A flexible and tractable class of one-factor copulas

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A nonparametric class of one-factor copulas to balance flexibililty and tractability

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Abstract

Copulas are a useful tool to model multivariate distributions. While there exist various families of bivariate copulas, the construction of flexible and yet tractable high dimensional copulas is much more challenging. This is even more true if one is concerned with extreme-value copulas. In this paper, we construct a nonparametric one-factor copula exhibiting a good balance between tractability and flexibility. This copula is built upon nonparametric generators, thus allowing to get away from restrictive parametric assumptions. Moreover, the analytical form of the copula allows to explicitly calculate the dependence coefficients and the associated extreme-value copula.

1 Introduction

The modelling of random multivariate events is a central problem in various scientific domains and the construction of multivariate distributions able to properly model the variables at play is challenging. To address this problem, the concept of copulas is a useful tool as it permits to impose a dependence structure on pre-determined marginal distributions.

While there are many copula families in the bivariate case, it is less clear how to construct higher dimensional copulas, especially if the data provide evidence of tail dependence or non Gaussian behaviors. To deal with such data, various copula models exist in the literature.

The class of Archimedean copulas [7, 12, 17] is very popular. Indeed, these copulas possess several advantages. First, they are explicit and tractable, a feature more and more important as the dimension increases. Second, they allow to model a different behavior in the lower and upper tails. For instance, Gumbel copulas are upper tail dependent but not lower tail dependent, and the opposite holds for Clayton copulas. However, Archimedean copulas also suffer from a severe drawback. Indeed, in practice, they are typically governed by only one or two parameters, hence the dependence structure is severly restricted. In particular, their dependence structure is exchangeable, meaning that all the pairs of variables have the same distribution.

Nested Archimedean copulas [10, 11] are a class of hierarchical copulas generalizing the class of Archimedean copulas. They allow to introduce asymmetry in the dependence structure but only between groups of variables. This hierarchical structure is not desirable when no prior knowledge of the random phenomenon under consideration is available. Furthermore, constraints on the parameters restrict the tractability of these copulas.

The class of elliptical copulas (see e.g. [16] Section 5) is radially symmetric. This means that the same statistical behavior is expected in the lower and upper tails, which may not be the case in applications.

Vines [1], or pair copula constructions, are flexible copula models but they are not easy to handle, as the modeling process is rather cumbersome. Indeed, modeling data with Vines demands to specify certain links between the variables *a priori*. Furthermore, the computation of the likelihood is challenging, and one is often obliged to require to a so called *simplifying assumption*, that is, to assume that the conditional pair-copulas depend on the conditioning variables only indirectly through the conditional margins. This simplifying assumption can be misleading [2].

Recently, a model involving latent factors has been proposed [13]. This copula model can be regarded as a truncated Vine copula to simplify the modeling process. The variables of interest are independent given the factors, thus allowing to simulate easily from this copula. When there is only one factor, this copula is parsimonious, as it requires only d parameters, where d is the number of variables at play. Still, this copula presents a flexible dependence structure, since various tail behaviors can be obtained. Nonetheless, only parametric factor copulas have been considered until now. This is a restriction, because one has to hope that a parametric family would fit the data in a reasonable way. Moreover, the copula expression, which writes as an univariate integral, could not be calculated, thus preventing a finer analysis of its dependence properties. Finally, this copula has not been proved to be an extreme-value copula, thus discarding it from the theoretical well grounded copulas suited for extreme-value analysis.

In this paper, we propose a nonparametric one-factor copula whose expression writes in closed form. This advantage permits to explicitely derive the dependence coefficients thus allowing to see that all types of dependence structures can be obtained. Moreover, we are able to derive the extreme-value copulas associated to the nonparametric one-factor model by computing the limiting distribution of the normalized maxima. Finally, even though the nonparametric one factor copula is not differentiable, we use recent theoretical results to propose to base the inference on the dependence coefficients, whose resulting estimator is proved to be consistent and asymptotically normal.

The remaining of this paper is as follows. Section 2 presents the nonparametric one-factor model, Section 3 deals with inference, and Section 4. The proofs are postponed to the Appendix.

2 A nonparametric class of one-factor copulas to balance flexibility and tractability

Since the proposed nonparametric one-factor copulas are special cases of the more general one-factor copulas, we introduce the later in Section 2.1. The construction and properties of the nonparametric one-factor copula is given in Section 2.2. Parametric examples are given in Section 2.3. The extreme-value copula associated to the nonparametric one-factor copula are derived in

Section 2.4.

2.1 One-factor copulas

Let $U_0, U_1, \ldots, U_d$ be standard uniform random variables such that the coordinates of $(U_1, \ldots, U_d)$ are independent given U_0. Let us write C_{0i} the distribution of (U_0, U_i) and $C_{i|0}(\cdot|u_0)$ the conditional distribution of U_i given $U_0 = u_0$ for $i = 1, \ldots, d$. It is easy to see that the distribution of $(U_1, \ldots, U_d)$ is given by

$$C(u_1, \ldots, u_d) = \int_0^1 C_{1|0}(u_1|u_0) \ldots C_{d|0}(u_d|u_0)\, du_0. \tag{1}$$

Since the role played by U_0 is that of a latent variable, or factor, this model has been called a *one factor model* in [13]. The copulas C_{0i} are called the *linking copulas* because they link the factor U_0 to the variables of interest U_i. The one-factor model has many advantages to address high dimensional problems. We recall and briefly discuss them below.

Nonexchangeability A copula C is said to be *exchangeable* if $C(u_1, \ldots, u_d) = C(u_{\pi(1)}, \ldots, u_{\pi(d)})$ for any permutation π of $(1, \ldots, d)$. This means in particular that all the bivariate marginals are restricted to exhibit the same statistical behaviour. For instance, the pairs (U_1, U_2) and (U_3, U_4) have the same distribution. As an example, Archimdean copulas are exchangeable copulas. Needless to say, this assumption may be too strong in practice. That is why the class of Archimedean copulas was extended to the wider class of nested Archimedean copulas (see [15] and [12] Section 4.2). The one-factor model has the advantage to be, in general, nonexchangeable.

Parsimony The one-factor model is parsimonious, a feature of great importance in high dimension. Indeed, only d linking copulas are involved in the construction of the one-factor model, and since they are typically governed by one parameter, the number of parameters in total is no more than d, which increases only linearly with the dimension.

Random generation The conditional independence property of the one factor model allows easily to generate data $(U_1, \ldots, U_d)$ from this model.

1. Generate $U_0 = u_0, V_1 = v_1, \ldots, V_d = v_d$ i.i.d. standard uniform random variables.
2. For $i = 1, \ldots, d$, put $U_i = C_{i|0}^{-1}(v_i|U_0 = u_0)$ where $v \mapsto C_{i|0}^{-1}(v|U_0 = u_0)$ denotes the inverse of $v \mapsto C_{i|0}(v|U_0 = u_0)$.

Properties of the one-factor model have been investigated in [13]. Regarding tail dependence, the authors studied how tail dependence properties are transmitted from the linking copulas to the one-factor copula bivariate marginals

$$C_{ij}(u_i, u_j) := C(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1).$$

A bivariate copula is *lower (upper) tail dependent* if its lower (upper) tail dependence coefficient is positive. Recall that the lower and upper tail dependence coefficients of a bivariate copula C are respectively given by

$$\lambda^{(L)} = \lim_{u \downarrow 0} \frac{C(u, u)}{u} \in [0, 1], \quad \lambda^{(U)} = \lim_{u \uparrow 1} \frac{1 - 2u + C(u, u)}{1 - u} \in [0, 1].$$

If the linking copulas C_{0i} and C_{0j} are lower (upper) tail dependent, then so is the one-factor copula bivariate marginal C_{ij}. Although this property is of interest, it would be better if one could say, for instance, *how much* the bivariate marginal is tail dependent. In other words, can we explicitly derive the tail dependence coefficients of the C_{ij}? Of course, the same question applies to the Spearman's rho and Kendall's tau dependence coefficients.

Another question related to extreme-value statistics is the following. Is the one-factor copula an extreme-value copula (the definition of extreme-value copulas is postponed to Section 2.4)? If not, can we calculate the limit

$$C^n(u_1^{1/n}, \ldots, u_d^{1/n})$$

as $n \to \infty$ to get an extreme-value copula?

These questions are part of a more general global problem of constructing a tail dependent copula in closed structure with the widest dependence structure possible while keeping the advantageous one-factor form. This copula may be nonparametric to get rid of restrictive parametric assumptions during the modeling process. We believe that the so called nonparametric one-factor copulas proposed in Section 2.2 permit to answer these questions.

2.2 Nonparametric one-factor copulas to balance tractability and flexibility

In this section, we construct a class of nonparametric one-factor copulas by choosing appropriate nonparametric linking copulas. The selected class of nonparametric copulas is referred to as the *Durante* class [4]. This class consists in the copulas of the form

$$C(u, v) = \min(u, v) f(\max(u, v)), \tag{2}$$

where $f : [0, 1] \to [0, 1]$ called the *generator* of C is a differentiable and increasing function such that $f(1) = 1$ and $t \mapsto f(t)/t$ is decreasing. The advantages of taking Durante copulas are twofold: we can calculate the integral (1) and the resulting multivariate copula is nonparametric.

Theorem 1. *Let C be defined by* (1) *and assume that C_{0i} belongs to the Durante class* (2) *with given generator f_i. Then*

$$\begin{aligned} C(u_1, \ldots, u_d) = u_{(1)} \Bigg[& \left(\prod_{j=2}^{d} u_{(j)} \right) \int_{u_{(d)}}^{1} \prod_{j=1}^{d} f_j'(x) dx + f_{(1)}(u_{(2)}) \left(\prod_{j=2}^{d} f_{(j)}(u_{(j)}) \right) \\ & + \sum_{k=3}^{d} \left(\prod_{j=2}^{k-1} u_{(j)} \right) \left(\prod_{j=k}^{d} f_{(j)}(u_{(j)}) \right) \int_{u_{(k-1)}}^{u_{(k)}} \prod_{j=1}^{k-1} f_{(j)}'(x) dx \Bigg], \end{aligned} \tag{3}$$

where $u_{(i)} := u_{\sigma(i)}$, $f_{(i)} := f_{\sigma(i)}$ and σ is the permutation of $(1, \ldots, d)$ such that $u_{\sigma(1)} \leq \cdots \leq u_{\sigma(d)}$.

The particularity of the nonparametric one-factor copula expression (3) is that it depends on the generators through their reordering underlain by the permutation σ. For instance, with $d = 3$ and $u_1 < u_3 < u_2$ we have $u_{(1)} = u_1$, $u_{(2)} = u_3$, $u_{(3)} = u_2$, $\sigma = \{1, 3, 2\}$ and $f_{(1)} = f_{\sigma(1)} = f_1$, $f_{(2)} = f_{\sigma(2)} = f_3$, $f_{(3)} = f_{\sigma(3)} = f_2$. This feature gives its flexibility to the model, referred to as NPOF from now on (NonParametric One-Factor model). Although the NPOF copula expression has the merit to be explicit, it is rather cumbersome. Hence, we shall continue its analysis through the prism of its bivariate marginals, which reveal to belong to the Durante class as well.

Proposition 1. *Let C_{ij} be a bivariate marginal of the NPOF copula* (3)*. Then it belongs to the Durante class* (2) *with generator*

$$f_{ij}(t) = f_i(t)f_j(t) + t\int_t^1 f_i'(x)f_j'(x)dx.$$

In other words,

$$C_{ij}(u_i, u_j) = C_{f_{ij}}(u_i, u_j) = \min(u_i, u_j)f_{ij}(\max(u_i, u_j)). \tag{4}$$

Thus, the NPOF model is stable in the sense that its bivariate marginals belong to the same class as that of the bivariate linking copulas. It can be viewed as a flexible generalization in higher dimension of the Durante class of bivariate copulas. Another generalization has been proposed in [5]:

$$C_f(u_1, \ldots, u_d) = u_{(1)}\prod_{i=2}^{d} f(u_{(i)}),$$

where f is a generator in the usual sense of the Durante class of bivariate copulas. Unfortunately, the fact that there is only one generator to determine the whole copula in arbitrary dimension restricts much of the applicability of this copula.

Since the bivariate marginals of the NPOF copula belong to the Durante class of bivariate copulas, we refer to the original paper [4] for a detailed account of their properties. Here, we shall only report the formulas for the dependence coefficients. The Spearman's rho ρ, the Kendall's tau τ, the lower $\lambda^{(L)}$ and upper $\lambda^{(U)}$ tail dependence coefficients of a bivariate copula belonging to the Durante class with generator f are respectively given by

$$\begin{aligned}\rho =& 12\int_0^1 x^2 f(x)dx - 3,\\ \tau =& 4\int_0^1 x f(x)^2 dx - 1,\\ \lambda^{(L)} =& f(0),\\ \lambda^{(U)} =& 1 - f'(1).\end{aligned}$$

Hence, to get the dependence coefficients of the NPOF bivariate marginals, it suffices to apply the above formulas and Proposition 1. The pairwise dependence coefficients of the NPOF copula are summarized in the next proposition.

Proposition 2. *The Spearman's rho, the Kendall's tau, the lower and upper tail dependence coefficients of the NPOF bivariate marginals C_{ij} are respectively* *given by*

$$\rho_{ij} = 12\int_0^1 x^2 f_i(x) f_j(x) dx + 3\int_0^1 x^4 f_i'(x) f_j'(x) dx - 3,$$
$$\tau_{ij} = 4\int_0^1 x\left(f_i(x)f_j(x) + x\int_x^1 f_i'(t)f_j'(t)dt\right)^2 dx - 1,$$
$$\lambda_{ij}^{(L)} = \lambda_i^{(L)}\lambda_j^{(L)} \text{ and}$$
$$\lambda_{ij}^{(U)} = \lambda_i^{(U)}\lambda_j^{(U)},$$

where $\lambda_i^{(L)} := f_i(0)$, $\lambda_i^{(U)} := 1 - f_i'(1)$, $i = 1, \ldots, d$ *are the lower and upper tail dependence coefficients of the bivariate linking copulas respectively.*

The ability to calculate the dependence coefficients in analytical form allows to better understand the tail dependence structure of the NPOF copula, and, in particular, how are passed tail dependence properties from the linking copulas to the NPOF bivariate marginals.

The results of Proposition 2 are in accordance with the results about general one-factor copulas. Recall that it was established that if the linking copulas C_{0i} and C_{0j} are lower (upper) tail dependent, then so is the bivariate marginal C_{ij}. It is easily seen that, here, if the linking copulas C_{0i} and C_{0j} are lower tail dependent, then $f_i(0) > 0$ and $f_j(0) > 0$, which happens if and only if $f_i(0)f_j(0) > 0$. In Section 2.3, various forms for the dependence coefficients are obtained by choosing parametric families for the linking copulas.

2.3 Examples of parametric families

The multivariate NPOF model is determined by d generators $(f_1, \ldots, f_d)$. In this section, we assume that each of these generators belong to a same parametric family whose parameter is real. We give four examples of such families, and we show that these families allow to get all possible types of dependence structure.

Example 1 (Cuadras-Augé generators)**.** *In* (3)*, let*

$$f_i(t) = t^{1-\theta_i},\ \theta_i \in [0,1]. \tag{5}$$

A copula belonging to the Durante class with generator given by (5) *gives rise to the well known Cuadras-Augé copula with parameter* θ_i *[3]. By Proposition 1, the generator for the bivariate marginal* C_{ij} *of the NPOF copula is given by*

$$f_{ij}(t) = \begin{cases} t^{2-\theta_i-\theta_j}\left(1 - \frac{(1-\theta_i)(1-\theta_j)}{1-\theta_i-\theta_j}\right) + t\frac{(1-\theta_i)(1-\theta_j)}{1-\theta_i-\theta_j} & \text{if } \theta_i + \theta_j \neq 1 \\ t(1-(1-\theta_i)(1-\theta_j)\log t) & \text{if } \theta_i + \theta_j = 1. \end{cases}$$

The Spearman's rho, the lower and upper tail dependence coefficients are respectively given by

$$\rho_{ij} = \frac{3\theta_i\theta_j}{5-\theta_i-\theta_j},\ \lambda_{ij}^{(L)} = 0 \text{ and } \lambda_{ij}^{(U)} = \theta_i\theta_j.$$

Example 2 (Fréchet generators)**.** *In* (3)*, let*

$$f_i(t) = (1-\theta_i)t + \theta_i,\ \theta_i \in [0,1]. \tag{6}$$

A copula belonging to the Durante class with generator given by (6) *gives rise to the well known Fréchet copula with parameter* θ_i *[8]. By Proposition 1, the generator for the bivariate marginal* C_{ij} *of the NPOF copula is given by*

$$f_{ij}(t) = (1 - \theta_i\theta_j)\, t + \theta_i\theta_j.$$

By noting that f_{ij} *is of the form* (6) *with parameter* $\theta_i\theta_j$*, one can see that the bivariate marginals of the NPOF copula based on Fréchet generators are still Fréchet copulas. The Spearman's rho, the lower and upper tail dependence coefficients are respectively given by*

$$\rho_{ij} = \lambda_{ij}^{(L)} = \lambda_{ij}^{(U)} = \theta_i\theta_j.$$

Example 3 (Sinus generators)**.** *In* (3)*, let*

$$f_i(t) = \frac{\sin(\theta_i t)}{\sin(\theta_i)}, \quad \theta_i \in (0, \pi/2]. \tag{7}$$

This generator was proposed in [4]. By Proposition 1, the generator for the bivariate marginal C_{ij} *of the NPOF copula is given by*

$$f_{ij}(t) = \frac{\sin(\theta_i t)\sin(\theta_j t)}{\sin(\theta_i)\sin(\theta_j)} + \frac{t\theta_i\theta_j}{2(\theta_j^2 - \theta_i^2)\sin(\theta_i)\sin(\theta_j)} \times$$
$$\Big\{ (\theta_i + \theta_j)\left[\sin\left((\theta_i - \theta_j)t\right) + \sin\left(\theta_j - \theta_i\right)\right]$$
$$+ (\theta_j - \theta_i)\left[\sin(\theta_i + \theta_j) - \sin\left((\theta_i + \theta_j)t\right)\right] \Big\} \text{ if } \theta_i \neq \theta_j.$$

The Spearman's rho, the lower and upper tail dependence coefficients are respectively given by

$$\rho_{ij} = 12(\sin\theta_i \sin\theta_j)^{-1} \int_0^1 x^2 \sin(\theta_i x)\sin(\theta_j x) + \frac{1}{4}\theta_i\theta_j x^4 \cos(\theta_i x)\cos(\theta_j x) dx - 3,$$
$$\lambda_{ij}^{(L)} = 0 \text{ and } \lambda_{ij}^{(U)} = \left(1 - \frac{\theta_i}{\tan(\theta_i)}\right)\left(1 - \frac{\theta_j}{\tan(\theta_j)}\right).$$

Example 4 (Exponential generators)**.** *In* (3)*, let*

$$f_i(t) = \exp\left(\frac{t^{\theta_i} - 1}{\theta_i}\right), \quad \theta_i > 0 \tag{8}$$

This generator was proposed in [4]. By Proposition 1, the generator for the bivariate marginal C_{ij} *of the NPOF copula is given by*

$$f_{ij}(t) = \exp\left(\frac{t^{\theta_i} - 1}{\theta_i} + \frac{t^{\theta_j} - 1}{\theta_j}\right) + t\int_t^1 \exp\left(\frac{x^{\theta_i} - 1}{\theta_i} + \frac{x^{\theta_j} - 1}{\theta_j}\right) x^{\theta_i + \theta_j - 2} dx.$$

The Spearman's rho, the lower and upper tail dependence coefficients are respectively given by

$$\rho_{ij} = 12\int_0^1 \exp\left(\frac{x^{\theta_i} - 1}{\theta_i} + \frac{x^{\theta_j} - 1}{\theta_j}\right)\left(x^2 + \frac{1}{4}x^{3+\theta_i+\theta_j}\right) dx - 3,$$
$$\lambda_{ij}^{(L)} = \exp\left(-\frac{1}{\theta_i} - \frac{1}{\theta_j}\right), \text{ and } \lambda_{ij}^{(U)} = 0.$$

The tail dependence coefficients are summarized in Table 1. One can see that all possible types of tail dependences can be obtained: the Cuadras-Augé and sinus families allow for upper but no lower tail dependence, the exponential family allows for lower but no upper tail dependence, and the Fréchet family allows for both.

family of generators	$\lambda_{ij}^{(L)}$	$\lambda_{ij}^{(U)}$
Cuadras-Augé	0	$\theta_i\theta_j$
Fréchet	$\theta_i\theta_j$	$\theta_i\theta_j$
Sinus	0	$(1-\frac{\theta_i}{\tan\theta_i})(1-\frac{\theta_j}{\tan\theta_j})$
Exponential	$\exp(-\frac{1}{\theta_i}-\frac{1}{\theta_j})$	0

Table 1: Lower $\lambda_{ij}^{(L)}$ and upper $\lambda_{ij}^{(U)}$ tail dependence coefficients for the four families presented in Section 2.3.

2.4 Extreme-value copulas attractors

In this section, we derive the extreme-value copulas attractors associated to the NPOF model. This is an important result for extreme-value statistics. Indeed, according to extreme-value theory, the proper copulas to be used for the analysis of extreme values should be the extreme-value copulas. A copula $C_{\#}$ is an *extreme-value copula* if there exists a copula C such that

$$C_{\#}(u_1,\ldots,u_d) = \lim_{n\uparrow\infty} C^n(u_1^{1/n},\ldots,u_1^{1/n}), \tag{9}$$

for every $(u_1,\ldots,u_d)\in[0,1]^d$. We say that $C_{\#}$ is the *attractor* of C and that C belongs to the *domain of attraction* of $C_{\#}$. A popular representation of extreme-value copulas involves the so called *stable tail dependence function* [9]. Let $S_{d-1}=\{(y_1,\ldots,y_d)\in[0,\infty)^d : \sum_j y_j = 1\}$ be the unit simplex in $\mathbb{R}^d$. Then there exists a finite Borel measure H on S_{d-1} such that

$$C_{\#}(u_1,\ldots,u_d) = \exp\left[-l(-\log u_1,\ldots,-\log u_d)\right], \tag{10}$$

where $l:[0,\infty)^d\to[0,\infty)$ is given by

$$l(x_1,\ldots,x_d) = \int_{S_{d-1}} \max_{j=1,\ldots,d}(y_jx_j)dH(y_1,\ldots,y_d), \quad (x_1,\ldots,x_d)\in[0,\infty)^d, \tag{11}$$

with

$$\int_{S_{d-1}} y_j dH(y_1,\ldots,y_d) = 1, \quad j=1,\ldots,d.$$

The function l is called the stable dependence function and completely determines the extreme-value copulas. Extreme-value copulas have no lower tail dependence. The dependence for extreme-value copulas is naturally measured with the upper tail dependence coefficient, or simply tail dependence coefficient.

The tail dependence coefficient λ of a bivariate extreme-value copula $C_\#$ takes the form

$$\lambda = 2 + \log C_\#(e^{-1}, e^{-1}).$$

The limit (9) can be calculated for the NPOF copula (3) leading to a new extreme-value copula (referred to as EVNPOF from now on) which benefits from most of the advantageous properties of the NPOF model.

Theorem 2. *Assume that the generators f_i of the NPOF copula are twice continuously differentiable on $[0,1]$. Then the attractor $C_\#$ of the NPOF copula exists and is given by*

$$C_\#(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_{(i)}^{\chi_i}, \tag{12}$$

where

$$\chi_i = \left(\prod_{j=1}^{i-1} (1 - \lambda_{(j)}) \right) \lambda_{(i)} + 1 - \lambda_{(i)},$$

with the convention that $\prod_{j=1}^{0}(1 - \lambda_{(j)}) = 1$ and where $\lambda_i = 1 - f_i'(1)$. As in (3)*, $u_{(i)} = u_{\sigma(i)}$ and $f_{(i)}'(1) = f_{\sigma(i)}'(1)$ where σ is the permutation of $(1, \ldots, d)$ such that $u_{(1)} \leq \cdots \leq u_{(d)}$. The stable tail dependence function of $C_\#$ is given by*

$$l(x_1, \ldots, x_d) = \sum_{i=1}^{d} x_{\sigma(i)} \chi_i, \quad x = (x_1, \ldots, x_d) \in [0, \infty)^d.$$

Note that, because of (10), we have $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(d)}$ if and only if $u_{\sigma(1)} \leq \cdots \leq u_{\sigma(d)}$.

Even though the copulas belonging to the domain of attraction of $C_\#$ are nonparametric, the attractor $C_\#$ is parametric, since it depends on the generator derivatives only at a single point. As for the NPOF model, we shall have a look at the bivariate marginals

$$C_{\#,ij} = C_\#(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1)$$

to study the dependence properties of this extreme-value copula.

Proposition 3. *Let $C_{\#,ij}$ is a bivariate marginal of the EVNPOF copula. Then $C_{\#,ij}$ is a Cuadras-Augé copula with parameter $\lambda_i \lambda_j$. In other words,*

$$C_{\#,ij}(u_i, u_j) = \min(u_i, u_j) \max(u_i, u_j)^{1 - \lambda_i \lambda_j}. \tag{13}$$

Hence, the bivariate marginals of the EVNPOF copula are Cuadras-Augé copulas, and, therefore, belong the Durante class of bivariate copulas. Moreover, the upper tail dependence coefficient of the EVNPOF copula bivariate marginal is the same as the upper tail dependence coefficient of the NPOF copula bivariate marginal with Cuadras-Augé (Example 1) and Fréchet (Example 2) generators.

The multivariate EVNPOF extreme-value copula can be regarded as a flexible generalization of the Cuadras-Augé copula. To demonstrate this, we shall compare it to other multivariate generalizations of the bivariate Cuadras-Augé copula. Let

$$A(u_1, \ldots, u_d) = u_{(1)} \prod_{i=2}^{d} u_{(i)}^{a_i},$$

where $(a_1 = 1, a_2, a_3, \ldots, a_d)$ is a d-monotone sequence of real numbers, that is, a sequence which satisfies $\nabla^{j-1} a_k \geq 0$, $k = 1, \ldots, d$, $j = 1, \ldots, d-k+1$ where $\nabla^j a_k = \sum_{i=0}^{j} (-1)^i \binom{j}{i} a_{k+i}$, $j, k \geq 1$ and $\nabla^0 a_k = a_k$. This exchangeable copula was proposed in [14]. In particular, the bivariate marginals write

$$A_{ij}(u_i, u_j) = \min(u_i, u_j) \max(u_i, u_j)^{a_2}.$$

One can see that all the bivariate marginals are governed by a single parameter a_2. This means that all of them exhibit the same statistical behavior. For instance, all the upper tail dependence coefficients are equal and are given by $1 - a_2$, which is far too restrictive for most applications.

Now let

$$B(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i^{1 - \sum_{j=1, j \neq i}^{d} \lambda_{ij}} \prod_{i<j} \min(u_i, u_j)^{\lambda_{ij}},$$

where $\lambda_{ij} \in [0, 1], \lambda_{ij} = \lambda_{ji}$ and

$$\sum_{j=1,\ldots,d;\, j \neq i} \lambda_{ij} \leq 1, \quad i = 1, \ldots, d. \tag{14}$$

This copula was proposed in [6]. It is easy to see that the bivariate marginals B_{ij} are Cuadras-Augé copulas with parameter (and thus upper tail dependence coefficient) λ_{ij}. Unlike the copula A, the tail dependence coefficients can take distinct values from each other. Unfortunately, the constraints (14) on the parameters λ_{ij} are quite restrictive, as it was already stressed by the original authors.

In the light of the above, one can see that the EVNPOF model achieves greater flexibility than its competitors. In particular, one can obtain distinct distributions for the bivariate marginals without imposing restrictive conditions on the parameters.

3 Parametric inference

Here the results of the working paper *weighted least square inference for multivariate copulas based on dependence coefficients* will be used.

4 Numerical illustrations

A Appendix

Proof of Theorem 1

Let $C_{j|0}(\cdot|u_0)$ be the conditional distribution of U_j given $U_0 = u_0$. The U_j's are conditionally independent given U_0, hence,

$$
\begin{aligned}
C(u) &= \int_0^1 \prod_{j=1}^d C_{j|0}(u_j|u_0)du_0 \qquad (15)\\
&= \int_0^1 \prod_{j=1}^d \frac{\partial C_{0j}(u_0,u_j)}{\partial u_0} du_0\\
&= \int_0^1 \prod_{j=1}^d \frac{\partial C_{0(j)}(u_0,u_{(j)})}{\partial u_0} du_0\\
&= \int_0^{u_{(1)}} \prod_{j=1}^d \frac{\partial C_{0(j)}(u_0,u_{(j)})}{\partial u_0} du_0\\
&\quad + \sum_{k=2}^d \int_{u_{(k-1)}}^{u_{(k)}} \prod_{j=1}^{k-1} \frac{\partial C_{0(j)}(u_0,u_{(j)})}{\partial u_0} \prod_{j=k}^{d} \frac{\partial C_{0(j)}(u_0,u_{(j)})}{\partial u_0}\\
&\quad + \int_{u_{(d)}}^1 \prod_{j=1}^d \frac{\partial C_{0(j)}(u_0,u_(j))}{\partial u_0} du_0.
\end{aligned}
$$

Since

$$
\frac{\partial C_{0j}(u_0,u_j)}{\partial u_0} = \begin{cases} f_j(u_j) \text{ if } u_0 < u_j \\ u_j f_j'(u_0) \text{ if } u_0 > u_j, \end{cases}
$$

(15) yields

$$
\begin{aligned}
C(u) =& u_{(1)} \prod_{j=1}^d f_j(u_j) + \sum_{k=2}^d \prod_{j=k}^d f_{(j)}(u_{(j)}) \int_{u_{(k-1)}}^{u_{(k)}} \prod_{j=1}^{k-1} u_{(j)} f_{(j)}'(u_0) du_0\\
&+ \prod_{j=1}^d u_j \int_{u_{(d)}}^1 f_j'(u_0) du_0.
\end{aligned}
$$

Putting $u_{(1)}$ in factor and noting that $\int_{u_{(1)}}^{u_{(2)}} f_{(1)}'(x)dx = f_{(1)}(u_{(2)}) - f_{(1)}(u_{(1)})$ finishes the proof.

Proof of Proposition1

It suffices to set all u_k equal to one but u_i and u_j in the formula (3).

Proof of Proposition 2

It suffices to apply the formula of the dependence coefficients for the Durante class of bivariate copulas given in Section 2.2. To compute the Spearman's rho, note that $\int_0^1 x^2 f_{ij}(x)dx = \int_0^1 x^2 f_i(x)f_j(x)dx + \int_0^1 x^3 \int_x^1 f_i'(z)f_j'(z)dzdx$. An integration by parts yields $\int_0^1 x^3 \int_x^1 f_i'(z)f_j'(z)dzdx = (1/4)\int_0^1 x^4 f_i'(x)f_j'(x)dx$ and the result follows.

Proof of Theorem 2

Fix $(u_1, \dots, u_d) \in [0,1]^d$ and let $n \geq 1$ an integer. Put

$$\begin{aligned} \alpha_n :=& u_{(1)}^{1/n} \prod_{j=1}^d \frac{f_j(u_j^{1/n})}{u_j^{1/n}}, \\ \beta_n :=& \int_{u_{(d)}^{1/n}}^1 \prod_{j=1}^d f_j'(u_0)du_0, \\ \gamma_n :=& \prod_{j=k}^d \frac{f_{(j)}(u_{(j)}^{1/n})}{u_{(j)}^{1/n}}, \\ \delta_{n,k} :=& \int_{u_{(k-1)}^{1/n}}^{u_{(k)}^{1/n}} \prod_{j=1}^{k-1} f_{(j)}'(u_0)du_0, \end{aligned}$$

and define

$$A_n := \alpha_n + \beta_n + \sum_{k=2}^d \gamma_{n,k}\delta_{n,k}.$$

We are going to derive equivalence at infinity for $\alpha_n, \beta_n\gamma_n$ and δ_n. Let $\sim$ denote the equivalent symbol at infinity (i.e., $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$). By using the well known formulas $e^x \sim 1+x$ (when $x \to 0$), $\log x \sim x-1$ (when $x \to 1$) and $f_j(x) \sim 1 + (x-1)f_j'(1)$ (when $x \to 1$) we get

$$\alpha_n \sim \left(1 + \frac{1}{n}\log u_{(1)}\right)\left(1 - \frac{1}{n}\log(u_1 \dots u_d)\right)\left(1 + \frac{1}{n}\sum_{j=1}^d \log u_{(j)} f_{(j)}'(1)\right) \text{ and}$$

$$\gamma_{n,k} \sim \left(1 - \frac{1}{n}\sum_{j=k}^d \log u_{(j)}\right)\left(1 + \frac{1}{n}\sum_{j=k}^d \log u_{(j)} f_{(j)}'(1)\right).$$

For β_n the equivalence is obtained as follows. Let $F(x)$ be a primitive of $\prod_{j=1}^d f_j'(x)$. Then $\beta_n = F(1) - F(u_{(d)}^{1/n})$. A Taylor expansion yields

$$F(u_{(d)}^{1/n}) = F(1) + (u_{(d)}^{1/n} - 1)F'(1) + \frac{(u_{(d)}^{1/n} - 1)^2}{2}F''(x_n)$$

where x_n is between $u_{(d)}^{1/n}$ and 1. Since F'' is assumed to be continuous on $[0,1]$, it is uniformly bouned on this set and therefore $(u_{(d)}^{1/n} - 1)^2 F''(x_n)/2 = o(1/n)$ where $o(1/n)$ is a quantity such that $no(1/n) \to 0$ as $n \to \infty$. Hence, since $u_{(d)}^{1/n} = \exp(\log(u_{(d)})/n) \sim 1 + \log(u_{(d)})/n$, we have as $n \to \infty$

$$F(1) - F(u_{(d)}^{1/n}) \sim -\frac{1}{n}\log(u_{(d)})F'(1).$$

The same arguments apply to get

$$\beta_n \sim -\frac{1}{n}\log u_{(d)} \prod_{j=1}^{d} f_j'(1)$$

$$\delta_{n,k} \sim \frac{1}{n}\log\left(\frac{u_{(k)}}{u_{(k-1)}}\right)\prod_{j=1}^{k-1} f_{(j)}'(1).$$

The quantity A_n is a polynomial with respect to n^{-1} of order at most three. In (16), the coefficients of order 0, 2, and 3 vanish at infinity. Only remain the terms of order 1, hence,

$$\lim_{n\uparrow\infty} n(A_n-1) = \log u_{(1)} - \log(u_1\dots u_d) + \sum_{j=1}^{d}\log u_{(j)} f_{(j)}'(1) - \log u_{(d)}\prod_{j=1}^{d} f_{(j)}'(1) + \sum_{k=2}^{d}\prod_{j=1}^{k-1} f_{(j)}'(1)\log\left(\frac{u_{(k)}}{u_{(k-1)}}\right).$$

By using Abel's identity, that is, $\sum_{i=1}^{d-1} a_i(b_{i+1}-b_i) = \sum_{i=1}^{d-1} b_i(a_{i-1}-a_i) + a_{d-1}b_d - a_1 b_1$ for two sequences (a_i) and (b_i) of real numbers, we can write

$$\lim_{n\uparrow\infty} n(A_n-1) = \sum_{k=1}^{d}\left(\underbrace{\left(\prod_{j=1}^{k-1} f_{(j)}'(1)\right)(1-f_{(k)}'(1)) + f_{(k)}'(1) - 1}_{=:\chi_k}\right)\log u_{(k)},$$

with the convention that $\prod_{j=1}^{0} f_{(j)}'(1) = 1$. From (3) we have

$$C^n(u_1^{1/n},\dots,u_d^{1/n}) = u_1\dots u_d \exp\left[n\log A_n\right]$$

which is equivalent at infinity to

$$u_1\dots u_d\exp\left[n(A_n-1)\right]. \tag{16}$$

Hence

$$C(u_1^{1/n},\dots,u_d^{1/n})^n \to \prod_{i=k}^{d} u_{(k)}^{\chi_k}$$

as $n\to\infty$.

To derive the expression for the stable tail dependence dependence function, it suffices to apply the definition. By (10) we have

$$\ell(-\log u_1,\dots,-\log u_d) = -\sum_{k=1}^{d}\chi_k\log u_{(k)} = -\sum_{k=1}^{d}\chi_k\log u_{\sigma(k)}$$

(recall that σ is the permutation of $(1,\dots,d)$ such that $u_{\sigma(1)}\le\dots\le u_{\sigma(d)}$). Putting $x_k = -\log u_k$ we get the result (observe that $x_{\sigma(1)}\ge\dots\ge x_{\sigma(d)}$).

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