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Approximation by Müntz spaces on positive intervals

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Abstract

The so-called Bernstein operators were introduced by S.N. Bernstein in 1912 to give a constructive proof of Weierstrass' theorem. We show how to extend his result to Müntz spaces on positive intervals. *To cite this article: R. Ait-Haddou and M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ??? (200?).*

Approximation par espaces de Müntz sur un intervalle positif. Résumé

En 1912, les opérateurs dits de Bernstein permirent à S.N. Bernstein de donner une preuve constructive du théorème de Weierstrass. Nous étendons ce résultat aux espaces de Müntz sur des intervalles positifs. *Pour citer cet article : R. Ait-Haddou and M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ??? (200?).*

1. Introduction

The famous Bernstein operator $\mathbb{B}_k$ of degree k on a given non-trivial interval $[a, b]$, associates with any $F \in C^0([a, b])$ the polynomial function

$$\mathbb{B}_k F(x) := \sum_{i=0}^k F\left(\left(1 - \frac{i}{k}\right)a + \frac{i}{k}b\right) B_i^k, \quad x \in [a, b], \quad (1)$$

where $(B_0^k, \dots, B_k^k)$ is the Bernstein basis of degree k on $[a, b]$, i.e., $B_i^k(x) := \binom{k}{i} \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{k-i}$. It reproduces any affine function U on $[a, b]$, in the sense that $\mathbb{B}_k U = U$. In [5], S.N. Bernstein proved that, for all function $F \in C^0([a, b])$, $\lim_{k \rightarrow +\infty} \|F - \mathbb{B}_k F\|_\infty = 0$. In Section 3 we show how this result extends to the class of Müntz spaces (i.e., spaces spanned by power functions) on a given positive interval $[a, b]$, see Theorem 3.1. Beforehand, in Section 2 we briefly remind the reader how to define operators of the Bernstein-type in Extended Chebyshev spaces.

2. Extended Chebyshev spaces and Bernstein operators

Throughout this section, $[a, b]$ is a fixed non-trivial real interval. For any $n \geq 0$, a given $(n+1)$ -dimensional space $\mathbb{E} \subset C^n([a, b])$ is said to be an *Extended Chebyshev space* (for short, EC-space) on $[a, b]$ when any non-zero element of $\mathbb{E}$ vanishes at most n times on $[a, b]$ counting multiplicities up to $(n+1)$.

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Let $\mathbb{E}$ be an $(n+1)$ -dimensional EC-space on $[a, b]$. Then, $\mathbb{E}$ possesses bases $(B_0, \dots, B_n)$ such that, for $i = 0, \dots, n$, B_i vanishes exactly i times at a and $(n-i)$ times at b and is positive on $]a, b[$. We say that such a basis is the *Bernstein basis relative to (a, b)* if it additionally satisfies $\sum_{i=0}^n B_i = \mathbb{I}$, where $\mathbb{I}$ is the constant function $\mathbb{I}(x) = 1$, $x \in [a, b]$. Let us recall that $\mathbb{E}$ possesses a Bernstein basis relative to (a, b) if and only if, firstly it contains constants, and secondly the n -dimensional space $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$ is an EC-space on $[a, b]$. Note that the second property is not an automatic consequence of the first one, see [8] and other references therein.

As an instance, given any pairwise distinct $\lambda_0, \dots, \lambda_k$, the so-called *Müntz space* $M(\lambda_0, \dots, \lambda_k)$, spanned over a given positive interval $[a, b]$ (i.e., $a > 0$) by the power functions x^{λ_i} , $0 \leq i \leq k$, is a $(k+1)$ -dimensional EC-space on $[a, b]$. If $\lambda_0 = 0$, since $D(M(\lambda_0, \dots, \lambda_k)) = M(\lambda_1 - 1, \dots, \lambda_k - 1)$, the space $M(\lambda_0, \dots, \lambda_k)$ possesses a Bernstein basis relative to (a, b) .

For the rest of the section we assume that $\mathbb{E} \subset C^n([a, b])$ contains constants and that $D\mathbb{E}$ is an $(n-1)$ -dimensional EC-space on $[a, b]$. We denote by $(B_0, \dots, B_n)$ the Bernstein basis relative to (a, b) in $\mathbb{E}$.

Definition 2.1 A linear operator $\mathbb{B} : C^0([a, b]) \rightarrow \mathbb{E}$ is said to be a *Bernstein operator based on $\mathbb{E}$* when, firstly it is of the form

$$\mathbb{B}F := \sum_{i=0}^k F(\zeta_i) B_i, \quad \text{for some } a = \zeta_0 < \zeta_1 < \dots < \zeta_n = b, \quad (2)$$

and secondly it reproduces a two-dimensional EC-space $\mathbb{U}$ on $[a, b]$, in the sense that $\mathbb{B}V = V$ for all $V \in \mathbb{U}$.

Any Bernstein operator $\mathbb{B}$ is positive (i.e., $F \geq 0$ implies $\mathbb{B}F \geq 0$) and shape preserving due to the properties of Bernstein bases in EC-spaces, see [8]. Everything concerning Bernstein-type operators in EC-spaces with no Bernstein bases can be deduced from Bernstein operators as defined above [8], [9].

Theorem 2.2 Given $n \geq 2$, let $\mathbb{E} \subset C^n([a, b])$ contain constants. Assume that $D\mathbb{E}$ is an n -dimensional EC-space on $[a, b]$. For a function $U \in \mathbb{E}$, expanded in the Bernstein basis relative to (a, b) as $U := \sum_{i=0}^n u_i B_i$, the following properties are equivalent:

- (i) $u_0, \dots, u_n$ form a strictly monotonic sequence;
- (ii) there exists a nested sequence $\mathbb{E}_1 \subset \mathbb{E}_2 \subset \dots \subset \mathbb{E}_{n-1} \subset \mathbb{E}_n := \mathbb{E}$, where $\mathbb{E}_1 := \text{span}(\mathbb{I}, U)$ and where, for $i = 1, \dots, n-1$, $\mathbb{E}_i$ is an $(i+1)$ -dimensional EC-space on $[a, b]$;
- (iii) there exists a Bernstein operator based on $\mathbb{E}$ which reproduces U .

In [8] it was proved that there exists a one-to-one correspondence between the set of all Bernstein operator based on $\mathbb{E}$ and the set of all two-dimensional EC-spaces $\mathbb{U}$ they reproduce. In particular, if (i) holds, then the unique Bernstein operator based on $\mathbb{E}$ reproducing U is defined by (2) with

$$\zeta_i := U^{-1}(u_i), \quad 0 \leq i \leq n. \quad (3)$$

Note that this is meaningful since (i) implies the strict monotonicity of U on $[a, b]$. Condition (ii) of Th. 2.2 yields the following corollary.

Corollary 2.3 Given an integer $n \geq 1$, consider a nested sequence

$$\mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \dots \subset \mathbb{E}_p \subset \mathbb{E}_{p+1} \subset \dots, \quad (4)$$

where $\mathbb{E}_n$ contains constants and for any $p \geq n$, $D\mathbb{E}_p$ is a p -dimensional EC-space on $[a, b]$. Let $U \in \mathbb{E}_n$ be a non-constant function reproduced by a Bernstein operator $\mathbb{B}_n$ based on $\mathbb{E}_n$. Then, U is also reproduced by a Bernstein operator $\mathbb{B}_p$ based on $\mathbb{E}_p$ for any $p > n$.

Remark 2.4 In the situation described in Corollary 2.3, a natural question arises: given $F \in C^0([a, b])$, does the sequence $\mathbb{B}_k F$, $k \geq n$, converges to F in $C^0([a, b])$ equipped with the infinite norm? Obviously,

for this to be true for any $F \in C^0([a, b])$, it is necessary that $\cup_{k \geq n} \mathbb{E}_k$ be dense in $C^0([a, b])$. The example of Müntz spaces proves that this is not always satisfied.

3. Müntz spaces over positive intervals

Throughout this section we consider a fixed positive interval $[a, b]$, a fixed infinite sequence of real numbers λ_k , $k \geq 0$, assumed to satisfy

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_k < \lambda_{k+1} < \dots, \quad \lim_{k \rightarrow +\infty} \lambda_k = +\infty. \quad (5)$$

We are interested with the corresponding nested sequence of Müntz spaces

$$M(\lambda_0) \subset M(\lambda_0, \lambda_1) \subset \dots \subset M(\lambda_0, \dots, \lambda_k) \subset M(\lambda_0, \dots, \lambda_k, \lambda_{k+1}) \subset \dots \quad (6)$$

Given any $n \geq 1$, for each $k \geq n$, we can select a Bernstein operator $\mathbb{B}_k$ based on $M(\lambda_0, \dots, \lambda_k)$. Assume the sequence $\mathbb{B}_k$, $k \geq n$, to satisfy

$$\lim_{k \rightarrow +\infty} \|F - \mathbb{B}_k F\|_\infty = 0 \quad \text{for any } F \in C^0([a, b]). \quad (7)$$

Then, the union of all spaces $M(\lambda_0, \dots, \lambda_k)$, $k \geq 0$, is dense in $C^0([a, b])$ equipped with the infinite norm. As is well-known, this holds if and only if the sequence (5) fulfils the so-called *Müntz density condition* below [4], [6],

$$\sum_{i \geq 1} \frac{1}{\lambda_i} = +\infty. \quad (8)$$

As an instance, the Müntz condition (8) is satisfied when $\lambda_k = \ell + 1$ for all $k \geq 1$. This case was addressed in [8]. Convergence – in the sense of (7) – was proved there under the assumption that each $\mathbb{B}_k$ reproduced the function x^{λ_1} . This convergence result includes the classical Bernstein operators [5] obtained with $\ell = 0$. Below we extend it to the general interesting situation of sequences of Müntz Bernstein operators $\mathbb{B}_k$ all reproducing the same two-dimensional EC-space (see Remark 2.4).

Theorem 3.1 *Given $n \geq 1$, let $\mathbb{E}_1 \subset M(\lambda_0, \dots, \lambda_n)$ be a two-dimensional EC-space reproduced by a Bernstein operator $\mathbb{B}_k$ based on $M(\lambda_0, \dots, \lambda_k)$ for any $k \geq n$. Then, if the Müntz density condition (8) holds, the sequence $\mathbb{B}_k$, $k \geq n$, converges in the sense of (7).*

Before starting the proof, let us introduce some notations. For $k \geq 1$, denote by $(B_{k,0}, \dots, B_{k,k})$ the Bernstein basis relative to (a, b) in the Müntz space $M(\lambda_0, \dots, \lambda_k)$. We consider the functions

$$U^*(x) = x^{\lambda_1}, \quad V_p(x) := x^{\lambda_p}, \quad p \geq 2, \quad x \in [a, b],$$

expanded in the successive Bernstein bases as

$$U^* = \sum_{i=0}^k u_{k,i}^* B_{k,i} \quad \text{for all } k \geq 1, \quad V_p = \sum_{i=0}^k v_{p,k,i} B_{k,i} \quad \text{for all } k \geq p. \quad (9)$$

With these notations, the key-point to prove Theorem 3.1 is the following lemma, for the proof of which we refer to [2], see also [1].

Lemma 3.2 *Assume that the Müntz density condition (8) holds. Then, we have*

$$\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} \left| (u_{k,i}^*)^{\frac{\lambda_p}{\lambda_1}} - v_{p,k,i} \right| = 0 \quad \text{for all } p \geq 2. \quad (10)$$

Proof of Theorem 3.1: • Let us start with the simplest example $n = 1$. Then, $\mathbb{E}_1 = \text{span}(\mathbb{1}, U^*)$. For each $k \geq 1$, the unique operator $\mathbb{B}_k^*$ which reproduces $\mathbb{E}_1$ is given by

$$\mathbb{B}_k^* F := \sum_{i=0}^k F(\zeta_{k,i}^*) B_{k,i}, \quad \text{with, for } i = 0, \dots, k, \quad \zeta_{k,i}^* := (u_{k,i}^*)^{\frac{1}{\lambda_1}} \quad (11)$$

According to Korovkin's theorem for positive linear operators [7], we just have to select a function F so that $\mathbb{I}, U^*, F$ span a three-dimensional EC-space on $[a, b]$ and prove that $\lim_{k \rightarrow +\infty} \|F - \mathbb{B}_k^* F\|_\infty = 0$ for this specific F . We can thus choose for instance $F := V_2$. Actually we will more generally prove the result with $F = V_p$, for any $p \geq 2$. Using (9) and (11), we obtain, for any $k \geq p$,

$$\|\mathbb{B}_k^* V_p - V_p\|_\infty = \left\| \sum_{i=0}^k (V_p(\zeta_{k,i}^*) - v_{p,k,i}) B_{k,i} \right\|_\infty \leq \max_{0 \leq i \leq k} |V_p(\zeta_{k,i}^*) - v_{p,k,i}|. \quad (12)$$

On account of (11), Lemma 3.2 yields the expected result

$$\lim_{k \rightarrow +\infty} \|\mathbb{B}_k^* V_p - V_p\|_\infty = 0 \quad \text{for each } p \geq 2.$$

• We now assume that $n > 1$. Select a strictly increasing function $U \in \mathbb{E}_1$. Condition (ii) of Theorem 2.2 enables us to select a function $V \in M(\lambda_0, \dots, \lambda_n)$ so that the functions $\mathbb{I}, U, V$ span a three-dimensional EC-space on $[a, b]$. For any $k \geq n$, expand U, V as

$$U = \sum_{i=0}^k u_{k,i} B_{k,i}, \quad V = \sum_{i=0}^k v_{k,i} B_{k,i}.$$

We know that, for each $k \geq n$, the sequence $(u_{k,0}, \dots, u_{k,k})$ is strictly increasing, and that the Bernstein operator $\mathbb{B}_k$ is defined by formula (2) with $\zeta_{k,i} := U^{-1}(u_{k,i})$ for $i = 0, \dots, k$. Via expansions of U and V in the basis $(\mathbb{I}, U^*, V_2, \dots, V_n)$ of the Müntz space $M(\lambda_0, \dots, \lambda_n)$, Lemma 3.2 readily proves that

$$\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |U(\zeta_{k,i}^*) - u_{k,i}| = 0 = \lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |V(\zeta_{k,i}^*) - v_{k,i}| \quad (13)$$

The left part in (13) can be written as $\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |U(\zeta_{k,i}^*) - U(\zeta_{k,i})| = 0$. On this account, the uniform continuity of the function $V \circ U^{-1}$ and the right part in (13) prove that $\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |V(\zeta_{k,i}) - v_{k,i}| = 0$, thus implying that $\lim_{k \rightarrow +\infty} \|\mathbb{B}_k V - V\|_\infty = 0$. By Korovkin's theorem, (7) is satisfied. $\square$

Remark 3.3 Given $n \geq 2$, one can apply Theorem 3.1 with $\mathbb{E}_1 := \text{span}(\mathbb{I}, V_n) = M(\lambda_0, \lambda_n)$, due to the nested sequence of Müntz spaces $M(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_n)$ for $1 \leq i \leq n$. Note that Theorem 3.1 contains in particular the *Bernstein-type result* expected in [3].

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