Approval Voting for Committee Elections: a General Family of Rules
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Résumé

Le vote par approbation est une procédure de vote utilisée, entre autres, pour élire des comités et qui permet aux votants de voter pour (“d’approuver”), le nombre de candidats qu’ils souhaitent. Deux règles de vote ont été particulièrement utilisées pour élire des comités à l’aide du vote par approbation. La règle usuelle, appelée aussi minisum, choisit l’ensemble des candidats (éventuellement soumis à une contrainte de cardinalité) ayant été le plus approuvés par les votants. La règle minimax élit un ensemble de candidats qui minimise le maximum, sur l’ensemble des votants, de la distance de Hamming à chaque vote.

Comme ces deux règles semblent trop extrêmes, nous les généralisons en un ensemble continu de règles de vote, par l’utilisation de l’opérateur de moyenne pondérée ordonnée (ordered weighted averaging OWA). Cette règle est paramétrée par un vecteur de poids, noté W, qui nous permet de modéliser des procédures de votes entre minisum et minimax. Nous nous intéressons aux vecteurs de poids non-décroissants, et en particulier, aux vecteurs de la forme W(i)=(0,..,0,1,..,1), où i est le nombre de 0’s. Nous étudions la complexité de la détermination d’un comité gagnant, et de l’ensemble des comités gagnants pour des règles associées aux vecteurs W(i). Nous montrons qu’il est difficile de trouver l’ensemble des comités gagnants pour ces règles, sauf pour minisum avec un nombre impair de votants pour laquelle cela est facile. Enfin, nous prouvons la manipulabilité de ces règles quand elles sont paramétrées par des vecteurs non-décroissants, et strictement croissants.

Mots Clef

Choix social computationnel, Vote d’approbation, Moyenne pondérée ordonnée, Complexité, Manipulation.

Abstract

Approval voting is a well-known voting procedure used, among others, for electing committees, where each voter casts a ballot consisting of a set of approved candidates (without any cardinality constraint). Two prominent rules for electing committees using approval voting are the standard rule (also called minisum), which selects the set of candidates (possibly subject to some cardinality constraint) with the highest number of approvals, and the minimax rule, where the set of elected candidates minimizes the maximum, over all voters, of the Hamming distance to the voter’s ballot.

As these two rules are in some way too extreme, we generalize them into a continuum of rules, by using ordered weighted averaging operators (OWA). The rule is parameterized by a weight vector W, which allows us to model voting procedures between minisum and minimax. We focus on non-decreasing weight vectors, and in particular, vectors of the form W(i)=(0,...,0,1,...,1), where i is the number of 0’s. We address the computational aspects of finding a winning committee and all the winning committees for rules associated with the W(i) vectors. We show that finding a winning committee for these rules is NP-hard whereas it is computationally easy for minisum. Finally, we address the issue of manipulating the rules when parameterized by non-decreasing and strictly increasing weight vectors.

Keywords

Computational social choice, Approval voting, Ordered weighted averaging, Complexity, Manipulation.

1 Introduction

Approval voting is a well-known voting procedure used for electing a single winner or a committee [2]. Voters cast approval ballots, which consist each of a set of approved candidates. An approval ballot can be seen as a binary vector indicating the candidates approved by a voter. Single-winner approval voting elects the candidate with the highest number of approvals (using some tie-breaking if necessary). An approval ballot can be seen as a binary vector indicating the candidates approved by a voter. Single-winner approval voting elects the candidate with the highest number of approvals (using some tie-breaking if necessary). As for multi-winner approval voting, there are several different ways of finding the winning committee, i.e., the winning set of candidates.

The most standard way consists in choosing the candidates according to their approval scores : in the case where the committee must be of size K, the candidates with the K highest number of approvals are elected ; or, if the size of the committee is not subject to any constraint, the candidates approved by a majority of voters are elected.

As argued in [3], this rule can be unfair to some voters,
who may completely disagree with the elected committee. To remedy this, they define minimax approval voting, that selects a committee that minimizes the maximum, over all voters, of the Hamming distance to the voter’s ballot, seen as a binary vector. The minimax outcome tends to be more widely acceptable than the minsum outcome, as it minimizes the disagreement of the least satisfied voter. Replacing max by sum in the definition of minimax approval voting (thus electing a committee that minimizes the sum of the Hamming distances to the ballots) leads back to the standard rule, that the authors rename minsum.

As these two rules (minimax and minsum) are in some way too extreme, we generalize them into a continuum of rules by using ordered weighted averaging (OWA). The OWA operator is an aggregation operator parameterized by a weight vector \( W \), which enables us to model various operators like maximum, minimum, or arithmetic average [15, 16]. In OWA aggregation, the weights are associated with ordered values of the input, and the aggregated score corresponds to the ordered weighted sum.

We introduce multi-winner OWA approval rules, denoted by \( AV_W \). A winning committee for \( AV_W \) is a committee minimizing the \( AV_W \) score, which corresponds to the aggregated score of OWA-W. We study a subset of \( AV_W \) parameterized by non-decreasing weight vectors which can model voting rules between minisum and minimax. First, we focus on a simple family of non-decreasing weight vectors \( W(i) = (0, \ldots, 0, 1, \ldots, 1) \), where \( i \) represents the number of 0’s, and present some properties verified by winning committees. We address the computational aspects of finding a winning committee and finding the co-winner set for \( AV_W \). Furthermore, we give manipulation results on \( AV_W \) parameterized by non-decreasing and strictly increasing weight vectors.

This work is related to at least three research streams. The first of these is a series of works in social choice theory that study multi-winner approval voting through the two solution concepts minisum and minimax, [4, 3, 14]. One can find a review of the procedures that can be used for electing committees using approval voting in [7]. More generally, our paper relates to computational aspects of multi-winner elections [13].

The second related stream is the study of ordered weighted averaging, introduced in [15]. OWA has been studied in several domains, especially in multi-criteria decision making and decision under uncertainty [17].

The third related research stream is a series of works that study manipulation issues and strategic behaviour in multi-winner elections [12]. The computational aspects of strategic behavior in standard multiwinner approval voting have been studied in [1]. The works [5, 10] both address the computational aspects of minimax approval voting (see Section 2 for more details) and study the conditions under which an approximation of minimax is (or not) sensitive to manipulation. It relates mainly to the section 5, where we explore manipulation of \( AV_W \).

Minimax approval voting also relates to belief merging, which aims at combining several pieces of information coming from different sources [8]. More details on this connection as well as on the computational aspects of multi-winner elections can be found in [9].

The paper is organized as follows. In Section 2 we introduce the necessary background on multi-winner approval voting and ordered weighted averaging. In Section 3 we give some properties on winning committees for \( AV_W \). In Section 4 we present some computational results on determining a winning committee for \( AV_W \). Section 5 addresses manipulation issues for \( AV_W \) parameterized by non-decreasing vectors. In Section 6 we present further research directions.

## 2 Preliminaries

We are given an election \( E \) with \( m \) candidates \( X = \{x_1, \ldots, x_m\} \) and \( n \) voters \( N = \{1, \ldots, n\} \). The approval ballot of a voter \( i \) is a subset of \( X \), represented by a binary vector \( v \in \{0, 1\}^m \), where the \( j \)-th bit of \( v \) is 1 if the voter \( i \) approves the candidate \( j \), 0 otherwise. An approval profile \( P = \{P_i\}_{i \in N} \) is a collection of approval ballots. For a multi-vector \( v \), we denote by \( v_i \), the \( i \)-th coordinate of the vector \( v \). For two binary vectors \( v, v' \in \{0, 1\}^m \), we denote by \( H(v, v') \) their Hamming distance, i.e. the number of bits on which they differ. Given a binary vector \( c \in \{0, 1\}^n \), called a committee, and an approval profile \( P \), the vector \( H(c, P) = (H(c, P_i))_{i \in N} \) denotes the vector of Hamming distances between \( c \) and each ballot of \( P \). The approval score of a candidate \( x \), denoted by \( app(x) \), is the number of voters who approve \( x \).

In multi-winner approval voting, the goal is to elect a committee, that is, a subset of \( X \). Sometimes the size of the committee is fixed to some integer value \( K \); sometimes it is not fixed and any committee is feasible (see [3]).

The minisum rule consists in electing a committee that minimizes the sum of the Hamming distances to the ballots. Formally, \( c^* \in \{0, 1\}^m \) is a winning committee for minisum if and only if \( \sum_{i \in N} H(c^*, P_i) = \min_{c \in \{0, 1\}^m} \sum_{i \in N} H(c, P_i) \). A majority voting committee is a committee containing all the candidates approved by strictly more than \( n/2 \) voters, and no candidates approved by strictly less than \( n/2 \) voters. It is known that a winning committee for minisum is equivalent to a majority voting committee [3]. Thus, given an approval profile \( P \), a winning committee for minisum can be easily computed.

The minimax rule elects a committee that minimizes the maximum of its Hamming distances to the ballots, i.e. \( c^* \) is a winning committee for minimax if and only if \( \max_{i \in N} H(c^*, P_i) = \min_{c \in \{0, 1\}^m} \{\max_{i \in N} H(c, P_i)\} \).
Given an approval profile $P$, finding a winning committee for minimax is NP-hard, since it is equivalent to the closest string problem in coding theory [6, 11].

The following example illustrates these two rules:

**Example** Consider an election $E$, with 6 voters \{1, 2, 3, 4, 5, 6\}, and 4 candidates \{x_1, x_2, x_3, x_4\}, and the following profile $P$:

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<tr>
<td>$P_1$</td>
<td>0110</td>
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<td>$P_2$</td>
<td>0010</td>
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<td>$P_3$</td>
<td>0011</td>
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<td>$P_4$</td>
<td>0001</td>
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<tr>
<td>$P_5$</td>
<td>1001</td>
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<tr>
<td>$P_6$</td>
<td>0001</td>
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Figure 1 shows the Hamming distance, the minisum score and the minimax score for some committees. One can verify that the committees that are not mentioned are not winning committees for neither minisum nor minimax. The

$$H(c, i) = \sum_{j=1}^{n} |c_j - i_j|$$

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<tr>
<td>$c_1$</td>
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<td>2</td>
<td>1</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>$c_2$</td>
<td>0001</td>
<td>3</td>
<td>2</td>
<td>1</td>
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<td>1</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>$c_3$</td>
<td>0101</td>
<td>2</td>
<td>1</td>
<td>0</td>
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<td>1</td>
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</table>

**Figure 1 – Minimun and minimax scores**

winning committees for minsum are $c_2$ and $c_3$, whereas they are $c_1$ and $c_3$ for minimax.

We generalize minsum and minimax into a continuum of rules by using ordered weighted averaging (OWA) for multi-winner approval voting. Given a vector $H$, we can order its coordinates in a non-decreasing way and the ordered vector will be denoted by $H'$. OWA is a family of functions, $O_W : \mathbb{R}^n \to \mathbb{R}$, parameterized by a vector $W$ of size $n$. It maps a $n$-vector of scores $H$ to an aggregated score, called OWA score: $H \mapsto O_W(H) = W \times H'$. Note that the definition of OWA differs from the original introduced in [15] in two respects: we use an opposite ordering of vectors $H$, and we do not require any normalization of weights.

We now introduce multi-winner OWA approval rules, denoted by $AV_W$. A committee $c$ is a winning committee for $AV_W$ if and only if $O_W(H(c, P)) = \min_{c \in \{0, 1\}}\{O_W(H(c, P))\}$. We call co-winner set the set of all winning committees for an approval profile $P$.

The following example illustrates OWA approval rules:

**Example** Consider the election of the previous example. Figure 2 shows the $AV_W$ score for some committees.

Thus, if we take $W = (1, 2, 3, 4, 5, 6)$, the scores of $c_1$, $c_2$ and $c_3$ are respectively 39, 37 and 35: we can check that there is no better committee and that the winning committee is $c_3$. If we take $W = (0, 0, 0, 1, 1, 1)$, then the scores of $c_1$, $c_2$ and $c_3$ are respectively 66, 66 and 66, all winning committees. If we take $W = (0, 0, 1, 1, 1)$, then the scores of $c_1$, $c_2$ and $c_3$ are respectively 8, 7, 7; $c_2$ and $c_3$ are all winning committees.

For fairness issues, we will focus on *non-decreasing weight vectors* $W$, which give more weight to more unsatisfied voters than to more satisfied voters. A vector $W$ is non-decreasing (respectively, strictly increasing) if $W_i \leq W_{i+1}$ (resp. $W_i < W_{i+1}$), for all $i = 1, \ldots, n - 1$. $AV_W$ parameterized by non-decreasing vectors allows us to model a range of voting rules between minsum and minimax, which can be represented respectively by the vectors $W = (1, \ldots, 1)$ and $W = (0, \ldots, 0, 1)$. An interesting family of vectors is the family $W(i)$ defined by $W(i) = (0, \ldots, 0, 1, \ldots, 1)$, where $i$ is the number of 0’s, for $i = 0, \ldots, n - 1$; this family ranges from minsum approval voting (corresponding to $W(0)$) to minimax approval voting (corresponding to $W(n - 1)$).

### 3 Winning committees for $AV_W(i)$

In this section, we focus on non-decreasing weight vectors $W(i) = (0, \ldots, 0, 1, \ldots, 1)$, where $i$ is the number of 0’s, in order to explore properties verified by winning committees.

First, we give simple sufficient conditions for a specific candidate to be a member of all winning committees, or not to be a member of any winning committee.

**Proposition 3.1** For $AV_W(i)$, $i \in N$, every candidate approved by at least $(n + i + 1)/2$ voters is included in all winning committees.

**Proof** Consider an election $E$ and an $AV_W(i)$ rule. Suppose there exists a candidate $x$ approved by at least $(n + i + 1)/2$ voters, such that $x$ is not included in a winning committee $c$. Then, the approval score of $x$ verifies the following inequalities:

$$\text{app}(x) \geq (n + i + 1)/2$$

$$2 \times \text{app}(x) - n \geq i + 1$$

$$\text{app}(x) - (n - \text{app}(x)) \geq i + 1$$

(1)

Now we study the $AV_W(i)$ score of the committee $c' = c \cup \{x\}$. Compared to $c$, there are $\text{app}(x)$ voters who have their Hamming distance reduced by 1, and $(n - \text{app}(x))$ voters who have their Hamming distance increased by 1. Since we consider the weight vector $W(i)$, it follows from inequality 1 that the $AV_W(i)$ score of $c'$ is strictly less than the $AV_W(i)$ score of $c$, a contradiction. [1]
With a similar argument to the proof of Proposition 3.1 we can prove the following proposition.

**Proposition 3.2** For $AV_{W_1}$, $i \in N$, every candidate approved by less than $(n - (i + 1))/2$ voters is not included in any winning committee.

Notice that Propositions 3.1 and 3.2 generalize the result stating that each majority voting committee is a winning committee for minisum [3]. These two propositions enable us to establish the following links between minisum and $AV_{W_1}$ and between $AV_{W_{n-2}}$ and minimax.

**Proposition 3.3** There always exists a committee $c$ such that $c$ is a winning committee for both minisum and $AV_{W_1}$.

**Proof** We distinguish two cases.

First consider an election $E$ with an odd number of voters. Then we build the following winning committee $c$ for the $AV_{W_1}$ rule:

- Candidates that are approved by more than $(n + 3)/2$ voters are in $c$. (prop. 3.1)
- Candidates that are approved by less than $(n - 3)/2$ voters are not in $c$. (prop. 3.2)
- Candidates that are approved by exactly $(n + 1)/2$ voters are in $c$. Clearly, since we consider $AV_{W_1}$, removing such candidates from the committee will not decrease its score.
- Candidates that are approved by exactly $(n - 1)/2$ voters are not in $c$. Clearly, since we consider $AV_{W_1}$, adding such candidates to the committee will not decrease its score.

Furthermore, $c$ is clearly a majority voting committee, hence $c$ is a winning committee for minisum.

Now consider an election $E$ with an even number of voters. Let $c$ be a winning committee for $AV_{W_1}$. Then it follows from Proposition 3.1 and 3.2 that:

- Candidates that are approved by more than $(n + 2)/2$ voters are in $c$.
- Candidates that are approved by less than $(n - 2)/2$ voters are not in $c$.

Thus, by definition, the committee $c$ is a majority voting committee, so $c$ is also a winning committee for minisum.

However, the result of Proposition 3.3 cannot be generalized to $AV_{W_1}$ and $AV_{W_2}$, as the following example (found by a computer program) shows.

**Example** Consider an election $E$ with 7 voters $\{1, \ldots, 7\}$, 5 candidates $\{x_1, \ldots, x_5\}$ and an approval profile defined by:

- $P_1 = (01111)$
- $P_2 = (01111)$
- $P_3 = (01110)$
- $P_4 = (11111)$
- $P_5 = (10000)$
- $P_6 = (01011)$

There is unique winning committee for $AV_{W_1}$ which is $(01111)$, and a unique winning committee for $AV_{W_2}$ which is $(11010)$.

Notice that we were not able to find an example for $AV_{W_1}$ and $AV_{W_2}$ when there is an even number of voters. Thus it remains open if the property holds for $AV_{W_1}$ and $AV_{W_2}$ and an even number of voters. Also, we conjecture that the counterexample generalizes to any pair $(i, i + 1)$ for $2 \leq i \leq n - 3$, that is, for all $i$ such that $2 \leq i \leq n - 3$, there exists a profile $P$ such that $AV_{W_1}(P) \cap AV_{W_1+i}(P) = \emptyset$. On the other hand, for $i = n - 2$ we get a positive result again:

**Proposition 3.4** There always exists a committee $c$ such that $c$ is a winning committee for both $AV_{W_{n-1}}$, that is to say minimax, and $AV_{W_{n-2}}$.

**Proof** Let $E$ be an election with $n$ voters and $m$ candidates. Let $c$ be a winning committee for minimax. Also, let $c'$ be a winning committee for $AV_{W_{n-2}}$ which maximizes the $(n-1)^{th}$ coordinate of the ordered vector of the Hamming distances.

We claim that $c'$ is a winning committee for minimax or $c$ is a winning committee for $AV_{W_{n-2}}$. Indeed, suppose that $c'$ is not a winning committee for minimax and $c$ is not a winning committee for $AV_{W_{n-2}}$. The fact that $c'$ is not a winning committee for minimax implies that the $AV_{W_{n-1}}$ score of $c'$ is strictly larger than the $AV_{W_{n-1}}$ score of $c$:

$$O_{W_{n-1}}(H(c', P)) > O_{W_{n-1}}(H(c, P))$$
$$H^\uparrow(c', P)_n \geq H^\uparrow(c, P)_n + 1 \quad (2)$$

Also, the fact that $c'$ is not a winning committee for $AV_{W_{n-2}}$ implies that the $AV_{W_{n-2}}$ score of $c$ is strictly larger than the $AV_{W_{n-2}}$ score of $c'$:

$$O_{W_{n-2}}(H(c, P)) > O_{W_{n-2}}(H(c', P))$$
$$n \sum_{j=n-1}^n H^\uparrow(c, P)_j \geq n \sum_{j=n-1}^n H^\uparrow(c', P)_j + 1 \quad (3)$$

From (2) and (3) we obtain:

$$H^\uparrow(c', P)_{n-1} + 2 \leq H^\uparrow(c, P)_{n-1},$$

Therefore, we have:

$$H^\uparrow(c', P)_{n-1} + 2 \leq H^\uparrow(c, P)_n,$$

This, together with (2), implies that:

$$H^\uparrow(c', P)_{n-1} + 2 \leq H^\uparrow(c', P)_{n-1} - 1,$$
$$H^\uparrow(c', P)_{n-1} + 3 \leq H^\uparrow(c', P)_n \quad (4)$$

Inequality 4 implies that there exists at least one candidate $x$ such that the voter corresponding to the $(n-1)^{th}$ largest
Hamming distance agrees with \( c' \) and the voter corresponding to the \( n \)th largest Hamming distance disagrees with \( c' \) on \( x \).

First assume that \( x \) is a member of \( c' \). Now if we consider the committee \( d = c' \setminus \{x\} \), then \( H^1(d, P)_{n-1} \) increases by one, and \( H^1(d, P)_{n} \) decreases by one compared to \( c' \). So, \( d \) is also a winning committee for \( AV_{W(n-2)} \), but \( H^1(d, P)_{n-1} > H^1(c', P)_{n-1} \) which contradicts the definition of \( c' \).

With the same argument, assuming that \( x \) is not a member of \( c' \) contradicts the definition of \( c' \).

4 Computational aspects

What we know about the complexity of computing winning committees for \( AV_W \) is that (a) finding a winning committee for minimax approval voting is polynomial, and it is also easy to give a simple polynomial characterization of all winning committees; (b) finding a winning committee for minimax approval voting is NP-hard. Our conjecture is that finding a winning committee for \( AV_W \) is hard for “most” vectors \( W \). We start by showing that, at least under the restriction to an odd number of voters, minimax is not the only \( AV_W \) committee election rule for which winner determination is polynomial: indeed, when the number of voters is odd, finding a winning committee is easy for \( AV_{W(1)} \).

**Proposition 4.1** When \( n \) is odd, finding a winning committee for \( AV_{W(1)} \) is polynomial time solvable.

**Proof** Consider an election \( E \), with an odd number of voters \( n \), and the following committee \( c = \{x \in X, such that app(x) \geq \frac{n+1}{2}\} \). As we have already shown in the proof of Proposition 3.3, \( c \) is a winning committee for \( AV_{W(1)} \).

The positive result of Proposition 4.1 could let us think that finding a winning committee is easy for \( AV_{W(1)} \) in general. But the proof of Proposition 4.1 is based on a case study that cannot be used when there is an even number of voters. Indeed in that case, there are candidates with exactly \( n/2 \) votes, who could have either a positive or a negative effect on the score of a committee. Thus the complexity of finding a winning committee for \( AV_{W(1)} \) remains an open question (and we do not have any conjecture about it). The complexity of finding a winning committee for \( AV_{W(k)} \) for any \( k \geq 2 \) is also open, but so far, the only NP-hardness we have is for values of \( k \) that are very close to \( n \):

**Proposition 4.2** For any fixed \( i \), finding a winning committee for \( AV_{W(n-i)} \) is NP-hard.

**Proof** Our proof is based on a reduction from winner determination in minimax approval voting. Consider an election \( E' \) for minimax approval voting, \( E' = (X', N', P') \) with \( |X'| = n', |N'| = n' \). We construct an election \( E \) for \( AV_{W(n-i)} \) with \( X = X', N = i \times N' \) and \( P = i \times P' \) meaning that we create \( i \) copies of each voter of \( N' \). We claim that a winning committee for \( E \) is winning committee for \( E' \) and vice-versa. To prove this claim, it is sufficient to show that the \( AV_{W(n-i)} \) score of a committee \( c \) in \( E \) is \( i \) times its minimax score in \( E' \). Clearly for each voter in \( N' \) with a Hamming distance \( h \), there exist \( i \) voters in \( N \) having the same Hamming distance \( h \).

We may wonder about the complexity of outputting all winning committees for \( AV_{W(i)} \). We start by remarking that clearly, for minisum with an even number of voters, the size of the co-winner set can be exponential in the number of candidates (consider an election where all the candidates are approved by half of the voters: then all the committees are winning committees). On the other hand, with an odd number of voters, there are no candidates approved by exactly half of voters, hence there is exactly one winning committee. We may wonder whether this last result extends to \( AV_{W(i)} \) for \( i \geq 1 \); actually, it does not:

**Proposition 4.3** There exists a collection of elections such that the size of the co-winner set for any \( i \geq 1 \), \( AV_{W(i)} \) is exponential in the number of candidates, even when \( n \) is odd.

**Proof** We study an infinite family of elections \( (E_{n \times m})_{n,m \in N} \), with an even number of voters, such that for each of its elements, the size of the co-winner set for any \( AV_{W(i)} \) is in \( \Theta(2^n) \). Consider the infinite family of elections \( (E_{n \times m})_{n,m \in N} = (X, N, P) \), where \( P \) is the approval profile defined as follows: \( n/2 \) voters approve all the candidates, and the other \( n/2 \) voters approve no candidates.

Remark that, by construction of such an election, the ordered vector of Hamming distances, \( H^1(c, P) \), of any committee \( c \) to the profile \( P \) verifies : \( H^1(c, P) = (\alpha_1, \ldots, \alpha_c, \beta_c, \ldots, \beta_c) \), with \( c, \beta_c \in N \), \( n/2 \) occurrences of each \( \alpha_c \) and \( \beta_c \), \( \alpha_c < \beta_c \), and \( \alpha_c + \beta_c = m \).

Now, consider an election \( e_{n \times m} \) of this family, a vector \( W(i) \), \( i = 1 \ldots n - 1 \), and a committee \( c \). We distinguish two cases:

- if \( i \geq n/2 \), then
  \[
  O_{W(i)}(H(c, P)) = (m - i) \times \beta_c
  \]
- if \( i < n/2 \), then
  \[
  O_{W(i)}(H(c, P)) = \begin{cases} 
  n/2 \times \beta_c + (n/2 - i) \times \alpha_c & \text{if } i \geq n/2, \\
  n/2 \times (\alpha_c + \beta_c) - i \times (m - \beta_c) & \text{if } i < n/2.
  \end{cases}
  \]

With these two results we see that finding the minimum of \( O_{W(i)}(H(c, P)) \) is equivalent to minimizing the the value of \( \beta_c \). The committees that have a minimal value of \( \beta_c \)
are the committees with \(m/2\) candidates if \(m\) is even, and 
\([m/2]\) or \([m/2]+1\) if \(m\) is odd. Thus, there are \(2^{m/2}\) of such committees.

With the same argument, we can construct a similar family of elections \((E_{n×m})_{n,m\in\mathbb{N}} = (X,V,P)\), with an odd number of voters, such that for each of its element, the size of the co-winner set for an \(AV_{W(i)}\) is in \(Θ(2^m)\), for \(i \geq 1\).

What this result does not say is whether we can find a simple, succinct characterization of all winning committees, as it is the case for minsum (recall that for minsum, the winning committees are all subsets of candidates that contain all candidates approved by a strict majority of voters and do not contain any candidate disapproved by a strict majority of voters). This question is left for further research.

5 Manipulability

In this section, we study the manipulability of the \(AV_W\) rules. Before starting, note that so far we consider the \(AV_W\) rules as nondeterministic rules, that is, mapping profiles into sets of winning committees. In this section, we consider deterministic rules (because studying the manipulability of nondeterministic rules requires much more background and definitions, that we do not have the space to expose here). A deterministic multi-winner OWA-approval rule is obtained as the composition of a nondeterministic \(AV_W\) rule and a tie-breaking mechanism; the obtained deterministic rule is denoted by \(AV_W^R\). As we do not want to sacrifice anonymity, a tie-breaking mechanism is a priority relation \(R\) over all possible committees (we can, in case there is more than one committee with a minimal score, the output of a deterministic rule \(AV_W^R\) will be the most priority committee (according to \(R\)) among the committees with minimal score.

We now discuss the manipulability of deterministic \(AV_W\) rules. A (deterministic) voting rule is said to be manipulable if a voter (or a group of voters) can change the result of the election in his/her favor by not expressing his/her true preferences. So far, we know that minimax approval voting is manipulable and that minsum approval voting is nonmanipulable. Further research obviously starts by establishing further complexity results, by trying to prove that “almost” all rules of our families are hard to compute.

\begin{proof}

Consider an election \(E\), with 4 voters \(\{1,2,3,4\}\), and 4 candidates \(\{x_1,x_2,x_3,x_4\}\), and the following profile \(P\):

\[ P_1 : (0100); P_2 : (0101); P_3 : (0110); P_4 : (0111) \]

Figure 3 shows the Hamming distance vectors for some committees. One can easily verify that committees that are not mentioned have \(AV_W\) scores larger than the \(AV_W\) scores of co-winning committees. The \(AV_W\) scores of these committees are equal and minimal, so there are 4 co-winning committees. Then, according to the tie-breaking rule, the committee (0111) is elected. The manipulation comes from voter 1 by voting (1000) instead of his true preferences. With this new vote, we have new \(AV_W\) scores summarized in Figure 4. Then, with a non-decreasing \(W\)

\begin{figure}[h]
\centering
\begin{tabular}{c|c}
\hline
\(c\) & \(W × H^3(c,P)\) \\
\hline
0100 & 0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4 \\
0101 & 0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4 \\
0110 & 0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4 \\
0111 & 0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4 \\
\hline
\end{tabular}
\caption{\(AV_W\) scores with true preferences}
\end{figure}

\begin{figure}[h]
\centering
\begin{tabular}{c|c}
\hline
\(c\) & \(W × H^3(c,P)\) \\
\hline
0100 & 1 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 2 \cdot W_4 \\
0101 & 0 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 3 \cdot W_4 \\
0110 & 0 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 3 \cdot W_4 \\
0111 & 0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 4 \cdot W_4 \\
\hline
\end{tabular}
\caption{\(AV_W\) scores with manipulation of voter 1}
\end{figure}

such that \(W_1 < W_n\), we obtain an unique winning committee (0100).
\end{proof}

This proof can be extended to any numbers of candidates by adding dummy candidates approved by all the voters, and to even numbers of voters by adding copies of voters 1 and 4.

6 Conclusion

We have generalized minsum and minimax approval voting by defining a continuous family of committee voting rules that contain minsum and minimax as special cases. We have established a few preliminary results about computational complexity of determining a winning committee. We have also shown that among our rules, only minsum is nonmanipulable. Further research obviously starts by establishing further complexity results, by trying to prove that “almost” all rules of our families are hard to compute.
Références


