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Approximation by Müntz spaces on positive intervals

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Abstract
The so-called Bernstein operators were introduced by S.N. Bernstein in 1912 to give a constructive proof of Weierstrass' theorem. We show how to extend his result to Müntz spaces on positive intervals. To cite this article: R. Ait-Haddou and M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ???? (200?).

Approximation par espaces de Müntz sur un intervalle positif. Résumé

1. Introduction

The famous Bernstein operator \( B_k \) of degree \( k \) on a given non-trivial interval \([a, b]\) associates with any \( F \in C^0([a, b]) \) the polynomial function

\[
B_k F(x) := \sum_{i=0}^{k} F \left( \frac{i}{k} \right) \left( \frac{x-a}{b-a} \right)^i \left( \frac{b-x}{b-a} \right)^{k-i}, \quad x \in [a, b],
\]

where \((B_0^k, \ldots, B_k^k)\) is the Bernstein basis of degree \( k \) on \([a, b]\), i.e., \( B_i^k(x) := \binom{k}{i} \left( \frac{x-a}{b-a} \right)^i \left( \frac{b-x}{b-a} \right)^{k-i} \). It reproduces any affine function \( U \) on \([a, b]\), in the sense that \( B_k U = U \). In [5], S.N. Bernstein proved that, for all function \( F \in C^0([a, b]) \), \( \lim_{k \to +\infty} \| F - B_k F \|_\infty = 0 \). In Section 3 we show how this result extends to the class of Müntz spaces (i.e., spaces spanned by power functions) on a given positive interval \([a, b]\), see Theorem 3.1. Beforehand, in Section 2 we briefly remind the reader how to define operators of the Bernstein-type in Extended Chebyshev spaces.

2. Extended Chebyshev spaces and Bernstein operators

Throughout this section, \([a, b]\) is a fixed non-trivial real interval. For any \( n \geq 0 \), a given \((n+1)\)-dimensional space \( \mathcal{E} \subset C^n([a, b]) \) is said to be an Extended Chebyshev space (for short, EC-space) on \([a, b]\) when any non-zero element of \( \mathcal{E} \) vanishes at most \( n \) times on \([a, b]\) counting multiplicities up to \((n+1)\).

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Let $E$ be an $(n+1)$-dimensional EC-space on $[a, b]$. Then, $E$ possesses bases $(B_0, \ldots, B_n)$ such that, for $i = 0, \ldots, n$, $B_i$ vanishes exactly $i$ times at $a$ and $(n-i)$ times at $b$ and is positive on $[a, b]$. We say that such a basis is the Bernstein basis relative to $(a, b)$ if it additionally satisfies $\sum_{i=0}^{n} B_i = 1$, where $1$ is the constant function $1(x) = 1$, $x \in [a, b]$. Let us recall that $E$ possesses a Bernstein basis relative to $(a, b)$ if and only if, firstly it contains constants, and secondly the $n$-dimensional space $DE := \{ DF := F' | F \in E \}$ is an EC-space on $[a, b]$. Note that the second property is not an automatic consequence of the first one, see [8] and other references therein.

As an instance, given any pairwise distinct $\lambda_0, \ldots, \lambda_k$, the so-called M"untz space $M(\lambda_0, \ldots, \lambda_k)$, spanned over a given positive interval $[a, b]$ (i.e., $a > 0$) by the power functions $x^{\lambda_i}$, $0 \leq i \leq k$, is a $(k+1)$-dimensional EC-space on $[a, b]$. If $\lambda_0 = 0$, since $D(M(\lambda_0, \ldots, \lambda_k)) = M(\lambda_1-1, \ldots, \lambda_k-1)$, the space $M(\lambda_0, \ldots, \lambda_k)$ possesses a Bernstein basis relative to $(a, b)$.

For the rest of the section we assume that $E \subset C^n([a, b])$ contains constants and that $DE$ is an $(n$-dimensional) EC-space on $[a, b]$. We denote by $(B_0, \ldots, B_n)$ the Bernstein basis relative to $(a, b)$ in $E$.

**Definition 2.1** A linear operator $B : C^0([a, b]) \to E$ is said to be a Bernstein operator based on $E$ when, firstly it is of the form

$$BF := \sum_{i=0}^{k} F(\zeta_i) B_i, \quad \text{for some } a = \zeta_0 < \zeta_1 < \cdots < \zeta_n = b,$$

and secondly it reproduces a two-dimensional EC-space $U$ on $[a, b]$, in the sense that $BV = V$ for all $V \in U$.

Any Bernstein operator $B$ is positive (i.e., $F \geq 0$ implies $BF \geq 0$) and shape preserving due to the properties of Bernstein bases in EC-spaces, see [8]. Everything concerning Bernstein-type operators in EC-spaces with no Bernstein bases can be deduced from Bernstein operators as defined above [8], [9].

**Theorem 2.2** Given $n \geq 2$, let $E \subset C^n([a, b])$ contain constants. Assume that $DE$ is an $n$-dimensional EC-space on $[a, b]$. For a function $U \in E$, expanded in the Bernstein basis relative to $(a, b)$ as $U := \sum_{i=0}^{n} u_i B_i$, the following properties are equivalent:

(i) $u_0, \ldots, u_n$ form a strictly monotonic sequence;

(ii) there exists a nested sequence $E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n := E$, where $E_1 := \text{span}(1, U)$ and where, for $i = 1, \ldots, n-1$, $E_i$ is an $(i+1)$-dimensional EC-space on $[a, b]$;

(iii) there exists a Bernstein operator based on $E$ which reproduces $U$.

In [8] it was proved that there exists a one-to-one correspondence between the set of all Bernstein operator based on $E$ and the set of all two-dimensional EC-spaces $U$ they reproduce. In particular, if (i) holds, then the unique Bernstein operator based on $E$ reproducing $U$ is defined by (2) with

$$\zeta_i := U^{-1}(u_i), \quad 0 \leq i \leq n.$$

Note that this is meaningful since (i) implies the strict monotonicity of $U$ on $[a, b]$. Condition (ii) of Th. 2.2 yields the following corollary.

**Corollary 2.3** Given an integer $n \geq 1$, consider a nested sequence

$$E_n \subset E_{n+1} \subset \cdots \subset E_p \subset E_{p+1} \subset \cdots,$$

where $E_n$ contains constants and for any $p \geq n$, $DE_p$ is a $p$-dimensional EC-space on $[a, b]$. Let $U \in E_n$ be a non-constant function reproduced by a Bernstein operator $B_{E_n}$ based on $E_n$. Then, $U$ is also reproduced by a Bernstein operator $B_{E_p}$ based on $E_p$ for any $p > n$.

**Remark 2.4** In the situation described in Corollary 2.3, a natural question arises: given $F \in C^0([a, b])$, does the sequence $B_k F$, $k \geq n$, converges to $F$ in $C^0([a, b])$ equipped with the infinite norm? Obviously,
for this to be true for any $F \in C^0([a, b])$, it is necessary that $\cup_{k \geq n} E_k$ be dense in $C^0([a, b])$. The example of Müntz spaces proves that this is not always satisfied.

3. Müntz spaces over positive intervals

Throughout this section we consider a fixed positive interval $[a, b]$, a fixed infinite sequence of real numbers $\lambda_k$, $k \geq 0$, assumed to satisfy
\[
0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k < \lambda_{k+1} < \ldots, \quad \lim_{k \to +\infty} \lambda_k = +\infty. \tag{5}
\]
We are interested with the corresponding nested sequence of Müntz spaces
\[
M(\lambda_0) \subset M(\lambda_0, \lambda_1) \subset \cdots \subset M(\lambda_0, \ldots, \lambda_k) \subset M(\lambda_0, \ldots, \lambda_k, \lambda_{k+1}) \subset \cdots \tag{6}
\]
Given any $n \geq 1$, for each $k \geq n$, we can select a Bernstein operator $B_k$ based on $M(\lambda_0, \ldots, \lambda_k)$. Assume the sequence $B_k$, $k \geq n$, to satisfy
\[
\lim_{k \to +\infty} \|F - B_k F\|_\infty = 0 \quad \text{for any } F \in C^0([a, b]). \tag{7}
\]
Then, the union of all spaces $M(\lambda_0, \ldots, \lambda_k)$, $k \geq 0$, is dense in $C^0([a, b])$ equipped with the infinite norm. As is well-known, this holds if and only if the sequence (5) fulfills the so-called Müntz density condition below [4], [6],
\[
\sum_{k \geq 1} \frac{1}{\lambda_k} = +\infty. \tag{8}
\]
As an instance, the Müntz condition (8) is satisfied when $\lambda_k = \ell + 1$ for all $k \geq 1$. This case was addressed in [8]. Convergence – in the sense of (7) – was proved there under the assumption that each $B_k$ reproduced the function $x^{\lambda_1}$. This convergence result includes the classical Bernstein operators [5] obtained with $\ell = 0$. Below we extend it to the general interesting situation of sequences of Müntz Bernstein operators $B_k$ all reproducing the same two-dimensional EC-space (see Remark 2.4).

Theorem 3.1 Given $n \geq 1$, let $E_1 \subset M(\lambda_0, \ldots, \lambda_n)$ be a two-dimensional EC-space reproduced by a Bernstein operator $B_k$ based on $M(\lambda_0, \ldots, \lambda_k)$ for any $k \geq n$. Then, if the Müntz density condition (8) holds, the sequence $B_k$, $k \geq n$, converges in the sense of (7).

Before starting the proof, let us introduce some notations. For $k \geq 1$, denote by $(B_{k,0}, \ldots, B_{k,k})$ the Bernstein basis relative to $(a, b)$ in the Müntz space $M(\lambda_0, \ldots, \lambda_k)$. We consider the functions
\[
U_*(x) = x^{\lambda_1}, \quad V_p(x) := x^{\lambda_p}, \quad p \geq 2, \quad x \in [a, b],
\]
expanded in the successive Bernstein bases as
\[
U_* = \sum_{k=0}^{\infty} u_{k,i}^* B_{k,i} \quad \text{for all } k \geq 1, \quad V_p = \sum_{i=0}^{p} v_{p,k,i} B_{k,i} \quad \text{for all } k \geq p. \tag{9}
\]
With these notations, the key-point to prove Theorem 3.1 is the following lemma, for the proof of which we refer to [2], see also [1].

Lemma 3.2 Assume that the Müntz density condition (8) holds. Then, we have
\[
\lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| \frac{u_{k,i}^*}{\lambda_p} - v_{p,k,i} \right| = 0 \quad \text{for all } p \geq 2. \tag{10}
\]

Proof of Theorem 3.1: Let us start with the simplest example $n = 1$. Then, $E_1 = \text{span}(1, U_*)$. For each $k \geq 1$, the unique operator $B_k^*$ which reproduces $E_1$ is given by
\[
B_k^* F := \sum_{i=0}^{k} F(\zeta_{k,i}^*) B_{k,i}, \quad \text{with, for } i = 0, \ldots, k, \quad \zeta_{k,i}^* := (u_{k,i}^*)^{1/\lambda_k} \tag{11}
\]
According to Korovkin’s theorem for positive linear operators \[7\], we just have to select a function \(F\) so that \(\mathbf{1}, U^*, F\) span a three-dimensional EC-space on \([a, b]\) and prove that \(\lim_{k \to +\infty} \|F - B^*_k F\|_\infty = 0\) for this specific \(F\). We can thus choose for instance \(F := V_2\). Actually we will more generally prove the result with \(F = V_p\), for any \(p \geq 2\). Using (9) and (11), we obtain, for any \(k \geq p\),

\[
\|\mathbb{B}^*_k V_p - V_p\|_\infty = \left\| \sum_{i=0}^{k} (V_p(\zeta^*_i) - v_{p,k,i}) B_{k,i} \right\|_\infty \leq \max_{0 \leq i \leq k} |V_p(\zeta^*_i) - v_{p,k,i}|. \tag{12}
\]

On account of (11), Lemma 3.2 yields the expected result

\[
\lim_{k \to +\infty} \|\mathbb{B}^*_k V_p - V_p\|_\infty = 0 \quad \text{for each } p \geq 2.
\]

- We now assume that \(n > 1\). Select a strictly increasing function \(U \in \mathcal{E}_1\). Condition (ii) of Theorem 2.2 enables us to select a function \(V \in M(\lambda_0, \ldots, \lambda_n)\) so that the functions \(\mathbf{1}, U, V\) span a three-dimensional EC-space on \([a, b]\). For any \(k \geq n\), expand \(U, V\) as

\[
U = \sum_{i=0}^{k} u_{k,i} B_{k,i}, \quad V = \sum_{i=0}^{k} v_{k,i} B_{k,i}.
\]

We know that, for each \(k \geq n\), the sequence \((u_{k,0}, \ldots, u_{k,k})\) is strictly increasing, and that the Bernstein operator \(\mathbb{B}_k\) is defined by formula (2) with \(\zeta_{k,i} := U^{-1}(u_{k,i})\) for \(i = 0, \ldots, k\). Via expansions of \(U\) and \(V\) in the basis \((\mathbf{1}, U^*, V_2, \ldots, V_n)\) of the Müntz space \(M(\lambda_0, \ldots, \lambda_n)\), Lemma 3.2 readily proves that

\[
\lim_{k \to +\infty} \max_{0 \leq i \leq k} |U(\zeta^*_i) - u_{k,i}| = 0 = \lim_{k \to +\infty} \max_{0 \leq i \leq k} |V(\zeta^*_i) - v_{k,i}| \tag{13}
\]

The left part in (13) can be written as \(\lim_{k \to +\infty} \max_{0 \leq i \leq k} |U(\zeta^*_i) - U(\zeta_{k,i})| = 0\). On this account, the uniform continuity of the function \(V \circ U^{-1}\) and the right part in (13) prove that \(\lim_{k \to +\infty} \max_{0 \leq i \leq k} |V(\zeta^*_i) - v_{k,i}| = 0\), thus implying that \(\lim_{k \to +\infty} \|\mathbb{B}_k V - V\|_\infty = 0\). By Korovkin’s theorem, (7) is satisfied. \(\square\)

**Remark 3.3** Given \(n \geq 2\), one can apply Theorem 3.1 with \(\mathcal{E}_1 := \operatorname{span}(\mathbf{1}, V_n) = M(\lambda_0, \lambda_n)\), due to the nested sequence of Müntz spaces \(M(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n)\) for \(1 \leq i \leq n\). Note that Theorem 3.1 contains in particular the Bernstein-type result expected in [3].

### References


