem 2.2. *For all* $q > 0$ *and* $p \geqslant 1$, *there exists a constant* $C_{q,p} > 0$ *such that*

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} \left|Y_t - Y_t^N\right|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T \left|Z_s - Z_s^N\right|^2 ds\right)^p\right] \leqslant \frac{C_{q,p}}{N^q}.$$

The proof of this theorem is given by Theorem 6.2 in [28] (see also Remark 5.5 in [40]).

Remark 2.2. The control of the above error in terms of any power of N^{-1} legitimates the choice to set $N := n^\alpha$ for some $\alpha > 0$.

The above result is strongly linked to the following estimate on Z, and on Z^N, proved e.g. in [40], stated here for later use.

Proposition 2.3. *Under our standard assumptions, for all $t \in [0,T]$ and all $N > 0$,*

$$\left|Z_t^N\right| + |Z_t| \leqslant C(1 + |X_t|).$$

Importantly, C does not depend on N.

We conclude this section by two technical lemmas.

Lemma 2.1. *Setting, for all $i < n$,*

$$\bar{Z}_i^N := \frac{1}{h_i}\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s^N ds\right] , \tag{2.2}$$

then

$$\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\bar{Z}_j^N\right|^2\right] \leqslant C \quad \text{and} \quad \left|\bar{Z}_i^N\right| \leqslant C\left(1 + \mathbb{E}_{t_i}\left[\sup_{t_i \leqslant s \leqslant t_{i+1}} |X_s|\right]\right) .$$

Proof. 1. For the first claim, we observe that, for $i \leqslant j < n$,

$$\mathbb{E}_{t_i}\left[\left|\bar{Z}_j^N\right|^2\right] \leqslant \frac{1}{h_j}\mathbb{E}_{t_i}\left[\int_{t_j}^{t_{j+1}} \left|Z_s^N\right|^2 \mathrm{d}s\right].$$

Summing over j the previous inequality and using Remark 2.1, we obtain

$$\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\bar{Z}_j^N\right|^2\right] \leqslant \mathbb{E}_{t_i}\left[\int_{t_i}^{T} \left|Z_s^N\right|^2 ds\right] \leqslant \left\|\int_0^\cdot Z_s^N dW_s\right\|_{BMO(\mathcal{F})} \leqslant C.$$

2. For the second claim, we compute

$$\left|\bar{Z}_i^N\right| = \frac{1}{h_i}\left|\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s^N ds\right]\right| \leqslant \mathbb{E}_{t_i}\left[\sup_{t_i \leqslant s \leqslant t_{i+1}} \left|Z_s^N\right|\right] \leqslant C\left(1 + \mathbb{E}_{t_i}\left[\sup_{t_i \leqslant s \leqslant t_{i+1}} |X_s|\right]\right)$$

where we used Proposition 2.3. □

Lemma 2.2. *We assume that $\alpha \leqslant 1/2$. Setting, for all $i < n$,*

$$\tilde{Z}_i^N := \mathbb{E}_{t_i}\left[Y_{t_{i+1}}^N \frac{(W_{t_{i+1}} - W_{t_i})^\top}{h_i}\right], \tag{2.3}$$

then

$$\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\tilde{Z}_j^N\right|^2\right] \leqslant C \quad \text{and} \quad \left|\tilde{Z}_i^N\right| \leqslant C\left(1 + \mathbb{E}_{t_i}\left[\sup_{t_i \leqslant s \leqslant t_{i+1}} |X_s|^4\right]^{1/2}\right).$$

Proof. 1. For the first claim, we observe that

$$\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\tilde{Z}_j^N\right|^2\right] \leqslant 2\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\bar{Z}_j^N\right|^2\right] + 2\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\bar{Z}_j^N - \tilde{Z}_j^N\right|^2\right].$$

The first term was already studied in Lemma 2.1. For the second term we compute, thanks to assumptions on f_N, Remark 2.1 and Cauchy-Schwarz inequality, for $i \leqslant j < n$,

$$\begin{aligned}
h_j \mathbb{E}_{t_i}\left[\left|\bar{Z}_j^N - \tilde{Z}_j^N\right|^2\right] &= h_j \mathbb{E}_{t_i}\left[\left|\mathbb{E}_{t_j}\left[\int_{t_j}^{t_{j+1}} f_N(X_s, Y_s^N, Z_s^N) ds \frac{W_{t_{j+1}} - W_{t_j}}{h_j}\right]\right|^2\right] \\
&\leqslant h_j \mathbb{E}_{t_i}\left[\int_{t_j}^{t_{j+1}} \left|f_N(X_s, Y_s^N, Z_s^N)\right|^2 ds\right] \\
&\leqslant C\left(h^2 + (1 + N^2 h)\mathbb{E}_{t_i}\left[\int_{t_j}^{t_{j+1}} \left|Z_s^N\right|^2 ds\right]\right).
\end{aligned}$$

Summing over j, we obtain

$$\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} h_j \left|\bar{Z}_j^N - \tilde{Z}_j^N\right|^2\right] \leqslant C\left(1 + \left\|\int_0^\cdot Z_s^N dW_s\right\|_{BMO(\mathcal{F})}^2\right) \leqslant C.$$

2. For the second claim, once again we have

$$\left|\tilde{Z}_i^N\right| \leqslant \left|\bar{Z}_i^N\right| + \left|\bar{Z}_i^N - \tilde{Z}_i^N\right|.$$

The first term is dealt with combining Lemma 2.1 and Cauchy-Schwarz inequality. For the second term, we compute, thanks to the growth assumption on f_N, Remark 2.1, Proposition 2.3 and Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\bar{Z}_i^N - \tilde{Z}_i^N\right| &\leqslant C\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} |f_N(X_s, Y_s^N, Z_s^N)| ds \frac{|W_{t_{i+1}} - W_{t_i}|}{h_j}\right] \qquad (2.4) \\
&\leqslant C\mathbb{E}_{t_i}\left[(1 + \sup_{t_i \leqslant s \leqslant t_{i+1}} |X_s|^2)|W_{t_{i+1}} - W_{t_i}|\right] \\
&\leqslant C h^{1/2}\left(1 + \mathbb{E}_{t_i}\left[\sup_{t_i \leqslant s \leqslant t_{i+1}} |X_s|^4\right]^{1/2}\right). \qquad (2.5)
\end{aligned}$$

□

2.2 Linearization of the BTZ scheme

Definition 2.1. *We consider the solution* $(Y_i, Z_i)_{0\leqslant i\leqslant n}$ *of the following BTZ scheme:*

(i) the terminal condition is given by $Y_n = \xi$ *for some* $\xi \in L^2(\mathcal{F}_T)$ *and* $Z_n = 0$*;*

(ii) for $0 \leqslant i < n$*, the transition from step* $i+1$ *to step* i *is given by*

$$\begin{cases} Y_i = \mathbb{E}_{t_i}\left[Y_{i+1} + h_i F_i(Y_i, Z_i)\right] \\ Z_i = \mathbb{E}_{t_i}\left[Y_{i+1} H_i\right], \end{cases}$$

with $(H_i)_{0\leqslant i<n}$ *some* $\mathbb{R}^{1\times d}$ *independent random vectors such that, for all* $0 \leqslant i < n$*,* H_i *is* $\mathcal{F}_{t_{i+1}}$ *measurable,* $\mathbb{E}_{t_i}[H_i] = 0$*,*

$$c_i I_{d\times d} = h_i \mathbb{E}\Big[H_i^\top H_i\Big] = h_i \mathbb{E}_{t_i}\left[H_i^\top H_i\right], \tag{2.6}$$

and

$$\frac{\lambda}{d} \leqslant c_i \leqslant \frac{\Lambda}{d}, \tag{2.7}$$

where λ*,* Λ *are positive constants. Let us remark that* (2.6) *and* (2.7) *imply that*

$$\lambda \leqslant h_i \mathbb{E}\big[|H_i|^2\big] = h_i \mathbb{E}_{t_i}\left[|H_i|^2\right] \leqslant \Lambda. \tag{2.8}$$

For the reader's convenience, we denote the above scheme by $\mathcal{E}[(F_i), \xi]$.

In the sequel, we use the following assumption on the coefficients of the scheme given in Definition 2.1.

Assumption (H1)

(i) Functions $F_i : \Omega \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$ are $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. They satisfy for some positive constants K_y and K_z^n which do not depend on i but K_z^n may depend on n,

- $F_i(0,0) \in L^2(\mathcal{F}_{t_i})$,
- $|F_i(y,z) - F_i(y',z')| \leqslant K_y|y-y'| + K_z^n|z-z'|$.

(ii) For a given $\varepsilon \in]0,1[$ which does not depend on n, we have that

$$hK_y < 1 - \varepsilon .$$

(iii) The following holds

$$\left(\sup_{0\leqslant i\leqslant n-1} h_i\,|H_i|\right) K_z^n < 1.$$

Observe that $(\mathbf{H}1)$(ii) guarantees the well-posedness of the scheme.
We now give a representation result for the difference of two BTZ scheme solutions. Let $(Y_i^1, Z_i^1)_{0\leqslant i\leqslant n}$ be the solution of $\mathcal{E}[(F_i^1),\xi^1]$ and $(Y_i^2, Z_i^2)_{0\leqslant i\leqslant n}$ be the solution of $\mathcal{E}[(F_i^2),\xi^2]$.
We denote $\delta Y_i = Y_i^1 - Y_i^2$, $\delta Z_i = Z_i^1 - Z_i^2$ and $\delta F_i = F_i^1(Y_i^2, Z_i^2) - F_i^2(Y_i^2, Z_i^2)$. Then, we have the following representation result.

Proposition 2.4 (Euler scheme linearization)**.** *Assume that F^1 satisfies $(\mathbf{H}1)$(i)-(ii). Setting, for $0\leqslant i\leqslant n$,*

$$E_i^\pi = \prod_{j=i}^{n-1}(1+h_jH_j\gamma_j) \textit{ and } B_i^\pi = \prod_{j=i}^{n-1}(1-h_j\beta_j)\,,$$

with

$$\beta_j = \frac{F_j^1(Y_j^1, Z_j^1) - F_j^1(Y_j^2, Z_j^1)}{Y_j^1 - Y_j^2}\mathbf{1}_{\{Y_j^1 - Y_j^2\neq 0\}}$$

and

$$\gamma_j = \frac{F_j^1(Y_j^2, Z_j^1) - F_j^1(Y_j^2, Z_j^2)}{\left|Z_j^1 - Z_j^2\right|^2}\left(Z_j^1 - Z_j^2\right)^\top \mathbf{1}_{\{Z_j^1 - Z_j^2\neq 0\}}\,,$$

then the following holds,

$$\delta Y_i = \mathbb{E}_{t_i}\!\left[E_i^\pi (B_i^\pi)^{-1}\left(\delta Y_n + \sum_{k=i}^{n-1} h_k B_{k+1}^\pi \delta F_k\right)\right]\,. \tag{2.9}$$

We used the convention $\prod_{j=n}^{n-1}\cdot = 1$.

Proof of Proposition 2.4. For $0\leqslant i\leqslant n-1$, we compute that

$$\delta Y_i = \mathbb{E}_{t_i}\left[\delta Y_{i+1} + h_i\beta_i\delta Y_i + h_i\delta Z_i\gamma_i + h_i\delta F_i\right]. \tag{2.10}$$

Observing that $\delta Z_i = \mathbb{E}_{t_i}[H_i\delta Y_{i+1}]$, we obtain

$$\begin{aligned}\delta Y_i &= \frac{1}{1-h_i\beta_i}\mathbb{E}_{t_i}\left[(1+h_iH_i\gamma_i)\delta Y_{i+1} + h_i\delta F_i\right]\\ &= \frac{1}{1-h_i\beta_i}\mathbb{E}_{t_i}\left[(1+h_iH_i\gamma_i)\left(\delta Y_{i+1} + h_i\delta F_i\right)\right].\end{aligned}$$

Under (**H**1)(ii), we observe that $1 - h_i\beta_i \neq 0$ and the previous equality is well defined. Using an easy induction argument we obtain

$$\delta Y_i = \mathbb{E}_{t_i}\left[E_i^\pi (B_i^\pi)^{-1}\left(\delta Y_n + \sum_{k=i}^{n-1} h_k (E_{k+1}^\pi)^{-1} B_{k+1}^\pi \delta F_k\right)\right] .$$

The proof is concluded using the tower property of conditional expectation and the fact that $\mathbb{E}_{t_{k+1}}[E_{k+1}^\pi] = 1$. □

The previous representation leads to the following comparison result for the BTZ scheme.

Corollary 2.5 (Comparison theorem)**.** *Assume that F^1 satisfies* (**H**1)*. If*

$$Y_n^1 \geqslant Y_n^2 \quad \text{and} \quad F_i^1(Y_i^2, Z_i^2) \geqslant F_i^2(Y_i^2, Z_i^2), \quad 0 \leqslant i \leqslant n-1,$$

then we have that

$$Y_i^1 \geqslant Y_i^2, \quad 0 \leqslant i \leqslant n.$$

Proof of Corollary 2.5. We will use the BTZ scheme linearization given in Proposition 2.4. Since $|\beta_i| \leqslant K_y$ and $|\gamma_i| \leqslant K_z^n$, the condition $(\sup_{0 \leqslant i < n} h_i\,|H_i|)K_z^n < 1$ combined with $hK_y < 1$, implies that the coefficients E_i^π, B_i^π are positive, for $i < n$. Moreover, we assume that

$$Y_n^1 \geqslant Y_n^2 \quad \text{and} \quad F_i^1(Y_i^2, Z_i^2) \geqslant F_i^2(Y_i^2, Z_i^2), \quad 0 \leqslant i \leqslant n-1,$$

so we have

$$\delta Y_n \geqslant 0 \quad \text{and} \quad \delta F_i \geqslant 0, \quad 0 \leqslant i \leqslant n-1.$$

Thus, (2.9) gives us for all $0 \leqslant i \leqslant n$

$$\delta Y_i = \mathbb{E}_{t_i}\left[E_i^\pi (B_i^\pi)^{-1}\left(\delta Y_n + \sum_{k=i}^{n-1} h_k B_{k+1}^\pi \delta F_k\right)\right] \geqslant 0.$$

□

Remark 2.3. (i) As for the classical comparison theorem, the previous result stays true if we replace the condition

$$F^1 \text{ satisfies } (\mathbf{H}1) \quad \text{and} \quad F_i^1(Y_i^2, Z_i^2) \geqslant F_i^2(Y_i^2, Z_i^2), \quad 0 \leqslant i \leqslant n-1,$$

with

$$F^2 \text{ satisfies } (\mathbf{H}1) \quad \text{and} \quad F_i^1(Y_i^1, Z_i^1) \geqslant F_i^2(Y_i^1, Z_i^1), \quad 0 \leqslant i \leqslant n-1.$$

(ii) The comparison result for BSΔEs is already proved in [14] but without using the scheme linearization.
(iii) The truncation of the generator is essential to make the comparison theorem hold: Example 4.1 in [13] shows that comparison fails for quadratic BSΔEs with bounded terminal condition.

2.3 A priori estimates (in the quadratic case)

In this part we establish some a priori estimates for the solution of the BTZ scheme given by Definition 2.1 with quadratic generator. More precisely we show that classical a priori estimates for quadratic BSDEs stay true for the corresponding BTZ scheme under suitable conditions. We consider schemes with essentially bounded terminal condition ξ and coefficients F satisfying more restrictive assumptions.

Assumption ($\mathbf{H}2$)

(i) $\xi \in L^\infty(\mathcal{F}_T)$ and $(F_i)_{0\leqslant i\leqslant n-1}$ satisfy ($\mathbf{H}1$),

(ii) $F_i(0,0) \in L^\infty(\mathcal{F}_{t_i})$ for all $0 \leqslant i \leqslant n-1$ and there exists a constant $\tilde{C}$ that does not depend on n and such that

$$\sup_{0\leqslant i\leqslant n} |F_i(0,0)| \leqslant \tilde{C},$$

(iii) there exist three positive constants K_y, $\tilde{L}$ and $\tilde{\Lambda}$ that do not depend on n and such that

$$|F_i(y,z)| \leqslant K_y|y| + \tilde{L}|z|^2 + \varsigma_i \quad \text{with } \mathbb{E}_{t_i}\left[\sum_{k=i}^n h_k|\varsigma_k|\right] \leq \tilde{\Lambda}. \tag{2.11}$$

The first key estimate is related to the uniform boundedness in n of $(Y_i)_{0\leqslant i\leqslant n}$.

Proposition 2.6. *Assume* ($\mathbf{H}2$)*(i)-(ii) holds true. Then,*

$$|Y_i| \leqslant \left(|\xi|_\infty + T \sup_{0\leqslant i\leqslant n-1} |F_i(0,0)|_\infty\right) e^{CK_y/\varepsilon} \leqslant \left(|\xi|_\infty + T\tilde{C}\right) e^{CK_y/\varepsilon}.$$

Proof of Proposition 2.6. We introduce $(Y_i^2, Z_i^2)_{0\leqslant i\leqslant n}$ the solution of the BTZ scheme $\mathcal{E}[(F_i^2), |\xi|_\infty]$ with $F_i^2(y,z) = |F_i(0,0)|_\infty + K_y\,|y|$. We observe that the terminal condition and the generator of this scheme are deterministic functions which implies that $Z_i^2 = 0$ for all $0 \leqslant i \leqslant n$. We are able to compare F_i and F_i^2 under ($\mathbf{H}2$)(i)-(ii):

$$F_i(Y_i^2, Z_i^2) = F_i(Y_i^2, 0) \leqslant |F_i(0,0)|_\infty + K_y\left|Y_i^2\right| = F_i^2(Y_i^2, Z_i^2).$$

Since $\xi \leqslant |\xi|_\infty$ we can apply the comparison theorem given in Corollary 2.5:

$$\begin{aligned} Y_i &\leqslant Y_i^2 = \frac{|\xi|_\infty}{\prod_{k=i}^{n-1}(1-h_k K_y)} + \sum_{j=i}^{n-1} \frac{h_j |F_j(0,0)|_\infty}{\prod_{k=i}^{j}(1-h_k K_y)} \\ &\leqslant |\xi|_\infty \left(1 + \frac{hK_y}{\varepsilon}\right)^{n-i} + \sum_{j=i}^{n-1} h_j |F_j(0,0)|_\infty \left(1+\frac{hK_y}{\varepsilon}\right)^{j-i+1} \\ &\leqslant \left(|\xi|_\infty + T \sup_{0\leqslant j\leqslant n-1} |F_j(0,0)|_\infty\right) e^{CK_y/\varepsilon}. \end{aligned}$$

Using similar arguments, we obtain that

$$Y_i \geqslant \left(-|\xi|_\infty - T \sup_{0\leqslant j\leqslant n-1} |F_j(0,0)|_\infty\right) e^{CK_y/\varepsilon}$$

which concludes the proof. □

The second estimate is related to $(Z_i)_{0\leqslant i\leqslant n}$.

Proposition 2.7. *Under* $(\mathbf{H}2)$*, we have that*

$$\mathbb{E}_{t_i}\left[\sum_{k=i}^{n-1} h_k |Z_k|^2\right] \leqslant C, \quad 0 \leqslant i \leqslant n-1.$$

Proof. Since $(\mathbf{H}2)$ holds, we can apply Proposition 2.6 and get

$$\sup_{0\leqslant i\leqslant n} |Y_i| \leqslant \left(|\xi|_\infty + T\tilde{C}\right) e^{CK_y/\varepsilon} := m.$$

We split the proof in two steps, depending on the value of m.
1. In this first step, we assume that

$$2m\tilde{L} \leqslant \frac{d}{2\Lambda} \ . \tag{2.12}$$

We observe that the BTZ scheme can be rewritten

$$Y_i = Y_{i+1} + h_i F_i(Y_i, Z_i) - h_i c_i^{-1} Z_i H_i^\top - \Delta M_i \ ,$$

where c_i is given by (2.6) and ΔM_i is an $\mathcal{F}_{t_{i+1}}$-measurable random variable satisfying $\mathbb{E}_{t_i}[\Delta M_i] = 0$, $\mathbb{E}_{t_i}\big[|\Delta M_i|^2\big] < \infty$ and $\mathbb{E}_{t_i}[\Delta M_i H_i] = 0$. Using the identity $|y|^2 = |x|^2 + 2x(y-x) + |y-x|^2$, we obtain, setting $x = Y_i$ and $y = Y_{i+1}$,

$$\begin{aligned} |Y_{i+1}|^2 &= |Y_i|^2 + 2Y_i\left(-h_i F_i(Y_i,Z_i) + h_i c_i^{-1} Z_i H_i^\top + \Delta M_i\right) \\ &\quad + \left|-h_i F_i(Y_i,Z_i) + h_i c_i^{-1} Z_i H_i^\top + \Delta M_i\right|^2 \ . \end{aligned}$$

Taking the conditional expectation w.r.t. $\mathcal{F}_{t_i}$ in the previous equality, we obtain using ($\mathbf{H}2$)(iii) and (2.6),

$$\begin{aligned}
\mathbb{E}_{t_i}\left[|Y_{i+1}|^2\right] &\geqslant |Y_i|^2 - 2Y_ih_iF_i(Y_i,Z_i) + \mathbb{E}_{t_i}\left[\left|h_ic_i^{-1}Z_iH_i^\top\right|^2\right]\\
&\geqslant |Y_i|^2 - 2mh_i\left(K_ym + \tilde{L}\,|Z_i|^2 + |\varsigma_i|\right) + h_i(c_i)^{-2}Z_ih_i\mathbb{E}_{t_i}\left[H_i^\top H_i\right]Z_i^\top\\
&\geqslant |Y_i|^2 - 2mh_i\left(K_ym + \tilde{L}\,|Z_i|^2 + |\varsigma_i|\right) + h_i(c_i)^{-1}\,|Z_i|^2\\
&\geqslant |Y_i|^2 - 2m^2K_yh_i + (\frac{d}{\Lambda} - 2m\tilde{L})h_i\,|Z_i|^2 - 2mh_i\,|\varsigma_i|\,.
\end{aligned}$$

Finally, an easy induction over i allows to obtain

$$\begin{aligned}
\mathbb{E}_{t_i}\left[\sum_{k=i}^{n-1}h_k|Z_k|^2\right] &\leqslant \frac{1}{d/\Lambda - 2m\tilde{L}}\left(\mathbb{E}_{t_i}\left[|Y_n|^2\right] - |Y_i|^2 + 2m^2K_yT + 2m\tilde{\Lambda}\right)\\
&\leqslant \frac{2m^2 + 2m^2K_yT + 2m\tilde{\Lambda}}{d/\Lambda - 2m\tilde{L}}.
\end{aligned}$$

Since the previous bound does not depend on n, the result is proved in this special case.
2.a. To prove the result in the general case, we use similar arguments as in [42]: we cut ξ and $(F_i(0,0))$ in pieces small enough such that we are able to use step 1. Let us set an integer $\kappa \in \mathbb{N}^*$ that does not depend on n and such that

$$\frac{4m\tilde{L}}{\kappa} \leqslant \frac{d}{2\Lambda}. \tag{2.13}$$

For each $a \in \{1,...,\kappa\}$, we denote $(Y_i^a, Z_i^a)_{0\leqslant i\leqslant n}$ the solution of $\mathcal{E}[(\Phi_i^a),\xi^a]$ with $\xi^a = \frac{\xi}{\kappa}$ and

$$\Phi_i^a(y,z) = F_i\left(y + \sum_{q=1}^{a-1}Y_i^q, z + \sum_{q=1}^{a-1}Z_i^q\right) - F_i\left(\sum_{q=1}^{a-1}Y_i^q, \sum_{q=1}^{a-1}Z_i^q\right) + \frac{F_i(0,0)}{\kappa}.$$

We observe that

$$Y_i = \sum_{a=1}^{\kappa}Y_i^a \quad \text{and} \quad Z_i = \sum_{a=1}^{\kappa}Z_i^a. \tag{2.14}$$

Since ($\mathbf{H}2$)(i)-(ii) holds true for (Φ_i^a) and ξ^a, we can apply Proposition 2.6 and remark that

$$\begin{aligned}
\sup_{0\leqslant i\leqslant n}|Y_i^a| &\leqslant \left(|\xi^a|_\infty + \sup_{0\leqslant i\leqslant n-1}|\Phi_i^a(0,0)|_\infty T\right)e^{CK_y/\varepsilon}\\
&\leqslant \left(\frac{|\xi|_\infty}{\kappa} + \frac{\sup_{0\leqslant i\leqslant n-1}|F_i(0,0)|_\infty}{\kappa}T\right)e^{CK_y/\varepsilon}\\
&\leqslant \frac{m}{\kappa}.
\end{aligned} \tag{2.15}$$

2.b. In this last step, we use an induction argument to show

$$\mathbb{E}_{t_i}\left[\sum_{k=i}^{n-1} h_k |Z_k^a|^2\right] \leqslant C, \qquad 0 \leqslant i < n, \tag{2.16}$$

for all $a \in \{1, ..., \kappa\}$. Combined with (2.14), this proves the proposition in the general case. We have proved in the first step that (2.16) is true for $a = 1$. Now let us assume that it is true up to $a < \kappa$. Then we compute that

$$\begin{aligned}|\Phi_i^{a+1}(y,z)| &\leqslant \left|F_i\left(y + \sum_{q=1}^a Y_i^q, z + \sum_{q=1}^a Z_i^q\right)\right| + \left|F_i\left(\sum_{q=1}^a Y_i^q, \sum_{q=1}^a Z_i^q\right)\right| + \frac{|F_i(0,0)|}{\kappa} \\ &\leqslant K_y|y| + 2\tilde{L}|z|^2 + \varsigma_i^a\end{aligned}$$

where $\varsigma_i^a = 2K_y|\sum_{q=1}^a Y_i^q| + 3\tilde{L}|\sum_{q=1}^a Z_i^q|^2 + 2|\varsigma_i| + |F_i(0,0)|_\infty/\kappa$. Assumption ($\mathbf{H}2$)(iii), bound (2.15) and the induction hypothesis yield that $\mathbb{E}_{t_i}[\sum_{k=i}^n h_k|\varsigma_k^a|] \leqslant C$ for all $0 \leqslant i < n$. Then, we have that Φ^{a+1} satisfies assumption ($\mathbf{H}2$) with $2\tilde{L}$ instead of $\tilde{L}$ and ς^a instead of ς. Since we have assumed that (2.13) holds true, then we can apply step 1. to obtain

$$\mathbb{E}_{t_i}\left[\sum_{k=i}^{n-1} h_k |Z_k^{a+1}|^2\right] \leqslant C, \qquad 0 \leqslant i < n,$$

which concludes the proof. □

We conclude this section by applying previous results to the scheme given in Definition 1.1.

Corollary 2.8. *Under assumptions of Theorem 1.1 the following holds true, for n large enough,*

$$\sup_{0 \leqslant i \leqslant n}\left(|Y_i^\pi| + \mathbb{E}_{t_i}\left[\sum_{k=i}^{n-1} |Z_k^\pi|^2 h_k\right]\right) \leqslant C.$$

Proof. We simply observe that with our special choice of parameters R and N, we have for n large enough

$$\left(\sup_{0 \leqslant i \leqslant n-1} h_i \left|H_i^R\right|\right) n^\alpha \leqslant \sqrt{h}\sqrt{d} R n^\alpha \leqslant \frac{C\sqrt{d}\log n}{n^{1/2-\alpha}} < 1,$$

and that the generator of the scheme given in Definition 1.1 satisfies ($\mathbf{H2}$) (with $K_z^n = N := n^\alpha$). The result follows then from a direct application of Proposition 2.6 and Proposition 2.7. □

Remark 2.4. In a slightly different framework, Gobet and Turkedjiev have already obtained the Corollary 2.8 in [25] by direct calculations without using the linearization technique.

2.4 Scheme stability

In this part we will establish some bounds on the difference between two schemes. Firstly, we introduce a perturbed version of the scheme given in Definition 2.1.

Definition 2.2. *(i) The terminal condition is given by $\tilde{Y}_n = \tilde{\xi}$ for some $\tilde{\xi} \in L^\infty(\mathcal{F}_T)$ and $\tilde{Z}_n = 0$;*

(ii) for $0 \leqslant i < n$

$$\begin{cases} \tilde{Y}_i = \mathbb{E}_{t_i}\left[\tilde{Y}_{i+1} + h_i F_i(\tilde{Y}_i, \tilde{Z}_i)\right] + \zeta_i^Y \\ \tilde{Z}_i = \mathbb{E}_{t_i}\left[\tilde{Y}_{i+1} H_i\right]. \end{cases}$$

Perturbations ζ_i^Y are $\mathcal{F}_{t_i}$-measurable and square integrable random variables. Moreover, we assume that

$$\sup_{0 \leqslant i < n} \mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} \left|\tilde{Z}_j\right|^2 h_j\right] < C. \tag{2.17}$$

2.4.1 Stability results for the Y component

Setting $\delta Y_i := Y_i - \tilde{Y}_i$ and $\delta Z_i := Z_i - \tilde{Z}_i$, we obtain a key stability result for the Y component.

Proposition 2.9. *Assume that assumption* ($\mathbf{H}1$) *holds true. Then, for all $0 \leqslant i \leqslant n$,*

$$|\delta Y_i| \leqslant C \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[|\delta Y_n| + \sum_{j=i}^{n-1} |\zeta_j^Y|\right]$$

where

$$\frac{\mathrm{d}\mathbb{Q}^\pi}{\mathrm{d}\mathbb{Q}} = E_0^\pi = \prod_{j=0}^{n-1}(1 + h_j H_j \gamma_j)$$

and

$$\gamma_j = \frac{F_j(\tilde{Y}_j, Z_j) - F_j(\tilde{Y}_j, \tilde{Z}_j)}{\left|Z_j - \tilde{Z}_j\right|^2} \left(Z_j - \tilde{Z}_j\right)^\top \mathbf{1}_{\{Z_j - \tilde{Z}_j \neq 0\}}. \tag{2.18}$$

Proof. Using the Euler scheme linearization given in Proposition 2.4 and observing $\delta F_k = \frac{-\zeta_k^Y}{h_k}$, it follows from (2.9) that

$$|\delta Y_i| \leqslant \mathbb{E}_{t_i}\left[|E_i^\pi|\,|B_i^\pi|^{-1}\left(|\delta Y_n| + \sum_{k=i}^{n-1} \left|B_{k+1}^\pi\right| \left|\zeta_k^Y\right|\right)\right].$$

Moreover,

$$\left|B_i^\pi\right|^{-1}\left|B_{k+1}^\pi\right| \leqslant \left(\frac{1}{1-hK_y}\right)^{k+1-i} \leqslant \left(1+\frac{hK_y}{\varepsilon}\right)^{k+1-i} \leqslant e^{\frac{CK_y}{\varepsilon}},$$

leading to

$$|\delta Y_i| \leqslant C\mathbb{E}_{t_i}\left[\left|E_i^\pi\right|\left(|\delta Y_n| + \sum_{k=i}^{n-1}\left|\zeta_k^Y\right|\right)\right].$$

Under ($\mathbf{H}1$)(iii), we get that $E_i^\pi > 0$ for all $0 \leqslant i \leqslant n$ and then

$$\left(\prod_{j=0}^{k}(1+h_j H_j \gamma_j)\right)_{0\leqslant k\leqslant n}$$

is a positive martingale with expectation equal to 1. The measure $\mathbb{Q}^\pi$ is thus a probability measure. □

2.4.2 Estimates on $\mathbb{Q}^\pi$

In order to retrieve nice estimates on the probability measure $\mathbb{Q}^\pi$, we need to introduce a new assumption.

Assumption ($\mathbf{H}3$)

(i) ($\mathbf{H}2$) holds true and $(\sup_{0\leqslant i\leqslant n-1} h_i \left|H_i\right|)K_z^n < 1-\varepsilon$ with ε a positive constant that does not depend on n,

(ii) F_i are $\tilde{L}$-locally Lipschitz functions with respect to z: $\forall y \in \mathbb{R}$, $\forall z, z' \in \mathbb{R}^{1\times d}$, $\forall 0 \leqslant i \leqslant n-1$,

$$\left|F_i(y,z) - F_i(y,z')\right| \leqslant \tilde{L}(1+|z|+\left|z'\right|)\left|z-z'\right|,$$

with $\tilde{L}$ a constant that does not depend on n.

Proposition 2.10. *Assume that* ($\mathbf{H}3$) *holds true. Then* $M_t := \sum_{t_i\leqslant t} h_i\gamma_i H_i$, *with* $(\gamma_i)_{0\leqslant i\leqslant n-1}$ *given by (2.18), is a BMO martingale for the discontinuous filtration* $\mathcal{F}^n$ *defined by* $\mathcal{F}_t^n := \mathcal{F}_{t_i}$ *when* $t_i \leqslant t < t_{i+1}$. *Moreover, there exists a constant* C *that does not depend on* n *such that*

$$\|M\|_{BMO(\mathcal{F}^n)} \leqslant C.$$

Proof. We have to show that there exists a constant C that does not depend on n such that, for all stopping time $S \leqslant T$,

$$\mathbb{E}\left[|M_T - M_{S^-}|^2 \,|\mathcal{F}_S\right] \leqslant C.$$

Thanks to remark (76.4) in chapter VII of [21], we know that it is sufficient to show that for all $0 \leqslant i < n$,

$$\mathbb{E}_{t_i}\left[\textstyle\sum_{j=i}^{n-1} |h_j H_j \gamma_j|^2\right] \leqslant C.$$

To prove this point we use the fact that F_i is a $\tilde{L}$-locally Lipschitz function with respect to z and (2.8):

$$\begin{aligned}\mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} |h_j H_j \gamma_j|^2\right] &\leqslant 3\tilde{L}^2 + 3\tilde{L}^2 \mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} |h_j H_j|^2 \left|\tilde{Z}_j\right|^2\right] + 3\tilde{L}^2 \mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} |h_j H_j|^2 |Z_j|^2\right] \\ &\leqslant 3\tilde{L}^2 + 3\tilde{L}^2 \Lambda \mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} \left|\tilde{Z}_j\right|^2 h_j\right] + 3\tilde{L}^2 \Lambda \mathbb{E}_{t_i}\left[\sum_{j=i}^{n-1} |Z_j|^2 h_j\right].\end{aligned}$$

The proof is concluded combining (2.17) with Proposition 2.7. □

Since M is a BMO martingale, we retrieve some strong properties for this process.

Proposition 2.11. *Assume that* (**H**3) *holds true. Then, the Doléans-Dade exponential* $E_t := \prod_{t_j \leqslant t}(1 + h_j H_j \gamma_j)$ *is a uniformly integrable martingale for the filtration* $\mathcal{F}^n$ *satisfying the "reverse Hölder inequality"*

$$\mathbb{E}_t\left[\frac{E_T^{p^*}}{E_t^{p^*}}\right] \leqslant C, \quad 0 \leqslant t \leqslant T,$$

for some $p^* > 1$ *and* $C > 0$ *that depend only on* $\|M\|_{BMO(\mathcal{F}^n)}$ *and* ε*. In particular, we can choose them independently of* n*. As a direct corollary, we have that* M *is a* L^{p^*} *bounded martingale.*

Proof. The first theorem in [31] states that $(E_t)_{0 \leqslant t \leqslant 1}$ is a uniformly integrable martingale satisfying the "reverse Hölder inequality" for some $p^* > 1$. We just have to check that we can choose C and p^* that only depend on $\|M\|_{BMO(\mathcal{F}^n)}$ and ε. Firstly, thanks to Theorem 2 in [30] we know that there exist positive constants a and K such that

$$\mathbb{E}_\tau\left[\left(\frac{E_T}{E_\tau}\right)^a\right] \leqslant K, \tag{2.19}$$

for any stopping time τ. By checking carefully the proof of this theorem, we remark that a is chosen such that

$$k_a := \frac{4a^2 + a}{\varepsilon^2} < \frac{1}{\|M\|_{BMO(\mathcal{F}^n)}}$$

and then K is set

$$K := \frac{1}{1 - k_a \|M\|^2_{BMO(\mathcal{F}^n)}}.$$

To conclude we use Lemma 3 in [31] that says that if M satisfies (2.19), then it satisfies a "reverse Hölder inequality". By checking carefully the proof of this lemma we can see that constants C and p^* in the "reverse Hölder inequality" are only obtained thanks to a, K and ε. $\square$

Combining the previous proposition with Proposition 2.9, we obtain, using Hölder's inequality, the following result.

Corollary 2.1. *Assume that* (**H**3) *holds true. Then there exist constants $C > 0$ and $q^* > 1$ that do not depend on n and such that, for all $0 \leqslant i \leqslant n$,*

$$|\delta Y_i| \leqslant C \left(\mathbb{E}_{t_i}\left[|\delta Y_n|^{q^*}\right]^{\frac{1}{q^*}} + \mathbb{E}_{t_i}\left[\Big(\sum_{j=i}^{n-1} |\zeta_j^Y|\Big)^{q^*}\right]^{\frac{1}{q^*}} \right).$$

q^* is the conjugate exponent of p^* given in Proposition 2.11.

Remark 2.5. If $\zeta_i^Y = \zeta_i^{Y,1} + \zeta_i^{Y,2}$, it is easy to see that one may just apply the Corollary 2.1 on the first part of the perturbation:

$$|\delta Y_i| \leq C \left(\mathbb{E}_{t_i}\left[|\delta Y_n|^{q^*}\right]^{\frac{1}{q^*}} + \mathbb{E}_{t_i}\left[\Big(\sum_{j=i}^{n-1} \left|\zeta_j^{Y,1}\right|\Big)^{q^*}\right]^{\frac{1}{q^*}} + \mathbb{E}^{\mathbb{Q}^\pi}_{t_i}\left[\sum_{j=i}^{n-1} |\zeta_j^{Y,2}|\right] \right), \quad 0 \leqslant i \leqslant n.$$

2.4.3 Stability result for the Z component

Proposition 2.12. *Assume that* (**H**3) *holds true. Then,*

$$\mathbb{E}\left[\sum_{i=0}^{n-1} h_i |\delta Z_i|^2\right] \leqslant C \left(\mathbb{E}\left[|\delta Y_n|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \frac{|\zeta_i^Y|^2}{h_i}\right] + \mathbb{E}\left[\sup_{0 \leqslant i \leqslant n-1} |\delta Y_i|^4\right]^{1/2} \right).$$

Proof. 1. As in the proof of Proposition 2.7, we first observe that equation (2.10) can be rewritten

$$\delta Y_i = \delta Y_{i+1} + h_i \beta_i \delta Y_i + h_i \delta Z_i \gamma_i + \zeta_i^Y - h_i c_i^{-1} \delta \mathrm{Z}_i H_i^\top - \delta \Delta M_i$$

where $\delta \Delta M_i$ is an $\mathcal{F}_{t_{i+1}}$ random variable satisfying $\mathbb{E}_{t_i}[\delta \Delta M_i] = 0$, $\mathbb{E}_{t_i}\big[|\delta \Delta M_i|^2\big] < \infty$ and $\mathbb{E}_{t_i}[\delta \Delta M_i H_i] = 0$. Using the identity $|y|^2 = |x|^2 + 2x(y-x) + |y-x|^2$ and taking the conditional expectation, we compute, setting $x = \delta Y_i$ and $y = \delta Y_{i+1}$,

$$\begin{aligned}\mathbb{E}_{t_i}\left[\left|\delta Y_{i+1}\right|^2\right] &\geqslant |\delta Y_i|^2 - 2\left|\delta Y_i\right|^2 h_i\beta_i - 2h_i\delta Y_i\delta Z_i\gamma_i \\ &\quad -2\delta Y_i\zeta_i^Y + c_i^{-1}h_i\delta Z_i c_i^{-1}h_i\mathbb{E}_{t_i}\left[H_i^\top H_i\right]\delta Z_i^\top.\end{aligned}$$

It follows from (2.6) and (2.7) applied to the previous inequality that

$$|\delta Y_i|^2 + \frac{d}{\Lambda}h_i\left|\delta Z_i\right|^2 \leqslant \mathbb{E}_{t_i}\left[\left|\delta Y_{i+1}\right|^2\right] + 2\delta Y_i\zeta_i^Y + 2h_i\delta Y_i\delta Z_i\gamma_i + 2\left|\delta Y_i\right|^2 h_i\beta_i$$

and Young inequality leads to

$$|\delta Y_i|^2 + \frac{d}{2\Lambda}h_i\left|\delta Z_i\right|^2 \leqslant \mathbb{E}_{t_i}\left[\left|\delta Y_{i+1}\right|^2\right] + h_i\left(1 + 2K_y + \frac{2\Lambda\left|\gamma_i\right|^2}{d}\right)\left|\delta Y_i\right|^2 + \frac{\left|\zeta_i^Y\right|^2}{h_i}.$$

Summing over i the previous inequality, we obtain

$$\mathbb{E}\left[\sum_{i=0}^{n-1} h_i\left|\delta Z_i\right|^2\right] \leqslant C\mathbb{E}\left[\left|\delta Y_n\right|^2\right] + C\mathbb{E}\left[\sum_{i=0}^{n-1} h_i(1+\left|\gamma_i\right|^2)\left|\delta Y_i\right|^2\right] + C\mathbb{E}\left[\sum_{i=0}^{n-1}\frac{\left|\zeta_i^Y\right|^2}{h_i}\right].$$

Applying Hölder inequality, we get

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1} h_i\left|\delta Z_i\right|^2\right] &\leqslant C\mathbb{E}\left[\left|\delta Y_n\right|^2\right] + C\mathbb{E}\left[\sum_{i=0}^{n-1}\frac{\left|\zeta_i^Y\right|^2}{h_i}\right] \\ &\quad + C\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\left|\delta Y_i\right|^4\right]^{1/2}\mathbb{E}\left[\left(1+\sum_{i=0}^{n-1}\left|\gamma_i\right|^2 h_i\right)^2\right]^{1/2}.\end{aligned}$$

To conclude the proof, we just have to show that

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1} h_i\left|\gamma_i\right|^2\right)^2\right] \leqslant C.$$

Using Burkholder-Davis-Gundy inequality for the discrete martingale $\left(\sum_{i=0}^{j} h_i H_i\gamma_i\right)_{0\leqslant j\leqslant n}$, the previous inequality holds true if we have

$$\mathbb{E}\left[\left(\sup_{0\leqslant j\leqslant n-1}\sum_{i=0}^{j} h_i H_i\gamma_i\right)^4\right] \leqslant C.$$

Thanks to Proposition 2.10 we know that $M_t = \sum_{t_i\leqslant t} h_i H_i\gamma_i$ is a BMO martingale with a BMO norm that does not depend on n. To conclude the proof, we use an energy inequality or the John-Nirenberg inequality, see e.g. Theorem 109 and inequality (109.5) in the chapter VI of [21], and obtain

$$\mathbb{E}\left[\left(\sup_{0\leqslant j\leqslant n-1}\sum_{i=0}^{j} h_i H_i \gamma_i\right)^4\right] \leqslant C$$

with C that depends only on $\|M\|_{BMO(\mathcal{F}^n)}$.

□

3 Convergence analysis of the discrete-time approximation

The aim of this part is to study the error between the solution (Y, Z) of the BSDE (1.2) and (Y^π, Z^π) the solution of the BTZ scheme given in Definition 1.1, recalling (1.9). Thanks to Theorem 2.2 we know that we just have to estimate the error between (Y^N, Z^N) and (Y^π, Z^π).
Let us first observe that we can apply results of the previous section to (Y^π, Z^π).

Lemma 3.1. *Under same assumptions as Theorem 1.1, the scheme given in Definition 1.1 satisfies* (**H**3).

Proof. With our special choice of parameters R and N, there exists $\varepsilon > 0$ such that for n big enough we have $K_{f,y}h \leqslant \frac{CK_{f,y}}{n} < 1-\varepsilon$. Moreover, we have also for n large enough

$$\left(\sup_{0\leqslant i\leqslant n-1} h_i \left|H_i^R\right|\right) n^\alpha \leqslant \sqrt{h} R n^\alpha \leqslant \frac{\sqrt{C}\log n}{n^{1/2-\alpha}} \leqslant 1-\varepsilon.$$

□

3.1 Expression of the perturbing error

We first observe that (Y^N, Z^N) can be rewritten as a perturbed BTZ scheme. Namely, setting $\tilde{\mathrm{Y}}_i := Y^N_{t_i}$, for all $i \leqslant n$, we have

$$\left\{\begin{array}{l} \tilde{\mathrm{Y}}_i = \mathbb{E}_{t_i}\left[\tilde{\mathrm{Y}}_{i+1} + h_i f_N(X_i^\pi, \tilde{\mathrm{Y}}_i, \tilde{\mathrm{Z}}_i)\right] + \zeta_i^Y \\ \tilde{\mathrm{Z}}_i = \mathbb{E}_{t_i}\left[\tilde{\mathrm{Y}}_{i+1} H_i^R\right], \end{array}\right. \tag{3.1}$$

with

$$\zeta_i^Y = \mathbb{E}_{t_i}\bigg[\int_{t_i}^{t_{i+1}} f_N(X_s, Y_s^N, Z_s^N) - f_N(X_i^\pi, Y_{t_i}^N, \tilde{\mathrm{Z}}_i) ds\bigg]. \tag{3.2}$$

The following lemma will allow us to use the results of the last section.

Lemma 3.2. *The perturbed scheme* $(\tilde{\mathrm{Y}}_i, \tilde{\mathrm{Z}}_i)_{i\leqslant n}$ *satisfies, for all* $0 \leqslant k \leqslant n-1$,

$$\mathbb{E}_{t_k}\left[\sum_{i=k}^{n-1} \left|\tilde{\mathrm{Z}}_i\right|^2 h_i\right] \leqslant C.$$

Proof. Observe that,

$$\mathbb{E}_{t_k}\left[\sum_{i=k}^{n-1} h_i \left|\tilde{\mathrm{Z}}_i\right|^2\right] \quad \leqslant \quad C\left(\mathbb{E}_{t_k}\left[\sum_{i\geqslant k}\left|\tilde{\mathrm{Z}}_i - \tilde{Z}_i^N\right|^2 h_i\right] + \mathbb{E}_{t_k}\left[\sum_{i\geqslant k}\left|\tilde{Z}_i^N\right|^2 h_i\right]\right) \tag{3.3}$$

where

$$\tilde{Z}_i^N := \mathbb{E}_{t_i}\left[Y_{t_{i+1}}^N \frac{\Delta W_i}{h_i}\right].$$

Applying Lemma 2.2, we obtain

$$\mathbb{E}_{t_k}\left[\sum_{i=k}^{n-1} h_i \left|\tilde{\mathrm{Z}}_i\right|^2\right] \quad \leqslant \quad C\left(1 + \mathbb{E}_{t_k}\left[\sum_{i\geqslant k}\left|\tilde{\mathrm{Z}}_i - \tilde{Z}_i^N\right|^2 h_i\right]\right). \tag{3.4}$$

Moreover, we compute

$$\begin{aligned}\mathbb{E}_{t_k}\left[\sum_{i\geqslant k}\left|\tilde{\mathrm{Z}}_i - \tilde{Z}_i^N\right|^2 h_i\right] \quad &= \quad \mathbb{E}_{t_k}\left[\sum_{i\geqslant k}\left|\mathbb{E}_{t_i}\left[\left(Y_{t_{i+1}}^N - Y_{t_i}^N\right)\left(H_i^R - \frac{\Delta W_i}{h_i}\right)\right]\right|^2 h_i\right] \\ &\leqslant \quad C\sum_{i\geqslant k}\mathbb{E}_{t_k}\Big[|Y_{t_{i+1}}^N - Y_{t_i}^N|^2\Big],\end{aligned}$$

where we used Cauchy-Schwartz inequality, recalling (2.8) .
We then compute, thanks to assumptions on f_N and Remark 2.1,

$$\begin{aligned}\mathbb{E}_{t_k}\Big[|Y_{t_{i+1}}^N - Y_{t_i}^N|^2\Big] &\leqslant C\left(h_i \mathbb{E}_{t_k}\Big[\int_{t_i}^{t_{i+1}} |f_N(X_s, Y_s^N, Z_s^N)|^2 \mathrm{d}s\Big] + \mathbb{E}_{t_k}\Big[\int_{t_i}^{t_{i+1}} |Z_s^N|^2 \mathrm{d}s\Big]\right) \\ &\leqslant C\left(h^2 + (1 + N^2 h)\mathbb{E}_{t_k}\Big[\int_{t_i}^{t_{i+1}} |Z_s^N|^2 \mathrm{d}s\Big]\right).\end{aligned}$$

Summing over i, recalling Remark 2.1, we obtain

$$\mathbb{E}_{t_k}\left[\sum_{i\geqslant k}\left|\tilde{\mathrm{Z}}_i - \tilde{Z}_i^N\right|^2 h_i\right] \leqslant C\left(1 + \left\|\int_0^\cdot Z_s^N dW_s\right\|_{BMO(\mathcal{F})}^2\right) \leqslant C. \tag{3.5}$$

The proof is concluded combining the above inequality with (3.4). $\square$

3.2 Regularity

In the followings, we need regularity results on (X, Y^N, Z^N). The specificity here is that we need the estimates under the probability measure $\mathbb{P}$ and $\mathbb{Q}^\pi$. The first result deals with the path regularity of Y under the probability measure $\mathbb{P}$. It is a mere generalization of Theorem 5.5 in [28].

Proposition 3.1. *(Y-part) For all $p \leqslant 1$, we have*

$$\sup_{0 \leqslant j \leqslant n-1} \mathbb{E}\left[\sup_{t_j \leqslant s \leqslant t_{j+1}} \left|Y_s^N - Y_{t_j}^N\right|^{2p}\right] \leqslant C_p h^p. \tag{3.6}$$

The second result is a slight modification of the well-known Zhang's path regularity theorem, whose proof is postponed to the Appendix.

Proposition 3.2. *(Z-part) For all $p \geqslant 1$ and $\eta > 0$, we have*

$$\mathbb{E}\left[\sup_{0 \leqslant i \leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left(\int_{t_j}^{t_{j+1}} \left|Z_s^N - \bar{Z}_j^N\right|^2 ds\right)^{1+\eta}\right]^p\right] \leqslant C_{\eta,p} h^{p(1+\eta)} .$$

Let us remark that the previous proposition stays true when we replace $\mathbb{Q}^\pi$ by $\mathbb{P}$: it is a mere generalization of Theorem 5.6 in [28].

3.3 Discretization error for the Y-component

Proposition 3.3. *There exists $q^* > 1$ and, for all $\eta > 0$ and $p \geqslant 1$, there exist constants C_p and $C_{\alpha,\eta,p}$ such that*

$$\begin{aligned}
\mathbb{E}\left[\sup_{0 \leqslant i \leqslant n} |Y_{t_i} - Y_i^\pi|^{2p}\right] \quad &\leqslant \quad C_{\alpha,\eta,p} h^{p(1-\eta)} + C_p \mathbb{E}\left[\sup_{0 \leqslant j \leqslant n} \left|X_{t_j} - X_j^\pi\right|^{2pq^*}\right]^{1/q^*} \\
&\quad + C_p \max_{0 \leqslant j \leqslant n-1}\left(\mathbb{E}\left[\left\|H_j^R - \frac{\Delta W_j}{h_j}\right\|\right]^{4p} + \mathbb{E}\left[\left\|H_j^R - \frac{\Delta W_j}{h_j}\right\|\right]^{2p}\right).
\end{aligned}$$

Before giving the proof, let us emphasize that q^* is the exponent given by Corollary 2.1 and so it is the conjugate exponent of p^* given by Proposition 2.11.

Proof. The proof is divided in several steps.

1. We first observe that

$$\mathbb{E}\left[\sup_{0 \leqslant i \leqslant n} |Y_{t_i} - Y_i^\pi|^{2p}\right] \leqslant C_p\left(\mathbb{E}\left[\sup_{0 \leqslant i \leqslant n} \left|Y_{t_i} - Y_{t_i}^N\right|^{2p}\right] + \mathbb{E}\left[\sup_{0 \leqslant i \leqslant n} \left|Y_{t_i}^N - Y_i^\pi\right|^{2p}\right]\right) \tag{3.7}$$

To bound the first term in the right-hand side of the above equation, we apply Theorem 2.2 and get:

$$\mathbb{E}\left[\sup_{0\leqslant i\leqslant n}\left|Y_{t_i}-Y^N_{t_i}\right|^{2p}\right]\leqslant C_{\alpha,p}h^p\ ,$$

recalling (1.9).

2. To control the error between the solution Y^N and the scheme Y^π, we will combine the stability results proved in the previous section with a careful analysis of the perturbation error $(\zeta_i^Y)_{0\leqslant i<n}$ given by (3.2). We first observe that

$$\begin{aligned}
\zeta_i^Y &= \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f_N(X_s,Y_s^N,Z_s^N)-f_N(X_i^\pi,Y_s^N,Z_s^N)ds\right]\\
&\quad+\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f_N(X_i^\pi,Y_s^N,Z_s^N)-f_N(X_i^\pi,Y_{t_i}^N,Z_s^N)ds\right]\\
&\quad+\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f_N(X_i^\pi,Y_{t_i}^N,Z_s^N)-f_N(X_i^\pi,Y_{t_i}^N,\bar{Z}_i^N)ds\right]\\
&\quad+\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f_N(X_i^\pi,Y_{t_i}^N,\bar{Z}_i^N)-f_N(X_i^\pi,Y_{t_i}^N,\tilde{Z}_i^N)ds\right]\\
&\quad+\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f_N(X_i^\pi,Y_{t_i}^N,\tilde{Z}_i^N)-f_N(X_i^\pi,Y_{t_i}^N,\tilde{\mathrm{Z}}_i)ds\right]\\
&:= \zeta_i^{Y,x}+\zeta_i^{Y,y}+\zeta_i^{Y,\bar{z}}+\zeta_i^{Y,\tilde{z}}+\zeta_i^{Y,w},
\end{aligned}$$

recalling (2.2) and (2.3).

Using Lemma 3.1 and Lemma 3.2, we apply Proposition 2.9 and Corollary 2.1 (see also Remark 2.5) to obtain

$$\begin{aligned}
\left|Y_{t_i}^N-Y_i^\pi\right| &\leqslant C\mathbb{E}_{t_i}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,x}\right|\right)^{q^*}\right]^{1/q^*}+C\mathbb{E}_{t_i}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,y}\right|\right)^{q^*}\right]^{1/q^*}\\
&\quad+C\mathbb{E}_{t_i}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,w}\right|\right)^{q^*}\right]^{1/q^*}+C\mathbb{E}_{t_i}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,\tilde{z}}\right|\right)^{q^*}\right]^{1/q^*}\\
&\quad+C\mathbb{E}_{t_i}\left[\left|Y_{t_n}^N-Y_n^\pi\right|^{q^*}\right]^{1/q^*}+C\mathbb{E}_{t_i}\left[\left\{\prod_{j=i}^{n-1}(1+h_jH_j^R\gamma_j^{N,n})\right\}\left\{\sum_{j=i}^{n-1}\left|\zeta_j^{Y,\bar{z}}\right|\right\}\right].
\end{aligned}$$

A convexity inequality and Doob maximal inequality allow us to write, for all $p\geqslant 1$,

$$\mathbb{E}\left[\sup_{0\leqslant i\leqslant n}\left|Y_{t_i}^N-Y_i^\pi\right|^{2p}\right]\leqslant C\Big(\mathcal{E}_p^x+\mathcal{E}_p^y+\mathcal{E}_p^w+\mathcal{E}_p^{\tilde{z}}+\mathcal{E}_p^{\bar{z}}\Big)\ , \tag{3.8}$$

with

$$\mathcal{E}_p^x := \mathbb{E}\left[\left|Y_{t_n}^N - Y_n^\pi\right|^{2pq^*}\right]^{1/q^*} + C\mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,x}\right|\right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of X by X^π in the terminal condition and the generator,

$$\mathcal{E}_p^y := \mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,y}\right|\right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of Y^N by $\sum_{i=0}^{n-1} Y_{t_i}^N \mathbf{1}_{t_i \leqslant t < t_{i+1}}$ in the generator,

$$\mathcal{E}_p^w := \mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,w}\right|\right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of ΔW_i by $h_i H_i$,

$$\mathcal{E}_p^{\tilde{z}} := \mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,\tilde{z}}\right|\right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of $\sum_{i=0}^{n-1} \bar{Z}_i^N \mathbf{1}_{t_i \leqslant t < t_{i+1}}$ by $\sum_{i=0}^{n-1} \tilde{Z}_i^N \mathbf{1}_{t_i \leqslant t < t_{i+1}}$ in the generator, and finally

$$\mathcal{E}_p^{\bar{z}} := n^p \mathbb{E}\left[\sup_{0 \leqslant i \leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left|\zeta_j^{Y,\bar{z}}\right|^2\right]^p\right],$$

due to the approximation of Z^N by $\sum_{i=0}^{n-1} \bar{Z}_i^N \mathbf{1}_{t_i \leqslant t < t_{i+1}}$ in the generator.
We will now bound these five terms.
2.a. Since g is Lipschitz continuous, we have

$$\mathbb{E}\left[\left|Y_{t_n}^N - Y_n^\pi\right|^{2pq^*}\right]^{1/q^*} \leqslant C_p \mathbb{E}\left[\left|X_n^\pi - X_T\right|^{2pq^*}\right]^{1/q^*}. \tag{3.9}$$

Similarly, since f_N is Lipschitz-continuous in its x-variable,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,x}\right|\right)^{2pq^*}\right]^{1/q^*} &\leqslant C_p \sup_{0 \leqslant j \leqslant n-1} \mathbb{E}\left[\left(\sup_{t_j \leqslant s \leqslant t_{j+1}}\left|X_s - X_j^\pi\right|\right)^{2pq^*}\right]^{1/q^*} \\
&\leqslant C_p \sup_{0 \leqslant j \leqslant n-1} \mathbb{E}\left[\sup_{t_j \leqslant s \leqslant t_{j+1}}\left|X_s - X_{t_j}\right|^{2pq^*}\right]^{1/q^*} \\
&\quad + C_p \sup_{0 \leqslant j \leqslant n-1} \mathbb{E}\left[\left|X_{t_j} - X_j^\pi\right|^{2pq^*}\right]^{1/q^*}.
\end{aligned} \tag{3.10}$$

Classical result on the path regularity of SDE's solutions yields

$$\sup_{0\leqslant j\leqslant n-1}\mathbb{E}\left[\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|X_s-X_{t_j}\right|^{2pq^*}\right]^{1/q^*}\leqslant C_ph^p. \tag{3.11}$$

Combining (3.9)-(3.10)-(3.11), we obtain

$$\mathcal{E}_p^x\leqslant C_ph^p+C_p\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\left|X_{t_j}-X_j^\pi\right|^{2pq^*}\right]^{1/q^*}. \tag{3.12}$$

2.b. We easily compute that

$$\mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,y}\right|\right)^{2pq^*}\right]^{1/q^*}\leqslant C_pn^{-1}\sum_{j=0}^{n-1}\mathbb{E}\left[\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|Y_s^N-Y_{t_j}^N\right|^{2pq^*}\right]^{1/q^*}.$$

Applying inequality (3.6), this leads to

$$\mathcal{E}_p^y\leqslant C_ph^p. \tag{3.13}$$

2.c. Using (**H**3)(ii) and Remark 2.1 we have

$$\begin{aligned}
\left|\zeta_j^{Y,w}\right| &\leqslant Ch_j\left(1+\left|\tilde{Z}_j^N\right|+\left|\tilde{\mathrm{Z}}_j\right|\right)\left|\tilde{Z}_j^N-\tilde{\mathrm{Z}}_j\right|\\
&\leqslant Ch_j\left(1+\left|\tilde{Z}_j^N\right|\right)\left(\left|\tilde{Z}_j^N-\tilde{\mathrm{Z}}_j\right|^2+\left|\tilde{Z}_j^N-\tilde{\mathrm{Z}}_j\right|\right)\\
&\leqslant Ch_j\left(1+\left|\tilde{Z}_j^N\right|\right)\left(\mathbb{E}_{t_j}\left[\left|Y_{t_{j+1}}^N\right|\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2+\mathbb{E}_{t_j}\left[\left|Y_{t_{j+1}}^N\right|\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]\right)\\
&\leqslant Ch_j\left(1+\left|\tilde{Z}_j^N\right|\right)\left(\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2+\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]\right),
\end{aligned}$$

and thus, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left|\zeta_j^{Y,w}\right|\right)^{2pq^*}\right]^{1/q^*}\\
&\leqslant C_p\max_{0\leqslant j\leqslant n-1}\left(\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2+\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]\right)^{2p}\left(1+\mathbb{E}\left[\max_{0\leqslant i\leqslant n-1}\left|\tilde{Z}_i^N\right|^{2pq^*}\right]^{1/q^*}\right).
\end{aligned}$$

Using Lemma 2.2, we compute

$$\begin{aligned}
\mathbb{E}\left[\max_{0\leqslant i\leqslant n-1}\left|\tilde{Z}_i^N\right|^{2pq^*}\right]^{1/q^*} &\leqslant C_p\left(1+\mathbb{E}\left[\max_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\sup_{0\leqslant s\leqslant T}|X_s|^4\right]^{pq^*}\right]^{1/q^*}\right)\\
&\leqslant C_p\left(1+\mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|X_s|^{4pq^*}\right]^{1/q^*}\right)\\
&\leqslant C_p,
\end{aligned}$$

where we used the Doob maximal inequality. Finally, we obtain

$$\mathcal{E}_p^w \leqslant C_p\max_{0\leqslant j\leqslant n-1}\left(\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^{4p}+\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^{2p}\right). \tag{3.14}$$

2.d. Using (**H**3)(ii), (2.5), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\left|\zeta_j^{Y,\tilde{z}}\right| &\leqslant Ch_j\left(1+\left|\tilde{Z}_j^N\right|+\left|\bar{Z}_j^N\right|\right)\left|\tilde{Z}_j^N-\bar{Z}_j^N\right|\\
&\leqslant Ch^{1/2}h_j\left(1+\left|\tilde{Z}_j^N\right|+\left|\bar{Z}_j^N\right|\right)\left(1+\mathbb{E}_{t_j}\left[\sup_{t_j\leqslant s\leqslant t_{j+1}}|X_s|^4\right]^{1/2}\right)\\
&\leqslant Ch^{1/2}h_j\left(1+\mathbb{E}_{t_j}\left[\sup_{t_j\leqslant s\leqslant t_{j+1}}|X_s|^4\right]\right).
\end{aligned}$$

Then by same arguments than in part 2.c we obtain

$$\mathcal{E}_p^{\tilde{z}} \leqslant C_p h^p. \tag{3.15}$$

2.e. The last term is the more involved. Since the functions f and f_N are locally Lipschitz with respect to z, compute $\left|\zeta_j^{Y,\bar{z}}\right|$:

$$\left|\zeta_j^{Y,\bar{z}}\right| \leqslant C\mathbb{E}_{t_j}\left[\left(1+\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|Z_s^N\right|+\left|\bar{Z}_j^N\right|\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|ds\right],$$

and so,

$$\left|\zeta_j^{Y,\bar{z}}\right|^2 \leqslant Ch_j\mathbb{E}_{t_j}\left[\left(1+\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|Z_s^N\right|^2+\left|\bar{Z}_j^N\right|^2\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2ds\right]. \tag{3.16}$$

Let us remark that in the previous bound, the term inside the conditional expectation is a $\mathcal{F}_{t_{j+1}}$-measurable random variable, so we have

$$\begin{aligned}
& \mathbb{E}_{t_j}\left[\left(1+\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|Z_s^N\right|^2+\left|\bar{Z}_j^N\right|^2\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right] \\
= \; & \mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\frac{1}{1+h_jH_j^R\gamma_j^{N,n}}\left(1+\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|Z_s^N\right|^2+\left|\bar{Z}_j^N\right|^2\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right] \\
\leqslant \; & \frac{1}{\varepsilon}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\left(1+\sup_{t_j\leqslant s\leqslant t_{j+1}}\left|Z_s^N\right|^2+\left|\bar{Z}_j^N\right|^2\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right] \\
\leqslant \; & \frac{1}{\varepsilon}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\left(1+\sup_{0\leqslant s\leqslant T}\left|Z_s^N\right|^2+\max_{0\leqslant i\leqslant n-1}\left|\bar{Z}_i^N\right|^2\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right]
\end{aligned}$$

since $1/(1+h_jH_j^R\gamma_j^{N,n})\leqslant 1/\varepsilon$ under (**H**3). Then (3.16) becomes

$$\left|\zeta_j^{Y,\bar{z}}\right|^2\leqslant Ch_j\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\left(1+\sup_{0\leqslant s\leqslant T}\left|Z_s^N\right|^2+\max_{0\leqslant i\leqslant n-1}\left|\bar{Z}_i^N\right|^2\right)\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right]. \tag{3.17}$$

Thanks to Proposition 2.3 and Lemma 2.1 we can simplify the first part of our estimate:

$$\sup_{0\leqslant s\leqslant T}\left|Z_s^N\right|\leqslant C(1+\sup_{0\leqslant s\leqslant T}|X_s|)$$

and

$$\max_{0\leqslant i\leqslant n-1}\left|\bar{Z}_i^N\right|\leqslant C\left(1+\max_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\sup_{t_i\leqslant s\leqslant t_{i+1}}|X_s|\right]\right)\leqslant C\left(1+\max_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\sup_{0\leqslant s\leqslant T}|X_s|\right]\right).$$

Inserting these two bounds into (3.17) we obtain

$$\mathcal{E}_p^{\bar{z}}\;\leqslant\; C\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\left(1+\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]\right)\sum_{j=i}^{n-1}\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right]^p\right],$$

and, using Hölder inequality and a convexity inequality, we get for any $\eta > 0$

$$\begin{aligned}
\mathcal{E}_p^{\bar{z}} &\leqslant C_{\eta,p}\left(1+\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{1+\eta}{\eta}}\right]^p\right]^{\frac{\eta}{1+\eta}}\right)\\
&\quad\times\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\left(\sum_{j=i}^{n-1}\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right)^{1+\eta}\right]^p\right]^{\frac{1}{1+\eta}}\\
&\leqslant C_{\eta,p}h^{-\frac{p\eta}{1+\eta}}\left(1+\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{1+\eta}{\eta}}\right]^p\right]^{\frac{\eta}{1+\eta}}\right)\\
&\quad\times\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left(\int_{t_j}^{t_{j+1}}\left|Z_s^N-\bar{Z}_j^N\right|^2 ds\right)^{1+\eta}\right]^p\right]^{\frac{1}{1+\eta}}. \qquad (3.18)
\end{aligned}$$

We can easily upper bound the first part of the last estimate. Indeed, thanks to Proposition 2.11 we are able to use once again Hölder inequality with p^* and q^*:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{1+\eta}{\eta}}\right]^p\right]^{\frac{\eta}{1+\eta}}\\
\leqslant\ &\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\prod_{j=i}^{n-1}(1+h_jH_j^R\gamma_j^{N,n})^{p^*}\right]^{p/p^*}\mathbb{E}_{t_i}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{q^*(1+\eta)}{\eta}}\right]^{p/q^*}\right]^{\frac{\eta}{1+\eta}}\\
\leqslant\ &C_{\eta,p}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{q^*(1+\eta)}{\eta}}\right]^{p/q^*}\right]^{\frac{\eta}{1+\eta}}\\
\leqslant\ &C_{\eta,p}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{q^*(1+\eta)}{\eta}}\right]^{2p}\right]^{\frac{\eta}{2q^*(1+\eta)}}.
\end{aligned}$$

To conclude now we just have to use Doob maximal inequality and classical estimates on X to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}\left[\max_{0\leqslant j\leqslant n}\mathbb{E}_{t_j}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{q^*(1+\eta)}{\eta}}\right]^{2p}\right]^{\frac{\eta}{2q^*(1+\eta)}}\\
\leqslant\ &C_{\eta,p}\mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|X_s|^{\frac{2pq^*(1+\eta)}{\eta}}\right]^{\frac{\eta}{2q^*(1+\eta)}}\leqslant C_{\eta,p}.
\end{aligned}$$

Finally (3.18) becomes

$$\mathcal{E}_p^{\bar{z}} \leqslant C_{\eta,p} h^{-\frac{p\eta}{1+\eta}} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left(\int_{t_j}^{t_{j+1}} \left|Z_s^N - \bar{Z}_j^N\right|^2 ds\right)^{1+\eta}\right]^p\right]^{\frac{1}{1+\eta}}. \tag{3.19}$$

Applying Proposition 3.2, we deduce from the last inequality

$$\mathcal{E}_p^{\bar{z}} \leqslant C_{\eta,p} h^{\frac{p}{1+\eta}} = C_{\eta,p} h^{p(1-\tilde{\eta})}, \tag{3.20}$$

with $\tilde{\eta} = 1 - 1/(1+\eta)$. Since (3.20) is true for all $\eta > 0$, then it is true for all $\tilde{\eta} > 0$ and then we can replace $\tilde{\eta}$ by η.

3. Inserting estimates (3.12)-(3.13)-(3.14)-(3.20) in (3.8) concludes the proof of the proposition. □

3.4 Discretization error for the Z-component

Proposition 3.4. *There exists $q^* > 1$ (the same as in Proposition 3.3) such that for all $\eta > 0$,*

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Z_s - Z_i^\pi|^2 ds\right] &\leqslant C_{\alpha,\eta} h^{1-\eta} + C\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\left|X_{t_j} - X_j^\pi\right|^{4q^*}\right]^{1/(2q^*)} \\ &\quad + C\max_{0\leqslant j\leqslant n-1}\left(\mathbb{E}\left[\left|H_j^R - \frac{\Delta W_j}{h_j}\right|\right]^4 + \mathbb{E}\left[\left|H_j^R - \frac{\Delta W_j}{h_j}\right|\right]^2\right).\end{aligned}$$

Proof. The proof is divided in several steps.
1. Firstly, thanks to Theorem 2.2 we know that we just have to estimate the error between Z^N and Z^π. We then observe

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|Z_s^N - Z_i^\pi\right|^2 ds\right] &\leqslant 4\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|Z_s^N - \bar{Z}_i^N\right|^2 ds\right] + 4\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|\bar{Z}_i^N - \tilde{Z}_i^N\right|^2 ds\right] \\ &\quad + 4\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|\tilde{Z}_i^N - \tilde{\mathrm{Z}}_i\right|^2 ds\right] + 4\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|\tilde{\mathrm{Z}}_i - Z_i^\pi\right|^2 ds\right].\end{aligned}$$

Applying Theorem 5.6 in [28] we obtain

$$\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|Z_s^N - \bar{Z}_i^N\right|^2 ds\right] \leqslant Ch.$$

Moreover, by using (2.5) and classical estimates en X, we directly have that

$$\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left|\bar{Z}_s^N - \tilde{Z}_i^N\right|^2 ds\right] \leqslant Ch.$$

Finally, by using the fact that Y^N is bounded uniformly in n (see Remark 2.1) we easily compute that

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|\tilde{Z}_i^N-\tilde{\mathrm{Z}}_i\right|^2 ds\right] &\leqslant \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}_{t_i}\left[\left|Y_{t_{i+1}}^N\right|\left|H_i^R-\frac{\Delta W_i}{h_i}\right|\right]^2 ds\right]\\ &\leqslant C\max_{0\leqslant j\leqslant n-1}\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|Z_s^N-Z_i^\pi\right|^2 ds\right] \leqslant Ch+C\max_{0\leqslant j\leqslant n-1}\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2\\ +\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|\tilde{\mathrm{Z}}_i-Z_i^\pi\right|^2 ds\right].\end{aligned}$$

2. Applying the stability results of Proposition 2.12, we obtain

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|\tilde{\mathrm{Z}}_i-Z_i^\pi\right|^2 ds\right] \leqslant C\mathbb{E}\left[\left|Y_{t_n}^N-Y_n^\pi\right|^2\right]+C\mathbb{E}\left[\sum_{i=0}^{n-1}\frac{\left|\zeta_i^Y\right|^2}{h_i}\right]\\ +C\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\left|Y_{t_i}^N-Y_i^\pi\right|^4\right]^{1/2}.\end{aligned}\tag{3.21}$$

Using the same arguments as in proof of Proposition 3.3 with the simpler setting $p=1$ and $\mathbb{Q}^\pi=\mathbb{P}$ (these arguments also require to show Proposition 3.2 with $\mathbb{Q}^\pi=\mathbb{P}$), one retrieves that

$$\begin{aligned}\mathbb{E}\left[\left|Y_{t_n}^N-Y_n^\pi\right|^2\right]+\mathbb{E}\left[\sum_{i=0}^{n-1}\frac{\left|\zeta_i^Y\right|^2}{h_i}\right] &\leqslant C_\eta h^{1-\eta}+C\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\left|X_{t_j}-X_j^\pi\right|^2\right]\\ &+C\max_{0\leqslant j\leqslant n-1}\left(\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^4+\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2\right).\end{aligned}$$

Plugging the last inequality in equation (3.21) and applying Proposition 3.3, with $p=2$, we obtain

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|\tilde{\mathrm{Z}}_i-Z_i^\pi\right|^2 ds\right] \leqslant C_\eta h^{1-\eta}+C\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\left|X_{t_j}-X_j^n\right|^{4q^*}\right]^{\frac{1}{2q^*}}\\ +C\max_{0\leqslant j\leqslant n-1}\left(\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^4+\mathbb{E}\left[\left|H_j^R-\frac{\Delta W_j}{h_j}\right|\right]^2\right).\end{aligned}$$

Combining this last inequality with step 1 concludes the proof of the proposition. □

3.5 Proof of Theorem 1.1

We have to combine Proposition 3.3 with $p = 1$, Proposition 3.4 with classical estimates on the Euler scheme for SDE, recall (1.8), and classical results about Gaussian distribution tails. Indeed, we compute that

$$\begin{aligned}\mathbb{E}\left[\left|H_i^R - \frac{\Delta W_i}{h_i}\right|\right] &\leqslant \mathbb{E}\left[\left|H_i^R - \frac{\Delta W_i}{h_i}\right|^2\right]^{1/2} \leqslant \left(\frac{2d}{h_i}\int_R^{+\infty} x^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx\right)^{1/2} \\ &\leqslant C\left(\frac{Re^{-R^2/2}}{h_i}\right)^{1/2} \leqslant C\left(\frac{\log n}{e^{\frac{1}{2}(\log n)^2 - \theta \log n}}\right)^{1/2} \leqslant \frac{C}{n}, \quad (3.22)\end{aligned}$$

recall (1.4). □

4 Numerical scheme

4.1 Definition and convergence

In this part, we propose a fully implementable numerical scheme based on a Markovian quantization method, see e.g. [26, 37] for general results about quantization and [2, 17] for a setting related to ours. To this end, given $\delta > 0$ and $\kappa \in \mathbb{N}^*$, we consider the bounded lattice grid:

$$\Gamma = \{x \in \delta\mathbb{Z}^d \mid |x^j| \leqslant \kappa\delta, 1 \leqslant j \leqslant d\} .$$

Observe that there are $(2\kappa)^d + 1$ points in Γ. We then introduce a projection operator Π on the grid Γ centered in X_0 given by, for $x \in R^d$,

$$(\Pi[x])^j = \begin{cases} \delta\lfloor \delta^{-1}(x^j - X_0^j) + \frac{1}{2}\rfloor + X_0^j, & \text{if } |x^j - X_0^j| \leqslant \kappa\delta, \\ \kappa\delta, & \text{if } x^j - X_0^j > \kappa\delta, \\ -\kappa\delta, & \text{if } x^j - X_0^j < \kappa\delta. \end{cases}$$

To compute the conditional expectation appearing in the scheme given in Definition 1.1, we use an optimal quantization of Gaussian random variables (ΔW_i). These random variables are approximated by a sequence of centered random variables $(\Delta\widehat{W}_i = \sqrt{h_i} G_M(\frac{\Delta W_i}{\sqrt{h_i}}))$ with discrete support. Here, G_M denotes the projection operator on the optimal quantization grid for the standard Gaussian distribution with M points in the support, see [26, 37] for details[2]. Moreover, it is shown in [26] that

$$\mathbb{E}\Big[|\Delta W_i - \Delta\widehat{W}_i|^p\Big]^{\frac{1}{p}} \leqslant C_{p,d}\sqrt{h} M^{-\frac{1}{d}} . \quad (4.1)$$

[2]The grids can be downloaded from the website: `http://www.quantize.maths-fi.com/` .

In this context, we introduce the following discrete/truncated version of the Euler scheme,

$$\left\{\begin{array}{l} \widehat{X}_0^\pi = X_0 \\ \widehat{X}_{i+1}^\pi = \Pi\left[\widehat{X}_i^\pi + h_i b(\widehat{X}_i^\pi) + \sigma(\widehat{X}_i^\pi)\Delta\widehat{W}_i\right]. \end{array}\right. \tag{4.2}$$

We observe that $\widehat{X}^\pi$ is a Markovian process living on Γ and satisfying $|\widehat{X}_i^\pi| \leqslant C(|X_0| + \kappa\delta)$, for all $i \leqslant n$.

We then adapt the scheme given in Definition 1.1 to this framework.

Definition 4.1. *We denote $(\widehat{Y}^\pi, \widehat{Z}^\pi)_{0\leqslant i\leqslant n}$ the solution of the BTZ-scheme satisfying*
(i) The terminal condition is $(\widehat{Y}_n^\pi, \widehat{Z}_n^\pi) = (g(\widehat{X}_n^\pi), 0)$
(ii) for $i < n$, the transition from step $i+1$ to step i is given by

$$\left\{\begin{array}{l} \widehat{Y}_i^\pi = \mathbb{E}_{t_i}\left[\widehat{Y}_{i+1}^\pi + h_i f_N(\widehat{X}_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi)\right] \\ \widehat{Z}_i^\pi = \mathbb{E}_{t_i}\left[\widehat{Y}_{i+1}^\pi \widehat{H}_i^R\right]. \end{array}\right. \tag{4.3}$$

The coefficients $(\widehat{H}_i^R)$ are defined, given $R > 0$, by

$$(\widehat{H}_i^R)^\ell = \frac{-R}{\sqrt{h_i}} \vee \frac{(\Delta\widehat{W}_i)^\ell}{h_i} \wedge \frac{R}{\sqrt{h_i}}, \quad 1 \leqslant \ell \leqslant d. \tag{4.4}$$

The parameters R and N are chosen as in (1.9)*.*

Proposition 4.1. *$(\widehat{Y}^\pi, \widehat{Z}^\pi)$ is a Markovian process. More precisely, for all $i \in \{0, ..., n\}$, there exist two functions $u^\pi(t_i, .) : \Gamma \to \mathbb{R}$ and $v^\pi(t_i, .) : \Gamma \to \mathbb{R}^{1\times d}$ such that*

$$\widehat{Y}^\pi = u^\pi(t_i, \widehat{X}_i^\pi) \quad \text{and} \quad \widehat{Z}_i^\pi = v^\pi(t_i, \widehat{X}_i^\pi).$$

These functions can be computed on the grid by the following backward induction: for all $i \in \{0, ..., n\}$ and $x \in \Gamma$,

$$\left\{\begin{array}{rcl} v^\pi(t_i, x) & = & \mathbb{E}\left[u^\pi\left(t_{i+1}, \Pi\left(x + h_i b(x) + \sqrt{h_i}\sigma(x)G_M(U)\right)\right)\frac{G_M^R(U)}{\sqrt{h_i}}\right] \\ u^\pi(t_i, x) & = & \mathbb{E}\left[u^\pi\left(t_{i+1}, \Pi\left(x + h_i b(x) + \sqrt{h_i}\sigma(x)G_M(U)\right)\right)\right] \\ & & +h f_N(t_i, x, u^\pi(t_i, x), v^\pi(t_i, x)), \quad \textit{for } i < n, \end{array}\right. \tag{4.5}$$

with $U \sim \mathcal{N}(0, 1)$ and $(G_M^R(.))^\ell = (-R) \vee (G_M(.))^\ell \wedge R$, for $\ell \in \{1, \ldots, d\}$.
The terminal condition is given by $u^\pi(t_n, x) = g(x)$ and $v^\pi(t_n, x) = 0$.

Remark 4.1. Observe that the above scheme is implicit in $u^\pi(t_i, x)$. We then use a Picard iteration to compute this term in practice, the error is very small because $hK_y \ll 1$ and we do not study it here.

Theorem 4.1. *For all $r > 0$ and $\eta > 0$, the following holds*

$$|Y_0 - \widehat{Y}_0^\pi| \leqslant C_{\alpha,\eta} h^{\frac{1}{2}-\eta} + C_r n(\kappa\delta)^{-r} + C(\delta n + n^{\alpha+\frac{1}{2}} M^{-\frac{1}{d}}) \ .$$

From the above theorem we straightforwardly deduce the following corollary.

Corollary 4.1. *Setting $\delta = n^{-\frac{3}{2}}$, $\kappa = n^{\frac{3}{2}+\tilde{\eta}}$ and $M = n^{(1+\alpha)d}$, we obtain*

$$|Y_0 - \widehat{Y}_0^\pi| \leqslant C_{\alpha,\eta,\tilde{\eta}} h^{\frac{1}{2}-\eta},$$

for all $\eta > 0$, $\tilde{\eta} > 0$ and $0 < \alpha < \frac{1}{2}$.

Proof of Theorem 4.1.
1. Error on Y: We first observe that

$$|Y_0 - \widehat{Y}_0^\pi| \leqslant |Y_0 - Y_0^\pi| + |Y_0^\pi - \widehat{Y}_0^\pi|.$$

Applying Theorem 1.1, we obtain

$$|Y_0 - \widehat{Y}_0^\pi| \leqslant C_{\alpha,\eta} h^{\frac{1}{2}-\eta} + |Y_0^\pi - \widehat{Y}_0^\pi| \ .$$

For the second term, we simply rewrite $(\widehat{Y}^\pi, \widehat{Z}^\pi)$ as a perturbation of the scheme given in Definition 1.1, namely

$$\widehat{Y}_i^\pi = \mathbb{E}_{t_i}\left[\widehat{Y}_{i+1}^\pi + h_i f_N\left(X_i^\pi, \widehat{Y}_i^\pi, \mathbb{E}_{t_i}\Big[\widehat{Y}_{i+1}^\pi H_i^R\Big]\right) + \zeta_i^Y\right]$$

with

$$\zeta_i^Y := h_i\left(f_N(\widehat{X}_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi) - f_N\left(X_i^\pi, \widehat{Y}_i^\pi, \mathbb{E}_{t_i}\Big[\widehat{Y}_{i+1}^\pi H_i^R\Big]\right)\right).$$

Applying Proposition 2.7 for the two schemes and the Corollary 2.1, we obtain for some $q > 1$,

$$|Y_0^\pi - \widehat{Y}_0^\pi| \leqslant C\left(\mathbb{E}\left[|X_n^\pi - \widehat{X}_n^\pi|^q\right]^{\frac{1}{q}} + \mathbb{E}\left[\Big(\sum_{i=0}^{n-1}|\zeta_i^{Y,x}|\Big)^q\right]^{\frac{1}{q}} + \mathbb{E}\left[\Big(\sum_{i=0}^{n-1}|\zeta_i^{Y,z}|\Big)^q\right]^{\frac{1}{q}}\right), \tag{4.6}$$

where

$$\zeta_i^{Y,x} := h_i\Big(f_N(\widehat{X}_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi) - f_N(X_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi)\Big)$$
$$\zeta_i^{Y,z} := h_i\left(f_N(X_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi) - f_N\left(X_i^\pi, \widehat{Y}_i^\pi, \mathbb{E}_{t_i}\Big[\widehat{Y}_{i+1}^\pi H_i^R\Big]\right)\right).$$

We easily compute that

$$\mathbb{E}\left[\Big(\sum_{i=0}^{n-1}|\zeta_i^{Y,x}|\Big)^q\right]^{\frac{1}{q}} \leqslant C\,\mathbb{E}\Big[\sup_i |X_i^\pi - \widehat{X}_i^\pi|^q\Big]^{\frac{1}{q}} \tag{4.7}$$

and

$$\mathbb{E}\left[\Big(\sum_{i=0}^{n-1}|\zeta_i^{Y,z}|\Big)^q\right]^{\frac{1}{q}} \leqslant C\,n^\alpha \sup_i \mathbb{E}\Big[|H_i^R - \widehat{H}_i^R|^q\Big]^{\frac{1}{q}}. \tag{4.8}$$

From (4.1), it follows that

$$\mathbb{E}\Big[|H_i^R - \widehat{H}_i^R|^q\Big]^{\frac{1}{q}} \leqslant Cn^{\frac{1}{2}}M^{-\frac{1}{d}}.$$

Combining the above estimations with (4.6), we obtain

$$|Y_0^\pi - \widehat{Y}_0^\pi| \leqslant C\left(\mathbb{E}\left[\sup_i |X_i^\pi - \widehat{X}_i^\pi|^q\right]^{\frac{1}{q}} + n^{\alpha+\frac{1}{2}}M^{-\frac{1}{d}}\right). \tag{4.9}$$

2. We now study the first term in the right hand side of the above equation, namely the error on the forward component.
Let $\tilde{X}^\pi$ denote the Euler scheme for X where we replace ΔW_i by $\Delta\widehat{W}_i$, i.e.

$$\tilde{X}_{i+1}^\pi = \tilde{X}_i^\pi + h_i b(\tilde{X}_i^\pi) + \sigma(\tilde{X}_i^\pi)\Delta\widehat{W}_i.$$

We then split the error into two terms:

$$\mathbb{E}\Big[\sup_i |X_i^\pi - \widehat{X}_i^\pi|^q\Big]^{\frac{1}{q}} \leqslant C\left(\mathbb{E}\Big[\sup_i |X_i^\pi - \tilde{X}_i^\pi|^{2q}\Big]^{\frac{1}{2q}} + \mathbb{E}\Big[\sup_i |\tilde{X}_i^\pi - \widehat{X}_i^\pi|^{2q}\Big]^{\frac{1}{2q}}\right).$$

2.a We now write $\tilde{X}^\pi$ as a perturbation of X^π, namely:

$$\tilde{X}_{i+1}^\pi \quad = \quad \tilde{X}_i^\pi + h_i b(\tilde{X}_i^\pi) + \sigma(\tilde{X}_i^\pi)\Delta W_i + \zeta_i^{\tilde{X}}$$

with

$$\zeta_i^{\tilde{X}} \quad = \quad \sigma(\tilde{X}_i^\pi)(\Delta\widehat{W}_i - \Delta W_i).$$

Applying Lemma A.1, we obtain

$$\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\Big|X_j^\pi - \tilde{X}_j^\pi\Big|^{2q}\right]^{1/2q} \quad \leqslant \quad C\mathbb{E}\left[\Big(\sum_{j=0}^{n}\Big|\zeta_j^{\tilde{X}}\Big|\Big)^{2q}\right]^{1/2q}.$$

Moreover, we compute

$$\mathbb{E}\left[\Big(\sum_{j=0}^{n}\left|\zeta_j^{\tilde{X}}\right|\Big)^{2q}\right] \leqslant n^{2q-1}\sum_{j=0}^{n}\mathbb{E}\left[\left|\zeta_j^{\tilde{X}}\right|^{2q}\right] \leqslant Cn^q M^{-\frac{2q}{d}}$$

since

$$\begin{aligned}\mathbb{E}\Big[\left|\zeta_j^{\tilde{X}}\right|^{2q}\Big] &\leqslant C\mathbb{E}\Big[(1+\left|\tilde{X}_j^{\pi}\right|)^{4q}\Big]^{\frac{1}{2}}\,\mathbb{E}\Big[\left|\Delta\widehat{W}_j-\Delta W_j\right|^{4q}\Big]^{\frac{1}{2}}\\ &\leqslant Ch^q M^{-\frac{2q}{d}}.\end{aligned}$$

Combining the above estimation, we obtain

$$\mathbb{E}\Big[\sup_{0\leqslant j\leqslant n}|X_j^{\pi}-\tilde{X}_j^{\pi}|^{2q}\Big]^{\frac{1}{2q}} \leqslant C\sqrt{n}M^{-\frac{1}{d}}.$$

2.b We now write $\widehat{X}^{\pi}$ as a perturbation of $\tilde{X}^{\pi}$, namely:

$$\widehat{X}_{i+1}^{\pi} \;=\; \widehat{X}_i^{\pi}+h_i b(\widehat{X}_i^{\pi})+\sigma(\widehat{X}_i^{\pi})\Delta\widehat{W}_i+\zeta_i^{\widehat{X}},$$

with

$$\zeta_i^{\widehat{X}} \;=\; \Pi\left[\check{X}_{i+1}\right]-\check{X}_{i+1} \quad\text{ and }\quad \check{X}_{i+1}:=\widehat{X}_i^{\pi}+h_i b(\widehat{X}_i^{\pi})+\sigma(\widehat{X}_i^{\pi})\Delta\widehat{W}_i.$$

Applying Lemma A.1, we get

$$\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\left|\tilde{X}_j^{\pi}-\widehat{X}_j^{\pi}\right|^{2q}\right]^{1/2q} \;\leqslant\; C\mathbb{E}\left[\Big(\sum_{j=0}^{n}\left|\zeta_j^{\widehat{X}}\right|\Big)^{2q}\right]^{1/2q}.$$

From the definition of the projection operator, we have that, for all $r>1$,

$$\left|\zeta_j^{\widehat{X}}\right| \leqslant \delta+\left|\check{X}_{i+1}\right|\mathbf{1}_{\{\left|\check{X}_{i+1}\right|>\kappa\delta\}} \leqslant \delta+\frac{\left|\check{X}_{i+1}\right|^{r+1}}{(\kappa\delta)^r}$$

which leads to

$$\mathbb{E}\left[\sup_{0\leqslant j\leqslant n}\left|\tilde{X}_j^{\pi}-\widehat{X}_j^{\pi}\right|^{2q}\right]^{1/2q} \;\leqslant\; Cn\left(\delta+\frac{1}{(\kappa\delta)^r}\mathbb{E}\Big[\sup_{0\leqslant j\leqslant n}\left|\check{X}_j\right|^{2q(r+1)}\Big]^{\frac{1}{2q}}\right).$$

The proof for this step is concluded observing that $\mathbb{E}\Big[\sup_j\left|\check{X}_j\right|^{2q(r+1)}\Big]^{\frac{1}{2q}} \leqslant C_r$.
3. The proof is concluded by inserting the above estimate in (4.9).

$\square$

4.2 A numerical example

We illustrate in this part the convergence of the algorithm given in Definition 4.1 with $d \in \{1, 2, 3\}$. To this end, we consider the following quadratic Markovian BSDE:

$$\left\{ \begin{array}{ll} X_t^\ell & = X_0^\ell + \int_0^t \nu X_s^\ell \mathrm{d}W_s^\ell \,,\ \ell \in \{1,2,3\} \\ Y_t & = g(X_1) + \int_t^1 \frac{a}{2} \|Z_s\|^2 \mathrm{d}s - \int_t^1 Z_s \mathrm{d}W_s \end{array} \right. , \qquad 0 \leqslant t \leqslant 1,$$

where a, ν, and $(X_0^\ell)_{\ell \in \{1,2,3\}}$ are given real positive parameters and $g : \mathbb{R}^d \to \mathbb{R}$ is a bounded Lipschitz function.

Applying Ito's formula, one can show that the solution is given by

$$Y_t = \frac{1}{a} \log \Big(\mathbb{E}_t\big[\exp\big(ag(X_1)\big)\big] \Big) , \quad t \leqslant 1 .$$

For any given g, ν and a, it is possible to estimate the solution Y_0 at time 0 using an approximation of the Gaussian distribution at time $T = 1$, since $X_1^\ell = X_0^\ell e^{-\frac{\nu^2}{2} + \nu W_1^\ell}$.

4.2.1 Illustration when $d = 2$

For our numerical illustration, g is given by

$$g : x \mapsto 3 \sum_{\ell=1}^{2} \sin^2(x^\ell) ,$$

and we set $\nu = 1$, $X_0^1 = X_0^2 = 1$.
Given n the number of time steps in the approximation grid, we consider

$$N(n) = n^{\frac{1}{4}} \text{ and } R(n) = \log(n) ,$$

recalling (1.9). We will refer to the scheme given in Definition 4.1 with this set of parameters (N, R) as the 'adaptive truncation' scheme. We discuss in Section 4.2.3 below the choice of α.

The graph on Figure 1 shows the convergence of the algorithm for time step varying from 5 to 40. In the simulation, we fixed M to be large enough ($M = 100$), so that the error in the space discretization can be neglected in the analysis.

The expected convergence rate should be between 0.5, that is to say the minimal rate proved in this paper, and 1 the general optimal rate for the Euler scheme, see e.g. [22, 11]. We found a rate 0.6 which then seems reasonable. Note that all the convergence rate estimated below are also in the predicted range.
On Figure 2, we illustrate qualitatively the importance of the truncation procedure.

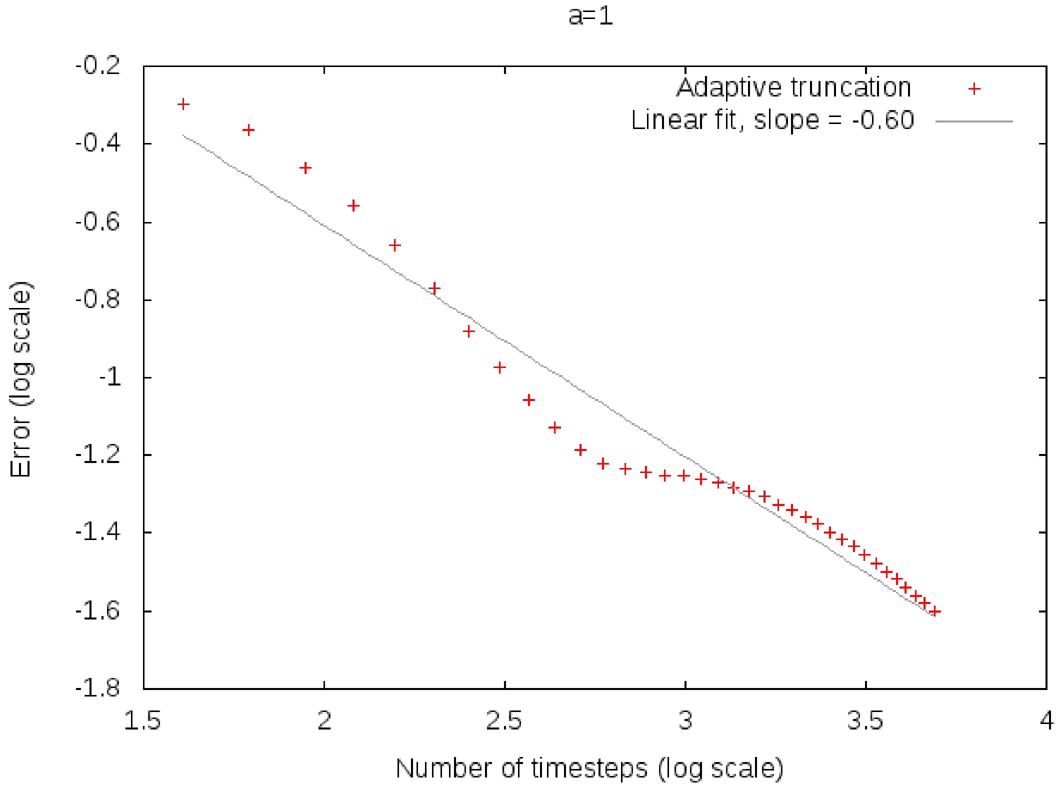

Figure 1: Empirical convergence of the scheme given in Definition 4.1

When $a = 1$, we already observed that the scheme given in Definition 4.1 is converging nicely. It appears that for this specific choice of parameters X_0, ν, g and a, the usual BTZ-scheme, referred to as 'no truncation' scheme, is also converging. But, when a becomes bigger, the usual BTZ-scheme becomes unstable.

On Figure 2, we consider $a = 3.5$. In this case, the behaviour of the usual BTZ-scheme is interesting. First, let us mention that we plot a truncated error which explains the flat alignment of some points. This shows that the scheme is not stable. It manages though to be stabilised when the number of time step is big enough (h small enough). We are not able to explain yet this behaviour. The detailed study of the numerical stability (or unstability) of the BTZ-scheme in the quadratic setting is outside the scope of this paper. These questions are left for further research. In the (more classical) Lipschitz case, we refer the reader to [12].

We also observe that the 'adaptive truncation' scheme is converging nicely, even for this large value of a.

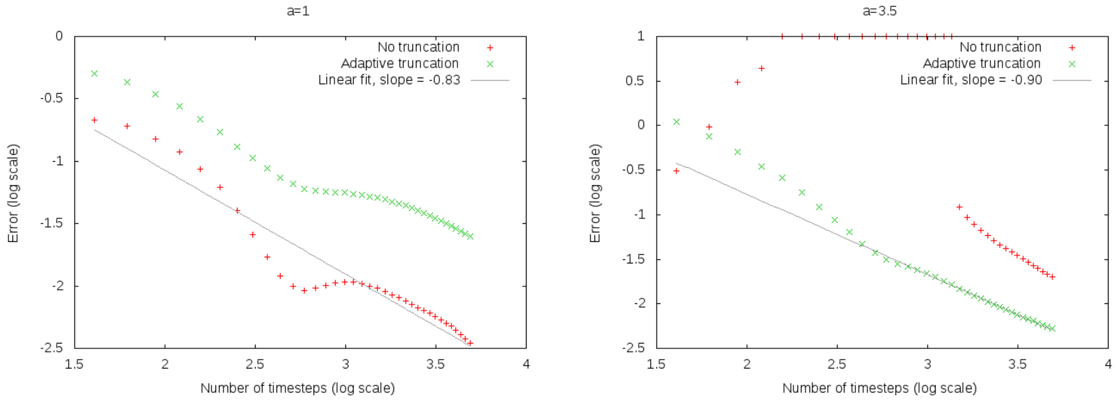

Figure 2: Comparison of schemes' convergence

4.2.2 Illustration when $d = 3$

For our numerical illustration, we tested the usual BTZ-scheme and the adaptively truncated scheme given in Definition 4.1 ($\alpha = 1/4$) for various models, i.e. various terminal conditions $g : \mathbb{R}^3 \to \mathbb{R}$ and values of a. In practice, we used the following parameters

- Model I: $g(x) = 3sin^2(\sum_{\ell=1}^{3} x^\ell)$ and $a = 5$.
- Model II: $g(x) = 3\sum_{\ell=1}^{3} sin^2(x^\ell)$ and $a = 5$.
- Model III: $g(x) = 4atan(\sum_{\ell=1}^{3} x^\ell)$ and $a = 5$.
- Model IV: $g(x) = 3 \wedge [x^1 - x^2]_+ + [2 - x^3]_+$ and $a = 4$.

We set the number of time steps $N = 12$.[3] We gather in the table below the results we obtained. The true value is estimated using the Cole-Hopf transform and we indicate, when relevant, the relative error between parenthesis.

Scheme/Model	*I*	*II*	*III*	*IV*
true value	2.67	7.53	5.38	3.96
No truncation	7.06×10^6	4.98×10^{59}	$5.31 (< 2\%)$	1.13×10^{29}
Adaptive truncation	$2.69\,(< 1\%)$	$7.29\,(\sim 3\%)$	$5.31\,(< 2\%)$	$4.37\,(\sim 10\%)$

For this large value of a, the adaptively truncated scheme is always able to compute good estimates of the true value. This is only the case for Model III when using the BTZ-scheme. For the other models, the usual BTZ-scheme is unstable.

[3] It takes 1/2 hour to obtain one value on an ultrabook with Intel Core i7-3667U CPU @ 2.00GHZ (4 cores).

4.2.3 Influence of the α parameter

To conclude this numerical illustration, we would like to comment on the choice of α. To do this, we work with $d = 1$ in order to be able to use quite a lot of time steps ($n = 250$). Moreover, we set $\nu = 0.4$, $a = 5$ and $g = 3\sin^2$. We plot on Figure 3 the convergence error of the scheme for $\alpha = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{5}{8}$ thus varying the truncation parameter $N = n^\alpha$. The theoretical convergence result of Corollary 4.1 states no dependence upon α for the convergence rate when $\alpha \in (0, \frac{1}{2})$. This is of course an asymptotic result. Nevertheless, we are able to observe this on Figure 3 for $\alpha = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}$ noticing small discrepancies for low n and some 'unstability' for $\alpha = 3/8$. For $\alpha = 0$ – meaning that the truncation is fixed to 1 – we observe that the scheme comes close to the correct value but then diverges, as expected. For $\alpha = \frac{5}{8}$, the scheme is unstable but manages to stabilize for large n. This numerical example is quite interesting as it illustrates the different behaviours of the scheme in terms of α. In general, the choice of α should depend on the various parameters of the problem X_0, ν, a and $\|g\|_\infty$ specially for small n. The optimal choice of α (balancing convergence error and stability) is an interesting question that requires a deeper understanding of the qualitative behaviour of the scheme in terms of the model parameters. These questions are left for further research.

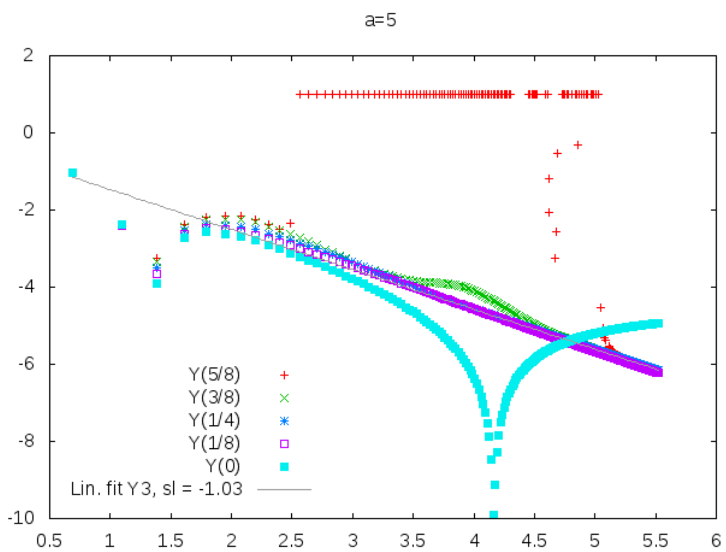

Figure 3: Convergence profile for different α - $Y(\alpha)$

A Appendix

A.1 Proof of Proposition 3.2

We have to study the quantity

$$A := \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left(\int_{t_j}^{t_{j+1}} \left|Z_s^N - \bar{Z}_j^N\right|^2 ds\right)^{1+\eta}\right]^p\right],$$

and show that $A \leqslant h^{p(1+\eta)}$. This is a slight modification of the well-known Zhang path regularity theorem, the main difference coming from the change of probability. To retrieve this result, we will mainly adapt the usual proof. In particular, we need to use a classical representation for Z based on the differentiability of (X, Y^N, Z^N) with respect to x (for the proof see e.g. [28, 6]).

Proposition A.1. *Suppose that b, σ, f and g are twice differentiable functions with respect to x, y, and z. Then for all $r \geqslant 2$ the process (X, Y^N, Z^N), solution of the system (1.1)-(2.1) belongs to $\mathcal{S}^r \times \mathcal{S}^r \times \mathcal{M}^r$ with norms bounded by constants that do not depend on N. Moreover, (X, Y^N, Z^N) is continuously differentiable with respect to the initial point x of the forward component. The derivative of X satisfies*

$$\nabla X_t = I + \int_0^t \nabla b(X_s)\nabla X_s ds + \int_0^t \nabla\sigma(X_s)\nabla X_s dW_s,$$

while the derivatives of (Y^N, Z^N) satisfy the linear BSDE

$$\begin{aligned}\nabla Y_t^N &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s^N dW_s + \int_t^T \nabla_x f_N(s, X_s, Y_s^N, Z_s^N)\nabla X_s \\ &\quad + \nabla_y f_N(s, X_s, Y_s^N, Z_s^N)\nabla Y_s^N + \nabla_z f_N(s, X_s, Y_s^N, Z_s^N)\nabla Z_s^N ds.\end{aligned}$$

The process $(\nabla X, \nabla Y^N, \nabla Z^N)$ belongs to $\mathcal{S}^r \times \mathcal{S}^r \times \mathcal{M}^r$ with norms bounded by constants that do not depend on N. Finally we have a continuous representation of Z^N given by

$$Z_t^N = \nabla Y_t^N (\nabla X_t)^{-1}\sigma(X_t).$$

We assume that b, σ, f, ρ_N and g are sufficiently smooth functions and thus we can apply Proposition A.1. When this is not the case, the result follows from standard approximation and stability results for quadratic BSDEs.
Working with the continuous version of Z^N we can consider $Z_{t_i}^N$ for all $0 \leqslant i < n$. We

have,

$$
\begin{aligned}
\left|Z_s^N - \bar{Z}_j^N\right|^2 &\leqslant 2\left|Z_s^N - Z_{t_j}^N\right|^2 + 2\left|Z_{t_j}^N - \bar{Z}_j^N\right|^2 \\
&\leqslant 2\left|Z_s^N - Z_{t_j}^N\right|^2 + 2\frac{1}{h_j}\mathbb{E}_{t_j}\left[\int_{t_j}^{t_{j+1}}\left|Z_u^N - Z_{t_j}^N\right|^2 du\right] \\
&\leqslant 2\left|Z_s^N - Z_{t_j}^N\right|^2 + \frac{2}{\varepsilon}\frac{1}{h_j}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\int_{t_j}^{t_{j+1}}\left|Z_u^N - Z_{t_j}^N\right|^2 du\right]
\end{aligned}
$$

because $1/\left|1 + h_j H_j^R \gamma_j^{N,n}\right| \leqslant 1/\varepsilon$ thanks to assumption ($\mathbf{H}3$). Inserting the previous estimate into the quantity A we get, by applying Hölder inequality and classical properties of the conditional expectation,

$$
\begin{aligned}
A &\leqslant C_{\eta,p}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left(\int_{t_j}^{t_{j+1}}\left|Z_s^N - Z_{t_j}^N\right|^2 ds\right)^{1+\eta}\right]^p\right] \\
&\quad + C_{\eta,p}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left(\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\int_{t_j}^{t_{j+1}}\left|Z_s^N - Z_{t_j}^N\right|^2 ds\right]\right)^{1+\eta}\right]^p\right] \\
&\leqslant C_{\eta,p}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\left(\int_{t_j}^{t_{j+1}}\left|Z_s^N - Z_{t_j}^N\right|^2 ds\right)^{1+\eta}\right]\right]^p\right] \\
&\leqslant C_{\eta,p}h^{p\eta}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\int_{t_j}^{t_{j+1}}\left|Z_s^N - Z_{t_j}^N\right|^{2(1+\eta)} ds\right]\right]^p\right]. \qquad (A.1)
\end{aligned}
$$

Then we use the continuous representation of Z^N given by Proposition A.1.

$$
\begin{aligned}
\left|Z_s^N - Z_{t_j}^N\right| &\leqslant \left|\nabla Y_s^N(\nabla X_s)^{-1}(\sigma(X_s) - \sigma(X_{t_j}))\right| \\
&\quad + \left|\nabla Y_s^N\left((\nabla X_s)^{-1} - (\nabla X_{t_j})^{-1}\right)\sigma(X_{t_j})\right| \\
&\quad + \left|\left(\nabla Y_s^N - \nabla Y_{t_j}^N\right)(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right| \\
&:= B_j^1 + B_j^2 + B_j^3.
\end{aligned}
$$

Then, inserting this into (A.1), we have

$$\begin{aligned} A &\leqslant C_{\eta,p} h^{p\eta} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_i}\left[\sum_{j=i}^{n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_j}\left[\int_{t_j}^{t_{j+1}} \left|B^1_j\right|^{2(1+\eta)} ds\right]\right]^p\right] \\ &\quad + C_{\eta,p} h^{p\eta} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_i}\left[\sum_{j=i}^{n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_j}\left[\int_{t_j}^{t_{j+1}} \left|B^2_j\right|^{2(1+\eta)} ds\right]\right]^p\right] \\ &\quad + C_{\eta,p} h^{p\eta} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_i}\left[\sum_{j=i}^{n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_j}\left[\int_{t_j}^{t_{j+1}} \left|B^3_j\right|^{2(1+\eta)} ds\right]\right]^p\right] \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

We now work on the first term. Thanks to Proposition 2.11 we know that $d\mathbb{Q}^\pi/d\mathbb{P}$ satisfies the reverse Hölder inequality and then

$$\begin{aligned} A_1 &\leqslant C_{\eta,p} h^{p\eta} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}^{\mathbb{Q}^\pi}_{t_i}\left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left|B^1_j\right|^{2(1+\eta)} ds\right]^p\right] \\ &\leqslant C_{\eta,p} h^{p\eta} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}\left[\left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left|B^1_j\right|^{2(1+\eta)} ds\right)^{q^*}\right]^{2p}\right]^{1/(2q^*)}. \end{aligned}$$

Applying Doob maximal inequality yields

$$A_1 \leqslant C_{\eta,p} h^{p\eta} \mathbb{E}\left[\left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left|B^1_j\right|^{2(1+\eta)} ds\right)^{2pq^*}\right]^{1/(2q^*)}$$

and, thanks to Hölder inequality and a convexity inequality, we get

$$A_1 \leqslant C_{\eta,p} h^{p\eta} \left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}\left[\left|B^1_j\right|^{4pq^*(1+\eta)}\right] ds\right)^{1/(2q^*)}.$$

Thanks to the Lipschitz regularity assumption on σ, Proposition A.1, a classical estimate on $(\nabla X)^{-1}$ and the estimate (3.11), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|B_j^1\right|^{4pq^*(1+\eta)}\right] \\
\leqslant\ &\mathbb{E}\left[\sup_{0\leqslant u\leqslant T}\left|\nabla Y_u^N(\nabla X_u)^{-1}\right|^{4pq^*(1+\eta)}\left(\sup_{t_j\leqslant u\leqslant t_{j+1}}\left|X_u - X_{t_j}\right|\right)^{4pq^*(1+\eta)}\right] \\
\leqslant\ &\mathbb{E}\left[\sup_{0\leqslant u\leqslant T}\left|\nabla Y_u^N(\nabla X_u)^{-1}\right|^{4pq^*(1+\eta)}\left(\sup_{t_j\leqslant u\leqslant t_{j+1}}\left|X_u - X_{t_j}\right|\right)^{4pq^*(1+\eta)}\right] \\
\leqslant\ &\mathbb{E}\left[\sup_{0\leqslant u\leqslant T}\left|\nabla Y_u^N(\nabla X_u)^{-1}\right|^{8pq^*(1+\eta)}\right]^{1/2}\mathbb{E}\left[\sup_{t_j\leqslant u\leqslant t_{j+1}}\left|X_u - X_{t_j}\right|^{8pq^*(1+\eta)}\right]^{1/2} \\
\leqslant\ &C_{\eta,p}h^{2pq^*(1+\eta)}.
\end{aligned}$$

Finally we have

$$A_1 \leqslant C_{\eta,p}h^{p(1+2\eta)},$$

with a constant C that does not depend on N. By the same type of arguments we can easily show that

$$A_2 \leqslant C_{\eta,p}h^{p(1+2\eta)}.$$

To handle the last term A_3 one needs to proceed with more care. Since $\left|1 + h_j H_j^R \gamma_j^{N,n}\right| \leqslant 1/\varepsilon$, we have that

$$\begin{aligned}
&A_3 \\
\leqslant\ &C_{\eta,p}h^{p\eta}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\int_{t_j}^{t_{j+1}}\left|\left(\nabla Y_s^N - \nabla Y_{t_j}^N\right)(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right|^{2(1+\eta)}ds\right]\right]^p\right] \\
\leqslant\ &C_{\eta,p}h^{p\eta}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left|(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right|^{2(1+\eta)}\right.\right. \\
&\qquad\left.\left.\times\int_{t_j}^{t_{j+1}}\mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\left|\nabla Y_s^N - \nabla Y_{t_j}^N\right|^{2(1+\eta)}\right]ds\right]^p\right] \\
\leqslant\ &C_{\eta,p}h^{p\eta}\mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1}\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1}\left|(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right|^{2(1+\eta)}\right.\right. \\
&\qquad\left.\left.\times\int_{t_j}^{t_{j+1}}\mathbb{E}_{t_j}\left[\left|\nabla Y_s^N - \nabla Y_{t_j}^N\right|^{2(1+\eta)}\right]ds\right]^p\right].
\end{aligned}$$

Writing the BSDE for the difference $\nabla Y_s^N - \nabla Y_{t_j}^N$ for $t_j \leqslant s \leqslant t_{j+1}$ and using the conditional Burkholder Davis Gundy inequality, we have, with $\Theta_r^N := (X_r, Y_r^N, Z_r^N)$,

$$\begin{aligned}
&\mathbb{E}_{t_j}\left[\left|\nabla Y_s^N - \nabla Y_{t_j}^N\right|^{2(1+\eta)}\right] \\
\leqslant\ & C_\eta \mathbb{E}_{t_j}\left[\left|\int_{t_j}^s \langle \nabla f_N(r,\Theta_r^N), \nabla\Theta_r^N\rangle dr\right|^{2(1+\eta)} + \left|\int_{t_j}^s \nabla Z_r^N dW_r\right|^{2(1+\eta)}\right] \\
\leqslant\ & C_\eta \mathbb{E}_{t_j}\left[\left(\int_{t_j}^{t_{j+1}} |\nabla f_N(r,\Theta_r^N)|\,|\nabla\Theta_r^N|\,dr\right)^{2(1+\eta)} + \left(\int_{t_j}^{t_{j+1}} |\nabla Z_r^N|^2\,dr\right)^{1+\eta}\right].
\end{aligned}$$

For the reader's convenience, we define the sum of the integrals inside the conditional expectation by $I_{[t_j,t_{j+1}]}$. Inserting the previous inequality into our last bound on A_3 and using again the fact that $\left|1 + h_j H_j \gamma_j^{N,n}\right| \leqslant 1/\varepsilon$, we obtain

$$\begin{aligned}
A_3 \ \leqslant\ & C_{\eta,p} h^{p(1+\eta)} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1} \mathbb{E}_{t_j}\left[\left|(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right|^{2(1+\eta)} I_{[t_j,t_{j+1}]}\right]\right]^p\right] \\
\leqslant\ & C_{\eta,p} h^{p(1+\eta)} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1} \mathbb{E}_{t_j}^{\mathbb{Q}^\pi}\left[\left|(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right|^{2(1+\eta)} I_{[t_j,t_{j+1}]}\right]\right]^p\right] \\
\leqslant\ & {}_{\eta,p}C h^{p(1+\eta)} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sum_{j=i}^{n-1} \left|(\nabla X_{t_j})^{-1}\sigma(X_{t_j})\right|^{2(1+\eta)} I_{[t_j,t_{j+1}]}\right]^p\right] \\
\leqslant\ & C_{\eta,p} h^{p(1+\eta)} \mathbb{E}\left[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\left[\sup_{0\leqslant t\leqslant T} \left|(\nabla X_t)^{-1}\sigma(X_t)\right|^{2(1+\eta)} \sum_{j=i}^{n-1} I_{[t_j,t_{j+1}]}\right]^p\right] \\
\leqslant\ & C_{\eta,p} h^{p(1+\eta)} \mathbb{E}\Bigg[\sup_{0\leqslant i\leqslant n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi}\Bigg[\sup_{0\leqslant t\leqslant T} \left|(\nabla X_t)^{-1}\sigma(X_t)\right|^{2(1+\eta)} \\
& \times \left\{\left(\int_0^T |\nabla f_N(r,\Theta_r)|\,|\nabla\Theta_r|\,dr\right)^{2(1+\eta)} + \left(\int_0^T |\nabla Z_r^N|^2\,dr\right)^{1+\eta}\right\}\Bigg]^p\Bigg].
\end{aligned}$$

Once again, thanks to Proposition 2.11 we know that $d\mathbb{Q}^\pi/d\mathbb{P}$ satisfies a reverse Hölder inequality and so we can get rid of the conditional expectation $\mathbb{E}_{t_i}^{\mathbb{Q}^\pi}$. Moreover we can get rid of the supremum by using the Doob maximal inequality. Finally, combining growth assumptions on f (true for f_N), and estimates given by Proposition A.1, we obtain

$$A_3 \leqslant C_{\eta,p} h^{p(1+\eta)}.$$

Collecting now the estimates on A_1, A_2 and A_3 we obtain that $A \leqslant C_{\eta,p} h^{p(1+\eta)}$. Since this estimate is true for all $\eta > 0$, the result is proved by taking $\eta := \frac{\tilde{\eta}}{1-\tilde{\eta}}$. □

A.2 Stability result for the Euler Scheme of an SDE

Lemma A.1. *Let us consider $q \geqslant 1$ and two forward schemes $(X_i)_{0\leqslant i\leqslant n}$ and $(\tilde{X}_i)_{0\leqslant i\leqslant n}$ given by*

$$\begin{aligned}
X_{i+1} &= X_i + h_i b(X_i) + \sigma(X_i)\sqrt{h_i}N_i,\\
\tilde{X}_{i+1} &= \tilde{X}_i + h_i b(\tilde{X}_i) + \sigma(\tilde{X}_i)\sqrt{h_i}N_i + \zeta_i,
\end{aligned}$$

with $(\zeta_i)_{0\leqslant i<n}$ some random variables in L^{2q} and $(N_i)_{0\leqslant i<n}$ some independent and centered random variables in L^{2q} such that N_i is $\mathcal{F}_{t_i}$ measurable for all $0 \leqslant i < n$ and $\mathbb{E}_{t_i}[N_i^2] = \mathbb{E}[N_i^2] \leqslant C$ with C that does not depend on n. Then, we have the following stability result:

$$\mathbb{E}\left[\sup_{0\leqslant k\leqslant n}\left|X_k - \tilde{X}_k\right|^{2q}\right] \leqslant C_q\left|X_0 - \tilde{X}_0\right|^{2q} + C_q\mathbb{E}\left[\left(\sum_{j=0}^{n-1}|\zeta_j|\right)^{2q}\right].$$

Proof. By considering the difference between the two schemes, we have

$$X_i - \tilde{X}_i = X_0 - \tilde{X}_0 + \sum_{j=0}^{i-1} h_j[b(X_j) - b(\tilde{X}_j)] + \sum_{j=0}^{i-1}\sqrt{h_j}[\sigma(X_j) - \sigma(\tilde{X}_j)]N_j + \sum_{j=0}^{i-1}\zeta_j,$$

and

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leqslant k\leqslant i}\left|X_k - \tilde{X}_k\right|^{2q}\right] \leqslant\; & C_q\left|X_0 - \tilde{X}_0\right|^{2q} + C_q\mathbb{E}\left[\left(\sum_{j=0}^{i-1}|\zeta_j|\right)^{2q}\right]\\
&+ C_q\mathbb{E}\left[\sup_{0\leqslant k\leqslant i}\left|\sum_{j=0}^{k-1} h_j[b(X_j) - b(\tilde{X}_j)]\right|^{2q}\right]\\
&+ C_q\mathbb{E}\left[\sup_{0\leqslant k\leqslant i}\left|\sum_{j=0}^{k-1}\sqrt{h_j}[\sigma(X_j) - \sigma(\tilde{X}_j)]N_j\right|^{2q}\right].
\end{aligned}$$

Recalling that b and σ are Lipschitz and by using a convexity inequality and the

Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leqslant k\leqslant i}\left|X_k-\tilde{X}_k\right|^{2q}\right] &\leqslant C_q\left|X_0-\tilde{X}_0\right|^{2q}+C_q\mathbb{E}\left[\left(\sum_{j=0}^{i-1}|\zeta_j|\right)^{2q}\right]\\
&\quad +C_q\sum_{j=0}^{i-1}h_j\mathbb{E}\left[\sup_{0\leqslant k\leqslant j}\left|X_k-\tilde{X}_k\right|^{2q}\right]\\
&\quad +C_q\mathbb{E}\left[\left(\sum_{j=0}^{i-1}h_j\left|X_j-\tilde{X}_j\right|^2\right)^{q}\right]\\
&\leqslant C_q\left|X_0-\tilde{X}_0\right|^{2q}+C_q\mathbb{E}\left[\left(\sum_{j=0}^{n-1}|\zeta_j|\right)^{2q}\right]\\
&\quad +C_q\sum_{j=0}^{i-1}h_j\mathbb{E}\left[\sup_{0\leqslant k\leqslant j}\left|X_k-\tilde{X}_k\right|^{2q}\right].
\end{aligned}$$

The proof is concluding by a direct application of the discrete Gronwall's Lemma. □

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