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Isometric and invertible composition operators on weighted Bergman spaces of Dirichlet series

Maxime Bailléul

Abstract

We show that a composition operator on weighted Bergman spaces \( A^p_\sigma \) is invertible if and only if it is Fredholm if and only if it is an isometry.

1 Introduction

In [8], the authors defined the Hardy space \( H^2 \) of Dirichlet series with square-summable coefficients. Thanks to the Cauchy-Schwarz inequality, it is easy to see that \( H^2 \) is a space of analytic functions on \( \mathbb{C}_+ := \{ s \in \mathbb{C}, \Re(s) > \frac{1}{2} \} \).

F. Bayart introduced in [3] the more general class of Hardy spaces of Dirichlet series \( H^p \) (\( 1 \leq p < +\infty \)). In another direction, McCarthy defined in [12] some weighted Bergman Hilbert spaces of Dirichlet series and these spaces have been generalized in [2].

In order to recall how these spaces are defined, we need to recall the principle of the Bohr’s point of view: let \( n \geq 2 \) be an integer, it can be written (in a unique way) as a product of prime numbers \( n = p_1^{a_1} \cdots p_k^{a_k} \) where \( p_1 = 2, p_2 = 3 \) etc . . . For \( s \in \mathbb{C} \), we consider \( z = (z_1, z_2, \ldots) = (p_1^{-s}, p_2^{-s}, \ldots) \). Then, writing

\[
 f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}
\]

we get

\[
 f(s) = \sum_{n=1}^{+\infty} a_n (p_1^{-s})^{a_1} \cdots (p_k^{-s})^{a_k} = \sum_{n=1}^{+\infty} a_n z_1^{a_1} \cdots z_k^{a_k}.
\]

So we can see a Dirichlet series as a Fourier series on the infinite-dimensional polytorus \( T^\infty = \{ (z_1, z_2, \cdots), |z_i| = 1, \forall i \geq 1 \} \). We shall denote this Fourier series \( D(f) \).

Let us fix now \( p \geq 1 \). The space \( H^p(T^\infty) \) is the closure of the set of analytic polynomials with respect to the norm of \( L^p(T^\infty, m) \) where \( m \) is the normalized Lebesgue measure on \( T^\infty \). Let \( f \) be a Dirichlet polynomial, \( D(f) \) is then an analytic polynomial on \( T^\infty \) by the Bohr’s point of view. By definition, \( \| f \|_{H^p} := \| D(f) \|_{L^p(T^\infty)} \) and \( H^p \) is the closure of the set of Dirichlet polynomials with respect to this norm. The spaces \( H^p \) and \( H^p(T^\infty) \) are then isometrically isomorphic.

We recall now how we can define the weighted Bergman spaces of Dirichlet series. For \( \sigma > 0 \), \( f_\sigma \) will be the translate of \( f \) by \( \sigma \), i.e. \( f_\sigma(s) := f(s + \sigma) \). We shall denote by \( \mathcal{P} \) the set of Dirichlet polynomials.

Let \( p \geq 1 \), \( P \in \mathcal{P} \) and \( \mu \) be a probability measure on \( (0, +\infty) \) such that \( 0 \in \text{Supp}(\mu) \). Then

\[
 \| P \|_{A^p_\sigma} := \left( \int_0^{+\infty} \| P_\sigma \|_{H^p}^p \, d\mu(\sigma) \right)^{1/p}.
\]
$\mathcal{A}^p_\nu$ is the completion of $\mathcal{P}$ with respect to this norm. When $d\mu(\sigma) = 2e^{-2\sigma} d\sigma$, these spaces are simply denoted by $\mathcal{A}^p$. It is shown in [2] that they are spaces of convergent Dirichlet series on $C_{1/2}$.

In [7], the bounded composition operators on $H^2$, in other words the analytic functions $\Phi : C_{-\frac{1}{2}} \rightarrow C_{\frac{1}{2}}$ such that for any $f \in H^2$, $f \circ \Phi \in H^2$, are characterized. In [3], F. Bayart generalized this result to the space $H^p$ when $p \geq 1$.

We denote by $D$ the set of functions $f$ which admit a representation by a convergent Dirichlet series in some half-plane and for $\theta \in \mathbb{R}$, $C_\theta$ will be the following half-plane $\{ s \in \mathbb{C}, \Re(s) > \theta \}$. We shall denote $C_+ \subseteq D$. On the spaces $\mathcal{A}^p_\nu$, the following theorems have been proved in [1]:

**Theorem 1** ([1], Th.1). Let $\Phi : C_{-\frac{1}{2}} \rightarrow C_{\frac{1}{2}}$ be an analytic function of the form $\Phi(s) = c_0 s + \varphi(s)$ where $c_0 \geq 1$ and $\varphi \in D$. Then $C_\Phi$ is bounded on $\mathcal{A}^p_\nu$ if and only if $\varphi$ converges uniformly in $C_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(C_+) \subseteq C_\frac{1}{2}$. Moreover in this case, $C_\Phi$ is a contraction.

**Theorem 2** ([1], Th.2). Let $\Phi : C_{-\frac{1}{2}} \rightarrow C_{\frac{1}{2}}$ be in $D$. Then

(i) If $C_\Phi$ is bounded on $\mathcal{A}^p_\nu$ then $\Phi$ converges uniformly in $C_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(C_+) \subseteq C_{1/2}$.

(ii) If $\Phi$ converges uniformly in $C_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(C_+) \subseteq C_{1/2+\eta}$ with some $\eta > 0$ then $C_\Phi$ is bounded on $\mathcal{A}^p_\nu$.

In the sequel, we assume that $\mu$ is a probability measure on $(0, +\infty)$ such that $d\mu(\sigma) = h(\sigma) d\sigma$ where $h$ is a positive continuous function.

**Example.** Let $\alpha > -1$, we denote $\mu_\alpha$ the probability measure defined on $(0, +\infty)$ by

$$d\mu_\alpha(\sigma) = \frac{2^{\alpha+1}}{\Gamma(\alpha + 1)} \sigma^\alpha e^{-2\sigma} d\sigma.$$ We denote the corresponding space $\mathcal{A}^p_{\mu_\alpha}$ instead of $\mathcal{A}^p_{\mu_\alpha}$.

**Main Theorem.** Let $1 \leq p < +\infty$ and $C_\Phi$ be a bounded composition operator on $\mathcal{A}^p_\nu$. The following assertions are equivalent:

(i) $C_\Phi$ is invertible.

(ii) $C_\Phi$ is Fredholm.

(iii) $C_\Phi$ is an isometry.

(iv) $\Phi$ is a vertical translation: there exists $\tau \in \mathbb{R}$ such that for every $s \in C_+$, $\Phi(s) = s + i\tau$.

We point out that the result is false on the spaces $H^p$. F. Bayart proved that (i), (ii), (iii) are still equivalent on $H^p$ but obtained a different characterization for the isometric composition operators on $H^p$ (see [3]). For example, if $\Phi$ is defined for every $s \in C_+$ by $\Phi(s) = c_0 s$ with $c_0 \geq 2$, then $C_\Phi$ is an isometry on $H^p$ but not on $\mathcal{A}^p_\nu$. The same phenomenon appears in the framework of composition operators on the unit disk (see [10]).

In order to prove the main theorem, it suffices to show that (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv). Indeed, (i) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are clear.
2 Background material

Let \( f \) be a Dirichlet series of form (1). We do not recall the definition of abscissa of simple (resp. absolute) convergence denoted by \( \sigma_c \) (resp. \( \sigma_a \)), see [13] or [14] for more details. We shall need the two other following abscissas:

\[
\begin{align*}
\sigma_u(f) &= \inf \{ a | \text{ The series (1) is uniformly convergent for } \Re(s) > a \} \\
&= \text{abscissa of uniform convergence of } f.
\end{align*}
\]

\[
\begin{align*}
\sigma_b(f) &= \inf \{ a | \text{ the function } f \text{ has an analytic, bounded extension for } \Re(s) > a \} \\
&= \text{abscissa of boundedness of } f.
\end{align*}
\]

It is easy to see that \( \sigma_c(f) \leq \sigma_u(f) \leq \sigma_b(f) \). An important result is that \( \sigma_u(f) \) and \( \sigma_b(f) \) coincide: this is the Bohr’s theorem [see [5]]. This result is really useful for the study of \( \mathcal{H}^{\infty} \), the algebra of bounded Dirichlet series on the right half-plane \( \mathbb{C}_+ \) (see [11]). We shall denote by \( \| \cdot \|_{\infty} \) the norm on this space:

\[
\| f \|_{\infty} := \sup_{\Re(s) > 0} |f(s)|.
\]

We shall make a crucial use of the point evaluation in the proof of the Main Theorem: for every \( p \geq 1 \), the spaces \( \mathcal{H}^p \) and \( \mathcal{A}_p^\mu \) are spaces of holomorphic functions on \( \mathbb{C}_{1/2} \) and more precisely if \( \delta_s \) is the operator of point evaluation at \( s \in \mathbb{C}_{1/2} \), then by [3], Th3:

\[
\| \delta_s \|_{(\mathcal{H}^p)^*} = \zeta(2\Re(s))
\]

and by [2], Th1 the point evaluation is also bounded on the spaces \( \mathcal{A}_p^\mu \). Moreover \( \sigma_b(f) \leq 1/2 \) for any \( f \in \mathcal{A}_p^\mu \). For example when \( \mu = \mu_0 \), it is shown in [2], Cor1 that there exists a positive constant \( c_{\alpha, p} \) such that for every \( s \in \mathbb{C}_{1/2} \):

\[
\| \delta_s \|_{(\mathcal{A}_p^\mu)^*} \leq c_{\alpha, p} \left( \frac{\Re(s)}{2\Re(s) - 1} \right)^{\frac{2 + \alpha}{p}}.
\]

When \( p = 2 \), \( \mathcal{A}_2^\mu \) is a Hilbert space and it is easy to see that

\[
\| f \|_{\mathcal{A}_2^\mu} = \left( \sum_{n=1}^{+\infty} |a_n|^2 w_h(n) \right)^{1/2}
\]

where for every \( n \geq 1 \),

\[
w_h(n) = \int_0^{+\infty} n^{-2\sigma} h(\sigma) d\sigma.
\]

Thanks of the boundedness of the point evaluation at \( s \in \mathbb{C}_{1/2} \), we consider the following reproducing kernels defined for every \( w \in \mathbb{C}_{1/2} \) by

\[
K_\mu(s, w) = \sum_{n=1}^{+\infty} \frac{n^{-\sigma - w}}{w_h(n)}.
\]

For every \( f \in \mathcal{A}_2^\mu \) and \( s \in \mathbb{C}_{1/2} \), one has

\[
f(s) = \langle f, K_\mu(s, \cdot) \rangle_{\mathcal{A}_2^\mu}.
\]
**Example.** On the space $\mathcal{A}^p_\mu$, we simply denote $(\omega_n^n)$ the corresponding weight and then for every $n \geq 1$,

$$w_n^n = \frac{1}{(\log(n) + 1)^{\alpha + 1}}.$$ 

Let $\Phi : \mathbb{C}_+ \to \mathbb{C}_+$ be an analytic function such that $\Phi(s) = c_0s + \varphi(s)$ where $c_0$ is a nonnegative integer and $\varphi \in \mathcal{D}$. We shall say that $\Phi$ is a symbol if $C_\Phi$ is bounded on the spaces $\mathcal{A}^p_\mu$.

For $\sigma > 0$, we denote $\Phi_\sigma$ the translate of $\Phi$ by $\sigma$: $\Phi_\sigma(s) := \Phi(s + \sigma)$.

When $c_0 \geq 1$, thanks to the Theorem 1 we know that $\Phi$ is a symbol if and only $\varphi$ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. In this case, it is easy to see that for every $\sigma > 0$, $\Phi_\sigma - \sigma$ is also a symbol: indeed let $\sigma > 0$ and $s \in \mathbb{C}_+$, then

$$\Re(\Phi_\sigma(s) - \sigma) = \Re(c_0(\sigma + s)) + \Re(\varphi(s + s)) - \sigma > \sigma(c_0 - 1) + \Re(s) > 0$$

because $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ and $s \in \mathbb{C}_+$. Point out that this result can be seen as the Schwarz’s lemma in this framework.

**3 Proof of (ii) $\Rightarrow$ (iv)**

With help of Proposition 4.2 from [7], F. Bayart proved the following useful lemma.

**Lemma 1 ([3], Lem11).** Let $\Phi$ be a symbol. If $\Phi$ is not a vertical translation then there exists $\varepsilon$ and $\eta > 0$ such that

$$\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}.$$ 

**Proof of (ii) $\Rightarrow$ (iv).** We follow ideas from [3], Th14. Assume that $\Phi$ is not a vertical translation. By the previous lemma, there exists $\varepsilon$ and $\eta > 0$ such that

$$\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}.$$ 

We remark that each element of $\text{Im}(C_\Phi)$ is defined and bounded on $\mathbb{C}_{1/2-\varepsilon}$: indeed $\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$ and if $f \in \mathcal{A}^p_\mu$, $f$ is bounded on $\mathbb{C}_{1/2+\eta}$ (because $\sigma_\mu(f) \leq 1/2$).

Now by lemma [3], Lem9 we know that there exists $f \in \mathcal{H}^p$ such that the line $\Re(s) = 1/2$ is both abscissa of convergence and natural boundary for $f$. Because of the inclusion $\mathcal{H}^p \subset \mathcal{A}^p_\mu$, $f$ belongs to $\mathcal{A}^p_\mu$. We consider the following infinite dimensional subspace of $\mathcal{A}^p_\mu$:

$$F = \text{span}\{n^{-s}f, n \geq 1\} = f\mathcal{P}.$$ 

We shall show that $F \cap \text{Im}(C_\Phi) = \{0\}$ and consequently $\text{Codim}(\text{Im}(C_\Phi)) = +\infty$ which is a contradiction with (ii).

Let $h \in F \cap \text{Im}(C_\Phi)$, there exists $P \in \mathcal{P}$ such that $h = Pf$. If $h \neq 0$, there exists $s_0$ such that $\Re(s_0) = 1/2$ and $P(s_0) \neq 0$. But in this case, $f$ extends beyond $\mathbb{C}_{1/2}$ and then we obtain a contradiction because the line $\Re(s) = 1/2$ is a natural boundary for $f$. Finally $F \cap \text{Im}(C_\Phi) = \{0\}$. \qed
4 Proof of (iii) ⇒ (iv)

First we shall show that if $C_{\Phi}$ is an isometry then $c_0 \geq 1$. We need the following result.

**Lemma 2.** $\lim_{\Re(s) \to +\infty} \|\delta_s\|_{(A^1_p)^*} = 1$.

*Proof.* Let $s \in C_1$. By the reproducing kernel property on $A^2_p$ (or just by a simple computation), for any Dirichlet polynomial we have

$$P(s) = \int_0^{+\infty} \frac{1}{2T} \int_{-T}^{T} P(\sigma + it)K_\mu(s, \sigma + it) \, d\sigma \, d\mu(\sigma).$$

Now by definition of the norm of Dirichlet polynomials in $H^1$ (see definition 1 from [3]), we have

$$\|P\|_{A^1_p} = \left( \int_0^{+\infty} \frac{1}{2T} \int_{-T}^{T} |P(\sigma + it)| \, d\sigma \, d\mu(\sigma) \right).$$

Consequently

$$|P(s)| \leq \|P\|_{A^1_p} \times \|K_\mu(s, \cdot)\|_{\infty}.$$ 

Now

$$\|K_\mu(s, \cdot)\|_{\infty} = \sup_{w \in C_1} \left| \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w(h(n))} \right| \leq \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w(h(n))},$$

and we point out that $w_h(1) = 1$ so

$$\lim_{\Re(s) \to +\infty} \sup_{w \in C_1} \|\delta_s\|_{(A^1_p)^*} \leq \lim_{\Re(s) \to +\infty} \left| \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w(h(n))} \right| = 1.$$

On the other hand, it is clear that $\|\delta_s\|_{(A^1_p)^*} \geq 1$ and then we obtain the result. \( \Box \)

**Proposition 1.** Let $\Phi$ be a symbol. If $C_{\Phi}$ is a contraction then $c_0 \geq 1$.

*Proof.* Let $s \in C_{1/2}$. For every $f \in A^p_\mu$ we have

$$|f \circ \Phi(s)| \leq \|\delta_s\|_{(A^p_\mu)^*} \|f \circ \Phi\| \leq \|\delta_s\|_{(A^p_\mu)^*} \|C_{\Phi}\| \|f\|$$

and then

$$\frac{\|\delta_{\Phi(s)}\|_{(A^p_\mu)^*}}{\|\delta_s\|_{(A^p_\mu)^*}} \leq \|C_{\Phi}\|.$$ 

By inclusion of the spaces $A^p_\mu$ and the fact that $H^p \subset A^p_\mu$ with $\|\cdot\|_{A^p_\mu} \leq \|\cdot\|_{H^p}$ we obtain:

$$\frac{\|\delta_{\Phi(s)}\|_{(H^p)^*}}{\|\delta_s\|_{(A^1_p)^*}} \leq \|C_{\Phi}\|.$$ 

By Theorem 3 from [3], we obtain

$$\zeta(2\Re(\Phi(s)))^{1/p} \times \frac{1}{\|\delta_s\|_{(A^1_p)^*}} \leq \|C_{\Phi}\|.$$ 

5
Now assume $c_0 = 0$, then $\Phi(s) = \varphi(s) = \sum_{n=1}^{+\infty} c_n n^{-s}$ and $\Re(c_1) > 1/2$ (see proof of Lemma 3.3 from [7]). Finally thanks to the Lemma 2, when $\Re(s)$ goes to infinity we get
\[
\|C_\Phi\| \geq \zeta(2\Re(c_1))^{1/p} > 1
\]
and consequently $C_\Phi$ is not a contraction.

**Remark.** In the previous Lemma we actually used that for every $s \in C_{1/2}$, $\delta_s \circ C_\Phi = \delta_{\Phi(s)}$.

**Proof of (iii) \Rightarrow (iv).** Assume that $C_\Phi$ is an isometry. By the last lemma, $c_0 \geq 1$ and then we know that $\Phi : C_+ \to C_+$ thanks to the Theorem 1. One has
\[
\|2^{-s}\|_{A_\mu^p} = \|2^{-\Phi}\|_{A_\mu^p}.
\]
Now by [2], Th6,
\[
\int_0^{+\infty} \|2^{-\sigma-\bullet}\|_{H^p}^p - \|2^{-\Phi(\sigma+\bullet)}\|_{H^p}^p \, d\mu(\sigma) = 0.
\]

But
\[
\|2^{-\Phi(\sigma+\bullet)}\|_{H^p} = \|2^{-\sigma-(\Phi(\sigma+\bullet)\sigma)}\|_{H^p} = \|C_{\Phi_{\sigma} - \sigma}(2^{-\sigma-\bullet})\|_{H^p}.
\]
Thanks to the Schwarz’s lemma in this framework (recall that $c_0 \geq 1$) we know that $\Phi_{\sigma} - \sigma : C_+ \to C_+$. So by the Theorem 1, $C_{\Phi_{\sigma} - \sigma}$ is a bounded composition operator on $H^p$ and $\|C_{\Phi_{\sigma} - \sigma}\| \leq 1$. Then
\[
\|2^{-\Phi(\sigma+\bullet)}\|_{H^p} \leq \|2^{-\sigma-\bullet}\|_{H^p}.
\]
Consequently $2^{-\sigma} = \|2^{-\sigma-\bullet}\|_{H^p} = \|2^{-\Phi(\sigma+\bullet)}\|_{H^p}$ for every $\sigma > 0$ (recall that $h$ is a positive continuous function). Now by Lemma 1, if $\Phi$ is not a vertical translation, there exists $\varepsilon$ and $\eta > 0$ such that $\Phi(C_{1/2-\varepsilon}) \subset C_{1/2+\eta}$ and then for every $\sigma > 1/2 - \varepsilon$,
\[
2^{-\sigma} = \|2^{-\Phi(\sigma+\bullet)}\|_{H^p} \leq \|2^{-\Phi(\sigma+\bullet)}\|_{H^{\infty}} \leq 2^{-1/2-\eta}
\]
and this is obviously false.

**Remark.** Let $\mu$ be a probability measure on $(0, +\infty)$ such that $0 \in \text{Supp}(\mu)$ and $d\mu = h d\sigma$ where $h$ is a nonnegative function. If there exists an open interval $I$ such that $h$ is positive on $I$ then the theorem still holds. It is a consequence of the following lemma and some easy adaptations of the previous proof.

**Lemma 3.** Let $\Phi$ be a symbol with $c_0 \geq 1$. If $\Phi$ is not a vertical translation then for every $\varepsilon > 0$, there exists $\eta = \eta_\varepsilon > 0$ such that $\Phi(C_\varepsilon) \subset C_{\varepsilon+\eta}$.

**Proof.** First we assume that $\varphi$ is non constant then $\varphi : C_+ \to C_+$ and by Proposition 4.2 from [7], there exists $\vartheta > 0$ such that $\varphi(C_\varepsilon) \subset C_\vartheta$ and consequently $\Phi(C_\varepsilon) \subset C_{c_0\varepsilon+\vartheta}$. In this case, it suffices to choose $\eta = (c_0 - 1)\varepsilon + \vartheta$ which is positive because $c_0 \geq 1$.

If $\varphi$ is constant equals to $i\tau$ ($\tau \in \mathbb{R}$) and $c_0 > 1$ then $\Phi(C_\varepsilon) \subset C_{c_0\varepsilon}$ and it suffices to choose $\eta = (c_0 - 1)\varepsilon$.

If $\varphi$ is constant and equals to $c_1 \in C_+$ and $c_0 \geq 1$, $\Phi(C_\varepsilon) \subset C_{c_0\varepsilon+\Re(c_1)}$ and it suffices to choose $\eta = (c_0 - 1)\varepsilon + \Re(c_1)$.
References


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