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Summary: The purpose of the present paper consists in proposing and discussing a double probabilistic representation for a porous media equation in the whole space perturbed by a multiplicative colored noise. For almost all random realizations ω, one associates a stochastic differential equation in law with random coefficients, driven by an independent Brownian motion. The key ingredient is a uniqueness lemma for a linear SPDE of Fokker-Planck type with measurable bounded (possibly degenerated) random coefficients.

Key words: stochastic partial differential equations, infinite volume, singular porous media type equation, double probabilistic representation, multiplicative noise, singular random Fokker-Planck type equation.

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1 Introduction

We consider a function $\psi : \mathbb{R} \to \mathbb{R}$ and real functions $e^1, \ldots, e^N$ on $\mathbb{R}$, for some strictly positive integer $N > 0$. In the whole paper, the following Assumption will be in force.

Assumption 1.1. • $\psi : \mathbb{R} \to \mathbb{R}$ *is such that its restriction to* $\mathbb{R}_+$ *is monotone increasing, with* $\psi(0) = 0$.

• $|\psi(u)| \leq \mathrm{const}|u|, \ u \geq 0$.
In particular, ψ *is right-continuous at zero and* $\psi(0) = 0$.

• *Let* $e^i \in C_{\mathrm{b}}^2(\mathbb{R}), 0 \leq i \leq N$, *such that they are* H^{-1}*-multipliers in the sense that the maps* $\varphi \mapsto \varphi e^i$ *are continuous in the* H^{-1}*-topology.* $\mathcal{C}(e^i)$ *denotes the norm of this operator and we will call it* **multiplier norm***.*

Let $T > 0$ and $(\Omega, \mathcal{F}, P)$, be a fixed probability space. Let $(\mathcal{F}_t, t \in [0, T])$ be a filtration fulfilling the usual conditions and we suppose $\mathcal{F} = \mathcal{F}_T$. Let $\mu(t, \xi), t \in [0, T], \xi \in \mathbb{R}$, be a random field of the type

$$\mu(t, \xi) = \sum_{i=1}^{N} e^i(\xi) W_t^i + e^0(\xi)t, \ t \in [0, T], \xi \in \mathbb{R},$$

where $W^i, 1 \leq i \leq N$, are independent continuous $\mathcal{F}_t$)-Brownian motions on $(\Omega, \mathcal{F}, P)$, which are fixed from now on until the end of the paper. For technical reasons we will sometimes set $W_t^0 \equiv t$. We focus on a stochastic partial differential equation of the following type:

$$\begin{cases} \partial_t X(t, \xi) &= \frac{1}{2} \partial_{xx}^2 (\psi(X(t, \xi)) + X(t, \xi)\partial_t \mu(t, \xi), \\ X(0, \mathrm{d}\xi) &= x_0(\mathrm{d}\xi), \end{cases} \tag{1.1}$$

which holds in the sense of Definition 2.4, where x_0 is a a given probability measure on $\mathbb{R}$. The stochastic multiplication above is of Itô type. We look for a solution of (1.1) with time evolution in $L^1(\mathbb{R})$.

Remark 1.2. *1. With* ψ *we can naturally associate an odd increasing function* $\psi_0 : \mathbb{R} \to \mathbb{R}$ *such that* $\psi_0(u) = -\psi_0(-u)$ *for every* $u \in \mathbb{R}$.

2. By the usual technique of filling the gap, ψ can be associated with a graph, i.e. a multivalued function $\mathbb{R} \mapsto 2^{\mathbb{R}}$, still denoted by the same letter, by setting $\psi(u) = [\psi(u-), \psi(u+)]$.

Since ψ restricted to $\mathbb{R}_+$ is monotone, Assumption 1.1 implies $\psi(u) = \Phi^2(u)u$, $u > 0$, $\Phi : \mathbb{R}_+^\star \to \mathbb{R}$ being a non-negative Borel function which is bounded on $\mathbb{R}_+^\star$. When $\psi(u) = |u|^{m-1}u$, $m > 1$, (1.1) and $\mu \equiv 0$, (1.1) is nothing else but the classical *porous media equation*. When ψ is a general increasing function (and $\mu \equiv 0$), there are several contributions to the analytical study of (1.1), starting from [11] for existence, [14] for uniqueness in the case of bounded solutions and [12] for continuous dependence on the coefficients. The authors consider the case where ψ is continuous, even if their arguments allow some extensions for the discontinuous case. Those are the classical references when the space variable varies on the real line. For equations in a bounded domain and Dirichlet boundary conditions, for simplicity, we only refer to monographs, e.g. [32, 30, 1, 2].

As far as the stochastic porous media is concerned, most of the work for existence and uniqueness concerned the case of bounded domain, see for instance [4, 5, 3]. The infinite volume case, i.e. when the underlying domain is $\mathbb{R}^d$, was fully analyzed in [25], when ψ is polynomially bounded (including the fast diffusion case) when the space dimension is $d \geq 3$, more precisely, see [25], Theorem 3.9, Proposition 3.1 and Exemple 3.4. To the best of our knowledge, except for [25] and our companion paper [6], this seems to be the only work concerning a stochastic porous type equation in infinite volume.

Definition 1.3. ● *We will say that equation (1.1) (or ψ) is* **non-degenerate** *if on each compact, there is a constant $c_0 > 0$ such that $\Phi \geq c_0$.*

● *We will say that equation (1.1) or ψ is* **degenerate** *if $\lim_{u \to 0_+} \Phi(u) = 0$ in the sense that for any sequence of non-negative reals (x_n) converging to zero, and $y_n \in \Phi(x_n)$ we have $\lim_{n \to \infty} y_n = 0$, see Remark 1.2 2.*

One of the typical examples of degenerate ψ is the case of ψ being **strictly increasing after some zero**. This notion was introduced in [7] and it

means the following. There is $0 \leq u_c$ such that $\psi_{[0,u_c]} \equiv 0$ and ψ is strictly increasing on $]u_c, +\infty[$.

Remark 1.4. *1. ψ is non-degenerate if and only if $\liminf_{u \to 0+} \Phi(u) (= \lim_{u \to 0+} \Phi(u)) > 0$.*

2. Of course, if ψ is strictly increasing after some zero, with $u_c > 0$ then ψ is degenerate. If ψ is degenerate, then $\psi^\kappa(u) = (\Phi^2(u) + \kappa)u$, for every $\kappa > 0$, is non-degenerate. In the sequel we will set $\Phi(0) := \lim_{u \downarrow 0} \Phi(u)$. In particular, if ψ is degenerate we have $\Phi(0) = 0$.

This paper will be devoted to both the case when ψ is non-degenerate and the case when ψ is degenerate.

One of the targets of the present paper concerns the probabilistic representation of solutions to (1.1) extending the results of [13, 7] which treated the deterministic case $\mu \equiv 0$. In the deterministic case, to the best of our knowledge the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a non-linear deterministic PDE was McKean [23], particularly in relation with the so called propagation of chaos. In his case, however, the coefficients were smooth. From then on the literature steadily grew and nowadays there is a vast amount of contributions to the subject, see the reference list of [13, 7]. A probabilistic representation when $\beta(u) = |u|u^{m-1}, m > 1$, was provided for instance in [10], in the case of the classical porous media equation. When $m < 1$, i.e. in the case of the fast diffusion equation, [8] provides a probabilistic representation of the so called **Barenblatt solution**, i.e. the solution whose initial condition is concentrated at zero.

[13, 7] discussed the probabilistic representation when $\mu = 0$ in the non-degenerate and degenerate case respectively, where ψ also may have jumps. In the sequel of this introduction we will suppose ψ to be single-valued.

In the case $\mu = 0$, the equation (1.1) models a non-linear phenomenon macroscopically. Let us denote by $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ the solution of that equation. The idea of the probabilistic representation is to find a process $(Y_t, t \in [0, T])$ whose law at time t has for density $u(t, \cdot)$.

Y turns out to be the weak solution of the non-linear stochastic differential equation

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(u(s, Y_s))dB_s, \\ \text{Law}(Y_t) &= u(t, \cdot), \quad t \geq 0, \end{cases} \tag{1.2}$$

where B is a classical Brownian motion. The behaviour of Y is the microscopic counterpart of the phenomenon described by (1.1), describing the evolution of a single particle, whose law behaves according to (1.1).

The idea of this paper is to consider the case when $\mu \neq 0$. This includes the case when the μ is not vanishing but it is deterministic; it happens when only e^0 is non-zero, and $e^i \equiv 0, 1 \leq i \leq n$. In this case our technique gives a sort of forward Feynman-Kac formula for non-linear PDEs.

We introduce a double stochastic representation (in a strong-weak probabilistic sense) by means of introducing an enlarged probability space on which one can represent the solution of (1.1) as the (generalized)-law (called μ-law) of a solution to a non-linear SDE. Intuitively, it describes the microscopic aspect of the SPDE (1.1) for almost all quenched ω. The terminology strong refers to the case that the probability space $(\Omega, \mathcal{F}, P)$ on which the SPDE is defined, will remain fixed.

We represent a solution X to (1.1) making use of another independent source of randomness described by another probability space based on some set Ω_1.

The analog of the process Y, obtained when μ is zero in [7, 13], is a doubly stochastic process, still denoted by Y defined on $(\Omega_1 \times \Omega, Q)$, for which, X constitutes the so-called family of μ-marginal laws of Y. More precisely, for fixed $\omega \in \Omega$, the μ-marginal law at time t of process Y is given by the positive finite Borel measure

$$A \mapsto E^{Q^\omega}\left(1_A(Y_t)\mathcal{E}_t\left(\int_0^\cdot \mu(ds, Y_s(\cdot, \omega))\right)\right), \tag{1.3}$$

$\mathcal{E}$ denoting the Doléans exponential, where

$$\int_0^t \mu(ds, Y_s(\cdot, \omega)) = \sum_{i=0}^N \int_0^t e^i(Y_s(\cdot, \omega))dW_s^i(\omega), t \in [0, T], \tag{1.4}$$

and where we assume that for some filtration $(\mathcal{G}_t)$ on $\Omega_1 \times \Omega$, Y is $(\mathcal{G}_t)$-adapted and $W^1, \ldots, W^N$ are $(\mathcal{G}_t)$-martingales on $\Omega_1 \times \Omega$. For fixed $t \in [0, T]$

we also say that the previous measure is the μ-*law* of Y_t. In the case $e^0 = 0$, the situation is the following. For each fixed $\omega \in \Omega$, (1.3) is a (random) non-negative measure which is not a (random) probability. However the expectation of its total mass is indeed 1.

The double probabilistic representation is based on a simple idea. Suppose there is a process Y defined on a suitably enlarged probability space $(\Omega_1 \times \Omega, Q)$ such that

$$\begin{cases} Y_t & = & Y_0 + \int_0^t \Phi(X(s, Y_s))dB_s, \\ \mu - \mathrm{Law}(Y_t) & = & X(t, \xi)\mathrm{d}\xi, \quad t \in]0, T], \\ \mu - \mathrm{Law}(Y_0) & = & x_0(\mathrm{d}\xi), \end{cases} \quad (1.5)$$

where B is a standard Brownian motion. Then X solves the SPDE (1.1). This is the object of Theorem 3.3. Vice versa, if X is a solution of (1.1) then there is a process Y solving (1.5), see Theorem 7.2.

Remark 1.5. *1. If X is a solution of (1.1), then (1.5) implies that $X \geq 0$ $dt \otimes dx \otimes dP$ a.e.*

2. Let $t \in [0, T]$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be Borel and bounded. Then

$$\int_{\mathbb{R}} \varphi(\xi) X(\omega)(t, \xi)\mathrm{d}\xi = E^{Q^\omega} \left(\varphi(Y_t(\omega))\mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s(\omega)) \right) \right).$$

So

$$\int_{\mathbb{R}} X(\omega)(t, \xi)d\xi = E^{Q^\omega} \left(\mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s(\omega)) \right) \right).$$

Even though for a.e. $\omega \in \Omega$, the previous expression is not necessarily a probability measure, of course,

$$\nu_\omega : \varphi \mapsto \frac{\int_{\mathbb{R}} \varphi(\xi) X(\omega)(t, \xi)d\xi}{\int_{\mathbb{R}} X(\omega)(t, \xi)d\xi}$$

is one. It can be expressed as

$$\nu_\omega(A) = \frac{E^{Q^\omega}(1_A(Y_t)\mathcal{E}_t(M(\cdot, \omega)))}{E^{Q^\omega}\mathcal{E}_t(M(\cdot, \omega))},$$

where $M_t(\cdot, \omega) = \int_0^t \mu(ds, Y_s(\cdot, \omega)), t \in [0, T]$, is defined in (1.4).

3. *Consider the particular case $e_0 = 0, e_1 = c$, c being some constant. In this case, the μ-marginal laws are given by*

$$
\begin{aligned}
A \quad &\mapsto \quad E^{Q^\omega}(1_A(Y_t)c\mathcal{E}_t(W)) = c\mathcal{E}_t(W)E^{Q^\omega}(1_A(Y_t)) \\
&= \quad c\mathcal{E}_t(W)\nu_\omega(t, A)
\end{aligned}
$$

and $\nu_\omega(t, \cdot)$ is the law of $Y_t(\cdot, \omega)$ under Q^ω.

Remark 1.6. *Item 2. of Remark 1.5 has a filtering interpretation, see e.g. [24] for a comprehensive introduction.*

Suppose $e^0 = 0$. Let $\hat{Q}$ be a probability on $(\Omega, \mathcal{G}_T, \hat{Q})$, and consider the non-linear diffusion problem (1.2) as a basic dynamical phenomenon. We suppose now that there are N observations $Y^1, \ldots, Y^N$ related to the process Y generating a filtration $(\mathcal{F}_t)$. We suppose in particular that $\mathrm{d}Y_t^i = \mathrm{d}W_t^i + e^i(Y_t), 1 \le i \le N$, and $W^1, \ldots, W^N$ be $(\mathcal{F}_t)$-Brownian motions. Consider the following dynamical system of non-linear diffusion type:

$$
\begin{cases}
Y_t &= \quad Y_0 + \int_0^t \Phi(X(s, Y_s))dB_s \\
\mathrm{d}Y_t^i &= \quad \mathrm{d}W_t^i + e^i(Y_s), 1 \le i \le N, \\
X(t, \cdot) &: \quad \text{conditional law under } \mathcal{F}_t.
\end{cases}
\tag{1.6}
$$

The third equality of (1.6) means, under $\hat{Q}$, that we have

$$
\int_{\mathbb{R}} \varphi(\xi)X(t, \xi)\mathrm{d}\xi = E(\varphi(Y_t)|\mathcal{F}_t).
\tag{1.7}
$$

We remark that, under the new probability Q defined by $\mathrm{d}Q = \mathrm{d}\hat{Q}\mathcal{E}(\int_0^T \mu(\mathrm{d}s, Y_s))$, $Y^1, \ldots, Y^N$ are standard $(\mathcal{F}_t)$-independent Brownian motions. Then (1.7) becomes

$$
\begin{aligned}
\int_{\mathbb{R}} \varphi(\xi)X(t, \xi)\mathrm{d}\xi &= \quad E^{\hat{Q}}(\varphi(Y_t)|\mathcal{F}_t) \\
&= \quad \frac{E^Q(\varphi(Y_t)\mathcal{E}_t(\int_0^\cdot \mu(ds, Y_s)|\mathcal{F}_t))}{E^Q(\mathcal{E}_t(\int_0^\cdot \mu(ds, Y_s)|\mathcal{F}_t))}.
\end{aligned}
$$

Consequently by Theorem 3.3 X will be the solution of the SPDE (1.1), with x_0 being the law of Y_0; so (1.1) constitutes the Zakai type equation associated with our filtering problem.

The present approach has some vague links with the topic of random irregular media. In this case the macroscopic equation is a linear random partial differential equation with diffusion being 1 and with a drift which is the realisation of a Brownian motion W. Here the equation has a random second term, and the diffusion term is non-linear. Our type of stochastic differential equation (1.2) is time inhomogeneous (and depending on the law of the solution), in contrast to the one of random media. The literature of random (even irregular) media is huge, see for instance [22, 18]. We mention that a stochastic calculus approach in this context was developed in [16, 17, 27].

The paper is organized as follows. After the present introduction, Section 2 is devoted to preliminaries, to the notion of a μ-law associated with a (doubly) stochastic process and to the notion of weak-strong solution of a doubly random stochastic differential equation. In Section 3 we define the notion of double probabilistic representation and we expose the main idea behind it. Section 4 shows that the μ-law of the solution of a *non-degenerate* weak-strong stochastic differential equation, always admits a density for a.e. $\omega \in \Omega$. In Section 5 a uniqueness theorem for an SPDE of Fokker-Planck type is formulated and proved, which is useful for the weak-strong probabilistic representation when ψ is non-degenerate, but it has an interest in itself. Section 6 shows existence and uniqueness of the double stochastic representation of (1.1) when ψ is non-degenerate and finally Section 7 provides the double probabilistic representation when ψ is degenerate and strictly increasing after some zero, in case ψ is Lipschitz. We believe that the latter assumption can be generalized, which is the subject of future work.

2   Preliminaries

First we introduce some basic recurrent notations. $C_0^\infty(\mathbb{R})$ is the space of smooth functions with compact support. $H^{-1}(\mathbb{R})$ is the classical Sobolev space. $\mathcal{M}(\mathbb{R})$ (resp. $\mathcal{M}_+(\mathbb{R})$) denotes the space of finite real (resp. non-negative) measures.
We recall that $\mathcal{S}(\mathbb{R})$ is the space of the Schwartz fast decreasing test functions. $\mathcal{S}'(\mathbb{R})$ is its dual, i.e. the space of Schwartz tempered distributions. On

$\mathcal{S}'(\mathbb{R})$, the map $(I - \Delta)^{\frac{s}{2}}, s \in \mathbb{R}$, is well-defined. For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the classical Sobolev space consisting of all functions $f \in \mathcal{S}'(\mathbb{R})$ such that $(I - \Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R})$. We introduce the norm

$$\|f\|_{H^s} := \|I - \Delta)^{\frac{s}{2}} f\|_{L^2},$$

where $\|\cdot\|_{L^p}$ is the classical $L^p(\mathbb{R})$-norm for $1 \leq p \leq \infty$. In the sequel, we will often simply denote $H^{-1}(\mathbb{R})$, by H^{-1} and $L^2(\mathbb{R})$ by L^2. Furthermore, $W^{r,p}$ denote the classical Sobolev space of order $r \in \mathbb{N}$ in $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$.

Definition 2.1. *Given a function e belonging to $L^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, we say that it is an H^{-1}-**multiplier**, if the map $\varphi \mapsto \varphi e$ is continuous from $\mathcal{S}(\mathbb{R})$ to H^{-1} with respect to the H^{-1}-topology on both spaces. We remark that φe is always a well-defined Schwartz tempered distribution, whenever φ is a fast decreasing test function.*

Of course, any constant function is an H^{-1}-**multiplier**. In the following lines we give some other sufficient conditions on a function e to be an H^{-1}-**multiplier**.

Lemma 2.2. *Let $e : \mathbb{R} \to \mathbb{R}$. If $e \in W^{1,\infty}$ (for instance if $e \in W^{2,1}$), then e is a $H^{-1}(\mathbb{R})$-multiplier.*

Proof. For the convenience of the reader we provide a proof. We observe that it is enough to show the existence of a constant $\mathcal{C}(e)$ such that

$$\|eg\|_{H^1} \leqslant \mathcal{C}(e) \|g\|_{H^1}, \ \forall \ g \in \mathcal{S}(\mathbb{R}). \tag{2.1}$$

In fact, if (2.1) holds, for every $f \in \mathcal{S}(\mathbb{R})$ we have

$$\|ef\|_{H^{-1}} = \sup_{\substack{g \in \mathcal{S}(\mathbb{R}) \\ \|g\|_{H^1} \leqslant 1}} \int (efg)(x) \mathrm{d}x \leqslant \|f\|_{H^{-1}} \sup_{\substack{g \in \mathcal{S}(\mathbb{R}) \\ \|g\|_{H^1} \leqslant 1}} \|eg\|_{H^1}$$

$$\leqslant \|f\|_{H^{-1}} \mathcal{C}(e),$$

which implies that e is a $H^{-1}(\mathbb{R})$-multiplier. We verify now (2.1).

For $g \in \mathcal{S}(\mathbb{R})$ we have

$$
\begin{aligned}
\|eg\|_{H^1}^2 &= \int (eg)^2(x)\mathrm{d}x + \int (eg)'^2(x)\mathrm{d}x \\
&\leqslant \|e\|_\infty^2 \|g\|_{L^2}^2 + 2\int (e'g)^2(x)\mathrm{d}x + 2\int (eg')^2(x)\mathrm{d}x \\
&\leqslant \left(\|e\|_\infty^2 + 2\|e'\|_\infty^2\right)\|g\|_{L^2}^2 + 2\|e\|_\infty^2 \|g'\|_{L^2}^2 \\
&\leqslant \mathcal{C}(e)\|g\|_{H^1}^2 ,
\end{aligned}
$$

where $\mathcal{C}(e) = \sqrt{2}\left(\|e\|_\infty^2 + \|e'\|_\infty^2\right)^{\frac{1}{2}}$. $\qquad\square$

As mentioned in the Introduction, we will consider a fixed filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$, where the $(\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration of a standard Brownian motion $(W^1, \ldots, W^N)$ enlarged with the σ-field generated by x_0. We suppose $\mathcal{F} = \mathcal{F}_T$.

Let $(\Omega_1, \mathcal{H})$ be a measurable space. In the sequel, we will also consider another filtered probability space $(\Omega_0, \mathcal{G}, Q, (\mathcal{G}_t)_{t\in[0,T]})$, where $\Omega_0 = \Omega_1 \times \Omega$, $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$.

Clearly any random element Z on $(\Omega, \mathcal{F})$ will be implicitly extended to $(\Omega_0, \mathcal{G})$ setting $Z(\omega^1, \omega) = Z(\omega)$. It will be for instance the case for the above mentioned processes $W^i, i = 1 \ldots N$.

Here we fix some conventions concerning measurability. Any topological space E is naturally equipped with its Borel σ-algebra $\mathcal{B}(E)$. For instance $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}([0,T])$) denotes the Borel σ-algebra of $\mathbb{R}$ (resp. $[0,T]$).

Given any probability space $(\Omega, \mathcal{F}, P)$, the σ-field $\mathcal{F}$ will always be omitted. When we will say that a map $T : \Omega \times E \to \mathbb{R}$ is measurable, we will implicitly suppose that the corresponding σ-algebras are $\mathcal{F} \otimes \mathcal{B}(E)$ and $\mathcal{B}(\mathbb{R})$.

All the processes on any generic measurable space $(\Omega_2, \mathcal{F}_2)$ will be considered to be measurable with respect to both variables (t, ω). In particular any processes on $\Omega^1 \times \Omega$ is supposed to be measurable with respect to $([0,T] \times \Omega^1 \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{H} \otimes \mathcal{F})$.

A function $(A, \omega) \mapsto Q(A, \omega)$ from $\mathcal{H} \times \Omega \to \mathbb{R}_+$ is called **random kernel** (resp. **random probability kernel**) if for each $\omega \in \Omega$, $Q(\cdot, \omega)$ is a finite

positive (resp. probability) measure and for each $A \in \mathcal{H}$, $\omega \mapsto Q(A, \omega)$ is $\mathcal{F}$-measurable. The finite measure $Q(\cdot, \omega)$ will also be denoted by Q^ω. To that random kernel we can associate a specific finite measure (resp. probability) denoted by Q on $(\Omega_0, \mathcal{G})$ setting $Q(A \times F) = \int_F Q(A, \omega) P(\mathrm{d}\omega) = \int_F Q^\omega(A) P(\mathrm{d}\omega)$, for $A \in \mathcal{H}, F \in \mathcal{F}$. The probability Q from above will be supposed here and below to be associated with a random probability kernel.

Definition 2.3. *If there is a measurable space $(\Omega_1, \mathcal{H})$ and a random kernel Q as before, then the probability space $(\Omega_0, \mathcal{G}, Q)$ will be called* **suitable enlarged probability space** *(of $(\Omega, \mathcal{F}, P)$).*

As said above, any random variable on $\Omega, \mathcal{F})$ will be considered as a random variable on $\Omega_0 = \Omega_1 \times \Omega$. Then, obviously, $W^1, \ldots, W^N$ are independent Brownian motions also $(\Omega_0, \mathcal{G}, Q)$.

Given a local martingale M on any filtered probability space, the process $Z := \mathcal{E}(M)$ denotes its Doléans exponential, which is a local martingale. In particular it is the unique solution of $\ dZ_t = Z_{t-} dM_t, \quad Z_0 = 1$. When M is continuous we have $Z_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$.

We go on discussing some basic probabilistic tools. We come back to the notations presented at the beginning of the Introduction, in particular concerning the random field μ.

Let $Z = (Z(s, \xi), s \in [0, T], \xi \in \mathbb{R})$ be a random field on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that $\int_0^T \left(\int_\mathbb{R} |Z(s, \xi)| \mathrm{d}\xi \right)^2 \mathrm{d}s < \infty$ a.s. and it is an $L^1(\mathbb{R})$-valued $(\mathcal{F}_s)$-progressively measurable process. Then the stochastic integral

$$\int_{[0,t] \times \mathbb{R}} Z(s, \xi) \mu(\mathrm{d}s, \xi) d\xi := \sum_{i=0}^N \int_0^t \left(\int_\mathbb{R} Z(s, \xi) e^i(\xi) \mathrm{d}\xi \right) \mathrm{d}W_s^i, \qquad (2.2)$$

is well-defined. The consistency of (2.2) and (1.4) can be seen as follows. Let for a moment $\langle \cdot, \cdot \rangle$ denote the dualization between measures and functions on $\mathbb{R}$, i.e.

$$\langle \nu, f \rangle := \int_\mathbb{R} f \mathrm{d}\nu,$$

whenever the right-hand side makes sense. Then, for $t \in [0, T]$

$$\int_{[0,t] \times \mathbb{R}} Z(s, \xi) \mu(\mathrm{d}s, \xi) \mathrm{d}\xi = \sum_{i=0}^N \int_0^t \langle Z(s, \xi) \mathrm{d}\xi, e^i \rangle \mathrm{d}W_s^i$$

and

$$\int \mu(\mathrm{d}s, Y_s(\cdot, \omega)) = \int_0^N \int_0^t \langle \delta_{Y_s(\cdot, \omega)}, e^i \rangle \mathrm{d}W_s^i,$$

where δ_x means Dirac measure with mass in $x \in \mathbb{R}$.

We discuss now in which sense the SPDE (1.1) has to be understood.

Definition 2.4. *A random field $X = (X(t, \xi, \omega), t \in [0, T]), \xi \in \mathbb{R}, \omega \in \Omega)$ is said to be a solution to (1.1) if P a.s. we have the following.*

- *$X \in C([0, T]; \mathcal{S}'(\mathbb{R})) \cap L^2([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}))$.*

- *X is an $\mathcal{S}'(\mathbb{R}))$ -valued $(\mathcal{F}_t)$-progressively measurable process.*

- *for any test function $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support, $t \in]0, T]$ we have*

$$\int_{\mathbb{R}} X(t, \xi)\varphi(\xi)\mathrm{d}\xi = \int_{\mathbb{R}} x_0(\mathrm{d}\xi)\varphi(\xi) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} \eta(s, \xi, \cdot)\varphi''(\xi)\mathrm{d}\xi$$

$$+ \int_{[0,t]\times\mathbb{R}} X(s, \xi)\varphi(\xi)\mu(\mathrm{d}s, \xi)\mathrm{d}\xi,$$

where η is an $L^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$-valued $(\mathcal{F}_t)$-progressuvely measurable process such that for any $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\int_0^T \mathrm{d}s \int_{\mathbb{R}} |\eta(s, \xi, \omega)\varphi(\xi)|\mathrm{d}\xi < \infty,$$

and $\eta(s, \xi, \omega) \in \psi(X(s, \xi, \omega))$, $\quad \mathrm{d}s\mathrm{d}\xi\mathrm{d}P$-a.e. $(s, \xi, \omega) \in [0, T] \times \mathbb{R} \times \Omega$.

Remark 2.5. *Clearly, if ψ is continuous then $\eta(s, \xi, \cdot) = \psi(X(s, \xi, \cdot))$.*

Definition 2.6. *Let $Y : \Omega_1 \times \Omega \times [0, T] \to \mathbb{R}$ be a measurable process, progressively measurable on $(\Omega_0, \mathcal{G}, Q, (\mathcal{G}_t))$, where $(\mathcal{G}_t)$ is some filtration on $(\Omega_0, \mathcal{G}, Q)$ such that $W^1, \ldots, W^N$ are $(\mathcal{G}_t)$-Brownian motions on $(\Omega_0, \mathcal{G}, Q)$. As we shall see below in Proposition 2.8, for every $t \in [0, T]$*

$$E^Q\left(\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)\right)\right) < \infty. \tag{2.4}$$

To Y, we will associate its **family of μ-marginal laws**, *i.e. the family of random kernels ($t \in [0, T]$)*

$$\Gamma_t = \left(\Gamma_t^Y(A, \omega), \ A \in \mathcal{B}(\mathbb{R}), \ \omega \in \Omega\right)$$

defined by

$$\varphi \mapsto E^{Q^\omega}\left(\varphi(Y_t(\cdot,\omega))\mathcal{E}_t\left(\int_0^\cdot \mu(\mathrm{d}s, Y_s(\cdot,\omega))\right)\right) = \int_{\mathbb{R}} \varphi(r)\Gamma_t^Y(\mathrm{d}r,\omega),$$

*where φ is a generic bounded real Borel function. We will also say that for fixed $t \in [0,T]$, Γ_t is **the μ- marginal law** of Y_t.*

We observe that, taking into account Lévy's characterization theorem, the assumption on $W^1, \ldots, W^N$ to be $(\mathcal{G}_t)$-Brownian motions can be replaced with $(\mathcal{G}_t)$-local martingales.

Remark 2.7. *i) If Ω is a singleton $\{\omega_0\}$, $e^i = 0$, $1 \leqslant i \leqslant N$, the μ-marginal laws coincide with the weighted laws*

$$\varphi \mapsto E^Q\left(\varphi(Y_t)\exp\left(\int_0^t e^0(Y_s)\mathrm{d}s\right)\right),$$

with $Q = Q^{\omega_0}$. In particular if $\mu \equiv 0$ then the μ-marginal laws are the classical laws.

ii) By (2.4), for any $t \in [0,T]$, for P almost all $\omega \in \Omega$,

$$E^{Q^\omega}\left(\mathcal{E}_t\left(\int_0^\cdot \mu(\mathrm{d}s, Y_s(\cdot,\omega))\right)\right) < \infty.$$

iii) The function $(t,\omega) \mapsto \Gamma_t(A,\omega)$ is measurable, for any $A \in \mathcal{B}(\mathbb{R})$, because Y is a measurable process.

Proposition 2.8. *Consider the situation of Definition 2.6. Then we have the following.*

i) The process $M_t := \mathcal{E}_t\left(\sum_{i=1}^N \int_0^\cdot e^i(Y_s)\mathrm{d}W_s^i\right)$ is a martingale.

ii) The quantity (2.4) is bounded by $\exp\left(T\left\|e^0\right\|_\infty\right)$.

iii) $E^Q(M_t^2) \leq \exp(3T\sum_{i=1}^N \|e^i\|_\infty^2), t \in [0,T]$. Consequently M is a uniformly integrable martingale.

iv) For P-a.e. $\omega \in \Omega$,

$$\sup_{0\leq t\leq T}\|\Gamma_t(\cdot,\omega)\|_{\mathrm{var}} < \infty.$$

Remark 2.9. *Proposition 2.8 ii) yields in particular that Y always admits μ-marginal laws.*

Proof.   i) The result follows since the Novikov condition

$$E\left(\exp\left(\frac{1}{2}\sum_{i=1}^{N}\int_{s}^{t}e^{i}(Y_{s})^{2}\mathrm{d}s\right)\right) < \infty$$

is verified, because the functions e^{i}, $i = 1\ldots N$, are bounded.

ii) This follows because $E^{Q}(M_{t}) = 1\ \forall t \in [0,T]$.

iii) M_{t}^{2} is equal to $N_{t}\exp\left(3\sum_{i=1}^{N}\int_{0}^{t}(e^{i})^{2}(Y_{s})\mathrm{d}s\right)$, where N is a positive martingale with $N_{0} = 1$.

iv) For $t \in [0,T]$,

$$\sup_{t\leqslant T}\|\Gamma_{t}(\,\cdot\,,\omega)\|_{\mathrm{var}} = \sup_{t\leqslant T}E^{Q^{\omega}}\left(M_{t}\exp\left(\int_{0}^{t}e^{0}(Y_{s})\mathrm{d}s\right)\right)$$

$$\leqslant \exp\left(T\left\|e^{0}\right\|_{\infty}\right)\sup_{t\leq T}E^{Q^{\omega}}\left(M_{t}\right).$$

Taking the expectation with respect to P it implies

$$E^{P}\left(\sup_{t\leqslant T}\left\|\Gamma_{t}^{Y}(\,\cdot\,,\omega)\right\|_{\mathrm{var}}\right) \leq \exp\left(T\left\|e^{0}\right\|_{\infty}\right)E^{P}\left(\sup_{t\leqslant T}E^{Q^{\omega}}\left(M_{t}\right)\right)$$

$$\leq \exp\left(T\left\|e^{0}\right\|_{\infty}\right)E^{P}\left(E^{Q^{\omega}}\left(\sup_{t\leqslant T}M_{t}\right)\right).$$

By the Burkholder-Davis-Gundy (BDG) inequality this is bounded by

$$3\exp\left(T\left\|e^{0}\right\|_{\infty}\right)E^{Q}\left(\langle M\rangle_{T}^{\frac{1}{2}}\right) \leq 3\exp\left(T\left\|e^{0}\right\|_{\infty}\right)E^{Q}\left(\left[\int_{0}^{T}\mathrm{d}s\sum_{i=1}^{N}M_{s}^{2}e^{i}(Y_{s})^{2}\right]^{\frac{1}{2}}\right)$$

$$\leq \mathrm{C}(e,N,T)E^{Q}\left(\int_{0}^{T}\mathrm{d}sM_{s}^{2}\right),$$

by Jensen's inequality; $\mathrm{C}(e,N,T)$ is a constant depending on N,T and e_{i}, $i = 0\ldots N$,. By Fubini's Theorem and item iii), we have

$$E^{Q}\left(\int_{0}^{T}\mathrm{d}sM_{s}^{2}\right) \leq T\exp(3T\sum_{i=1}^{N}\|e^{i}\|_{\infty}).$$

$\square$

We go on introducing the concept of weak-strong existence and uniqueness of a stochastic differential equation. Let $\gamma : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)$-progressively measurable random fields and x_0 be a probability on $\mathcal{B}(\mathbb{R})$.

Definition 2.10.   *a) We say that (DSDE)(γ, x_0) admits* **weak-strong existence** *if there is a suitable extended probability space $(\Omega_0, \mathcal{G}, Q)$, i.e. a measurable space $(\Omega_1, \mathcal{H})$, a probability kernel $(Q(\cdot\,, \omega), \omega \in \Omega)$ on $\mathcal{H} \times \Omega$, two Q-a.s. continuous processes Y, B on $(\Omega_0, \mathcal{G})$ where $\Omega_0 = \Omega_1 \times \Omega$, $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ such that the following holds.*

> *1) For almost all ω, $Y(\cdot, \omega)$ is a (weak) solution to*
>
> $$\begin{cases} Y_t(\cdot\,, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot\,, \omega), \omega) \mathrm{d}B_s(\cdot\,, \omega), \\ Law(Y_0) = x_0, \end{cases} \tag{2.6}$$
>
> *with respect to Q^ω, where $B(\cdot\,, \omega)$ is a Q^ω-Brownian motion for almost all ω.*
>
> *2) We denote $(\mathcal{Y}_t)$ the canonical filtration associated with $(Y_s, 0 \leq s \leq t)$ and $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset, \Omega_1\} \otimes \mathcal{F}_t)$. We suppose that $W^1, \ldots, W^N$ is a $(\mathcal{G}_t)$-martingale under Q.*
>
> *3) For every $0 \leq s \leq T$, for every bounded continuous $F : C([0, s]) \to \mathbb{R}$, the r.v. $\omega \mapsto E^{Q^\omega}(F(Y_r(\cdot, \omega), r \in [0, s]))$ is $\mathcal{F}_s$-measurable.*

b) We say that (DSDE)(γ, x_0) admits **weak-strong uniqueness** *if the following holds. Consider a measurable space $(\Omega_1, \mathcal{H})$ (resp. $(\widetilde{\Omega}_1, \widetilde{\mathcal{H}})$), a probability kernel $(Q(\cdot\,, \omega), \omega \in \Omega)$ (resp. $(\widetilde{Q}(\cdot\,, \omega), \omega \in \Omega)$), with processes (Y, B) (resp. $(\widetilde{Y}, \widetilde{B})$) such that (2.6) holds (resp. (2.6) holds with $(\Omega_0, \mathcal{G}, Q)$ replaced with $(\widetilde{\Omega}_0, \widetilde{\mathcal{G}}_0, \widetilde{Q})$, $\widetilde{Q}$ being associated with $(\widetilde{Q}(\cdot\,, \omega))$). Moreover we suppose that item 2. is verified for Y and $\tilde{Y}$. Then $(Y, W^1, \ldots, W^N)$ and $(\widetilde{Y}, W^1, \ldots, W^N)$ have the same law.*

c) A process Y fulfilling items 1) and 2) under (a) will be called **weak-strong solution** *of (DSDE)(γ, x_0).*

Remark 2.11.   *a) Since for almost all $\omega \in \Omega$, $B(\cdot, \omega)$ is a Brownian motion under Q^ω, it is clear that B is a Brownian motion under Q, which is independent of $\mathcal{F}_T$, i.e. independent of $W^1, \ldots, W^N$.*

Indeed let $G : C([0,T]) \to \mathbb{R}$ be a continuous bounded functional, and denote by $\mathcal{W}$ the Wiener measure. Let F be a bounded $\mathcal{F}_T$-measurable r.v. Since for each ω, $B(\cdot, \omega)$ is a Wiener process with respect to Q^ω, we get

$$E^Q(FG(B)) = \int_\Omega F E^{Q^\omega}(G(B(\cdot, \omega))) \mathrm{d}P(\omega) = \int_\Omega F(\omega) \mathrm{d}P(\omega) \int_{\Omega_1} G(\omega_1) d\mathcal{W}(\omega_1)$$

$$= \int_{\Omega_0} F(\omega) \mathrm{d}Q(\omega_0) \int_{\Omega_0} G(\omega_1) \mathrm{d}Q(\omega_0).$$

This shows that $(W^1, \ldots, W^N)$ and B are independent. Taking $F = 1_\Omega$ in previous expression, the equality between the left-hand side and the third term, shows that B is a Brownian motion under Q.

b) Since for any $1 \le i, j \le N$,

$$[W^i, W^j]_t = \delta_{ij} t, \quad [W^i, B] = 0, \quad [B, B]_t = t, \tag{2.7}$$

Lévy's characterization theorem, implies that $(W^1, \ldots, W^N, B)$ is a Q-Brownian motion.

c) By item a) 2) of Definition 2.10, by Lévy's characterization theorem and again by (2.7), it follows that $W^1, \ldots, W^N$ are $(\mathcal{G}_t)$-Brownian motions with respect to Q.

d) An equivalent formulation to 1) in item a) of Definition 2.10 is the following. For P a.e., $\omega \in \Omega$, $Y(\cdot, \omega)$ solves the Q^ω-martingale problem with respect to the (random) PDE operator

$$L_t^\omega f(\xi) = \frac{1}{2} \gamma^2(t, \xi, \omega) f''(\xi),$$

and initial distribution x_0.

The lemma below shows that, whenever weak-strong uniqueness holds, then the marginal laws of any weak solution Y are uniquely determined.

Lemma 2.12. *Let Y (resp. $\tilde{Y}$) be a process on a suitable enlarged probability space $(\Omega_0, \mathcal{G}, Q)$ (resp. $(\tilde{\Omega}_0, \tilde{\mathcal{G}}, \tilde{Q})$). Set $W = (W^1, \ldots, W^N)$. Suppose that the law (Y, W) under Q and the law of $(\tilde{Y}, W)$ under $\tilde{Q}$ are the same. Then, the μ-marginal laws of Y under Q coincide a.s. with the μ-marginal laws of $\tilde{Y}$ under $\tilde{Q}$.*

Proof. Let $0 \leq t \leq T$. Using the assumption, we deduce that for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, and every $F \in \mathcal{F}_t$, we have

$$E^Q \left(1_F f(Y_t) \mathcal{E}_t \left(\sum_{i=0}^{N} \int_0^{\cdot} e^i(Y_s) dW_s^i \right) \right) = E^{\tilde{Q}} \left(1_F f(\tilde{Y}_t) \mathcal{E}_t \left(\sum_{i=0}^{N} \int_0^{\cdot} e^i(\tilde{Y}_s) dW_s^i \right) \right).$$
(2.8)

To show this, using classical regularization properties of Itô integral, see e.g. Theorem 2 in [29], and uniform integrability arguments, we first observe that

$$\mathcal{E}_t \left(\sum_{i=0}^{N} \int_0^{\cdot} e^i(Y_s) dW_s^i \right)$$

is the limit in $L^2(\Omega_0, Q)$ of

$$\mathcal{E}_t \left(\sum_{i=0}^{N} \int_0^{\cdot} e^i(Y_s) \frac{W_{s+\varepsilon}^i - W_s^i}{\varepsilon} ds \right).$$

A similar approximation property arises replacing Y with $\tilde{Y}$ and Q with $\tilde{Q}$. Then (2.8) easily follows.

To conclude, it will be enough to show the existence of a countable well-chosen family $(f_j)_{j \in \mathbb{N}}$ of bounded continuous real functions for which, for P almost all $\omega \in \Omega$, for any $j \in \mathbb{N}$, we have $R_j = \tilde{R}_j$ where

$$R_j(\omega) = E^{Q^\omega} \left(f_j(Y_t(\cdot, \omega)) \mathcal{E}_t \left(\sum_{i=0}^{N} \int_0^{\cdot} e^i(Y_s(\cdot, \omega)) dW_s^i \right) \right)$$

$$R_j(\omega) = E^{\tilde{Q}^\omega} \left(f_j(\tilde{Y}_t(\cdot, \omega)) \mathcal{E}_t \left(\sum_{i=0}^{N} \int_0^{\cdot} e^i(\tilde{Y}_s(\cdot, \omega)) dW_s^i \right) \right).$$

This will follow, since applying (2.8), for any $F \in \mathcal{F}_t$, we have $E^P(1_F R_j) = E^P(1_F \tilde{R}_j)$. $\square$

Proposition 2.13. *Let Y be a process as in Definition 2.10 a). We have the following.*

1. *Y is a $(\mathcal{G}_t)$-martingale on the product space $(\Omega_0, \mathcal{G}, Q)$.*

2. *$[Y, W^i] = 0, \ \forall 1 \leq i \leq N$.*

Proof. Let $0 \leq s < t \leq T$, $F_s \in \mathcal{F}_s$ and $G : C([0, s]) \to \mathbb{R}$ be continuous and bounded. We will prove below that, for $1 \leq i \leq N + 1$, setting $W_t^{N+1} = 1$, for all $t \geq 0$,

$$E^Q(Y_t W_t^i G(Y_r, r \leq s)1_{F_s}) = E^Q(Y_s W_s^i 1_{F_s} G(Y_r, r \leq s)). \tag{2.10}$$

Then (2.10) with $i = N+1$ shows item 1. Considering (2.10) with $1 \leq i \leq N$, shows that YW^i is a $(\mathcal{G}_t)$-martingale, which shows item 2. Therefore, it remains to show (2.10).

The left-hand side of that equality gives

$$\int_\Omega \mathrm{d}P(\omega) \qquad W_t^i(\omega)1_{F_s}(\omega)E^{Q^\omega}\left(Y_t(\cdot, \omega)G(Y_r(\cdot, \omega), r \leq s)\right)$$

$$= \int_\Omega \mathrm{d}P(\omega)1_{F_s}(\omega)W_t^i(\omega)E^{Q^\omega}\left(Y_s(\cdot, \omega)G(Y_r(\cdot, \omega), r \leq s)\right),$$

because $Y(\cdot, \omega)$ is a Q^ω-martingale for P-almost all ω. To obtain the right-hand side of (2.10) it is enough to remember that W^i are $(\mathcal{G}_t)$-martingales and that item a) 3) in Definition 2.10 holds. This concludes the proof of Proposition 2.13. $\square$

3 The concept of double stochastic non-linear diffusion.

We come back to the notations and conventions of the introduction and of Section 2. Let x_0 be a probability on $\mathbb{R}$.

Definition 3.1. *1) We say that the double stochastic non-linear diffusion (DSNLD) driven by Φ (on the space $(\Omega, \mathcal{F}, P)$ with initial condition x_0, related to the random field μ (shortly (DSNLD)(Φ, μ, x_0)) admits* **weak existence** *if there is a measurable random field $X : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ with the following properties.*

a) The problem (DSDE)(γ, x_0) with $\gamma = \chi$ for some measurable $\chi : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\chi(t, \xi, \omega) \in \Phi(X(t, \xi, \omega))\mathrm{d}t\mathrm{d}\xi\mathrm{d}P$ a.e. admits weak-strong existence.

b) $X = X(t, \xi, \cdot)\mathrm{d}\xi, t \in]0, T]$, is the family of μ-marginal laws of Y. In other words X constitutes the densities of those μ-marginal laws.

2) *A couple (Y, X), such that Y is a (weak-strong) solution to the (DSDE)(χ, x_0), with χ as in item 1) a), which also fulfills 1) b), is called* **weak solution** *to the (DSNLD)(Φ, μ, x_0). Y is also called double stochastic representation of the random field X.*

3) *Suppose that, given two measurable random fields $X^i : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}, i = 1, 2$ on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$, and Y^i, on extended probability space $(\Omega_0^i, Q^i), i = 1, 2$, such that (Y^i, X^i) is a weak-strong solution of (DSDE)$(\chi^i, x_0), i = 1, 2$ where $\chi^i \in \Phi(X^i)$ dtdξdP a.e., we always have that $(Y^1, W^1, \ldots, W^N)$ and $(Y^2, W^1, \ldots, W^N)$ have the same law. Then we say that the (DSNLD)(Φ, μ, x_0) admits* **weak uniqueness**.

Remark 3.2. *If (DSNLD)(Φ, μ, x_0) admits* **weak uniqueness** *then the μ-marginal laws of Y is uniquely determined, P-a.s., see Lemma 2.12.*

The first connection between (1.1) with $\psi(u) = \Phi^2(u)u$ and (DNSLD)(Φ, μ, x_0) is the following.

Theorem 3.3. *Let (Y, X) be a solution of (DSNLD)(Φ, μ, x_0). Then X is a solution to the SPDE (1.1).*

Proof. Let B denote the Brownian motion associated to Y as a solution to (DSDE)(χ, x_0), mentioned in item a)1) of Definition 3.1, with $\gamma = \chi$. For $t \in [0, T]$, we set

$$Z_t = \mathcal{E}_t \left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s) \right),$$

$$M_t = Z_t \exp \left(- \int_0^t e^0(Y_s)\mathrm{d}s \right), \; t \in [0, T].$$

1. We first prove that the first item of Definition 2.4 is verified. By Proposition 2.8, $(M_t, t \in [0, T])$ is a uniformly integrable martingale. Consequently $t \mapsto Z_t$ is continuous in $L^1(\Omega, Q)$. On the other hand the process Y is continuous. This implies that P a.e. $\omega \in \Omega$, $X \in C([0, T]; \mathcal{M}(\mathbb{R}))$, where $\mathcal{M}(\mathbb{R})$ is equipped with the weak topology. This implies that $X \in C([0, T]; \mathcal{S}'(\mathbb{R}))$. Furthermore, for P a.e. $\omega \in \Omega$, and $t \in]0, T]$, $X(t, \cdot, \omega) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} X(t, \xi, \omega)\mathrm{d}\xi = \|\Gamma(t, \cdot, \omega)\|_{\mathrm{var}}$. By

item iv) of Proposition 2.8, it follows that P-a.s.

$$X \in L^{\infty}([0,T]; L^1(\mathbb{R})) \subset L^2([0,T]; L^1_{\text{loc}}(\mathbb{R})).$$

2. We prove now the validity of the third item of Definition 2.4. Let $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support. For simplicity of the formulation we suppose here ψ to be single-valued. Taking into account Proposition 2.13, we apply Itô's formula to get

$$\varphi(Y_t)Z_t = \varphi(Y_0) + \int_0^t \varphi'(Y_s)Z_s \mathrm{d}Y_s$$

$$+ \int_0^t \varphi(Y_s)Z_s \left(\mu(\mathrm{d}s, Y_s) - \frac{1}{2}\sum_{i=1}^N (e^i(Y_s))^2 \mathrm{d}s \right)$$

$$+ \frac{1}{2}\int_0^t \varphi''(Y_s)\Phi^2(X(s,Y_s))Z_s \mathrm{d}s$$

$$+ \frac{1}{2}\int_0^t \varphi(Y_s)Z_s \left(\sum_{i=1}^N (e^i(Y_s))^2 \right) \mathrm{d}s.$$

Indeed we remark that

$$\int_0^t \varphi'(Y_s)\mathrm{d}[Z,Y]_s = 0,$$

because

$$[Z,Y]_t = \sum_{i=1}^N \int_0^t e^i(Y_s)Z_s \mathrm{d}[W^i,Y]_s = 0;$$

in fact $[W^i,Y] = 0$ by Proposition 2.13. So

$$\varphi(Y_t)Z_t = \varphi(Y_0) + \int_0^t \varphi'(Y_s)Z_s\Phi(X(s,Y_s))\mathrm{d}B_s$$

$$+ \int_0^t \varphi(Y_s)Z_s\mu(\mathrm{d}s,Y_s)$$

$$+ \frac{1}{2}\int_0^t \varphi''(Y_s)\Phi^2(X(s,Y_s))Z_s \mathrm{d}s.$$

Taking the expectation with respect to Q^ω we get $\ \mathrm{d}P$-a.s.,

$$\int_{\mathbb{R}} \mathrm{d}\xi\varphi(\xi)X(t,\xi) = \int_{\mathbb{R}} \varphi(\xi)x_0(\mathrm{d}\xi) + \sum_{i=0}^N \int_0^t \mathrm{d}W^i_s \left(\int_{\mathbb{R}} \mathrm{d}\xi\varphi(\xi)e^i(\xi)X(s,\xi) \right)$$

$$+ \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}\xi\varphi''(\xi)\Phi^2(X(s,\xi))X(s,\xi),$$

which implies the result. Indeed, in the previous equality, we have used the lemma below.

$\square$

Lemma 3.4. *Let* $1 \leq i \leq N$. *For* P *a.e.* $\omega \in \Omega$, *we have*

$$E^{Q^\omega} \left(\int_0^t \varphi(Y_s) Z_s e^i(Y_s) \mathrm{d}W_s^i \right)(\cdot, \omega) = \int_0^t \mathrm{d}W_s^i(\omega) \int_{\mathbb{R}} \varphi(\xi) e^i(\xi) X(s, \xi, \omega) \mathrm{d}\xi.$$

Proof. Since $E^Q(\int_0^T (\varphi(Y_s) Z_s)^2 \mathrm{d}s) < \infty$, again the usual regularization properties of the Itô integral (see e.g. Theorem 2, [29]), give

$$\lim_{\varepsilon \to 0} E^Q \left| \int_0^T \frac{W_{s+\varepsilon} - W_s}{\varepsilon} \varphi(Y_s) Z_s e^i(Y_s) \mathrm{d}s - \int_0^T \varphi(Y_s) Z_s e^i(Y_s) \mathrm{d}W_s^i \right| = 0.$$

This implies the existence of a sequence (ε_ℓ) such that P a.e. $\omega \in \Omega$,

$$\lim_{\ell \to \infty} E^{Q^\omega} \quad \left| \int_0^T \frac{W_{s+\varepsilon_\ell}^i(\omega) - W_s^i(\omega)}{\varepsilon_\ell} \varphi(Y_s(\cdot, \Omega)) Z_s(\cdot, \omega) e^i(Y_s(\cdot, \omega)) \mathrm{d}s \right.$$

$$\left. - \int_0^T \varphi(Y_s(\cdot, \omega)) Z_s(\cdot, \omega) e^i(Y_s(\cdot, \omega)) \mathrm{d}W_s^i(\omega) \right| = 0.$$

So P-a.e $\omega \in \Omega$,

$$\lim_{\ell \to \infty} E^{Q^\omega} \quad \left(\int_0^T \frac{W_{s+\varepsilon_\ell}^i(\omega) - W_s^i(\omega)}{\varepsilon_\ell} \varphi(Y_s(\cdot, \omega)) Z_s(\cdot, \omega) e^i(Y_s(\cdot, \omega)) \mathrm{d}s \right.$$

$$\left. - \int_0^T \varphi(Y_s(\cdot, \omega)) Z_s(\cdot, \omega) e^i(Y_s(\cdot, \omega)) \mathrm{d}W_s^i(\omega) \right) = 0.$$

The left-hand side (3.1), by Fubini's, gives

$$\int_0^T \frac{W_{s+\varepsilon_\ell}^i(\omega) - W_s^i(\omega)}{\varepsilon_l} E^{Q^\omega} (\varphi(Y_s(\cdot, \omega)) Z_s(\cdot, \omega) e^i(Y_s(\cdot, \omega)))$$

$$= \int_0^T \frac{W_{s+\varepsilon_\ell}^i(\omega) - W_s^i(\omega)}{\varepsilon_\ell} \int \varphi(\xi) e^i(\xi) X(s, \xi, \omega) \mathrm{d}\xi$$

$$= \int_0^T \mathrm{d}W_s^i \int \varphi(\xi) e^i(\xi) X(s, \xi, \omega) \mathrm{d}\xi,$$

again by Theorem 2 of [29].

$\square$

4 The densities of the μ-marginal laws

This section constitutes an important step towards the double probabilistic representation of a solution to (1.1), when ψ is non-degenerate. Let x_0 be a fixed probability on $\mathbb{R}$. We remind that a process Y (on a suitable enlarged probability space $(\Omega_0, \mathcal{G}, Q)$), which is a weak solution to the (DSNLD)(Φ, μ, x_0), is in particular a weak-strong solution of a (DSDE)(γ, x_0) where $\gamma : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is some suitable progressively measurable random field on $(\Omega, \mathcal{F}, P)$. The aim of this section is twofold.

A) To show that whenever γ is a.s. bounded and non-degenerate, (DSDE)(γ, x_0) admit weak-strong existence and uniqueness.

B) The marginal μ-laws of the solution to (DSDE)(γ, x_0) admits a density for P ω a.s.

A) We start discussing well-posedness.

Proposition 4.1. *We suppose the existence of random variables A_1, A_2 such that*

$$0 < A_1(\omega) \leqslant \gamma(t, \xi, \omega) \leqslant A_2(\omega) \quad \mathrm{d}P\text{-a.s.}.$$

Then (DSDE)(γ, x_0) admits weak-strong existence and uniqueness.

Proof. Uniqueness. This is the easy part. Let Y and $\widetilde{Y}$ be two solutions. Then for ω outside a P-null set $N_0, Y(\cdot\,, \omega)$ and $\widetilde{Y}(\cdot\,, \omega)$ are solutions to the same one-dimensional classical SDE with measurable bounded and non-degenerate coefficients. Then, by Exercise 7.3.3 of [31] the law of $Y(\cdot\,, \omega)$ equals the law of $\widetilde{Y}(\cdot\,, \omega)$. Then obviously the law of Y equals the law of $\widetilde{Y}$.

Existence. This point is more delicate. In fact one needs to solve the random SDE for P almost all ω but in such a way that the solution produce bimeasurable processes Y and B.

First we regularize the coefficient γ. Let ϕ be a mollifier with compact support; we set

$$\phi_n(x) = n\phi(nx), \; x \in \mathbb{R} \,, \; n \in \mathbb{N}.$$

We consider the random fields $\gamma_n : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ by $\gamma_n(t,x,\omega) := \int_{\mathbb{R}} \gamma(t, x-y, \omega) \phi_n(y) \mathrm{d}y$.

Let $(\widetilde{\Omega}_1, \widetilde{\mathcal{H}}_1, \widetilde{P})$ be a probability space where we can construct a random variable Y_0 distributed according to x_0 and an independent Brownian motion B.

In this way on $(\widetilde{\Omega}_1 \times \Omega, \widetilde{\mathcal{H}}_1 \otimes \mathcal{F}, \widetilde{P} \otimes P)$ we dispose of a random variable Y_0 and a Brownian motion independent of $\{\phi, \Omega\} \otimes \mathcal{F}$. By usual fixed point techniques, it is possible to exhibit a (strong) solution of $(\mathrm{DSDE})(\gamma_n, x_0)$ on the overmentioned product probability space. We can show that there is a unique solution $Y = Y^n$ of

$$Y_t = Y_0 + \int_0^t \gamma_n(s, Y_s, \cdot) \mathrm{d}B_s.$$

In fact, the maps

$$\Gamma_n : Z \mapsto \int_0^{\cdot} \gamma_n(s, Z_s, \omega) \mathrm{d}B_s + Y_0,$$

where $\Gamma_n : L^2(\widetilde{\Omega}_1 \times \Omega; \widetilde{P} \otimes P) \to L^2(\widetilde{\Omega}_1 \times \Omega, \widetilde{P} \otimes P)$ are Lipschitz; by usual Picard fixed point arguments one can show the existence of a unique solution $Z = Z^n$ in $L^2(\widetilde{\Omega}_1 \times \Omega; \widetilde{P} \otimes P)$. We observe that, by usual regularization arguments for Itô integral as in Lemma 3.4, for ω-a.s., $Y(\cdot, \omega)$ solves for P a.e. $\omega \in \Omega$, equation

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma_n(s, Y_s(\cdot, \omega), \omega) \mathrm{d}B_s, \tag{4.1}$$

on $(\widetilde{\Omega}_1, \widetilde{\mathcal{H}}_1, \widetilde{P})$. We consider now the measurable space $\Omega_0 = \Omega_1 \times \Omega$, where $\Omega_1 = C([0,T], \mathbb{R}) \times \mathbb{R}$, equipped with product σ-field $\mathcal{G} = \mathcal{B}(\Omega_1) \otimes \mathcal{F}$. On that measurable space, we introduce the probability measures Q_n where $Q_n = \int_{\Omega} \mathrm{d}P(\omega) Q_n(\cdot, \omega)$ and $Q_n(\cdot, \omega)$ being the law of $Y^n(\cdot, \omega)$ for almost all fixed ω.

We set $Y_t(\omega_1, \omega) = \omega_1(t)$, where $\omega_1(t) = \omega_1^0(t) + a$, if $\omega^1 = (\omega_1^0, a)$. We denote by $(\mathcal{Y}_t, \ t \in [0,T])$ (resp. $(\mathcal{Y}_t^1)$) the canonical filtration associated with Y on Ω_0 (resp. Ω_1). The next step will be the following.

Lemma 4.2. *For almost all ω dP a.s. $Q_n(\omega, \cdot)$ converges weakly to $Q(\omega, \cdot)$,*

where under $Q(\cdot, \omega)$, $Y(\cdot, \omega)$ solves the SDE

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) \mathrm{d}B_s(\cdot, \omega),$$

where $B(\cdot, \omega)$ is an $(\mathcal{Y}_t^1)$-Brownian motion on Ω_1.

Proof. It follows directly from Proposition A.4 of the Appendix.   □

Remark 4.3.   *1) Since $Q_n(\cdot, \omega)$ converges weakly to $Q(\cdot, \omega)$, ω dP a.s., then the limit (up to an obvious modification) is a measurable random kernel.*

   2) This also implies that $Y_n(\cdot, \omega)$ converges stably to $Q(\cdot, \omega)$. For details about the stable convergence the reader can consult [19, section VIII 5. c].

The considerations above allow to conclude the proof of Proposition 4.1 By Lemma 4.2, $Q^\omega = Q(\cdot, \omega)$ is a random kernel, being a limit of random kernels. Let us consider the associated probability measure on the suitable enlarged probability space $(\Omega_0, \mathcal{G}, Q)$. We observe that Y on $(\Omega_0, \mathcal{G})$ is obviously measurable, because it is the canonical process $Y(\omega_1, \omega) = \omega_1$. Setting

$$B_t(\cdot, \omega) = \int_0^t \frac{\mathrm{d}Y_s}{\gamma(s, Y_s, \omega)},$$

we get $[B]_t(\cdot, \omega) = t$ under $Q(\cdot, \omega)$, so, by Lévy characterization theorem, it is a Brownian motion. Moreover B is bimeasurable. The last point to check is that $W^1, \ldots, W^N$ are $(\mathcal{G}_t)$-martingales, where $\mathcal{G}_t = (\mathcal{F}_t \otimes \{\emptyset, \Omega_1\}) \vee \mathcal{Y}_t$, $0 \leq t \leq T$.
Indeed, we justify this immediately. Condider $0 < s \leq t \leq T$. Taking into account monotone class arguments, given $F \in \mathcal{F}_s$, $G \in \mathcal{Y}_s^1$, $1 \leq i \leq N$, it is enough to prove that

$$E^Q(FGW_t^i) = E^Q(FGW_s^i). \tag{4.2}$$

We first observe that the r.v. $\omega \mapsto E^{Q^\omega}(G)$ is $\mathcal{F}_s$-measurable. This happens because Y is, under Q^ω, a martingale with quadratic variation $\left(\int_0^t \gamma_(^2 s, Y_s(\cdot, \omega), \omega) \mathrm{d}s, 0 \leq t \leq T \right)$, i.e. with (random) coefficient which is

$(\mathcal{F}_t)$-progressively measurable.

Consequently, also using the fact that W^i is an $(\mathcal{F}_t)$-martingale and that $E^{Q^\omega}(G)$ is $\mathcal{F}_s$-measurable by item a) 3) of Definition 2.10, the left-hand side of previous equality gives

$$E^P(FW_t^i E^{Q^\omega}(G)) = E^P(FW_s^i E^{Q^\omega}(G)),$$

which constitutes the right-hand side of (4.2). This concludes the proof of the proposition. □

We go on now with step B) of the beginning of Section 4.

Proposition 4.4. *We suppose the existence of r.v. A_1, A_2 such that*

$$0 < A_1(\omega) \leqslant \gamma(t, \xi, \omega) \leqslant A_2(\omega), \forall (t, \xi) \in [0, T] \times \mathbb{R}, \qquad a.s.$$

Let Y be a weak-strong solution to $(DSDE)(\gamma, x_0)$ and we denote by $(\nu_t(dy, \cdot), t \in [0, T])$, the μ-marginal laws of process Y.

1. *There is a measurable function $q : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ such that $dtdP$ a.e., $\nu_t(dy, \cdot) = q_t(y, \cdot)dy$. In other words the μ-marginal laws admit densities.*

2.
$$\int_{[0,T] \times \mathbb{R}} q_t^2(y, \cdot) dt dy < \infty \qquad dP\text{-}a.s.$$

3. *q is an $L^2(\mathbb{R})$-valued progressively measurable process.*

Proof. By 3) of Definition 2.10, the μ-marginal laws constitute an $\mathcal{S}'(\mathbb{R})$-valued progressively measurable process. Consequently 3. holds if 1. and 2. hold.

Let

$$B_t := \int_0^t \frac{dY_s}{\gamma(s, Y_s, \omega)}.$$

We denote again $Q^\omega := Q(\cdot, \omega)$ according to Definition 2.10, $\omega \in \Omega$.

Let $\omega \in \Omega$ be fixed. Let $\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with

compact support. We need to evaluate

$$E^{Q^\omega}\left(\int_0^T \varphi(s, Y_s) Z_s \mathrm{d}s\right),\tag{4.3}$$

where $Z_s = \exp\left(\int_0^s e^0(Y_r)\mathrm{d}r\right)$. M_s and $M_s = \mathcal{E}_s\left(\sum_{i=1}^N \int_0^\cdot e^i(Y_r)\mathrm{d}W_r^i\right)$. M_s is smaller or equal than

$$\exp\left(\sum_{j=1}^N \int_0^s e^j(Y_r)\,\mathrm{d}W_r^j\right)$$

$$= \exp\left(\sum_{j=1}^N \left\{W_s^j e^j(Y_s) - \int_0^s W_r^j (e^j)'(Y_r)\mathrm{d}Y_r\right\}\right)$$

$$\times \exp\left(-\frac{1}{2}\int_0^s \sum_{j=1}^N \left\{W_r^j(e^j)''(Y_r)\gamma^2(r, Y_r, \cdot)\mathrm{d}r - \frac{1}{2}W_r^j(e^j)'(Y_r)\gamma^2(r, Y_r, \cdot)\mathrm{d}r\right\}\right),$$

taking into account the fact that $[Y, W^j] = 0$ for any $1 \le j \le n$, by Proposition 2.13.

Denoting $\|g\|_\infty := \sup_{t \in [0,T]} |g(t)|$, for a function $g : [0, T] \to \mathbb{R}$, (4.3) is smaller or equal than

$$\exp\left(\sum_{j=1}^N \|W^j\|_\infty(\|e^j\|_\infty + \frac{T}{2}\|(e^j)''\|_\infty A_2^2(\omega))\right)\exp\left(-\int_0^s \left[\sum_{j=1}^N W_r^j(e^j)'(Y_r)\gamma(r, Y_r, \omega)\right]\mathrm{d}B_r\right).$$

So (4.3) is bounded by

$$\varrho(\omega)E^{Q^\omega}\left(\int_0^T |\varphi|(s, Y_s) R_s \mathrm{d}s\right),\tag{4.4}$$

where

$$\begin{aligned}\varrho(\omega) &= \exp\Bigg(T\|e_0\|_\infty + \sum_{i=1}^N \|W^i\|_\infty \|e^i\|_\infty \\ &+ T\frac{A_2^2(\omega)}{2}\sum_{i=1}^N (\|W^i\|_\infty^2 \|(e^i)'\|_\infty^2 + \|W^i\|_\infty\|(e^i)''\|_\infty)\Bigg)\end{aligned}$$

and R is the Q^ω-exponential martingale

$$\begin{aligned}R_t(\,\cdot\,, \omega) = \exp\Big(&- \int_0^t \delta(r, \,\cdot\,, \omega)\mathrm{d}B_r \\ &- \frac{1}{2}\int_0^t \delta^2(r, \,\cdot\,, \omega)\mathrm{d}r\Big).\end{aligned}$$

where

$$\delta(r, \, \cdot \, , \omega) = \sum_{j=1}^{N} W_r^j (e^j)' \left(Y_r(\, \cdot \, , \omega) \right) \gamma \left(r, Y_r(\, \cdot \, , \omega), \omega \right).$$

So there is a random (depending on $(\Omega, \mathcal{F})$) constant

$$\varrho_1(\omega) := \mathrm{const} \left(T, W^j, \left\| e^j \right\|_\infty, \left\| e^{j\prime} \right\|_\infty, \left\| e^{j\prime\prime} \right\|_\infty, \ 1 \leqslant j \leqslant N, \ A_2(\omega) \right), \tag{4.5}$$

so that (4.4) is smaller than

$$\varrho_1(\omega) E^{Q^\omega} \left(\int_0^T |\varphi(s, Y_s(\, \cdot \, , \omega))| \mathrm{d}s R_T(\, \cdot \, , \omega) \right). \tag{4.6}$$

By Girsanov theorem,

$$\widetilde{B}_t(\cdot, \omega) = B_t(\cdot, \omega) + \int_0^t \delta(r, \, \cdot \, , \omega) \mathrm{d}r$$

is a $\widetilde{Q}^\omega$-Brownian motion with

$$\mathrm{d}\widetilde{Q}^\omega = R_T(\, \cdot \, , \omega) \mathrm{d}Q^\omega.$$

At this point, the expectation in (4.6) gives

$$E^{\widetilde{Q}^\omega} \left(\int_0^T |\varphi|(s, Y_s(\, \cdot \, , \omega)) \mathrm{d}s \right), \tag{4.7}$$

where

$$Y_t(\, \cdot \, , \omega) = Y_0 + \int_s^t \gamma(s, Y_s(\, \cdot \, , \omega), \omega) \mathrm{d}\widetilde{B}_s$$

$$- \int_0^t \gamma(s, Y_s(\, \cdot \, , \omega), \omega) \delta(s, \, \cdot \, , \omega) \mathrm{d}s.$$

For fixed $\omega \in \Omega$, δ is bounded by a random constant $\varrho_2(\omega)$ of the type (4.5). Moreover we keep in mind assumption (4.1) on γ. By Exercise 7.3.3 of [31], (4.7) is bounded by

$$\varrho_3(\omega) \left\| \varphi \right\|_{L^2([0,T] \times \mathbb{R})}.$$

where $\varrho_3(\omega)$ again depends on the same quantities as in (4.5) and Φ. So for ω dP-a.s., the map $\varphi \mapsto E^{Q^\omega} \left(\int_0^T \varphi(s, Y_s(\, \cdot \, , \omega)) \widetilde{M}_s(\, \cdot \, , \omega) \mathrm{d}s \right)$ prolongates to $L^2([0,T] \times \mathbb{R})$. Using Riesz theorem it is not difficult to show the existence of an $L^2([0,T] \times \mathbb{R})$ function $(s, y) \mapsto q_s(y, \omega)$ which constitutes indeed the density of the family of the μ-marginal laws. $\qquad\square$

5 On the uniqueness of a Fokker-Planck type SPDE

The theorem below plays the analogous role as Theorem 3.8 in [13] or Theorem 3.1 in [9]. It has an interest in itself since it is a Fokker-Planck SPDE with possibly degenerate measurable coefficients.

Theorem 5.1. *Let z^1, z^2 be two measurable random fields belonging ω a.s. to $C([0,T], \mathcal{S}'(\mathbb{R}))$ such that $z :]0,T]\times\Omega \to \mathcal{M}(\mathbb{R})$. Let $a : [0,T]\times\mathbb{R}\times\Omega \to \mathbb{R}_+$ be a bounded measurable random field such that, for any $t \in [0,T]$, $a(t,\cdot)$ is $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable. We suppose moreover the following.*

i) $z = z^1 - z^2 \in L^2([0,T] \times \mathbb{R})$ *a.s.*

ii) $t \mapsto z(t,\cdot)$ *is $(\mathcal{F}_t)$-progressively measurable $\mathcal{S}'(\mathbb{R})$-valued process.*

iii) z^1, z^2 *are solutions to*

$$\begin{cases} \partial_t z(t,\xi) = \partial^2_{\xi\xi}((az)(t,\xi)) + z(t,\xi)\mu(\mathrm{d}t,\xi), \\ z(0,\,\cdot\,) = z_0, \end{cases} \tag{5.1}$$

where z_0 is some distribution in $\mathcal{S}'(\mathbb{R})$.

Then $z^1 \equiv z^2$.

Remark 5.2.   *a) By solution of equation (5.1) we intend, as expected, the following: for every $\varphi \in \mathcal{S}(\mathbb{R})$, $\forall t \in [0,T]$,*

$$\int_{\mathbb{R}} \varphi(x)z(t,\mathrm{d}\xi) = \langle x_0, \varphi \rangle + \int_0^t \mathrm{d}s \int_{\mathbb{R}} a(s,\xi)\varphi''(\xi)z(s,\mathrm{d}\xi)$$

$$+ \int_{[0,t]\times\mathbb{R}} \mu(\mathrm{d}s,\xi)z(s,\mathrm{d}\xi)\varphi(\xi) \quad a.s.$$

b) Since $z(\cdot,\omega)$ is ω a.s. in $L^2([0,T];L^2(\mathbb{R})) \subset L^2([0,T];H^{-1}(\mathbb{R}))$, then $\int_0^t \mu(\mathrm{d}s,\,\cdot\,)z(s,\cdot)$ belongs ω a.s. to $C([0,T];H^{-1}(\mathbb{R}))$. On the other hand $\int_0^t (az)''(s,\cdot)\mathrm{d}s$ can be seen as Bochner integral in $H^{-2}(\mathbb{R})$ and so $t \mapsto \int_0^t \mu(\mathrm{d}s,\,\cdot\,)z(s,\cdot)$ belongs to $C([0,T];H^{-2}(\mathbb{R}))$ ω a.s. In particular any solutions z^1, z^2 to (5.1) are such that $z = z^1 - z^2$ admits a modification whose paths belong (a.s.) to $C([0,T];H^{-2}(\mathbb{R}))\cap L^2([0,T];L^2(\mathbb{R}))$.

Since $z^i, i = 1, 2$, are continuous with values in $\mathcal{S}'(\mathbb{R})$, then their difference is indistinguishable with the mentioned modification.

Consequently for ω a.s. $z(t, \cdot) \in C([0, T]; H^{-2}(\mathbb{R}))$. Then, outside a P-null set N_0, for $\omega \in N_0$ we have (in $\mathcal{S}'(\mathbb{R})$ and $H^{-2}(\mathbb{R})$))

$$z(t, \cdot) = \int_0^t (az)''(s, \cdot)\mathrm{d}s + \int_0^t \mu(\mathrm{d}s, \cdot)z(s, \cdot). \tag{5.2}$$

c) By assumption i), possibly enlarging the P-null set N_0 we get the following. For $\omega \notin N_0$, for almost all $t \in]0, T]$, $\left(\int_0^t (az)(s, \cdot)\mathrm{d}s \right)'' \in H^{-1}(\mathbb{R})$ and so $\int_0^t (az)(s, \cdot)\mathrm{d}s \in H^1$ $\mathrm{d}t$ a.e.

Proof of Theorem 5.1. We fix the null set N_0 and so ω will always lie outside N_0 related to Remark 5.2 c). Let ϕ be a mollifier with compact support and $\phi_\varepsilon = \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon})$ be a generalized sequence of mollifiers converging to the Dirac delta function. We set

$$g_\varepsilon(t) = \|z_\varepsilon(t)\|_{H^{-1}}^2 = \int_{\mathbb{R}} \mathrm{d}\xi z_\varepsilon(t, \xi)((I - \Delta)^{-1}z_\varepsilon)(t, \xi),$$

where $z_\varepsilon(t, \xi) = \int_{\mathbb{R}} \phi_\varepsilon(\xi - y)z(t, \mathrm{d}y)$. Since $t \mapsto z(t, \cdot)$ is continuous in $H^{-2}(\mathbb{R})$, then $t \mapsto z_\varepsilon(t, \cdot)$ is continuous in $L^2(\mathbb{R})$ and so also in $H^{-1}(\mathbb{R})$. We look at the equation fulfilled by z_ε. The identity (5.2) produces the following equality in $L^2(\mathbb{R})$ and so in $H^{-1}(\mathbb{R})$:

$$z_\varepsilon(t, \cdot) = \int_0^t \mathrm{d}s \left\{ [(a(s, \cdot)z(s, \cdot)) \star \phi_\varepsilon]'' - (a(s, \cdot)z(s, \cdot)) \star \phi_\varepsilon \right\} \mathrm{d}s$$

$$+ \int_0^t \mathrm{d}s(a(s, \cdot)z(s, \cdot)) \star \phi_\varepsilon \tag{5.3}$$

$$+ \sum_{i=1}^N \int_0^t \mathrm{d}W_s^i(e_i z)(s, \cdot) \star \phi_\varepsilon.$$

We apply $(I - \Delta)^{-1}$ and we get

$$(I - \Delta)^{-1}z_\varepsilon(t, \cdot) = -\int_0^t \mathrm{d}s(a(s, \cdot)z(s, \cdot)) \star \phi_\varepsilon \tag{5.4}$$

$$+ \int_0^t \mathrm{d}s(I - \Delta)^{-1}\left[(a(s, \cdot)z(s, \cdot)) \star \phi_\varepsilon\right]$$

$$+ \sum_{i=1}^N \int_0^t \mathrm{d}W_s^i(I - \Delta)^{-1}(e^i z)(s, \cdot) \star \phi_\varepsilon.$$

We apply Itô's formula for stochastic calculus, with values in the Hilbert space $H^{-1}(\mathbb{R})$. For a general introduction to infinite dimensional Hilbert valued calculus, see [15] or [26]. We evaluate the H^{-1}-norm of $z_\varepsilon(t)$. Taking into account, (5.3), (5.4) and that $\langle f, g \rangle_{H^{-1}} = \langle f, (I - \Delta)^{-1} g \rangle_{L^2}$, it gives

$$g_\varepsilon(t) = 2 \int_0^t \langle z_\varepsilon(s, \cdot), \, \mathrm{d}z_\varepsilon(s, \cdot) \rangle_{H^{-1}} \tag{5.5}$$

$$+ \sum_{i=1}^N \int_0^t \mathrm{d}s \left\langle (z(s, \cdot)e^i) \star \phi_\varepsilon, \, (z(s, \cdot)e^i) \star \phi_\varepsilon \right\rangle_{H^{-1}}$$

$$= -2 \int_0^t \langle z_\varepsilon(s, \cdot), (a(s, \cdot)z(s, \cdot)) \star \phi_\varepsilon \rangle_{L^2} \, \mathrm{d}s$$

$$+ 2 \int_0^t \mathrm{d}s \left\langle z_\varepsilon(s, \cdot), \, (I - \Delta)^{-1} \left((a(s, \cdot)z(t, \cdot)) \star \phi_\varepsilon \right) \right\rangle_{L^2}$$

$$+ \sum_{i=1}^N \int_0^t \mathrm{d}s \left\langle (z(s, \cdot)e^i) \star \phi_\varepsilon, \, (z(s, \cdot)e^i) \star \phi_\varepsilon \right\rangle_{H^{-1}}$$

$$+ 2 \int_0^t \mathrm{d}s \left\langle z_\varepsilon(s, \cdot), (ze^0)(s, \cdot) \star \phi_\varepsilon \right\rangle_{H^{-1}} + M_t^\varepsilon$$

where

$$M_t^\varepsilon = 2 \sum_{i=1}^N \int_0^t \left\langle z_\varepsilon(s, \cdot), \, (e^i z)(s, \cdot) \star \phi_\varepsilon \right\rangle_{H^{-1}} \mathrm{d}W_s^i. \tag{5.6}$$

Below we will justify that (5.6) is well-defined. We summarize (5.5) into

$$g_\varepsilon(t) = \widetilde{g}_\varepsilon(t) + M_t^\varepsilon, t \in [0, T].$$

We remark that

$$\sum_{i=1}^N \int_0^T \left(\langle z(s, \cdot), \, e^i z(s, \cdot) \rangle_{H^{-1}} \right)^2 \mathrm{d}s = \sum_{i=1}^N \int_0^T \left(\left\langle e^i z(s, \cdot), \, (I - \Delta)^{-1} z(s, \cdot) \right\rangle_{L_2} \right)^2 \mathrm{d}s$$

$$\tag{5.7}$$

$$\leq \sum_{i=1}^N \int_0^T \| e^i z(s, \cdot) \|_{L^2}^2 \| z(s, \cdot) \|_{H^{-2}}^2 \mathrm{d}s$$

$$\leq \sum_{i=1}^N \| e^i \|_\infty^2 \sup_{s \in [0,T]} \| z(s, \cdot) \|_{H^{-2}}^2 \int_0^T \| z(s, \cdot) \|_{L^2}^2 \mathrm{d}s,$$

because $z : [0, T] \to H^{-2}$ is a.s. continuous by Remark 5.2 b).

Consequently

$$M_t = \sum_{i=1}^{N} \int_0^t \left\langle z(s, \cdot), \, e^i z(s, \cdot) \right\rangle_{H^{-1}} \mathrm{d}W_s^i$$

is a well-defined local martingale. It is also not difficult to show that for $\varepsilon > 0$,

$$\int_0^T \left\{ \sum_{i=1}^{N} \left\langle z_\varepsilon(s, \cdot), \, (e^i z(s, \cdot)) \star \phi_\varepsilon \right\rangle_{H^{-1}}^2 \right\}^2 \mathrm{d}s < \infty,$$

and so M^ε is a local martingale.

By assumption we have of course $(\omega \notin N_0)$

$$\int_{[0,T] \times \mathbb{R}} (z_\varepsilon(s, \xi) - z(s, \xi))^2 \mathrm{d}s \mathrm{d}\xi \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{5.8}$$

$$\int_{[0,T] \times \mathbb{R}} ((az) \star \phi_\varepsilon - az)^2 (s, \xi) \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{5.9}$$

$$\int_{[0,T] \times \mathbb{R}} ((z(s, \cdot)e_i) \star \phi_\varepsilon - z(s, \cdot)e_i)^2 (\xi) \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{5.10}$$

for every $1 \leq i \leq N$, because $z, az, e_i z \in L^2([0, T] \times \mathbb{R}), 1 \leq i \leq N$. Using (5.8) and (5.10), it is not difficult to show that $(\omega \notin N_0)$

$$\sum_{i=0}^{N} \int_0^T \left(\left\langle z_\varepsilon(s, \cdot), (e^i z(s, \cdot) \star \phi_\varepsilon \right\rangle - \left\langle z(s, \cdot), e^i z(s, \cdot) \right\rangle \right)^2 \mathrm{d}s \tag{5.11}$$

converge to zero. As a consequence of (5.10), $(\omega \notin N_0)$,

$$\sum_{i=0}^{N} \int_0^T \left(\left\| z(s, \cdot)e^i) \star \phi_\varepsilon \right\|_{H^{-1}}^2 - \left\| z(s, \cdot)e^i \right\|_{H^{-1}}^2 \right) \mathrm{d}s \tag{5.12}$$

converges to zero. Taking into account (5.8), (5.9), (5.11) and (5.12) we

obtain ($\omega \notin N_0$), that $\lim\limits_{\varepsilon \to 0} \widetilde{g}_\varepsilon(t) = \widetilde{g}(t)$, $t \in [0,T]$, where

$$\widetilde{g}(t) = -2 \int_0^t \langle z(s,\,\cdot\,), a(s,\,\cdot\,)z(s,\,\cdot\,)\rangle_{L^2}\, \mathrm{d}s \tag{5.13}$$

$$+ 2 \int_0^t \mathrm{d}s \,\langle z(s,\,\cdot\,), (I-\Delta)^{-1}(a(s,\,\cdot\,)z(s,\,\cdot\,))\rangle_{L^2}$$

$$+ 2 \int_0^t \mathrm{d}s \,\langle z(s,\,\cdot\,), z(s,\,\cdot\,)e^0\rangle_{H^{-1}}$$

$$+ \sum_{i=1}^N \int_0^t \langle z(s,\,\cdot\,)e^i, z(s,\,\cdot\,)e^i\rangle_{H^{-1}}\, \mathrm{d}s.$$

The convergence of the second term in the right-hand side of (5.5) to the second term of the right hand sides of (5.13) works again using (5.8) and (5.9) cutting the difference in two pieces and using Cauchy-Schwarz. On the other hand the convergence of (5.11) to zero implies that $M^\varepsilon \to M$ ucp, so that the ucp limit of $\widetilde{g}_\varepsilon(t) + M_t^\varepsilon$ gives $\widetilde{g}(t) + M_t$. So after a possible modification of the P-null set N_0, setting $g(t) := \|z(t,\,\cdot\,)\|_{H^{-1}}^2$, for $\omega \notin N_0$, we have

$$g(t) + 2 \int_0^t \langle z(s,\,\cdot\,), a(s,\,\cdot\,)z(s,\,\cdot\,)\rangle_{L^2}\, \mathrm{d}s \tag{5.14}$$

$$= 2 \int_0^t \mathrm{d}s \,\langle (I-\Delta)^{-1}z(s,\,\cdot\,), a(s,\,\cdot\,)z(s,\,\cdot\,)\rangle_{L^2}$$

$$+ 2 \int_0^t \mathrm{d}s \,\langle z(s,\,\cdot\,), e^0 z(s,\,\cdot\,)\rangle_{H^{-1}}$$

$$+ \sum_{i=1}^N \int_0^t \langle z(s,\,\cdot\,)e^i, z(s,\,\cdot\,)e^i\rangle_{H^{-1}}\, \mathrm{d}s$$

$$+ M_t.$$

By the inequality

$$2bc \leqslant \frac{b^2}{\|a\|_\infty} + c^2 \|a\|_\infty,$$

$b, c \in \mathbb{R}$, it follows,

$$2 \int_0^t < (I - \Delta)^{-1} z(s, \, \cdot \,), (az)(s, \, \cdot \,) >_{L^2} \mathrm{d}s$$

$$\leqslant \|a\|_\infty \int_0^t \left\| (I - \Delta)^{-1} z(s, \, \cdot \,) \right\|_{L^2}^2 \mathrm{d}s$$

$$+ \frac{1}{\|a\|_\infty} \int_0^t < (az)(s, \, \cdot \,), (az)(s, \, \cdot \,) >_{L^2} \mathrm{d}s$$

$$\leqslant \|a\|_\infty \int_0^t \| z(s, \, \cdot \,) \|_{H^{-2}}^2 \mathrm{d}s$$

$$+ \frac{1}{\|a\|_\infty} \|a\|_\infty \int_0^t \langle z(s, \, \cdot \,), az(s, \, \cdot \,) \rangle_{L^2} \mathrm{d}s.$$

Since $\| \cdot \|_{H^{-2}} \leqslant \| \cdot \|_{H^{-1}}$, (5.14) gives now ($\omega \notin N_0$),

$$g(t) + \int_0^t \langle z(s, \, \cdot \,), (az)(s, \, \cdot \,) \rangle_{L^2} \mathrm{d}s$$

$$\leqslant M_t + \sum_{i=1}^N \int_0^t \mathrm{d}s \left\langle z(s, \, \cdot \,)e^i, z(s, \, \cdot \,)e^i \right\rangle_{H^{-1}}$$

$$+ 2 \int_0^t \mathrm{d}s \left\langle z(s, \, \cdot \,), z(s, \, \cdot \,)e^0 \right\rangle_{H^{-1}}$$

$$+ \|a\|_\infty \int_0^t \| z(s, \, \cdot \,) \|_{H^{-1}}^2 \mathrm{d}s.$$

Since $e^i, \; 0 \leqslant i \leqslant N$ are H^{-1}-multipliers, we obtain the existence of a constant $\mathcal{C} = \mathcal{C}(e_i, \, 1 \leqslant i \leqslant n, \, \|a\|_\infty)$ such that ($\omega \notin N_0$)

$$g(t) + \int_0^t \langle z(s, \, \cdot \,), (az)(s, \, \cdot \,) \rangle_{L^2} \mathrm{d}s \tag{5.15}$$

$$\leqslant M_t + \mathcal{C} \int_0^t \| z(s, \, \cdot \,) \|_{H^{-1}}^2 \mathrm{d}s$$

$$= M_t + \mathcal{C} \int_0^t g(s) \mathrm{d}s, \; \forall \, t \in [0, T].$$

We proceed now via localization which is possible because $\int_0^T \| z(s, \, \cdot \,) \|^2 \mathrm{d}s$ and $\sup_{t \in [0,T]} \| z(s, \cdot) \|_{H^{-2}}$ are P a.s. finite. Let (ς^ℓ) be the sequence of stopping times

$$\varsigma^\ell := \inf \{ t \in [0, T] | \int_0^t \mathrm{d}s \, \| z(s, \, \cdot \,) \|_{L^2}^2 \geq \ell, \| z(s, \cdot) \|_{H^{-2}}^2 \geqslant \ell \}.$$

If $\{\ \} = \emptyset$ we convene that $\varsigma^\ell = +\infty$. Clearly the stopped processes M^{ς^ℓ} are (square integrable) martingales starting at zero. We evaluate (5.15) at $t \wedge \varsigma^\ell$. Taking the expectation we get

$$E(g(t \wedge \varsigma^\ell)) \leqslant \underbrace{E(M_{\varsigma^\ell \wedge t})}_{=0} + \mathcal{C}E\left(\int_0^{t \wedge \varsigma^\ell} g(s)\mathrm{d}s\right)$$

$$\leqslant \mathcal{C}\int_0^t \mathrm{d}s E(g(s \wedge \varsigma^\ell)).$$

By Gronwall lemma it follows

$$E(g(t \wedge \varsigma_\ell)) = 0 \quad \forall\, \ell \in \mathbb{N}^\star.$$

Since g is a.s. continuous and $\lim_{\ell \to \infty} t \wedge \varsigma_\ell = T$ a.s., for every $t \in [0, T]$, by Fatou's lemma we get

$$E(g(t)) = E\left(\liminf_{\ell \to \infty} g(t \wedge \varsigma_\ell)\right) \leqslant \liminf_{\ell \to \infty} E\left(g(t \wedge \varsigma_\ell)\right) = 0.$$

Finally the result follows. $\qquad\square$

6 The non-degenerate case

We are now able to discuss the double probabilistic representation of a solution to the (1.1) when ψ is non-degenerate provided that its solution fulfills some properties. We remark that up to now we have not used the first item of Assumption 1.1. We remind that the functions $e_i.0 \leq i \leq N$, are H^{-1}-multipliers.

Theorem 6.1. *We suppose the following assumptions.*

1. *x_0 is a real probability measure.*

2. *ψ is non-degenerate.*

3. *There is only one random field $X : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ solution of (1.1) (see Definition 2.4) such that*

$$\int_{[0,T]\times\mathbb{R}} X^2(s, \xi)\mathrm{d}s\mathrm{d}\xi < \infty \quad a.s. \tag{6.1}$$

Then there is a unique weak solution to the (DSNLD)(Φ, μ, x_0).

Remark 6.2.   *1. Suppose that $e^i, 1 \leq i \leq d$, belong to $W^{1,\infty}$. In Theorem 3.4 of [6], we show that (even if x_0 belongs to $H^{-1}(\mathbb{R})$)), when ψ is Lipschitz, there is a solution to (1.1) such that*

$$E\left(\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi\right) < \infty.$$

According to Thereom B.1, that solution is unique.
In particular item 3. in Theorem 6.1 statement holds.

2. Theorem 6.1 constitutes the converse of Theorem 3.3 when ψ is non-degenerate.

3. Again for simplicity of the formulation, without restriction of generality, in the proof we will suppose ψ to be single-valued and Φ admitting a continuous extension to $\mathbb{R}$. Otherwise one can adopt the techniques of [13].

4. As side-effect of the proof of the weak-strong existence Proposition 4.1, the space $(\Omega_0, \mathcal{G}, Q)$ can be chosen as $\Omega_0 = \Omega_1 \times \Omega$, $\Omega_1 = C([0,T]; \mathbb{R}) \times \mathbb{R}$, $\mathcal{G} = \mathcal{B}(\Omega_1) \times \mathcal{F}$, $Q(H \times F) = \int_{\Omega_1 \times \Omega} \mathrm{d}P(\omega)1_F(\omega)Q(\mathrm{d}\omega_1, \omega)$.

Proof. 1) We set $\gamma(t,\xi,\omega) = \Phi(X(t,\xi,\omega))$. According to Proposition 4.1 there is a weak-strong solution Y to $(DSDE)(\gamma, x_0)$. By Proposition 4.4 ω a.s. the μ-marginal laws of Y admit densities $(q_t(\xi,\omega), t \in]0,T], \xi \in \mathbb{R}, \omega \in \Omega)$ such that $\mathrm{d}P$-a.s.

$$\int_{[0,T]\times\mathbb{R}} \mathrm{d}s\mathrm{d}\xi q_s^2(\xi, \cdot) < \infty \quad \text{a.s.}$$

2) Setting

$$\nu_t(\xi,\omega) = \left(\begin{array}{ccc} q_t(\xi,\omega)\mathrm{d}\xi & : & t \in]0,T], \\ x_0 & : & t = 0, \end{array}\right.$$

ν is a solution to (5.1) with $\nu_0 = x_0$, $a(t,\xi,\omega) = \Phi^2(X(t,\xi,\omega))$. This can be shown applying Itô's formula similarly as in the proof of Theorem 3.3.

3) On the other hand X is obviously also a solution of (5.1), which in particular verifies (6.1). Consequently $z^1 = \nu$, $z^2 = X$ verify items i), ii), iii) of Theorem 5.1. So Theorem 5.1 implies that $\nu \equiv X$; this shows that Y provides a solution to (DSNLD)(Φ, μ, x_0).

4) Concerning uniqueness, let Y^1, Y^2 be two solutions to the (DSNLD) related to (Φ, μ, x). The corresponding random fields X^1, X^2 constitute the μ-marginal laws of Y^1, Y^2 respectively.

Now Y^i, $i = 1, 2$, is a weak-strong solution of (DSDE)(γ_i, x) with $\gamma_i(t, \xi, \omega) = \Phi(X_i(t, \xi, \omega))$, so by Proposition 4.4 X_i, $i = 1, 2$ fulfills (6.1). By Theorem 3.3, X_1 and X_2 are solutions to (1.1). By assumption 3. of the statement, $X_1 = X_2$. The conclusion follows by Proposition 4.1, which guarantees the uniqueness of the weak-strong solution of (DSDE)(γ, x_0) with $\gamma_1 = \gamma_2$. $\quad\square$

Remark 6.3. *One side-effect of Theorem 6.1 is the following. Suppose ψ to be non-degenerate. Let $X : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a solution such that* dP*-a.s.*

$$\int\limits_{[0,T]\times\mathbb{R}} X^2(s, \xi) \mathrm{d}s\mathrm{d}\xi < \infty \quad a.s.$$

*We have the following for ω *dP-a.s.*.*

i) $X(t, \cdot, \omega) \geqslant 0$ a.e. $\forall t \in [0, T]$

ii) $E\left(\int\limits_{\mathbb{R}} X(t, \xi)\mathrm{d}\xi\right) = 1$, $\forall t \in [0, T]$ if $e_0 = 0$.

Remark 6.4. *If (1.1) has a solution, not necessarily unique then (DSNLD) with respect to (Φ, μ, x_0) still admits existence.*

7 The degenerate case

The idea consists in proceeding similarly to [7], which treated the case $\mu = 0$ and the case when x_0 is absolutely continuous with bounded density. ψ will be assumed to be strictly increasing after some zero $u_c \geq 0$, see Definition 1.3. We recall that if ψ is degenerate, then necessarily $\Phi(0) := \lim_{\varepsilon \to 0} \Phi(x) = 0$.

Remark 7.1. *i) If $u_c > 0$ then ψ is necessarily degenerate and also Φ restricted to $[0, u_c]$ vanishes.*

ii) Let x_0 is a probability on $\mathbb{R}$. Suppose the existence of a solution X to (1.1) such that

$$E\left(\int_{[0,T]\times\mathbb{R}} \mathrm{d}t\mathrm{d}\xi X^2(t,\xi)\right) < \infty. \tag{7.1}$$

We recall that, by Definition 2.4, a.s. it belongs to $C\left([0,T],\mathcal{S}'(\mathbb{R})\right)$. In this case a.s. $\int_0^t \psi(X(s,\,\cdot\,))\mathrm{d}s \in H^1(\mathbb{R})$ for every $t \in [0,T]$. See Remark B.3 vi).

iii) If X is a solution such that (7.1) is verified and $x_0 \in H^{-2}$, then $X \in C([0,T];H^{-2})$ a.s., see Remark B.3 vi).

iv) (7.1) implies in particular that if X is a solution of (1.1), then

$$E\left(\int_0^T \mathrm{d}s\, \|X(s,\,\cdot\,)\|^2_{H^{-1}}\right) < \infty.$$

v) If ψ is Lipschitz, we remind (Remark 6.2 1.) and $x_0 \in L^2$, there is a unique solution to (1.1) such that (7.1) is fulfilled, at least if we suppose that all the e^i belong to $H^1(\mathbb{R})$, see Theorem 3.4 of [6] and Theorem B.1.

Theorem 7.2. *We suppose the following.*

1. *The functions $e_i.1 \leq i \leq N$ belong to $H^1(\mathbb{R})$.*

2. *We suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is non-decreasing, Lipschitz and strictly increasing after some zero.*

3. *Let x_0 belong to $L^2(\mathbb{R})$.*

Then there is a weak solution to the $(DSNLD)(\Phi,\mu,x_0)$.

Proof. 1) We proceed by approximation rendering Φ non-degenerate. Let $\kappa > 0$. We define $\Phi_\kappa : \mathbb{R} \to \mathbb{R}_+$ by

$$\Phi_\kappa(u) = \sqrt{\Phi^2(u) + \kappa}$$

$$\psi_\kappa(u) = \Phi_\kappa^2(u) \cdot u$$

Let X^κ be the solution so (1.1) with ψ_κ instead of ψ. According to Theorem 6.1 and Remark 6.2 4., setting $\Omega_1 = C([0,T],\mathbb{R}) \times \mathbb{R}$, $\mathcal{H}$ its Borel σ-algebra, $Y(\omega^1,\omega) = \omega^1$, there are families of probability kernels Q^κ on $\mathcal{H} \times \Omega_1$, and processes B^κ on Ω_0 such that

i) $B^\kappa(\,\cdot\,,\omega)$ is a $Q^\kappa(\,\cdot\,,\omega)$-Brownian motion;

ii)

$$Y_t = Y_0 + \int_0^t \Phi_\kappa(X^\kappa(s,Y_s,\omega))\mathrm{d}B_s^\kappa; \tag{7.2}$$

iii) Y_0^κ is distributed according to $x_0 = X^\kappa(0,\,\cdot\,)$.

iv) The μ-marginal laws of Y under Q^κ are $(X^\kappa(t,\,\cdot\,))$.

We need to show the existence of a probability kernel Q on $\mathcal{H} \times \Omega_1$, a process B on Ω_0 such that the following holds.

i) $B(\,\cdot\,,\omega)$ is a $Q(\,\cdot\,,\omega)$-Brownian motion.

ii)

$$Y_t = Y_0 + \int_0^t \Phi(X(s,Y_s,\omega))\mathrm{d}B_s^\kappa.$$

iii) Y_0 is distributed according to x_0.

iv) For every $t \in]0,T]$, $\varphi \in C_b(\mathbb{R})$,

$$\int_{\mathbb{R}} X(t,\xi)\varphi(\xi)\mathrm{d}\xi = E^{Q^\omega}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^\cdot \mu(\mathrm{d}s,Y_s)X(s,Y_s)\right)\right),$$

where $Q^\omega = Q(\,\cdot\,,\omega)$.

2) We need to show that X^κ approaches X in some sense when $\kappa \to 0$, where X is the solution to (1.1). This is given in the Lemma 7.3 below.

Lemma 7.3. *Under the the assumptions of Theorem 7.2, let X (resp. X^κ) be a solution of (1.1) verifying (7.1) with $\psi(u) = u\Phi^2(u)$ (resp. $\psi_\kappa(u) = u(\Phi^2(u) + \kappa))$, for $u > 0$. We have the following.*

a) $\lim_{\kappa \to 0} \sup_{t \in [0,T]} E\left(\|X^\kappa(t,\,\cdot\,) - X(t,\,\cdot\,)\|_{H^{-1}}^2\right) = 0$;

b) $\lim_{\kappa \to 0} E\left(\int_0^T \mathrm{d}t\, \|\psi\left(X^\kappa(t,\,\cdot\,)\right) - \psi\left(X(t,\,\cdot\,)\right)\|_{L^2}^2\right) = 0$;

c) $\lim_{\kappa \to 0} \kappa E\left(\int_{[0,T]\times\mathbb{R}} \mathrm{d}t\mathrm{d}\xi\, (X^\kappa(t,\xi) - X(t,\xi))^2\right) = 0$.

Remark 7.4. *1) a) implies of course*

$$\lim_{\kappa \to 0} E\left(\int_0^T \mathrm{d}t\, \|X^\kappa(t,\,\cdot\,) - X(t,\,\cdot\,)\|_{H^{-1}}^2\right) = 0.$$

2) In particular Lemma 7.3 b) implies that for each sequence $(\kappa_n) \to 0$ there is a subsequence still denoted by the same notation that

$$\int_{[0,T]\times\mathbb{R}} (\psi(X^{\kappa_n}(t,\xi)) - \psi(X(t,\xi)))^2 \, \mathrm{d}t\mathrm{d}\xi \xrightarrow[n\to\infty]{} 0$$

a.s.

3) For every $t \in [0,T]$

$$X(t,\,\cdot\,) \geqslant 0 \quad \mathrm{d}\xi \otimes dP \, a.e.$$

Indeed, for this it will be enough to show that a.s.

$$\int_{\mathbb{R}} \mathrm{d}\xi\varphi(\xi)X(t,\xi) \geq 0 \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}), \tag{7.3}$$

for every $t \in [0,T]$. Since $X \in C\left([0,T];\, \mathcal{S}'(\mathbb{R})\right)$ it will be enough to show (7.3) for almost all $t \in [0,T]$. This follows since item 1) in this Remark 7.4, implies the existence of a sequence (κ_n) such that

$$\int_0^T \mathrm{d}t\, \|X^{\kappa_n}(t,\cdot) - X(t,\,\cdot\,)\|_{H^{-1}}^2 \xrightarrow[n\to\infty]{} 0, \quad \text{a.s.}$$

4) Since ψ is strictly increasing after u_c, for P almost all ω, for almost all $(t,\xi) \in [0,T] \times \mathbb{R}$, there is a sequence (κ_n) such that

$$(X^{\kappa_n}(t,\xi) - X(t,\xi))\, \mathbb{1}_{\{X(t,\xi)>u_c\}}(t,\xi) \xrightarrow[n\to\infty]{} 0.$$

This follows from item 2) of Remark 7.4.

Since $\Phi^2(u) = 0$ for $0 \leqslant u \leqslant u_c$ and X is a.e. non-negative, this implies that $\mathrm{d}t\mathrm{d}\xi dP$ a.e. we have

$$\Phi^2\left(X(t,\xi)\right)\left(X^{\kappa_n}(t,\xi) - X(t,\xi)\right) \xrightarrow[n\to\infty]{} 0. \tag{7.4}$$

Proof (of Lemma 7.3). Proceeding similarly as in Theorem 5.1, we can write dP-a.s. the following $H^{-2}(\mathbb{R})$-valued equality.

$$(X^{\kappa} - X)(t, \, \cdot \,) = \int_0^t \mathrm{d}s \, (\psi_{\kappa}(X^{\kappa}(s, \, \cdot \,)) - \psi(X(s, \, \cdot \,)))''$$

$$+ \sum_{i=0}^N \int_0^t (X^{\kappa}(s, \, \cdot \,) - X(s, \, \cdot \,)) \, e^i \mathrm{d}W_s^i.$$

So

$$(I - \Delta)^{-1} (X^{\kappa} - X)(t, \, \cdot \,) = - \int_0^t \mathrm{d}s \, (\psi_{\kappa}(X^{\kappa}(s, \, \cdot \,)) - \psi(X(s, \, \cdot \,)))$$

$$+ \int_0^t \mathrm{d}s (I - \Delta)^{-1} (\psi_{\kappa}(X^{\kappa}(s, \, \cdot \,)) - \psi(X(s, \, \cdot \,)))$$

$$+ \sum_{i=0}^N \int_0^t (I - \Delta)^{-1} \left(e^i (X^{\kappa}(s, \, \cdot \,) - X(s, \, \cdot \,)) \right) \mathrm{d}W_s^i.$$

After regularization and application of Itô calculus with values in H^{-1}, setting $g^{\kappa}(t) = \|(X^{\kappa} - X)(t, \, \cdot \,)\|_{H^{-1}}^2$, similarly to the proof of Theorem 5.1, we obtain

$$g^{\kappa}(t) = \sum_{i=1}^N \int_0^t \left\| e^i (X^{\kappa} - X)(s, \, \cdot \,) \right\|_{H^{-1}}^2 \mathrm{d}s \tag{7.5}$$

$$-2 \int_0^t \left\langle (X^{\kappa} - X)(s, \, \cdot \,), \psi_{\kappa}(X^{\kappa}(s, \, \cdot \,)) - \psi(X(s, \, \cdot \,)) \right\rangle_{L^2}$$

$$+2 \int_0^t \mathrm{d}s \left\langle (X^{\kappa} - X)(s, \, \cdot \,), (I - \Delta)^{-1} (\psi_{\kappa}(X^{\kappa}(s, \, \cdot \,)) - \psi(X(s, \, \cdot \,))) \right\rangle_{L^2}$$

$$+2 \int_0^t \mathrm{d}s \left\langle (X^{\kappa} - X)(s, \, \cdot \,), (I - \Delta)^{-1} e^0 (X^{\kappa} - X)(s, \, \cdot \,) \right\rangle_{L^2}$$

$$+ M_t^{\kappa},$$

where M^κ is the local martingale

$$M^\kappa_t = 2 \sum_{i=1}^{N} \int_0^t \left\langle (I - \Delta)^{-1} (X^\kappa - X)(s, \,\cdot\,), (X^\kappa - X)(s, \,\cdot\,)e^i \right\rangle_{L^2} \mathrm{d}W^i_s.$$

Indeed, M^κ is a local martingale because, taking into account (B.1) and Remark 7.1 iii), acting similarly as for the proof of (5.7), see also (2.1) in Appendix B, we can prove that

$$\sum_{i=1}^{N} \int_0^t \left| \left\langle (X^\kappa - X|)(s, \,\cdot\,), (I - \Delta)^{-1} (X^\kappa - X)(s, \,\cdot\,)e^i \right\rangle_{L^2} \right|^2 \mathrm{d}s < \infty.$$

(7.5) gives

$$g^\kappa(t) + 2 \int_0^t \left\langle (X^\kappa - X)(s, \,\cdot\,), \, \psi(X^\kappa(s, \,\cdot\,)) - \psi(X(s, \,\cdot\,)) \right\rangle_{L^2} \mathrm{d}s$$

$$+ 2\kappa \int_0^t \left\langle (X^\kappa - X)(s, \,\cdot\,), \, (X^\kappa - X)(s, \,\cdot\,) \right\rangle_{L^2} \mathrm{d}s$$

$$\leqslant - 2\kappa \int_0^t \mathrm{d}s \left\langle (X^\kappa - X)(s, \,\cdot\,), \, X(s, \,\cdot\,) \right\rangle_{L^2} \mathrm{d}s$$

$$+ \sum_{i=1}^{N} \int_0^t \left\| e^i (X^\kappa - X)(s, \,\cdot\,) \right\|^2_{H^{-1}} \mathrm{d}s$$

$$+ 2 \int_0^t \mathrm{d}s \left\langle (I - \Delta)^{-1} (X^\kappa - X)(s, \,\cdot\,), \, \psi(X^\kappa(s, \,\cdot\,)) - \psi(X(s, \,\cdot\,)) \right\rangle_{L^2}$$

$$+ 2\kappa \int_0^t \mathrm{d}s \left\langle (I - \Delta)^{-1} (X^\kappa - X)(s, \,\cdot\,), \, (X^\kappa - X)(s, \,\cdot\,) \right\rangle_{L^2}$$

$$+ 2\kappa \int_0^t \mathrm{d}s \left\langle (I - \Delta)^{-1} (X^\kappa - X)(s, \,\cdot\,), \, X(s, \,\cdot\,) \right\rangle_{L^2}$$

$$+ 2 \int_0^t \mathrm{d}s \left\langle (I - \Delta)^{-1} (X^\kappa - X)(s, \,\cdot\,), \, \left(e^0 (X^\kappa - X)(s, \,\cdot\,) \right) \right\rangle_{L^2} + M^\kappa_t.$$

We use Cauchy-Schwarz and the inequality

$$2\sqrt{\kappa}b\sqrt{\kappa}c \leqslant \kappa b^2 + \kappa c^2$$

with first

$$b = \|X^\kappa(s, \,\cdot\,) - X(s, \,\cdot\,)\|_{L^2}, \quad c = \|X(s, \,\cdot\,)\|_{L^2}$$

and then

$$b = \|X^\kappa(s,\cdot) - X(s,\cdot)\|_{H^{-2}}, \quad c = \|X(s, \,\cdot\,)\|_{L^2}.$$

We also take into account the property of H^{-1}-multiplier for e^i, $0 \leqslant i \leqslant N$. Consequently there is a constant $\mathrm{C}(e)$ depending on $(e^i,\ 0 \leqslant i \leqslant N)$ such that

$$g^\kappa(t) + 2\int_0^t \langle (X^\kappa - X)(s, \,\cdot\,),\ \psi(X^\kappa(s, \,\cdot\,)) - \psi(X(s, \,\cdot\,))\rangle_{L^2}\,\mathrm{d}s \quad (7.6)$$

$$+ 2\kappa \int_0^t \|X^\kappa(s, \,\cdot\,) - X(s, \,\cdot\,)\|_{L^2}^2\,\mathrm{d}s$$

$$\leqslant \kappa \int_0^t \|(X^\kappa - X)(s, \,\cdot\,)\|_{L^2}^2\,\mathrm{d}s$$

$$+ \kappa \int_0^t \mathrm{d}s\, \|X(s, \,\cdot\,)\|_{L^2}^2$$

$$+ \mathrm{C}(e) \int_0^t \mathrm{d}s\, \|X^\kappa(s, \,\cdot\,) - X(s, \,\cdot\,)\|_{H^{-1}}^2$$

$$+ 2\int_0^t \|(X^\kappa - X)(s, \,\cdot\,)\|_{H^{-2}} \|\psi(X^\kappa(s, \,\cdot\,)) - \psi(X(s, \,\cdot\,))\|_{L^2}$$

$$+ 2\kappa \int_0^t \mathrm{d}s g^\kappa(s)$$

$$+ \kappa \int_0^t \mathrm{d}s\, \|(X^\kappa - X)(s, \,\cdot\,)\|_{H^{-2}}^2 + \kappa \int_0^t \mathrm{d}s\, \|X(s, \,\cdot\,)\|_{L^2}^2$$

$$+ M_t^\kappa.$$

Since ψ is Lipschitz, it follows

$$\left(\psi(r) - \psi(r_1)\right)\left(r - r_1\right) \geqslant \alpha\left(\psi(r) - \psi(r_1)\right)^2,$$

for some $\alpha > 0$. Consequently, the inequality

$$2bc \leqslant b^2\alpha + \frac{c^2}{\alpha},$$

with $b, c \in \mathbb{R}$ and the fact that $\|\cdot\|_{H^{-2}} \leqslant \|\cdot\|_{H^{-1}}$ give

$$2\int_0^t \mathrm{d}s \left\|(X^\kappa - X)(s,\,\cdot\,)\right\|_{H^{-2}} \left\|\psi\left(X^\kappa(s,\,\cdot\,)\right) - \psi\left(X(s,\,\cdot\,)\right)\right\|_{L^2}$$

$$\leqslant \int_0^t \mathrm{d}s\,\alpha g^\kappa(s,\,\cdot\,) + \int_0^t \mathrm{d}s \left\langle \psi\left(X^\kappa(s,\,\cdot\,)\right) - \psi\left(X(s,\,\cdot\,)\right),\, X^\kappa(s,\,\cdot\,) - X(s,\,\cdot\,)\right\rangle_{L^2}.$$

So (7.6) yields

$$g^\kappa(t) + \int_0^t \left\langle X^\kappa(s,\,\cdot\,) - X(s,\,\cdot\,),\, \psi\left(X^\kappa(s,\,\cdot\,)\right) - \psi\left(X(s,\,\cdot\,)\right)\right\rangle_{L^2} \mathrm{d}s \tag{7.7}$$

$$+ \kappa \int_0^t \mathrm{d}s \left\|X^\kappa(s,\,\cdot\,) - X(s,\,\cdot\,)\right\|_{L^2}^2 \mathrm{d}s$$

$$\leqslant 2\kappa \int_0^t \mathrm{d}s \left\|X(s,\,\cdot\,)\right\|_{L^2}^2$$

$$+ M_t^\kappa$$

$$+ (\mathrm{C}(e) + \alpha + 3\kappa) \int_0^t g^\kappa(s)\mathrm{d}s.$$

Taking the expectation we get

$$E(g^\kappa(t)) \leq (\mathrm{C}(e) + \alpha + 3\kappa) \int_0^t E(g^\kappa(s))\mathrm{d}s$$

$$+ 2\kappa \int_0^t E(\|X(s,\cdot)\|_{L^2}^2)\mathrm{d}s,$$

for every $t \in [0, T]$. By Gronwall lemma we get

$$E\left(g_\kappa(t)\right) \leqslant 2\kappa E \left\{ \int_0^T \mathrm{d}s \, \|X(s, \, \cdot \,)\|_{L^2}^2 \right\} e^{(\mathrm{C}(e) + \alpha + 3\kappa)T}, \quad \forall \, t \in [0, T]. \quad (7.8)$$

Taking the supremum and letting $\kappa \to 0$, item a) of Lemma 7.3 is now established.

We go on with item b). Since ψ is Lipschitz, (7.7) implies that, for $t \in [0, T]$,

$$\int_0^t \mathrm{d}s \, \|\psi\left(X^\kappa(s, \, \cdot \,)\right) - \psi\left(X(s, \, \cdot \,)\right)\|_{L^2}^2$$

$$\leqslant \frac{1}{\alpha}\mathrm{d}s \left\langle \psi\left(X^\kappa(s, \, \cdot \,)\right) - \psi\left(X(s, \, \cdot \,)\right), \, X^{(\kappa)}(s, \, \cdot \,) - X(s, \, \cdot \,) \right\rangle_{L^2}$$

$$\leqslant \frac{\kappa}{2\alpha} \int_0^t \mathrm{d}s \, \|X(s, \, \cdot \,)\|_{L^2}^2$$

$$+ \mathrm{C}(e, \alpha) \int_0^t g_\kappa(s)\mathrm{d}s + M_t^\kappa,$$

where $\mathrm{C}(e, \alpha)$ is a constant depending on $e^i, 0 \leqslant i \leqslant N$ and α. Taking the expectation for $t = T$, we get

$$E\left(\int_0^T \mathrm{d}s \, \|\psi\left(X^\kappa(s, \, \cdot \,)\right) - \psi\left(X(s, \, \cdot \,)\right)\|_{L^2}^2\right)$$

$$\leqslant \frac{\kappa}{2\alpha} E\left(\int_0^T \mathrm{d}s \, \|X(s, \, \cdot \,)\|_{L^2}^2\right) + \mathrm{C}(e, \alpha) \int_0^T E(g_\kappa(s))\mathrm{d}s.$$

Taking $\kappa \to 0$, (7.1) and (7.8) provide the conclusion of item b) of Lemma 7.3.

c) Coming back to (7.7), and $t = T$, we have

$$\kappa \int_0^T \mathrm{d}s \, \|X^\kappa(s, \, \cdot \,) - X(s, \, \cdot \,)\|_{L^2}^2$$

$$\leqslant 2\kappa \int_0^T \mathrm{d}s \, \|X(s, \, \cdot \,)\|_{L^2}^2 + M_T^\kappa$$

$$+ (\mathrm{C}(e) + \alpha + 3\kappa) \int_0^T \mathrm{d}s g^\kappa(s).$$

Taking the expectation we have

$$\kappa E \left(\int_0^T \mathrm{d}s \, \|X^\kappa(s, \, \cdot \,) - X(s, \, \cdot \,)\|_{L^2}^2 \right)$$

$$\leqslant 2\kappa E \left(\int_0^T \mathrm{d}s \, \|X(s, \, \cdot \,)\|_{L^2}^2 \right)$$

$$+ (\mathrm{C}(e) + \alpha + 3\kappa) E \left(\int_0^T g^\kappa(s) \mathrm{d}s \right).$$

Using item a) and the fact that

$$E \left(\int_{[0,T] \times \mathbb{R}} X^2(s, \xi) \mathrm{d}s \mathrm{d}\xi \right) < \infty,$$

the result follows. Lemma 7.3 is finally completely established.

$\square$

We need now another intermediate lemma concerning the paths of a solution to (1.1).

Lemma 7.5. *For almost all $\omega \in \Omega$, almost all $t \in [0, T]$,*

1) $\xi \mapsto \psi(X(t, \xi, \omega)) \in H^1(\mathbb{R})$,

2) $\xi \mapsto \Phi(X(t, \xi, \omega))$ is continuous.

Proof. Item 1) is established in [6], see Definition 3.2 and Theorem 3.4. 1) implies that $\xi \mapsto \psi(X(t, \xi, \omega))$ is continuous. By the same arguments as in Proposition 4.22 in [7], we can deduce item 2). $\qquad\square$

3) We go on with the proof of Theorem 7.2. We keep in mind i), ii), iii), iv) near (7.2). Since Φ is bounded, using Burkholder-Davies-Gundy inequality are obtains

$$E^{Q^\kappa(\,\cdot\,,\omega)}(Y_t - Y_s)^4 \leqslant \mathrm{const}(t-s)^2.$$

On the other hand, for all $Q^\kappa(\,\cdot\,,\omega)$, Y_0 is distributed according to x_0. By Kolmogorov-Centsov criterion, see for instance an easy adaptation of [21], Problem 4.11 of Section 2.4, for $\omega \in \Omega$ a.s., the probabilities $Q^\kappa(\,\cdot\,,\omega)$, $\kappa > 0$ are tight. Consequently there is a sequence $Q^{\kappa_n}(\,\cdot\,,\omega)$, (κ_n) depending on ω, converging weakly to some probability $Q(\,\cdot\,,\omega)$ on $C([0,T]; \mathbb{R})$. By Skorohod's theorem there is a new probability space $\left(\Omega_1^\omega, \mathcal{H}^\omega, \tilde{Q}^\omega\right)$ and processes $Y^n(\,\cdot\,,\omega)$ distributed as $Y(\,\cdot\,,\omega)$ under $Q^{\kappa_n}(\,\cdot\,,\omega)$, converging to some process $Y^\infty(\,\cdot\,,\omega)$ ucp under $\tilde{Q}^\omega$. From now on we will denote again $Q^\omega := \tilde{Q}^\omega$. In particular $Y^n(\,\cdot\,,\omega)$ are local martingales with respect to their own filtrations such that $[Y^n(\,\cdot\,,\omega)]_t = \int_0^t \Phi_{\kappa_n}^2(X^{\kappa_n}(r, Y_r^n, \omega))\,\mathrm{d}r$. We remark that $Y^n(\,\cdot\,,\omega) - Y_0^n(\cdot,\omega)$ are even square integrable martingales.

4) Let X be the solution to the SPDE (1.1). The next step consists in showing that for P almost ω, $Y^\infty(\,\cdot\,,\omega)$ is a weak solution to the equation

$$Y_t^\infty(\,\cdot\,,\omega) = Y_0^\infty(\,\cdot\,,\omega) + \int_0^t \Phi(X(s, Y_s^\infty, \omega))\,\mathrm{d}\beta_s^\omega, \qquad (7.9)$$

for some Brownian motion β^ω. We need here a technical lemma.

Lemma 7.6. *For ω $\mathrm{d}P$-a.s., the random variables*

$$\int_0^T \left(\Phi_{\kappa_n}(X^{\kappa_n}(r, Y_r^n, \omega)) - \Phi(X(r, Y_r^\infty, \omega))\right)^2 \mathrm{d}r \qquad (7.10)$$

converge to zero in $L^p(\Omega_1^\omega, Q^\omega)$, $\forall\, p \geqslant 1$, and consequently in probability.

Proof. It is of course enough to show that the E^{Q^ω} expectation of (7.10) goes to zero up to a subsequence. This is bounded by $I_1(n) + I_2(n)$ where

$$I_1(n) = 2E^{Q^\omega} \left(\int_0^T dr \left(\Phi_{\kappa_n} \left(X^{\kappa_n}(r, Y_r^n, \omega) \right) - \Phi \left(X(r, Y_r^n, \omega) \right)^2 \right) \right)$$

$$I_2(n) = 2E^{Q^\omega} \left(\int_0^T dr \left(\Phi \left(X(r, Y^n, \omega) \right) - \Phi \left(X(r, Y^\infty, \omega) \right)^2 \right) \right).$$

Since $\Phi \left(X(r, \cdot) \right)$ is continuous for almost all $(r, \omega_1) \in [0, T] \times \Omega_1^\omega$, by Lemma 7.5, then $I_2(n)_{n \to \infty} 0$ by an easy application of Lebesgue's dominated convergence theorem. Concerning $I_1(n)$, it is also enough to show the existence of a subsequence (κ_{n_ℓ}) such that

$$\left(\Phi_{\kappa_n} \left(X^{\kappa_n}(r, Y_r^n, \omega) \right) - \Phi \left(X(r, Y_r^n, \omega) \right) \right)^2 \xrightarrow[n \to \infty]{} 0,$$

$dQ^\omega \times dr$ a.e.. Since the Doléans exponential is strictly positive, this will be guaranteed if we show that

$$\left(\Phi_{\kappa_n} \left(X^{\kappa_n}(r, Y_r^n, \omega) \right) - \Phi \left(X(r, Y_r^n, \omega) \right) \right)^2 \mathcal{E}_r \left(\int_0^\cdot \mu(ds, Y_s^n) \right) \xrightarrow[n \to \infty]{} 0,$$

$dQ^\omega \times dr$ a.e. Clearly, this will be verified, if we show that, for any $\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ continuous, with compact support, we have

$$E^{Q^\omega} \left(\int_0^T \varphi(r, Y_r^n) \mathcal{E}_r \left(\int_0^\cdot \mu(ds, Y_s^n) \right) \left(\Phi_{\kappa_n} \left(X^{\kappa_n}(r, Y_r^n, \omega) \right) - \Phi \left(X(r, Y_r^n, \omega) \right) \right)^2 dr \right)$$

goes to zero, eventually up to a subsequence.

Since X^{κ_n} constitute the μ-marginal laws of Y^n, previous expression gives

$$\int_0^T dr \int_{\mathbb{R}} \varphi(r, y) \left(\Phi_{\kappa_n} \left(X^{\kappa_n}(r, y, \omega) \right) - \Phi \left(X(r, y, \omega) \right) \right)^2 X^{\kappa_n}(r, y, \omega) dy$$

$$\leqslant I_{11}(n) + I_{12}(n) + I_{13}(n) + I_{14}(n)$$

where

$$I_{11}(n) = \int\limits_0^T \mathrm{d}r \int\limits_{\mathbb{R}} \mathrm{d}y |\varphi(r,y)||\psi\left(X^{\kappa_n}(r,y,\omega)\right) - \psi\left(X(r,y,\omega)\right)|,$$

$$I_{12}(n) = \int\limits_0^T \mathrm{d}r \int\limits_{\mathbb{R}} \mathrm{d}y |\varphi(r,y)||\Phi^2\left(X^{\kappa_n}(r,y,\omega)\right)||(X - X^{\kappa_n})(r,y,\omega)|,$$

$$I_{13}(n) = \int\limits_0^T \mathrm{d}r \int\limits_{\mathbb{R}} \mathrm{d}y |\varphi(r,y)|\kappa_n |X^{\kappa_n} - X|(r,y,\omega),$$

$$I_{14}(n) = \int\limits_0^T \mathrm{d}r \int\limits_{\mathbb{R}} \mathrm{d}y \kappa_n |X(r,y,\omega)||\varphi(r,y)|.$$

By Cauchy-Schwarz, $I_{11}^2(n)$ is bounded by

$$\|\varphi\|_{L^2([0,T]\times\mathbb{R})}^2 \int\limits_0^T \mathrm{d}r \int\limits_{\mathbb{R}} \left(\psi\left(X^{\kappa_n}(r,y,\omega)\right) - \psi\left(X(r,y,\omega)\right)\right)^2 \mathrm{d}y.$$

This converges to zero according to Remark 7.4 2), after extracting a further subsequence (not depending on ω). The square of $I_{12}(n)$ is bounded by

$$\|\varphi\|_{L^2([0,T]\times\mathbb{R})}^2 \int\limits_{[0,T]\times\mathbb{R}} \mathrm{d}r\mathrm{d}y \Phi^4\left(X(r,y,\omega)\right)|X^{\kappa_n} - X|^2(r,y,\omega).$$

This goes to zero because of (7.4) in Remark 7.4 4).
$I_{13}^2(n)$ is bounded by

$$\kappa_n \|\varphi\|_{L^2([0,T]\times\mathbb{R})}^2 \kappa_n \int\limits_{[0,T]\times\mathbb{R}} \mathrm{d}r\mathrm{d}y |X^{\kappa_n} - X|^2(r,y,\omega).$$

After extracting a subsequence, previous expression converges to zero because of Lemma 7.3 c). Finally $I_{14}(n)_{n\to\infty} 0$ by Cauchy-Schwarz and the fact that $\int\limits_{[0,T]\times\mathbb{R}} \mathrm{d}r\mathrm{d}y X^2(r,y,\omega) < \infty$ $\mathrm{d}P$-a.s.
This establishes the proof of Lemma 7.6. $\qquad\square$

4) We go on with the proof of Theorem 7.2. We want to prove that Y^∞ is a weak-strong solution of

$$Y_t = Y_0 + \int\limits_0^t \Phi\left(X(s, Y_s, \, \cdot \,)\right) \mathrm{d}B_s.$$

According to Remark 2.11 d) item 1) of Definition 2.10, it is enough to show that for $\mathrm{d}P$-a.s.ω $Y := Y^\infty(\,\cdot\,, \omega)$ is a solution of the following (local) martingale problem. For every $f \in C^{1,2}([0, T] \times \mathbb{R})$ with compact support, the process

$$Z_t^f \quad := \quad f(t, Y_t) - f(0, Y_0) - \frac{1}{2} \int\limits_0^t f''(Y_s)\Phi^2\left(X(s, Y, \omega)\right) \mathrm{d}s$$

$$- \quad \in_0 t\partial_s f(s, Y_s)$$

is a (local) martingale under Q^ω.

For this it is enough to prove that under Q^ω, Y is a local martingale with quadratic variation $[Y^\infty(\,\cdot\,, \omega)]_t = \int\limits_0^t \Phi^2\left(X(s, Y_s^\infty(\,\cdot\,, \omega))\right) \mathrm{d}s.$ According to Proposition A.1 of the Appendix, it is enough to show that

$$\int\limits_0^T \mathrm{d}t |\Phi_{\kappa_n}^2\left(X(s, Y_s^n(\,\cdot\,, \omega)) - \Phi^2\left(X(s, Y_s^\infty(\,\cdot\,, \omega))\right)| \mathrm{d}s \qquad (7.13)$$

$$\xrightarrow[n\to\infty]{} 0,$$

in Q^ω probability. Now, the expectation of (7.13) is bounded by

$$2\left\|\Phi\right\|_\infty E^{Q^\omega}\left(\int\limits_0^T \mathrm{d}s |\Phi_{\kappa_n}\left(X(s, Y_s^n(\,\cdot\,, \omega)) - \Phi\left(X(s, Y_s^\infty(\,\cdot\,, \omega))\right)|\right)$$

$$\leqslant 2\left\|\Phi\right\|_\infty \left\{E^{Q^\omega}\left(\int\limits_0^T \mathrm{d}s |\Phi_{\kappa_n}\left(X(s, Y_s^n(\,\cdot\,, \omega)) - \Phi\left(X(s, Y_s^\infty(\,\cdot\,, \omega))\right)|^2\right)\right\}^{\frac{1}{2}}$$

This converges to zero by Lemma 7.6.

This proves (7.13) and that $Y^\infty(\,\cdot\,, \omega)$ is a weak solution of (7.10).

5) The solution $Y^\infty(\,\cdot\,,\omega)$ of (7.9) lives on a space $\Omega_1^\omega \times \Omega$ where, $(\Omega_1^\omega, \mathcal{H}^\omega, Q^\omega)$ is a probability space depending on ω.

We choose now $\Omega_1 = C([0,T]; \mathbb{R}) \times \mathbb{R} \times C([0,T]; \mathbb{R})$ and we select $Q^\omega := Q(\,\cdot\,,\omega)$ being the law of $Y^\infty(\,\cdot\,,\omega)$ on Ω_1. Again we set $Y_t(\omega_1, \omega) = \omega_1^0(t) + a$, this time with $\omega_1 = (\omega_1^0, a, \omega_1^1)$.

We have to show that Y is a weak-strong solution of

$$Y_t = Y_0 + \int\limits_0^t \Phi\left(X(s, Y_s, \,\cdot\,)\right) \mathrm{d}B_s.$$

For the moment we have shown that $Y_t(\cdot,\omega) - Y_0(\cdot,\omega)$ is a martingale under Q^ω for almost all ω with Q^ω-quadratic variation given by $\int_0^t \Phi^2(X(s, Y(\cdot,\omega)_s)) \mathrm{d}s$. We need to construct a process B on $\Omega \times \Omega_1$, such that for almost all ω, B is a Q^ω-Brownian motion and (2.6) holds for $\gamma(t,\cdot,\omega) = \Phi(X(t,\cdot,\omega))$. Let $\beta_t(\omega_1) = \omega_1^1(t)$, a supplementary Brownian motion on Ω_1 which is Q^ω-independent on Y and we remind that $Y_t(\omega_1) = \omega_1^0(t) + a$. β can be also considered as a Brownian motioon on $\Omega \times \Omega_1$ which is Q-independent of Y and $(\mathcal{F}_t)$.

We define

$$B_t(\cdot,\omega) = \int_0^t \mathrm{d}Y_s(\cdot,\omega) 1_{\{\gamma(s,\xi,\omega)\neq 0\}} \frac{1}{\gamma(s,\xi,\omega)} + \int_0^t 1_{\{\gamma(s,\xi,\omega)=0\}} \mathrm{d}\beta_s.$$

Now for Q^ω-a.s. the quadratic variation of the Q^ω-martingale $B(\cdot,\omega)$ is t, so that, by Lévy characterization theorem, $B(\cdot,\omega)$ is a Brownian motion under Q^ω.

It remains to show items 2) and 3) of the definition of weak-strong solution. Let $(\mathcal{Y}_t)$ be the canonical filtration of the process $Y(\cdot,\omega)$. Item 3) follows because of item 1) and because $\gamma(t,\cdot,\omega) = \Phi(X(t,\cdot,\omega))$ is progressively measurable. Concerning item 2) we see that under Q defined by P and the kernel $Q(\cdot,\omega)$, $W^1,\ldots,W^N$ are Q-martingales with $(\mathcal{G}_t)$ as defined in Definition 2.10. Indeed let F be a bounded $\mathcal{F}_s$-measurable random variable and A be a bounded $\mathcal{Y}_s$-measurable r.v. Let $1 \leq i \leq N$. By item 3) $E^{Q^\omega}(A)$ is $\mathcal{F}_s$-mesurable, so

$$E^Q((W_t^i - W_s^i)FA) = E^P((W_t^i - W_s^i)FE^{Q^\omega}(A)) = 0,$$

since W^i is an $\mathcal{F}_s$-martingale.

6) The final step consists in proving that X is the family of μ-marginal laws of Y, under Q defined by $\int_\Omega Q(\cdot, \omega)dP(\omega)$. Let $\omega \in \Omega$ outside some P-null set.

By step 1) of the proof of this Theorem 7.2, we know that X^κ fulfills, for almost all ω,

$$\int_{\mathbb{R}} d\xi X^\kappa(t,\xi)\varphi(\xi) = E^{Q^\kappa(\cdot,\omega)}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^{\cdot} \mu(ds, Y_s)\right)\right),$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. We recall that, according to the lines before step 3) of this proof, after use of Skorohod theorem and a change of probability space (depending on ω), there is a sequence (κ_n) and processes $Y^n(\cdot, \omega)$ converging ucp to $Y^\infty(\cdot, \omega)$ such that

$$\int_{\mathbb{R}} d\xi X^{\kappa_n}(t,\xi)\varphi(\xi) = E^{Q^\omega}\left(\varphi(Y^n)\mathcal{E}_t\left(\int_0^{\cdot} \mu(ds, Y^n)\right)\right), \qquad (7.14)$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$. It remains to show, for every $t \in [0,T]$, $\varphi \in \mathcal{S}(\mathbb{R})$, that

$$\int_{\mathbb{R}} d\xi \varphi(\xi) X(t,\xi,\omega) = E^{Q^\omega}\left(\varphi(Y_t^\infty)\mathcal{E}_t\left(\int_0^{\cdot} \mu(ds, Y_s^\infty)\right)\right). \qquad (7.15)$$

Let $\omega \in \Omega$ outside a P-null set.

Since $t \mapsto X(t, \cdot)$ is continuous from $[0,T]$ to $\mathcal{S}'(\mathbb{R})$ and the right-hand side is continuous on $[0,T]$ for fixed $\varphi \in \mathcal{S}(\mathbb{R})$, it is enough to show (7.15) for almost all $t \in [0,T]$.

Now for almost all t, the left-hand side of (7.15) is approached by the left-hand side of (7.14). It remains to show that the right-hand side of (7.15) is the limit of the right-hand side of (7.14).

According to Proposition A.3 in the Appendix it will be enough to show the following.

i)

$$\sum_{i=1}^{N} \int_0^t e^i(Y_s^n)dW_s^i - \frac{1}{2}\int_0^t e^i(Y_s^n)^2 ds \qquad (7.16)$$

converges in law (with respect to Q^ω) to

$$\sum_{i=1}^{N} \left(\int_0^t e^i(Y_s) \mathrm{d}W_s^i - \frac{1}{2} \int_0^t e^i(Y_s)^2 \mathrm{d}s \right). \qquad (7.17)$$

ii) $\varphi(Y_t^n)\mathcal{E}\left(\int_0^\cdot \mu(\mathrm{d}s, Y_s^n) \right)$ is a sequence which is uniformly integrable.

We check now those properties.

i) (7.16) equals

$$J_1(n) + J_2(n)$$

where

$$J_1(n) = \sum_{i=1}^{N} \left\{ W_s^i e^i(Y_s^n) - \frac{1}{2} \int_0^t e^i(Y_s^n)^2 \mathrm{d}s \right.$$

$$\left. - \frac{1}{2} \int_0^t W_s^i(e^i)''(Y_s^n) \Phi_{\kappa_n}^2 \left(X^{\kappa_n}(s, Y_s^n, \omega) \right) \mathrm{d}s \right\}$$

$$J_2(n) = -\sum_{i=1}^{N} \int_0^t W_s^i(e^i)'(Y_s^n) \mathrm{d}Y_s^n.$$

Lemma 7.6 implies that

$$\int_0^T |\Phi_{\kappa_n}^2 \left(X^{\kappa_n}(s, Y_s^n, \omega) \right) - \Phi^2 \left(X(s, Y_s, \omega) \right)| \mathrm{d}s \underset{n \to \infty}{\longrightarrow} 0$$

in probability. Consequently, since $Y^n \to Y$ ucp, it follows that $J_1(n)$ converges in probability to

$$J_1 := \sum_{i=1}^{N} \left(W_t^i e^i(Y_t) - \frac{1}{2} \int_0^t e^i(Y_s)^2 \mathrm{d}s \right.$$

$$\left. - \frac{1}{2} \int_0^t W_s^i(e^i)''(Y_s) \Phi^2 \left(X(s, Y_s, \omega) \right) \mathrm{d}s. \right.$$

According to [20], $J_2(n) \to J_2$ in law, where $J_2 = \int_0^t W_s^i(e^i)'(Y_s)\mathrm{d}Y_s$. This implies that (7.16) converges in law to (7.17) and item i) is established.

To prove ii) we only need to prove that

$$\sup_n E^{Q^\omega}\left(\mathcal{E}\left(\int_0^t \mu(\mathrm{d}s, Y_s^n)\right)^2\right) < \infty.$$

The integrand of previous expectation gives

$$\exp\left(2(J_1(n) + J_2(n))\right).$$

For each ω, $\exp\left(2(J_1(n))\right)$ is bounded, so it remains to prove that, for every $0 \leqslant i \leqslant N$

$$\sup_n E^{Q^\omega}\left(\exp\left(-2\int_0^t W_s^i(e^i)'(Y_s^n)\mathrm{d}Y_s^n\right)\right) < \infty. \qquad (7.18)$$

Since $-2\int_s^t W_s^i(e^i)'(Y_s^n)\mathrm{d}Y_s^n$ is a Q^ω-martingale,

$$\mathcal{E}_t^n := \exp\left(-2\int_0^t W_s^i(e^i)'(Y_s^n)\mathrm{d}Y_s^n - 2\int_0^t (W^i)_s^2(e^i)'^2(Y_s^n)\Phi_{\kappa_n}^2\left(X^{\kappa_n}(s, Y_s^n, \omega)\right)\mathrm{d}s\right)$$

is an (exponential) martingale. Consequently (7.18) is bounded by

$$\sup_n E^{Q^\omega}\left(\mathcal{E}_t^n \exp\left(2\int_0^t (W^i)_s^2(e^{i'})^2(Y_s^n)\Phi_{\kappa_n}^2(X^{\kappa_n}(s, Y_s^n, \omega))\mathrm{d}s\right)\right)$$

$$\leqslant \exp\left(2\left\|e^{i'}\right\|_\infty^2\left(\|\Phi\|_\infty^2 + \kappa_n\right)\int_0^T (W_s^i)^2\mathrm{d}s\right).$$

This quantity is bounded for each ω, ii) is now established and so is the step 6) of Theorem 7.2. $\qquad \square$

A   Technicalities

Proposition A.1. *Let $(\Omega, \mathcal{H}, Q_1)$ be a probability space. Let (Y^n) be a sequence of continuous local martingales such that $Y^n \to Y$ ucp; we suppose*

the existence of a adapted continuous process A, adapted to the canonical filtration associated with Y such that the total variation of $[Y^n]_t - A_t$ goes to zero in probability for every $t \in [0, T]$.

Then Y is a local martingale whose quadratic variation is A.

Remark A.2.   *i)* *A local martingale is in particular a local martingale with respect to its own filtration.*

 ii) *This proposition should be known in the literature but for the moment we cannot find the reference. The difficulty is that there is no filtration specified.*

Proof of Proposition A.1. After a localization procedure, we can suppose that the process Y is bounded and Y^n are bounded by the same constant. By classical approximation techniques we only need to show that

$$f(Y_t) - \frac{1}{2} \int_0^t f''(Y_s) \mathrm{d}s \qquad (A.1)$$

is a martingale, for every $f \in C^2$ with compact support. Let $0 \leqslant s < t \leqslant T$ and $\Theta : C([0, s]) \to \mathbb{R}$ be a bounded and continuous functional. We need to show that

$$E\left((f(Y_t) - f(Y_s) - \frac{1}{2} \int_s^t f''(Y_r) \mathrm{d}A_r) \Theta_s \right) = 0, \qquad (A.2)$$

where $\Theta_s = \Theta(Y_r : r \leqslant s)$. The left-hand side of (A.2) equals

$$I_1(n) + I_2(n),$$

where

$$I_1(n) = E\left((f(Y_t) - f(Y_s)) \Theta_s - (f(Y_t^n) - f(Y_s^n)) \Theta_s^n \right)$$

and

$$\Theta_s^n = \Theta(Y_r^n : r \leqslant s), \ \Theta_s = \Theta(Y_r : r \leqslant s),$$

$$I_2(n) = E\left(\left(f(Y_t^n) - f(Y_s^n) - \frac{1}{2}\int_s^t f''(Y_r^n)\mathrm{d}[Y^n]_r\right)\Theta_s^n\right),$$

$$I_3(n) = E\left(\frac{1}{2}\int_s^t f''(Y_r^n)\mathrm{d}[Y^n]_r\left(\Theta_s^n - \Theta_s\right)\right)$$

$$I_4(n) = E\left(\left\{\frac{1}{2}\int_s^t (f''(Y_r^n) - f''(Y_r))\mathrm{d}[Y^n]_r\right\}\Theta_s\right),$$

$$I_5(n) = E\left(\left\{\frac{1}{2}\int_s^t (f''(Y_r))\mathrm{d}([Y^n] - A_r)\right\}\Theta_s\right).$$

Observe that $I_2(n) = 0$ since Y^n is a martingale.

Taking into account the ucp convergence of Y^n to Y, it is not difficult to show that

$$I_i(n) \to 0, \ i = 1, 3.$$

By BDG inequality there is a constant $C > 0$ such that

$$E([Y^n]_T) \le CE(\sup_{t \le [0,T]} |Y_t^n|). \tag{A.3}$$

As far as $I_5(n)$ is concerned, we have $[Y^n] \to A$ ucp, taking into account the fact that A is a continuous process, the convergence in probability and a Dini type argument, see e.g. Lemma 3.1 of [28]. So after extraction of subsequences we get

$$\int_s^t Z_r(\mathrm{d}[Y^n] - \mathrm{d}A)_r \to 0, \tag{A.4}$$

in probability, for the continuous process $Z_r = f''(Y_r)$. On the other hand

$$E(\int_s^t Z_r\mathrm{d}[Y^n]_r)^2 \le C\|f''\|_\infty E([Y^n]T),$$

is uniformly bounded by (A.3) and so the family of r.v. in (A.4) is uniformly integrable. Finally (A.4) also holds in L^1.

Concerning $I_4(n)$, since $f''(Y^n)$ converges to $f''(Y)$ a.s. uniformly, by (A.3), the integral inside the expectation converges to zero a.s. Then, by Lebesgue dominated convergence theorem its expectation, i.e. $I_4(n)$ converges to zero.

Finally the result follows. $\qquad\square$

Proposition A.3. *Let (Z_n) be a sequence of random elements with values in a Banach space E converging in law to some random element Z still with values in E. Let $\psi : E \to \mathbb{R}$ continuous, such that $(\psi(Z_n))$ are uniformly integrable. Then $\lim_{n\to\infty} E\left(\psi(Z_n)\right) = E\left(\psi(Z)\right).$*

Proof of Proposition A.3. According to Skorohod theorem, there is a new probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{P})$, random elements $\widetilde{Z}_n$(resp. $\widetilde{Z}$) on $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{P})$ with the same distribution as Z_n(resp. Z) and $\widetilde{Z}_n \to \widetilde{Z}$ a.s. So $\psi(\widetilde{Z}_n) \to \psi(\widetilde{Z})$ a.s. Clearly the sequence $\psi(\widetilde{Z}_n)$ is also uniformly integrable; finally $\psi(\widetilde{Z}_n) \to \psi(\widetilde{Z})$ in $L^1(\widetilde{P})$. In particular $E^{\widetilde{P}}(\psi(\widetilde{Z}_n))_{n \to \infty} E^{\widetilde{P}}(\psi(\widetilde{Z}))$, and so the result follows. $\square$

Proposition A.4. *Let Y_0 be distributed according to x_0. Let $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Borel function such there are $0 < c < C$ with*

$$c \leqslant a(s,\xi) \leqslant C, \quad \forall\, (s,\xi) \in [0,T] \times \mathbb{R}. \tag{A.5}$$

We fix $0 \leq r \leq t \leq T$. We set $a_n(t,x) = \int_{\mathbb{R}} \rho_n(x-y)a(t,y)\mathrm{d}y$ where (ρ_n) is the usual sequence of mollifiers converging to the Dirac delta. The unique solutions S^n to

$$S_t^n = Y_0 + \int_r^t a_n(s, S_s^n)\mathrm{d}B_s,$$

B being a classical Wiener process, converges in law to the (weak unique solution) of

$$S_t = Y_0 + \int_r^t a(s, S_s)\mathrm{d}B_s,$$

Proof.   i) Let $r \in [0,T[$, $y \in \mathbb{R}$. According to Problem 7.3.3 of [31], the equation

$$S_t = y + \int_r^t a(s, S_s)\mathrm{d}B_s, \tag{A.6}$$

admits a solution, which is unique in law. We denote by $P^{r,y}$ the law on $C([r,T])$ of the corresponding canonical process. The equation

$$S_t = y + \int_r^t a^n(s, S_s)\mathrm{d}B_s$$

admits even a strong solution since a^n is Lipschitz with linear growth. We denote with $P_n^{r,y}$ the corresponding law. Moreover, those processes are Markovian.

ii) By the same mentioned problem 7.3.3 of [31], there is a constant C_1 only depending on c, C in (A.5) and T such that

$$E^{P_n^{r,y}} \left(\int_{[r,T] \times \mathbb{R}} f(r, S_r) \mathrm{d}r \right) \leqslant C_1 \|f\|_{L^2([0,T] \times \mathbb{R})}, \qquad (A.7)$$

for every $n \in \mathbb{N} \cup \{\infty\}, (r, y) \in [0, T[\times \mathbb{R}$, and every bounded function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ with compact support. For convenience in (A.7) we set $P^{r,y} := P_\infty^{r,y}$.

iii) From ii), there are Borel functions $q_n : [0, T]^2 \times \mathbb{R} \to \mathbb{R}, n \in \mathbb{N} \cup \{\infty\}$ such that $(t, z) \to q_n(r, t, y, z) \in L^2([r, T] \times \mathbb{R})$ for every $(r, y) \in [0, T] \times \mathbb{R}$ and the law of S_t under $P_n^{r,y}$ equals $q_n(r, t; y, \cdot) \, dt$ a.e. Moreover

$$\int_{[r,T]} q_n^2(r, t; y.z) \mathrm{d}t \mathrm{d}z \leqslant C_1^2.$$

iv) For $0 \leqslant r < t \leqslant T, \ y \in \mathbb{R}$ the laws $(P_n^{r,y})$ are tight. In fact, by Burkholder-Davies-Gurdy inequality, there are constants $C_2, C_3 > 0$ such that

$$E^{P_n^{r,y}} \left((S_t - S_r)^4 \right) \leqslant$$

$$C_2 E^{P_n^{r,y}} \left(\left(\int_r^t a_n^2(s, S_s) \mathrm{d}s \right)^2 \right) \leqslant C_3(t - r)^2.$$

A slight adaptation of Problem 4.11 associated with Theorem 4.10 Chapter 2 of [21] implies the tightness.

v) In particular, for each subsequence there is a subsubsequence converging weakly.

vi) In fact for every $0 \leqslant r < t \leqslant T, \ y \in \mathbb{R}$, the sequence $(P_n^{r,y})$ converges weakly to $P^{r,y}$.

To prove this, by point v) and the uniqueness in law of (A.6), it is enough to show that the limit of a weakly converging subsequence of $(P_n^{r,y})$ (still denoted in the same manner) fulfills the martingale problem related to (A.6). Let $Q^{r,y}$ be such a limit. Using the Markov property we only need to show that for every $f \in C_0^\infty([0, T] \times \mathbb{R})$

$$E^{Q^{r,y}} \left(f(t, S_t) - f(0, y) - \int_r^t \partial_{xx} f(s, S_s) \frac{a^2}{2}(s, S_s) \mathrm{d}s \right) = 0$$

For this, taking into account the fact that $P_n^{r,y} \to Q^{r,y}$ and the fact that

$$E^{P_n^{r,y}}\left(f(t, S_t) - f(0, y) - \int_r^t \partial_{xx} f(s, S_s) \frac{a_n^2}{2}(s, S_s) \mathrm{d}s \right),$$

it will be enough to show that

$$E^{P_n^{r,y}}\left(\int_r^t \partial_{xx} f(s, S_s) \frac{a_n^2}{2}(s, S_s) \mathrm{d}s \right), \qquad (A.8)$$

$$\underset{n\to\infty}{\to} E^{Q^{r,y}}\left(\int_r^t \partial_{xx} f(s, S_s) \frac{a^2}{2}(s, S_s) \mathrm{d}s \right).$$

The left hand side of (A.8) equals $I_1(n) + I_2(n)$ where

$$I_1(n) = E^{P_n^{r,y}}\left(\int_r^t \partial_{xx} f(s, S_s) \frac{a_n^2 - a^2}{2}(s, S_s) \mathrm{d}s \right)$$

$$I_2(n) = E^{P_n^{r,y}}\left(\int_r^t \partial_{xx} f(s, S_s) \frac{a^2}{2}(s, S_s) \mathrm{d}s \right)$$

$$- E^{Q^{r,y}}\left(\int_r^t \partial_{xx} f(s, S_s) \frac{a^2}{2}(s, S_s) \mathrm{d}s \right).$$

By item iii) $I_1(n)$ gives

$$\int_{[r,T]\times\mathbb{R}} \mathrm{d}t\mathrm{d}z \partial_{xx} f(t, z) \frac{a_n^2 - a^2}{2}(t, z) q(r, t; y, z).$$

Since $\partial_{xx} f$ has compact support, $|a_n^2 - a^2| \leqslant 2C_1$, together with $a_n^2 \to a^2 \mathrm{d}t\mathrm{d}z$ a.e., Cauchy-Schwarz and Lebesgue dominated convergence theorem imply that (A.1) goes to zero when $n \to \infty$. Concerning $I_2(n)$, we only need to prove that for every $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ bounded measurable with compact support verifies

$$\lim_{n\to\infty} E^{P_n^{r,y}}\left(\int_r^T f(s, S_s) \mathrm{d}s \right) = E^{Q^{r,y}}\left(\int_r^T f(s, S_s) \mathrm{d}s \right). \qquad (A.9)$$

Indeed we can prove (A.9) holds for $f \in L^2([0, T] \times \mathbb{R})$. In fact, by the convergence in law, (A.9) holds for every $f \in C^0(r, T) \times \mathbb{R})$ with compact support. (A.7) and Banach-Steinhaus allow to establish (A.9) and therefore the conclusion.

$$\square$$

B   Uniqueness for the porous media equation with noise

We state here a general uniqueness lemma when the coefficient $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Since the paper concerns one-space dimension porous media type equation, we remain in that framework. However, the theorem below easily extends to the multi-dimensional case.

We consider here an infinite number of modes for the random field μ, i.e. $\mu(t,\xi) = \sum\limits_{i=0}^{\infty} e^i(\xi) W_t^i$ where W^i are independent Brownian motions, $e^i : \mathbb{R}^d \to \mathbb{R} \in L^1(\mathbb{R}^d)$ being H^{-1} multipliers with norm $\mathcal{C}(e^i)$, $W_t^0 = t$.

Theorem B.1. *Let $x_0 \in \mathcal{S}'(\mathbb{R}^d)$ and make the following assumptions.*

  i) ψ is Lipschitz,

  ii) $\sum_{i=1}^{\infty} \left(\mathcal{C}(e_i)^2 + \|e^i\|_\infty^2 \right) < \infty$.

Then equation (1.1) *admits at most one solution among the random fields $X :]0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that*

$$\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi < \infty \quad a.s. \tag{B.1}$$

Remark B.2. *We observe that condition ii) is compatible with (3.1) of [6].*

Remark B.3. *Let X be a solution of* (1.1).

  i) There is a P null set N_0, so that for $\omega \notin N_0$, $X(t, \cdot) \in L^2(\mathbb{R})$ for almost all $t \in [0,T]$.

  ii) Condition (B.1) *also implies that*

$$\int_0^T \|X(s, \cdot)\|_{H^{-1}}^2 \, \mathrm{d}s < \infty \quad a.s.$$

  iii) Since ψ is Lipschitz and $\psi(0) = 0$, (B.1) *implies that $\int_0^T \|\psi(X(r,\cdot))\|_{L^2}^2 \, dr < \infty$. So $\int_0^t \mathrm{d}s\psi(X(s,\cdot))$ is a Bochner integral with values in $L^2(\mathbb{R})$.*

iv) Consequently, $t \mapsto \Delta \left(\int_0^t \mathrm{d}s \psi(X(s, \, \cdot \,)) \right)$ is continuous from $[0,T]$ to $H^{-2}(\mathbb{R})$ and so also in $\mathcal{S}'(\mathbb{R})$; since $e^i, 1 \leq i \leq N$, are H^{-1}-multipliers, then $t \mapsto \int_0^t \mu(\mathrm{d}s, \cdot) X(s, \cdot)$ belongs to $C \left([0,T]; H^{-1}(\mathbb{R}) \right)$; since $x_0 \in \mathcal{S}'(\mathbb{R})$ and $X \in C([0,T]; \mathcal{S}'(\mathbb{R}))$ a.s., it follows that for ω not belonging to a null set, we have

$$X(t, \, \cdot \,) = x_0 + \Delta \left(\int_0^t \psi(X(s, \, \cdot \,)) \mathrm{d}s \right) + \int_0^t \mu(\mathrm{d}s, \cdot) X(s, \cdot) \quad t \in [0,T],$$

as an identity in $\mathcal{S}'(\mathbb{R})$.

v) If $x_0 \in H^{-1}$ then $X \in C \left([0,T]; H^{-2} \right)$, for $\omega \notin N_0$, N_0 a P-null set. Moreover, if $x_0 \in L^2$ or ψ is non-degenerate, then, by Theorem 3.4 of [6], then $t \mapsto \int_0^t \psi(X(s, \cdot)) ds \in C([0,T]; H^1(\mathbb{R}))$.

vi) If $x_0 \in H^{-s}$ for some $s \geq 2$ then $X \in C \left([0,T]; H^{-s} \right)$, for $\omega \notin N_0$, N_0 a P-null set.

vii) Let $\varepsilon > 0$ and consider a sequence of mollifiers (ϕ_ε) converging to the Dirac measure. Then $X^\varepsilon(t, \, \cdot \,) = X(t, \cdot) \star \phi_\varepsilon$ belongs a.s. to $C \left([0,T]; L^2(\mathbb{R}) \right)$.

viii) All the previous items hold provided there exists a solution X verifying (B.1).

ix) Since ψ is Lipschitz, there is $\alpha > 0$ such that

$$(\psi(r) - \psi(\bar{r})) (r - \bar{r}) \geqslant \alpha (\psi(r) - \psi(\bar{r}))^2.$$

Proof. Let $(\phi_\varepsilon, \varepsilon > 0)$ be a sequence of mollifiers as in Remark B.3 vii). Let X^1, X^2 be two solutions of (1.1). For $i = 1, 2$, we set $(X^i)^\varepsilon(t, \cdot) = X^i(t, \cdot) \star \phi_\varepsilon$. We denote $X = X_1 - X_2$ and $X^\varepsilon = (X^1)^\varepsilon - (X^2)^\varepsilon$ which a.s. belongs to $C \left([0,T], L^2(\mathbb{R}) \right) \subset C \left([0,T]; H^{-1} \right)$. We expand

$$g_\varepsilon(t) := \|X^\varepsilon(t, \, \cdot \,)\|_{H^{-1}}^2$$

$$= \int_{\mathbb{R}} \left((I - \Delta)^{-1} X^\varepsilon(t, \, \cdot \,) \right)(\xi) X^\varepsilon(t, \xi) \mathrm{d}\xi.$$

$\square$

Itô formula gives

$$g_\varepsilon(t) = 2 \int_s^t < X^\varepsilon(s, \, \cdot \,), X^\varepsilon(\mathrm{d}s, \, \cdot \,) >_{H^{-1}} \qquad (\mathrm{B.2})$$

$$+ \sum_{i=1}^\infty \int_s^t \mathrm{d}s \left\| (e^i X)(s, \cdot) \star \phi_\varepsilon \right\|_{H^{-1}}^2 .$$

On the other hand we have

$$X^\varepsilon(t, \, \cdot \,) = \int_0^t \mathrm{d}s \Delta \left[\left\{ \psi(X^1(s, \, \cdot \,)) - \psi(X^2(s, \, \cdot \,)) \right\} \star \phi_\varepsilon \right] \qquad (\mathrm{B.3})$$

$$+ \int_0^t (\mu(\mathrm{d}s, \, \cdot \,)X) \star \phi_\varepsilon.$$

$$(I - \Delta)^{-1} X^\varepsilon(t, \, \cdot \,) = - \int_0^t \left(\psi(X^1(s, \, \cdot \,)) - \psi(X^2(s, \, \cdot \,)) \right) \star \phi_\varepsilon \qquad (\mathrm{B.4})$$

$$+ \int_0^t \mathrm{d}s (I - \Delta)^{-1} \left(\psi(X^1(s, \, \cdot \,)) - \psi(X^2(s, \, \cdot \,)) \right) \star \phi_\varepsilon$$

$$+ \sum_{i=0}^\infty \int_0^t \mathrm{d}W_s^i \left[(I - \Delta)^{-1}(e^i X(s, \, \cdot \,)) \right] \star \phi_\varepsilon.$$

We define

$$M_t = \sum_{i=1}^\infty \int_0^t < (I - \Delta)^{-1} X(s, \, \cdot \,), e^i X(s, \, \cdot \,) >_{L^2} \mathrm{d}W_s^i.$$

We observe that previous M is well-defined and it is a local martingale. Indeed, by Remark B.3 iv), $X \in C([0, T]; H^{-2})$, so by similar arguments as in (5.7),

$$\sum_{i=1}^\infty \int_0^t < (I - \Delta)^{-1} X(s, \, \cdot \,), e^i X(s, \, \cdot \,) >_{L^2}^2 \mathrm{d}s \; \leq \; \sup_{s \in [0,T]} \| X(s, \cdot) \|_{H^{-2}}^2 \sum_{i=1}^\infty \| e^i \|_\infty^2$$

$$\int_0^T \| X(s, \, \cdot \,) \|_{L^2}^2 \mathrm{d}s < \infty.$$

Using (B.2), (B.3) and (B.4) we get

$$g_\varepsilon(t) = \sum_{i=1}^{\infty} \int_0^t \mathrm{d}s \, \left\| (e^i X)(s, \cdot) \star \phi_\varepsilon \right\|_{H^{-1}}^2 \tag{B.5}$$

$$- 2 \int_0^t < X^\varepsilon(s, \cdot), \left[\psi(X^1(s, \cdot)) - \psi(X^2(s, \cdot)) \right] \star \phi_\varepsilon >_{L^2}$$

$$+ 2 \int_0^t < X^\varepsilon(s, \cdot), (I - \Delta)^{-1} \left[\psi(X^1(s, \cdot)) - \psi(X^2(s, \cdot)) \right] \star \phi_\varepsilon >_{L^2}$$

$$+ 2 \int_0^t < X^\varepsilon(s, \cdot), (I - \Delta)^{-1} \left[e^0 X(s, \cdot) \right] \star \phi_\varepsilon >_{L^2}$$

$$+ M_t^\varepsilon,$$

where M^ε is the local martingale defined by

$$M_t^\varepsilon = \sum_{i=1}^{\infty} \int_0^t < X^\varepsilon(s, \cdot), (I - \Delta)^{-1} | \left(e^i X(s, \cdot) \right) \star \phi_\varepsilon >_{L^2} dW_s^i,$$

which is again well-defined by similar arguments as for the proof of (2.1). Taking into account (B.1) and the Lipschitz property for ψ, we can take the limit when $\varepsilon \to 0$ in (B.5) and for $g(t) := \| X(t, \cdot) \|_{H^{-1}}^2$, we obtain

$$g(t) + 2 \int_0^t \mathrm{d}s \, \left\langle X(s, \cdot), \, \psi\left(X^1(s, \cdot)\right) - \psi\left(X^2(s, \cdot)\right) \right\rangle_{L^2} \tag{B.6}$$

$$= \sum_{i=1}^{\infty} \int_0^t \mathrm{d}s \, \left\| e^i X(s, \cdot) \right\|_{H^{-1}}^2$$

$$+ 2 \int_0^t \mathrm{d}s \, \left\langle (I - \Delta)^{-1} X(s, \cdot), \, \psi\left(X^1(s, \cdot)\right) - \psi\left(X^2(s, \cdot)\right) \right\rangle_{L^2}$$

$$+ 2 \int_0^t \mathrm{d}s \, \left\langle X(s, \cdot) \, e^0 X(s, \cdot) \right\rangle_{H^{-1}} + M_t.$$

The convergence $M^\varepsilon \to M$ when $\varepsilon \to 0$ holds ucp since

$$\sum_{i=1}^{\infty} \int_0^t \mathrm{d}s \, \left| \left\langle (X^\varepsilon - X)(s, \cdot), \, e^i X(s, \cdot) \right\rangle_{H^{-1}} \right|^2 \underset{\varepsilon \to 0}{\longrightarrow} 0$$

because of property ii).

We take into account the inequality

$$2ab \leqslant \frac{a^2}{\alpha} + b^2 \alpha,$$

for $a, b \in \mathbb{R}$, α being the constant appearing at item vii) of Remark B.3.
Then the second term of the right-hand side of equality (B.6) is bounded by

$$\alpha^2 \int_0^t \mathrm{d}s \left\| (I - \Delta)^{-1} X(s, \, \cdot \,) \right\|_{L^2}^2 + \frac{1}{\alpha^2} \int_0^t \left(\psi \left(X^1(s, \, \cdot \,) \right) - \psi \left(X^2(s, \, \cdot \,) \right) \right)_{L^2}^2 \mathrm{d}s$$

$$\leqslant \alpha^2 \int_0^t \mathrm{d}s \left\| X(s, \, \cdot \,) \right\|_{L^2}^2 + \int_0^t \mathrm{d}s \left\langle \psi \left(X^1(s, \, \cdot \,) \right) - \psi \left(X^2(s, \, \cdot \,) \right), \, X(s, \, \cdot \,) \right\rangle_{L^2}.$$

This together with (B.6) gives $\mathrm{d}P$-a.s.

$$g(t) + \int_0^t \mathrm{d}s \left\langle X(s, \, \cdot \,), \, \psi \left(X^1(s, \, \cdot \,) \right) - \psi \left(X^2(s, \, \cdot \,) \right) \right\rangle_{L^2}$$

$$\leqslant 2 \int_0^t \mathrm{d}s \left\langle X(s, \, \cdot \,) \, e^0 X(s, \, \cdot \,) \right\rangle_{H^{-1}}$$

$$+ \sum_{i=1}^{\infty} \int_0^t \mathrm{d}s \left\| (e^i X)(s, \cdot) \right\|_{H^{-1}}^2 + M_t.$$

Since e^i, $i \in \mathbb{N}$ are H^{-1}-multipliers and taking into account Hypothesis ii),
we get

$$g(t) \leqslant M_t + \left(2 + \sum_{i=1}^{\infty} \mathcal{C}(e_i)^2 \right) \int_0^t \mathrm{d}s g(s). \tag{B.7}$$

We proceed now by localization. Let (ς^ℓ) be a sequence of stopping times
defined by

$$\varsigma^\ell = \inf \{ t \in [0, T] | \int_0^t \mathrm{d}s \left\| X(s, \, \cdot \,) \right\|_{L^2}^2 \geqslant \ell, \left\| X(s, \cdot) \right\|_{H^{-2}} \geq \ell \},$$

with the convention that $\varsigma^\ell = \infty$ if $\{ \, \} = \emptyset$. Since $\int_0^T \left\| X(s, \, \cdot \,) \right\|_{H^{-1}}^2 \mathrm{d}s < \infty$
a.s. we have $\Omega = \bigcup_{\ell=1}^{\infty} \{ \varsigma^\ell > T \}$ up to a null set. Clearly the stopped processes
M_ℓ^ς are martingales starting from zero. We evaluate (B.7) at $t \wedge \varsigma^\ell$ and we
take the expectation wich gives

$$E \left(g(t \wedge \varsigma_\ell) \right) \leqslant \mathcal{C} E \left(\int_0^{t \wedge \varsigma_\ell} g(s) \mathrm{d}s \right).$$

Since $\int\limits_0^{t \wedge \varsigma_\ell} g(s) \mathrm{d}s \leqslant \int_0^{\varsigma_\ell} g(s) \mathrm{d}s \leqslant \ell$, $E \left(g(t \wedge \varsigma_\ell) \right)$ is finite for every $\ell > 0$.
Consequently

$$E \left(g(t \wedge \varsigma_\ell) \right) \leqslant \mathcal{C} \int_0^t \mathrm{d}s E \left(g(s \wedge \varsigma_\ell) \right)$$

and by Gronwall lemma it follows

$$E\left(g(t \wedge \varsigma_\ell)\right) = 0 \quad \forall\, \ell \in \mathbb{N}.$$

By (B.6) g is a.s. continuous. On the other hand, for every $t \in [0, T]$, $\lim_{\ell \to 0} t \wedge \varsigma_\ell = t$ implies that

$$E\left(g(t)\right) = E\left(\liminf_{\ell \to 0} g(t \wedge \varsigma_\ell)\right)$$

$$\leq \liminf_{\ell \to \infty} E\left(g(t \wedge \varsigma_\ell)\right) = 0$$

by Fatou's lemma. This concludes the proof.

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