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Odd permutations are nicer than even ones[☆]

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Abstract

We give simple combinatorial proofs of some formulas for the number of factorizations of permutations in \mathbb{S}_n as a product of two n -cycles, or of an n -cycle and an $(n - 1)$ -cycle.

Dedicated to Antonio Machì, on his 70th birthday

1. Introduction

The parameter number of cycles plays a central role in the algebraic theory of the symmetric group, however there are very few results giving a relationship between the number of cycles of two permutations and that of their product.

The first results on the subject go back to O. Ore, E. Bertram, R. Stanley (see [13], [1], [15]), who proved some existence theorems. These results allowed to obtain informations on the structure of the commutator subgroup of the alternating group.

After these pioneering works, enumerative results were investigated probably for the first time by D. Walkup (in [17]). In order to obtain enumeration formulas, different techniques were used. Some of them are purely combinatorial, while others use character theory. The key point of this second approach is a theorem, often attributed to Frobenius, expressing linearization coefficients for the product of conjugacy classes in the group algebra, as a sum of products of characters of irreducible representations.

The use of character theory was very fruitful to many authors and gave rise to many papers by R. Stanley, D. Jackson, D. Zagier, G. Jones, A. Goupil, G. Schaeffer, D. Poulalhon, P. Biane ([15], [9], [10], [18], [8], [14], [2]).

The main results using combinatorial methods were obtained by G. Boccara in a paper [3] containing many results, but using also some integrals of polynomials with a lot of computations, that can be considered as *ugly* by pure combinatorists. The most surprising result in Boccara's paper is that for any odd permutation in \mathbb{S}_n , the number

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of different ways to write it as a product of an n -cycle and an $(n - 1)$ -cycle is $2(n - 2)!$. This formula was puzzling, and M. P. Schützenberger asked for a bijective proof for it. Antonio Machì found such a proof (in [12]), giving a bijection, in which cycles of length 3 play a central role, while generally combinatorialists use products of transpositions (i.e. cycles of length 2) to represent permutations. His approach was applied by some of his students (see [5] and [6]), obtaining new enumerative results.

G. Boccara's method was recently used by R. Stanley (in [16]) in order to solve a conjecture of M. Bóna (mentioned in [4]), and to give another proof of a result obtained by D. Zagier (see [18], application 3 of Theorem 1). In his paper R. Stanley asks for purely combinatorial proofs of these results. Our paper gives some answers to his questions. Another approach to the same problem was developed independently by V. Feray and E. Vassilieva (see [7]).

Our approach is purely combinatorial, it consists in rederiving combinatorially Boccara's main result, and using this central theorem to derive many other results. The paper is organized as follows: after some definitions and notation in the first part, the second one gives a bijective proof of Boccara's main result which is different from Machì's one. In the third section we show how to deduce results for even permutations from those for odd ones, thus illustrating the title of our article, we also give a new proof of Zagier's result. The forth section is devoted to a new, purely combinatorial proof of Bóna's conjecture. We conclude in the fifth section with a new proof for a formula of Boccara (see [3] Corollary 4.8) enumerating factorizations of even permutations.

Notation and definitions

A permutation α of \mathbb{S}_n will be considered here as a bijection from $\{1, 2, \dots, n\}$ on to itself, so that $\alpha(i)$ will denote the image of i by the permutation α . The convention we use is that the product $\alpha\beta$ of the two permutations α and β is the permutation mapping i on $\alpha(\beta(i))$. Permutations will be represented by their cycles like:

$$\alpha = (i_1, \alpha(i_1), \dots)(i_2, \alpha(i_2), \dots) \cdots (i_m, \alpha(i_m), \dots).$$

The number of cycles of a permutation α will be denoted $z(\alpha)$. Hence α is an n -cycle of \mathbb{S}_n if and only if $z(\alpha) = 1$; moreover if β is an $(n - 1)$ -cycle of \mathbb{S}_n then $z(\beta) = 2$. Recall that a permutation α of \mathbb{S}_n is odd if it is the product of an odd number of transpositions, or equivalently if $n + z(\alpha)$ is an odd number, it is even otherwise.

Definition 1. *A factorization of a permutation σ of \mathbb{S}_n into two large cycles is a pair of permutations α, β such that: $\sigma = \alpha\beta$, the permutation α is an n -cycle, and the permutation β is an n -cycle when σ is even and an $(n - 1)$ -cycle when σ is odd.*

Example 1. *There are 4 factorizations of the permutation $\sigma = (1, 2, 3, 4)$ into two large cycles, they are:*

α	β	α	β
$(1, 3, 2, 4)$	$(1, 3, 2) (4)$	$(1, 2, 4, 3)$	$(2, 4, 3) (1)$
$(1, 4, 2, 3)$	$(1, 4, 3) (2)$	$(1, 3, 4, 2)$	$(1, 4, 2) (3)$

We use the following classical notation and definitions on partitions of an integer:

A *partition* of the integer n is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ such that:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0, \quad \text{and} \quad \sum_{i=1}^m \lambda_i = n.$$

We will write $\lambda \vdash n$ and define the length of λ as $\ell(\lambda) = m$. A partition $\lambda \vdash n$ will be said to be *odd* if $n + \ell(\lambda)$ is an odd number it to be *even* otherwise.

The cyclic type of a permutation is the sequence of the lengths of its cycles written in weakly decreasing order, defining hence a partition of n , whose length is the number of cycles of the permutation. Note that a permutation has the same parity as the partition given by its cyclic type.

Definition 2. An *ELC-factorization* of type λ (even factorization into large cycles) is a pair of n -cycles (α, β) such that $\alpha\beta = \sigma$ is of cyclic type λ .

An *OLC-factorization* of type λ (odd factorization into large cycles) is a pair (α, β) such that, α is an n -cycle, β an $(n-1)$ -cycle and $\alpha\beta = \sigma$ is of cyclic type λ .

The sets of ELC-factorizations and of OLC-factorizations of type λ are denoted $E(\lambda)$ and $O(\lambda)$ respectively. We will denote by E_n the set of all even factorizations into two large cycles of \mathbb{S}_n , and O_n that of all odd factorizations into two large cycles of \mathbb{S}_n . Clearly the number $|E_n|$ of elements in E_n is equal to $(n-1)!^2$ and $|O_n| = n!(n-2)!$.

2. Odd factorizations into two large cycles

In this section we give a new combinatorial proof of the following central result:

Theorem 1. For any odd permutation σ in \mathbb{S}_n the number of OLC-factorizations (α, β) such that $\alpha\beta = \sigma$ is equal to $2(n-2)!$.

The proof we give here uses a technique introduced by A. Lehman in order to enumerate the number of maps with one vertex and one face embedded on an orientable surface [11]¹. The key point is to represent factorizations (α, β) of a permutation σ in 2 large cycles by using a sequence of integers $u = b_0, b_1, b_2, \dots, b_{n-1}$, which represents the permutation β , and a directed graph $G_{\sigma, u}$ which partially represents the permutation α .

A permutation α of \mathbb{S}_n is classically represented by a (directed) graph, which we denote G_α , it has $\{1, 2, \dots, n\}$ as vertex set and contains n arcs, each one has head i and tail $\alpha(i)$ for $1 \leq i \leq n$. The circuits of G_α correspond to the cycles of α .

When $\sigma = \alpha\beta$ where $\beta = (b_1, b_2, \dots, b_{n-1})(b_n)$ is an $(n-1)$ -cycle, then (since $\alpha = \sigma\beta^{-1}$), the graph G_α may be obtained from σ , by joining b_i to $\sigma(b_{i-1})$ for $i = 2, \dots, n-1$, then adding an arc from b_1 to $\sigma(b_{n-1})$ and an arc from b_n to $\sigma(b_n)$. This remark leads to an algorithm allowing to compute factorizations of σ into two large cycles. Lehman's contribution consists in adding an element b_0 , allowing to obtain enumeration formulas.

¹This is a manuscript he gave to his students which we obtained thanks to Timothy Walsh

Definition 3. Let σ be a permutation of \mathbb{S}_n , and let $u = b_0, b_1, \dots, b_k$ be a sequence of positive integers not greater than n and such that b_1, b_2, \dots, b_k are all distinct. The graph $G_{\sigma,u}$ associated to σ, u has for vertices the integers $\{1, 2, \dots, n\}$ and an arc for each j , $1 \leq j \leq k$ with head b_j and tail $\sigma(b_{j-1})$.

Definition 4. A sequence of integers $u = b_0, b_1, \dots, b_{n-1}$ is a Lehman sequence of the permutation σ in \mathbb{S}_n , if no two b_i for $i \geq 1$ are equal and if the graph $G_{\sigma,u}$ has no circuit.

Let u be a Lehman sequence of the permutation σ in \mathbb{S}_n , then the graph $G_{\sigma,u}$ has n vertices and $n - 1$ arcs, each vertex b_1, b_2, \dots, b_{n-1} is the head of an arc and each vertex $\sigma(b_1), \sigma(b_2), \dots, \sigma(b_{n-2})$ is the tail of an arc. Notice that no vertex can be the head of two arcs, however $\sigma(b_0)$ may be the tail of either one or two arcs.

Example 2. The 6 Lehman sequences of the permutation $(1, 2, 3)$ are given below with their graphs:

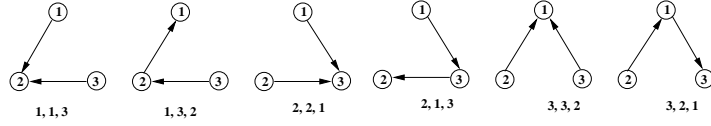


Figure 1: The graphs of the 6 Lehman sequences of $\sigma = (1, 2, 3)$

To any Lehman sequence $u = b_0, b_1, b_2, \dots, b_{n-1}$ one associates an $(n - 1)$ -cycle $\beta = (b_1, b_2, \dots, b_{n-1})(b_n)$ where b_n is the element of $\{1, 2, \dots, n\}$ distinct from all the $b_i, 1 \leq i \leq n - 1$. We will denote $\beta = \Lambda(u)$.

Proposition 1. Let σ be an odd permutation of \mathbb{S}_n , let $u = b_0, b_1, \dots, b_{n-1}$ be a Lehman sequence of σ and $\beta = \Lambda(u)$, then the permutation $\alpha = \sigma\beta^{-1}$ is an n -cycle.

Proof : Let $G'_{\sigma,u}$ be the graph obtained from $G_{\sigma,u}$ deleting the arc $(b_1, \sigma(b_0))$. In $G'_{\sigma,u}$ the vertices b_1 and b_n are the head of no arc, and $\sigma(b_n)$ and $\sigma(b_{n-1})$ are the tail of no arc. The other vertices are the head of exactly one arc, and the tail of exactly one. This graph is the union of two paths, where one may be reduced to a single vertex. We have thus one of the two situations depicted in Figure 2:



Figure 2: The two situations in the proof of Proposition 1.

In the first case $\alpha(b_1) = \sigma\beta^{-1}(b_1) = \sigma(b_{n-1})$, and $\alpha(b_n) = \sigma\beta^{-1}(b_n) = \sigma(b_n)$ hence the graph G_α which is obtained by adding the two arcs $(b_1, \sigma(b_{n-1}))$, and $(b_n, \sigma(b_n))$

to $G'_{\sigma,u}$ has two circuits, and the permutation α has two cycles. This implies that σ is the product of two permutations each one having two cycles, contradicting the fact that σ is odd. Hence only the second case has to be considered and $\alpha(b_1) = \sigma(b_{n-1})$, $\alpha(b_n) = \sigma(b_n)$, proving that α has only one cycle. \square

Proposition 2. *The number of Lehman sequences of a permutation σ of \mathbb{S}_n is equal to $n!$.*

Proof : One may build all the Lehman sequences of σ using the following algorithm, which proceeds as an iteration of n steps:

- Choose any b_0 among $\{1, 2, \dots, n\}$, and b_1 different from $\sigma(b_0)$ in the same set,
- For $i = 2, \dots, n-1$ choose b_i in $\{1, 2, \dots, n\}$ different from b_1, b_2, \dots, b_{i-1} and from b'_i , the vertex which is the tail of the longest path with head $\sigma(b_{i-1})$ in the graph $G_{\sigma, b_0 b_1 \dots b_{i-1}}$.

Notice that at each step i the number of possible choices for b_i is $n-i$. Indeed it is clear that there are n possible choices for b_0 and $n-1$ choices for b_1 ; moreover since b'_i is a vertex of out-degree 0 of $G_{\sigma, b_0 b_1 \dots b_{i-1}}$, it cannot be equal to any of the b_j for $1 \leq j < i$, as these vertices are all of out-degree 1. Hence at each step the number of possible choices decreases by one, then the number of sequences built by the algorithm is $n!$.

Since the algorithm closely follows the definition of a Lehman sequence it is clear that it builds exactly all the Lehman sequences of σ . \square

Proposition 3. *Let σ be an odd permutation of \mathbb{S}_n and let (α, β) be a factorization of σ into two large cycles. The number of Lehman sequences u of σ such that $\beta = \Lambda(u)$ is $\frac{n(n-1)}{2}$.*

Proof : Let α, β be a factorization of σ into two large cycles and let b_n be the fixed point of β . The Lehman sequences u of σ such that $\Lambda(u) = \beta$ are all obtained from β in the following way:

- Choose b_1 as any element of the large cycle of β ,
- Write this cycle of β as $(b_1, b_2, \dots, b_{n-1})$, set $v = b_1, b_2, \dots, b_{n-1}$ and build the graph $G_{\sigma, v}$,
- Choose b_0 such that the arc $(b_1, \sigma(b_0))$ does not create a circuit in the graph $G_{\sigma, v}$.

Now it is necessary to examine how $G_{\sigma, v}$ is built from G_α , the latter being a circuit with n vertices and n arcs. Clearly two arcs are deleted, the arc $(b_n, \sigma(b_n))$ is one of them, then by the choice of b_1 , the arc $(b_1, \sigma(b_{n-1}))$ is the other.

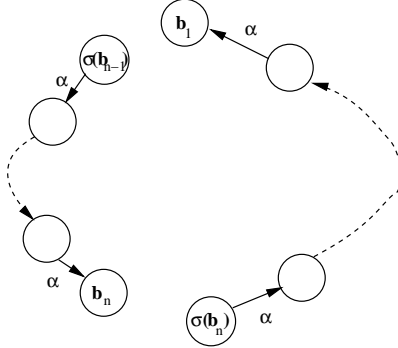


Figure 3: Deletion of two arcs in G_α

So that the graph obtained has shape given in Figure 3.

For each choice of b_1 the number of vertices of the path P from $\sigma(b_{n-1})$ to b_n takes any value from 1 to $(n-1)$, and all these values are obtained. Now b_0 has to be chosen such that the arc $(b_1, \sigma(b_0))$ does not create a circuit in the graph; hence $\sigma(b_0)$ must be among the vertices of this path P . This gives a total number of choices equal to sum of the lengths of these paths for all the possible choices of b_1 , that is $1 + 2 + \dots + (n-1)$. Hence the number of Lehman sequences u such that $\Lambda(u) = \beta$ is $\frac{n(n-1)}{2}$. \square

The proof of Theorem 1 then follows from Propositions 1, 2 and 3.

3. From odd factorizations to even ones

In this section we show how to build O_{n+1} from E_n . We express for a given odd partition $\mu \vdash n+1$, the set $O(\mu)$ as a union of sets obtained from the $E(\lambda)$ for some partitions $\lambda \vdash n$ closely related to μ . We will then use this result to determine the number of factorisations of even permutations with k cycles, giving a new combinatorial proof of a result of D. Zagier.

3.1. Building factorizations of even permutations

We will use transpositions, i. e. permutations of \mathbb{S}_n with $n-2$ fixed points and one cycle of length 2; a transposition will be denoted (i, j) , where i, j are the two elements which are not fixed.

Remark 1. Let α be a permutation in \mathbb{S}_n and let i be an integer such that $1 \leq i \leq n$, then $\alpha(i, n+1)$ is a permutation of \mathbb{S}_{n+1} obtained from α inserting $n+1$ after i in its cyclic representation and $(i, n+1)\alpha$ is a permutation of \mathbb{S}_{n+1} obtained from α inserting $n+1$ before i in its cyclic representation.

Let (α, β) be an element of E_n and let i, j be two integers such that $1 \leq i \leq n$ and $1 \leq j \leq n+1$. We define the factorization $\Phi_{i,j}(\alpha, \beta) = (\alpha', \beta')$ in O_{n+1} in the following way:

- Insert $n+1$ in the cyclic representation of α immediately before i giving a permutation α_1

- Let $\beta_1 \in \mathbb{S}_{n+1}$ be equal to the permutation β to which is added $n+1$ as a fixed point.
- Exchange j and $n+1$ in α_1 and in β_1 obtaining α' and β' .

More precisely we have, using the fact that exchanging $n+1$ and j is obtained by conjugation by $(j, n+1)$:

$$\alpha' = (j, n+1)(i, n+1)\alpha(j, n+1), \quad \beta' = (j, n+1)\beta(j, n+1) \quad (1)$$

Definition 5. Let λ be a partition of n and μ a partition of $n+1$; λ is covered by μ , if they have the same length and if $\lambda_i = \mu_i$ for all $1 \leq i \leq \ell(\mu)$, except for only one value of i . This will be denoted by $\lambda < \mu$

Remark 2. When $\lambda < \mu$ then for the unique k such that $\lambda_k \neq \mu_k$ we have $\mu_k = \lambda_k + 1$. Moreover, for a given μ the number of different λ such that $\lambda < \mu$ is equal to the number of different values taken by the μ_i that are larger than 1.

For instance $\mu = (4, 3, 3, 1)$ covers 2 partitions: $\lambda = (3, 3, 3, 1)$ and $\lambda = (4, 3, 2, 1)$.

Theorem 2. Let (α, β) be an even factorization of type $\lambda \vdash n$. For any $1 \leq i \leq n$ and $1 \leq j \leq n+1$, $\Phi_{i,j}(\alpha, \beta)$ is a factorization which type μ is such that $\lambda < \mu$.

Conversely let $(\alpha', \beta') \in O(\mu)$ with $\mu \vdash n+1$, then there exist two integers i, j and a factorization $(\alpha, \beta) \in E_n$ such that $1 \leq i \leq n$, $1 \leq j \leq n+1$ and $(\alpha', \beta') = \Phi_{i,j}((\alpha, \beta))$. Moreover i, j, α, β are uniquely determined.

Proof : Considering the first part of the theorem, let $\sigma = \alpha\beta$ and $\Phi_{i,j}((\alpha, \beta)) = (\alpha', \beta')$. It is easy to check that α' is an $(n+1)$ -cycle, since $(i, n+1)\alpha$ is an $(n+1)$ -cycle, and conjugation does not modify the cyclic type. For the same reason β' is an n -cycle, which fixed point is j . The type of $\Phi_{i,j}(\alpha, \beta)$ is the cyclic type of the permutation $(j, n+1)(i, n+1)\alpha\beta(j, n+1)$ but, since conjugation does not modify the cyclic type of a permutation, this is also the cyclic type of $\sigma' = (i, n+1)\alpha\beta$. But from Remark 1 this last permutation is obtained from $\alpha\beta$ inserting $n+1$ before i in its cycle, showing that its cyclic type is a partition μ covering λ .

We now prove the converse. Let $(\alpha', \beta') \in O(\mu)$ and let j be the point fixed by β' . Consider the formula giving α' in (1):

$$\alpha' = (j, n+1)(i, n+1)\alpha(j, n+1)$$

We get $\alpha'(j) = n+1$ if $i = j$, and $\alpha'(j) = i$ if $j \neq i$; this allows to determine uniquely i from $\alpha'^{-1}(j)$. We then obtain (α, β) by equation (1). Since $n+1$ is a fixed point in both α and β , these two permutations may be considered as elements of \mathbb{S}_n .

□

3.2. Permutations with a given number of cycles

We are now able to give a combinatorial proof of the following result due to D. Zagier (see [18], application 3 of Theorem 1.)

Corollary 1. *Let n, k be two integers such that $1 \leq k \leq n$ and $n - k$ is even. The probability that the product of two n -cycles of \mathbb{S}_n (taken randomly) has k cycles is equal to the probability that an odd permutation of \mathbb{S}_{n+1} has k cycles.*

Proof : We compute the number $e_{n,k}$ of factorizations $(\alpha, \beta) \in E_n$ such that $\alpha\beta$ has k cycles, for that we consider first the number of elements (α', β') in O_{n+1} such that $\alpha'\beta'$ has the same number k of cycles. By Theorem 1, this number is equal to $2(n-1)!$ times the number of elements of \mathbb{S}_{n+1} with k cycles. That is:

$$2(n-1)!s_{n+1,k}$$

where $s_{n,p}$ is the unsigned Stirling number of the first kind.

By Theorem 2, for any factorization of $(\alpha, \beta) \in E_n$ we obtain by $\Phi_{i,j}$, $n(n+1)$ different factorizations $(\alpha', \beta') \in O_{n+1}$. For all of them, $\alpha'\beta'$ has k cycles, and any element of O_{n+1} with k cycles is obtained in this way. Hence:

$$e_{n,k} = \frac{2(n-1)!s_{n+1,k}}{n(n+1)}$$

Since the number of elements of E_n is $(n-1)!^2$, the probability for the product of two n -cycles of \mathbb{S}_n (taken randomly) has k cycles is:

$$\frac{e_{n,k}}{(n-1)!^2} = \frac{2s_{n+1,k}}{(n+1)!}$$

Now the number of odd permutations in \mathbb{S}_{n+1} is $\frac{(n+1)!}{2}$ and $s_{n+1,k}$ of them have k cycles, proving the result. \square

4. Permutations in which two given elements are in the same cycle

In this section we consider the following question conjectured by M. Bóna and solved by R. Stanley [16]; Proposition 5 below was obtained in collaboration with Valentin Feray and Amarpreet Rattan.

Definition 6. *A factorization (α, β) into two large cycles of \mathbb{S}_n for $n \geq 2$ is said to be connecting if 1 and 2 are in the same cycle of $\alpha\beta$.*

Let p_n be the number of connecting factorizations of E_n and q_n be the number of those in O_n . It is easy to compute q_n :

Proposition 4. *The number of connecting factorizations in O_n is given by:*

$$q_n = (n-2)! \left(\frac{n!}{2} + (-1)^n (n-2)! \right) \quad (2)$$

Proof : We first begin by computing the numbers a_n and b_n of even and odd permutations of \mathbb{S}_n such that 1 and 2 are in the same cycle. Any even permutation of \mathbb{S}_n is obtained either from an even one in \mathbb{S}_{n-1} adding n as a fixed point or from an odd one in \mathbb{S}_{n-1} adding n inside a cycle. Since this last operation can be done in $n-1$ ways, and since a similar statement holds for odd permutations of \mathbb{S}_n we have for $n \geq 3$:

$$a_n = a_{n-1} + (n-1)b_{n-1}, \quad b_n = b_{n-1} + (n-1)a_{n-1}$$

the difference of these two numbers is equal to:

$$a_n - b_n = (2-n)(a_{n-1} - b_{n-1})$$

Multiplication by the transposition $(1, 2)$ is a bijection between permutations in \mathbb{S}_n where 1 and 2 are in the same cycle and those where they are not, hence giving;

$$a_n + b_n = \frac{n!}{2}$$

Each odd permutation with 1 and 2 in the same cycle gives $2(n-2)!$ connecting factorizations in O_n hence:

$$q_n = 2(n-2)!b_n = 2(n-2)! \left(\frac{n!}{4} + \frac{(-1)^n(n-2)!}{2} \right)$$

□

We again find results for even factorizations using our result for odd ones.

Proposition 5. *The number of odd connecting factorizations in \mathbb{S}_{n+1} and that of the even ones in \mathbb{S}_n are related by:*

$$q_{n+1} = (n+2)(n-1)p_n + 2(n-1)!^2 \quad (3)$$

Proof : We divide the set of connecting odd factorizations (α', β') in \mathbb{S}_{n+1} into three subsets X_{n+1} , Y_{n+1} and Z_{n+1} according to the value of the fixed point j of β' . Those in X_{n+1} , Y_{n+1} have $j = 1$ and $j = 2$ respectively, and those in Z_{n+1} are such that $j > 2$.

We use Theorem 2 to compute the number z_{n+1} of elements in Z_{n+1} . Each factorization (α', β') in this set is equal to a $\Phi_{i,j}(\alpha, \beta)$ such that (α, β) is a connecting even factorization in E_n , and $j = 3 \leq j \leq n+1$. This gives $n-1$ possible values for j and since there are n possible values for i we have:

$$z_{n+1} = n(n-1)p_n.$$

The computation of the numbers x_{n+1}, y_{n+1} (which are equal by symmetry) of elements in X_{n+1} and Y_{n+1} is more intricate. Notice that if $(\alpha', \beta') = \Phi_{i,j}(\alpha, \beta)$ is such that $\beta'(1) = 1$ then $j = 1$ and (α, β) is not necessarily connecting.

Nevertheless we can obtain an inductive formula for x_{n+1} . We have for $(\alpha', \beta') \in X_{n+1}$:

$$\alpha' \beta' = (1, n+1)(i, n+1)\alpha\beta(1, n+1)$$

for a given i . In order that (α', β') be connecting we must have one of the two following conditions:

- $i = 2$
- $i \neq 2$ and (α, β) is such that i and 2 are in the same cycle of $\alpha\beta$.

If $i = 2$, any factorization in E_n will give a connecting factorization, the number of such is $(n-1)!^2$.

If $i \neq 2$, we have to choose a factorization such that i and 2 are in the same cycle of $\alpha'\beta'$, and the number of such factorizations is p_n by symmetry. Since there are $n-1$ possible choices for i this gives $x_{n+1} = (n-1)!^2 + (n-1)p_n$.

Hence using $x_{n+1} = y_{n+1}$ we get:

$$q_{n+1} = x_{n+1} + y_{n+1} + z_{n+1} = 2(n-1)!^2 + 2(n-1)p_n + n(n-1)p_n.$$

□

Corollary 2. *Taking two n -cycles at random the probability that their product contains 1 and 2 in the same cycle is $\frac{1}{2}$ if n is odd and $\frac{1}{2} - \frac{2}{(n+2)(n-1)}$ if n is even.*

Proof : From the two Propositions above we get:

$$(n+2)(n-1)p_n = \frac{(n-1)!^2}{2} (n^2 + n - 2(-1)^n - 4)$$

Since $n^2 + n - 2(-1)^n - 4 = (n+2)(n-1) - 2((-1)^n + 1)$ we obtain:

$$p_n = \frac{(n-1)!^2}{2} \left(1 - 2 \frac{(-1)^n + 1}{(n+2)(n+1)} \right)$$

The probability we are seeking for is $\frac{p_n}{(n-1)!^2}$ completing the proof.

□

5. A formula for the number of elements of $E(\lambda)$

We have proved in Section 2 that the number of factorizations of an odd permutation of \mathbb{S}_n into two large cycles depends only on n . For an even permutation σ this number depends on the conjugacy class of σ which is determined by a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of n . A formula was given for these numbers by G. Boccara (in [3] Corollary 4.8), we give a new proof of this formula here based on Theorem 2.

We give first some notation for such a partition and introduce the function $P(\lambda)$ which is central in this enumeration.

- Let I be the set of indexes of the elements of λ namely: $I = \{1, 2, \dots, m\}$.
- For any subset X of I , denote $s_X(\lambda)$ the sum of the parts of λ whose indexes are in X :

$$s_X(\lambda) = \sum_{i \in X} \lambda_i$$

- The number $P(\lambda)$ is the sum given below, where $|X|$ is the number of elements of X :

$$P(\lambda) = \sum_{X \subseteq I} (-1)^{s_X(\lambda) + |X|} s_X(\lambda)! (n - s_X(\lambda))! \quad (4)$$

Let c_λ denotes the number of permutations of cyclic type λ and e_λ the number of elements in $E(\lambda)$.

Theorem 3. *For any even partition λ :*

$$e_\lambda = \frac{c_\lambda P(\lambda)}{n(n+1)} \quad (5)$$

5.1. Some properties of the function $P(\lambda)$

In order to prove this Theorem we need to prove some properties of the function $P(\lambda)$.

Lemma 1. *If λ has length 1 we have:*

$$P((n)) = 2n! \text{ if } n \text{ is odd, and } P((n)) = 0 \text{ otherwise.} \quad (6)$$

Let $\lambda \vdash n$ be an even partition with at least one part equal to 1, and let $\lambda' \vdash n-1$ be the partition obtained from it deleting one of these parts. Then:

$$P(\lambda) = (n+1)P(\lambda') \quad (7)$$

Proof : For $\lambda = (n)$ the sum giving $P(\lambda)$ reduces to two elements with $X = \emptyset$ and $X = I$, for each of these we have: $s_\emptyset(\lambda) = 0$ and $s_I(\lambda) = n$ giving (6).

To prove (7) we notice that the sum giving $P(\lambda)$ have twice the number of terms than that giving $P(\lambda')$. Each X giving the term $u'_X = (-1)^{s_X(\lambda') + |X|} s_X(\lambda')! (n-1 - s_X(\lambda'))!$ in $P(\lambda')$ gives the two terms $u_X = s_X(\lambda')! (n - s_X(\lambda'))!$ and $v_X = (s_X(\lambda') + 1)! (n - s_X(\lambda') - 1)!$ in $P(\lambda)$ with the same sign as u'_X . These two terms sum up to:

$$u_X + v_X = s_X(\lambda')! (n-1 - s_X(\lambda'))! (n - s_X(\lambda') + s_X(\lambda') + 1) = (n+1)u'_X.$$

Hence ending the proof. □

Lemma 2. *Let $\mu \vdash (n+1)$ be an odd partition of length m , where all the m parts are strictly greater than 1. Let for $1 \leq i \leq m$, $\lambda^{(i)}$ be the sequence obtained from μ deleting one element in part i . Then for any non empty X strictly included in $\{1, 2, \dots, m\}$ we have*

$$\sum_{i=1}^m (-1)^{s_X^i} \mu_i s_X^i! (n - s_X^i)! = 0 \quad (8)$$

where $s_X^i = \sum_{j \in X} \lambda_j^{(i)}$.

Proof : Let $p = s_X(\mu)$, since $\lambda_i^{(i)} = \mu_i - 1$ and $\lambda_j^{(i)} = \mu_j$ for $i \neq j$ we have:

$$s_X^i = \begin{cases} p & \text{if } i \notin X \\ (p-1) & \text{if } i \in X \end{cases}$$

Hence

$$\begin{aligned} \sum_{i=1}^m (-1)^{s_X^i} \mu_i s_X^i! (n - s_X^i)! &= \sum_{i \notin X} (-1)^{s_X^i} \mu_i s_X^i! (n - s_X^i)! + \sum_{i \in X} (-1)^{s_X^i} \mu_i s_X^i! (n - s_X^i)! \\ &= (-1)^p (p!(n-p)! \sum_{i \notin X} \mu_i - (p-1)!(n-p+1)! \sum_{i \in X} \mu_i) \end{aligned}$$

But since $p = \sum_{j \in X} \mu_j$ we have $\sum_{j \notin X} \mu_j = n+1-p$ and the result. \square

Proposition 6. *Let μ be an odd partition of $n+1$ then:*

$$\sum_{\lambda \prec \mu} c_\lambda u(\lambda, \mu) P(\lambda) = 2n! c_\mu \quad (9)$$

Proof : We distinguish two cases:

- If μ has no part of length 1. Then we use Lemma 2 and add equations (8) multiplied by $(-1)^{|X|}$ for all subsets X of $I = \{1, 2, \dots, m\}$. For $X = I$ we have for all i : $s_X^i = n$ and $|X| + s_X^i = n+m$, which is even. For $X = \emptyset$ we have $s_X^i = 0$ and $|X| + s_X^i = 0$, hence we get (since $\sum_{i=1}^m \mu_i = n+1$):

$$\sum_{X \subset I} \sum_{i=1}^m (-1)^{s_X^i + |X|} \mu_i s_X^i! (n - s_X^i)! = 2(n+1)!$$

Reordering any sequence $\lambda^{(i)}$ we obtain a partition λ such that $\lambda \prec \mu$. Moreover for a given λ such that $\lambda \prec \mu$, the number of the sequences $\lambda^{(j)}$ which give λ after reordering it, is equal to n_i , the number of occurrences of μ_i in μ (where i is the index such that $\lambda_i = \mu_i - 1$).

Hence reversing the order of the two sums in the equation above we obtain:

$$\sum_{\lambda \prec \mu} n_i \mu_i P(\lambda) = 2(n+1)!$$

Multiplying by c_μ we get

$$\sum_{\lambda \prec \mu} c_\mu n_i \mu_i P(\lambda) = 2c_\mu (n+1)!$$

Since μ and $\lambda \prec \mu$ differ only by 1 in position i we have:

$$n_i \mu_i c_\mu = (n+1) u(\lambda, \mu) c_\lambda$$

and the result.

- If μ has k parts equal to 1 then we proceed by induction. Above we have proved $k = 0$. Let μ be a partition with k parts equal to 1. Let μ' be obtained from μ deleting one part equal to 1. By the induction hypothesis we have

$$\sum_{\lambda' \prec \mu'} c_{\lambda'} u(\lambda', \mu') P(\lambda') = 2(n-1)! c_{\mu'}$$

But there are as many partitions λ' such that $\lambda' \prec \mu'$ as those λ such that $\lambda \prec \mu$ each of those λ' is obtained from a λ deleting one part equal to 1. For these we have $c_{\lambda'} = \frac{c_{\lambda} k}{n}$. Moreover $c_{\mu'} = \frac{c_{\mu} k}{n+1}$ and $u(\lambda', \mu') = u(\lambda, \mu)$.

Giving:

$$\sum_{\lambda' \prec \mu'} \frac{c_{\lambda} k}{n} u(\lambda, \mu) P(\lambda') = \frac{c_{\mu} k}{n+1} 2(n-1)!$$

Then by Lemma 1, we have $(n+1)P(\lambda') = P(\lambda)$ and the result. □

5.2. Proof of Theorem 3

Let λ be an even partition of n and let μ be such that $\lambda \prec \mu$, where $\mu_k = \lambda_k + 1$ then we denote by $u(\lambda, \mu)$ the sum of the parts of size λ_k in λ , that is

$$u(\lambda, \mu) = i_{\lambda_k} \lambda_k$$

where i_p denotes the number of λ_j equal to p . Then we will use the following Lemma giving an elementary proof of a formula allowing to compute the $e_{\lambda} = |E(\lambda)|$ due to V. Feray and E. Vassilieva [7]. This lemma may be considered as a Corollary of Theorem 2.

Lemma 3. *Let μ be an odd partition of $n+1$ then:*

$$2c_{\mu}(n-1)! = (n+1) \sum_{\lambda \prec \mu} u(\lambda, \mu) e_{\lambda} \quad (10)$$

Where c_{μ} is the number of permutations with cyclic type μ and the range of the sum in the right hand side consists of all the partitions λ covered by μ .

Proof: The left hand side of the above equality is the number of factorizations (α', β') in O_{n+1} such that $\alpha'\beta'$ has cyclic type μ . By Theorem 2, each of such factorization is the image by some $\Phi_{i,j}$ of a factorization (α, β) in E_n , such that the cyclic type of $\alpha\beta$ is a partition λ covered by μ .

Conversely let λ be a partition covered by μ such that $\lambda_k + 1 = \mu_k$ and let $(\alpha, \beta) \in E(\lambda)$. Then $\Phi_{i,j}(\alpha, \beta) \in O(\mu)$ if and only if i belongs to a cycle of length λ_k in the permutation $\alpha\beta$. Since there are $u(\lambda, \mu)$ elements in $\{1, 2, \dots, n\}$ belonging to a cycle of length λ_k of $\alpha\beta$ the number of possible values for i is $u(\lambda, \mu)$ and the number of possible values for j is $n+1$. Hence the number of factorization in $O(\mu)$ obtained as a $\Phi_{i,j}(\alpha, \beta)$ for $(\alpha, \beta) \in E(\mu)$ is $(n+1)u(\lambda, \mu)a_{\lambda}$ proving the result. □

Consider equations (10) for all $\mu \vdash (n+1)$ of length m . For $m = 1$ there is only one equation allowing to obtain for n odd:

$$e_{(n)} = \frac{2(n-1)!^2}{n+1}.$$

For a given $m > 1$ each partition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ of length m gives an equation:

$$(n+1) \sum_{\lambda \prec \mu} u(\lambda, \mu) e_\lambda = 2c_\mu (n-1)!.$$

In this equation all the partitions λ appearing in the sum, except one of them, contain the same number of highest parts equal to μ_1 . Using induction on the value of the highest part λ_1 of λ , this proves that these equations enable to obtain uniquely the e_λ for the even partitions λ of length m .

Now let us compare with what we obtained for $P(\lambda)$ in Proposition 6:

$$\sum_{\lambda \prec \mu} c_\lambda u(\lambda, \mu) P(\lambda) = 2n!c_\mu$$

This clearly shows that the $\frac{c_\lambda P(\lambda)}{n(n+1)}$ satisfy the same set of equations as the e_λ . Since this set of equations has a unique solution this shows

$$\frac{c_\lambda P(\lambda)}{n(n+1)} = e_\lambda$$

□

The following result may be useful for the computation of e_λ when some of the λ_i are equal to 1.

Remark 3. Let λ be an even partition of length m with $k > 0$ parts equal to 1, let λ' be obtained from λ deleting a part of size 1, then the following relation allows to compute e_λ inductively:

$$e_\lambda = e_{\lambda'} \frac{n(n-1)}{k}$$

Proof : Notice that from any factorization (α', β') in $E(\lambda')$ one builds $n-1$ factorizations in $E(\lambda)$ inserting n before an i in the cycle of β' and after that element i in the cycle of α' . The factorizations (α, β) obtained in this way are such that n is a fixed point of $\alpha\beta$. Moreover any factorization (α, β) of cyclic type λ and such that n is a fixed point of $\alpha\beta$ can be obtained in that way. This proves that the number of these factorizations is equal to $(n-1)e_{\lambda'}$. Now given k integers $1 \leq i_1, i_2, \dots, i_k \leq n$, the number of factorizations (α, β) in $E(\lambda)$ such that i_1, i_2, \dots, i_k are fixed points of $\alpha\beta$ is independent of i_1, i_2, \dots, i_k giving :

$$\binom{n-1}{k-1} e_\lambda = \binom{n}{k} (n-1) e_{\lambda'}$$

and the result follows from : $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$.

□

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References

- [1] E. Bertram. Even permutations as a product of two conjugate cycles. *J. Comb. Theory (A)*, 12:368—380, 1972.
- [2] P. Biane. Nombre de factorisations d'un grand cycle. *Seminaire Lotharingien de Combinatoire*, 51:B51a, 2004.
- [3] G. Boccara. Nombre de représentations d'une permutation comme produit de deux cycles de longueurs données. *Discrete Math.*, 29:105—134, 1980.
- [4] M. Bóna and R. Flynn. The average number of block interchanges needed to sort a permutation and a recent result of Stanley. *Preprint, arXiv*, 081.0740, 2008.
- [5] L. Cangelmi. Factorizations of an n -cycle into two n -cycles. *Eur. J. Comb.*, 24:849—853, 2003.
- [6] L. Cangelmi and P. Cellini. A representation of even permutations. *Eur. J. Comb.*, 19:419—422, 1998.
- [7] V. Feray and E. Vassilieva. Linear coefficients of Kerov's polynomials: bijective proof and refinement of Stanley's result. In *Proceedings FPSAC 2010, San Francisco*, 2010.
- [8] A. Goupil and G. Schaeffer. Factoring n -cycles and counting maps of given genus. *Eur. J. Combinatorics*, 1:819—834, 1998.
- [9] D. M. Jackson. Counting cycles in permutations by group characters with an application to a topological problem. *Trans. Amer. Math. Soc.*, 299:785—801, 1987.
- [10] G. Jones. Trees, permutations and characters. Unpublished manuscript, University of Southampton, 1997.
- [11] A. Lehman. Enumeration of one-vertex one-face rooted maps. Unpublished manuscript, University of Toronto, 1973.
- [12] A. Machì. On the representation of odd permutations. *European Journal of Combinatorics*, 13:273—277, 1992.
- [13] O. Ore. Some remarks on commutators. *Proc. Amer. Math. Soc.*, 2:307—314, 1951.
- [14] D. Poulalhon and G. Schaeffer. Factorizations of large cycles in the symmetric group. *Discrete Math.*, 254:433—458, 2002.
- [15] R. P. Stanley. Factorization of permutations into n -cycles. *Discrete Maths.*, 37:255—262, 1981.
- [16] R. P. Stanley. Two enumerative results on cycles of permutations. *Eur. J. of Comb.*, 32:937—943, 2011.
- [17] D. Walkup. How many ways can a permutation be factored into two n -cycles. *Discrete Math.*, 28:315—319, 1979.
- [18] D. Zagier. On the distribution of the number of cycles of elements in symmetric groups. *Nieuw Arch. Wisk.*, 13:489—495, 1995.