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Jean Marc Vanel, Roger Peyret, Patrick Bontoux. A pseudo-spectral solution of vorticity-stream function equations using the influence matrix technique. Morton K W and Baines M J. Numerical Methods for Fluid Dynamics II, Clarendon Press Oxford, pp.463-475, 1986. hal-00974966

**HAL Id: hal-00974966**

**<https://hal.science/hal-00974966>**

Submitted on 7 Apr 2014

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# Numerical methods for fluid dynamics II

CLARENDON PRESS · OXFORD · 1986

## A PSEUDO-SPECTRAL SOLUTION OF VORTICITY-STREAM FUNCTION EQUATIONS USING THE INFLUENCE MATRIX TECHNIQUE

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### 1. INTRODUCTION

In this paper we present a fast and accurate method for computing unsteady two-dimensional incompressible flows. The method makes use of the vorticity and the stream function as dependent variables approximated by means of Chebyshev polynomial expansions. When using these variables the delicate question is the derivation of proper boundary conditions for the vorticity. The treatment of these conditions must preserve accuracy, stability and should avoid the use of an iterative solution procedure. The method developed here makes use of the influence matrix technique. This technique which has been found successful for the velocity-pressure equations (Kleiser and Schumann, 1980; Le Quéré and Alziary de Roquefort, 1982) has been applied until now to vorticity-stream function equations only in cases which reduce to one-dimensional problems (Tuckerman, 1983; Dennis and Quartapelle, 1983). Our purpose is to give a two-dimensional formulation of the method, to point out the theoretical and numerical difficulties associated with it, and finally to present numerical results illustrating the properties of the method.

### 2. EQUATIONS AND NUMERICAL APPROXIMATION

The Navier-Stokes equations within the vorticity-stream function formulation are

$$\frac{\partial \omega}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \omega = - \vec{V} \cdot \nabla \omega + f \equiv F(\psi, \omega; f) \quad (2.1)$$

$$\nabla^2 \psi + \omega = 0 \quad (2.2)$$

where the stream function  $\psi$  and the vorticity  $\omega$  are connected to the velocity  $\vec{V}$  by

$$\vec{V} = (u, v) = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right), \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (2.3)$$

These equations are solved in  $D$  ( $-1 < x, y < 1$ ) with the boundary conditions on  $\Gamma = \partial D$ .

$$\psi = g, \quad \frac{\partial \psi}{\partial \nu} = h \quad (\vec{\nu} = \text{unit normal to } \Gamma). \quad (2.4a, b)$$

The initial condition at  $t = 0$  is

$$\vec{V} = \vec{V}^0, \quad \text{then } \omega = \omega^0. \quad (2.5)$$

In (2.1),  $Re$  is the Reynolds number and  $f$  is a given forcing term.

The time-discretization is a combination of the 2nd-order Backward Euler scheme for the viscous term with an Adams-Bashforth type evaluation of the convective term, i.e:

$$\frac{1}{2\Delta t} (3\omega^{n+1} - 4\omega^n + \omega^{n-1}) - \frac{1}{Re} \nabla^2 \omega^{n+1} = 2F^n - F^{n-1} \quad (2.6)$$

$$\nabla^2 \psi^{n+1} + \omega^{n+1} = 0 \quad (2.7)$$

where  $n$  refers to the time  $t_n = n\Delta t$ . This second order scheme is preferred to the usual Crank-Nicolson/Adams-Bashforth scheme for its better stability properties:

- (1) the higher Fourier or Chebyshev modes are better damped,
- (2) for moderate  $Re$  the critical time-step is larger, and for small  $Re$  the scheme is unconditionally stable, as found by Zakaria (1985) for the advection-diffusion equation solved with a Collocation-Chebyshev method.

So, at each time-step the following Stokes-type problem has to be solved:

$$\nabla^2 \omega - \lambda \omega = G \quad (2.8)$$

$$\nabla^2 \psi + \omega = 0 \quad (2.9)$$

$$\psi = g, \quad \frac{\partial \psi}{\partial \nu} = h \text{ on } \Gamma \quad (2.10a,b)$$

where  $\omega$  and  $\psi$  are written for  $\omega^{n+1}$  and  $\psi^{n+1}$ , respectively.

The approximation of (2.8) - (2.10) makes use of the Chebyshev polynomial expansion:

$$\begin{pmatrix} \omega \\ \psi \end{pmatrix} = \sum_{n=0}^N \sum_{m=0}^M \begin{pmatrix} \hat{\omega}_{n,m} \\ \hat{\psi}_{n,m} \end{pmatrix} T_n(x) T_m(y) \quad (2.11)$$

The Chebyshev coefficients of  $G$  in (2.8) are calculated by the pseudo-spectral technique which consists in performing the products in the physical space and derivations in the spectral space, these two spaces being connected through a FFT algorithm.

The unknowns  $\hat{\omega}_{n,m}$ ,  $\hat{\psi}_{n,m}$  are obtained through the Tau-Method (see Gottlieb-Orszag, 1977) associated with the matrix diagonalization technique of Haidvogel and Zang (1979) for the solution of the systems.

The classical difficulty associated with the formulation lies in the determination of boundary conditions for the vorticity. The technique which consists in using  $\omega = -\nabla^2 \psi$  on the boundary has been employed in association with a Tau-Chebyshev method by Elie et al. (1983). We present here another solution to this problem which makes use of the influence matrix method. The advantage of this method lies in the strong coupling at the same time-level  $(n+1)$  between  $\omega$  and  $\psi$  on the boundary, leading to good accuracy in time and stability without requiring any iterative procedure.

### 3. INFLUENCE MATRIX METHOD

#### 3.1 Description of the Method

Thanks to the linearity of the problem, the solution of (2.8) - (2.10) can be written as

$$\begin{pmatrix} \omega \\ \psi \end{pmatrix} = \begin{pmatrix} \tilde{\omega} \\ \tilde{\psi} \end{pmatrix} + \sum_{i=1}^J \gamma_i \begin{pmatrix} \omega_i \\ \psi_i \end{pmatrix} \quad (3.1.1)$$

In this formula,  $(\tilde{\omega}, \tilde{\psi})$  is the solution of Problem A:

$$\begin{cases} \nabla^2 \tilde{\omega} - \lambda \tilde{\omega} = G & \text{in } D \\ \tilde{\omega} = \tilde{\omega}_0 \text{ (arbitrary)} & \text{on } \Gamma \end{cases} \quad (3.1.2)$$

$$\begin{cases} \nabla^2 \tilde{\psi} = -\tilde{\omega} & \text{in } D \\ \tilde{\psi} = g & \text{on } \Gamma \end{cases} \quad (3.1.3)$$

In the summation in the equation (3.1.1) the number  $J$  is connected with the number of collocation points  $(\xi_j, \eta_j)$  on  $\Gamma$ .

These  $(\omega_i, \psi_i)$  are solutions of Problem B:

$$\begin{cases} \nabla^2 \omega_i - \lambda \omega_i = 0 & \text{in } D \\ \omega_{ij} = \delta_{ij} & \text{on } \Gamma \end{cases} \quad \begin{cases} \nabla^2 \psi_i = -\omega_i & \text{in } D \\ \psi_i = 0 & \text{on } \Gamma \end{cases} \quad (3.1.4)$$

where  $j$  refers to the points  $(\xi_j, \eta_j)$  and  $\delta_{ij}$  is the Kronecker symbol. Then the  $J$  coefficients  $\gamma_i$  are determined such that  $\psi$  satisfies the boundary conditions (2.10b), i.e.

$$\left( \frac{\partial \psi}{\partial \nu} \right)_j = h_j = \left( \frac{\partial \tilde{\psi}}{\partial \nu} \right)_j + \sum_{i=1}^J \gamma_i \left( \frac{\partial \psi_i}{\partial \nu} \right)_j \quad (3.1.5)$$

This equation, written for all  $j$ , yields an algebraic system for the  $\gamma_i$ 's. The matrix  $A = [(\partial \psi_i / \partial \nu)_j]$  of this system is the "influence matrix".

Note that: (1) Problem B is independent of the time-level considered. Consequently, the calculation of  $\omega_i, \psi_i, A$  and its inverse  $A^{-1}$  can be done once and for all before starting the time-integration. (2) It is more economical to define the boundary values of  $\omega$

$$\omega_j = \tilde{\omega}_{0j} + \gamma_j \quad \text{on } \Gamma \quad (3.1.6)$$

and to solve a problem A [with (3.1.6) as boundary condition for  $\omega$ ] rather than to store the  $J$  fields  $(\omega_i, \psi_i)$  and to perform the linear combination in (3.1.1).

### 3.2 Problem of corners and determination of $J$

The presence of corners in the domain  $D$  leads to difficulties. Two methods have been considered to surmount these.

Firstly, we assume that  $f, g$  and  $h$  are such that  $\psi \in C^2$  so that  $\omega$  at a corner  $C$  can be expressed by:

$$\omega|_C = -\nabla^2 \psi|_C \quad (3.2.1)$$

which is known from the condition (2.10a).

So, in Method 1, we prescribe

$$\tilde{\omega}|_C = \tilde{\omega}_0|_C = -\nabla^2 \psi|_C \quad (3.2.2)$$

$$\omega_i|_C = 0 \quad (3.2.3)$$

and the equation (3.1.5) is written at each collocation point on  $\Gamma$  except the corners; hence  $J = 2(N + M - 2)$ .

In Method 2,  $f, g$  and  $h$  are assumed such that  $\nabla^4 \psi \in C^0$  so that the equation (2.8) is satisfied up to the corner  $C$ . Now, together with (3.2.2) and (3.2.3), we can prescribe

$$\nabla^2 \tilde{\omega}|_C = -\lambda \nabla^2 \psi|_C + G|_C \quad (3.2.4)$$

$$\nabla^2 \omega|_C = 0 \quad (3.2.5)$$

Because of these additional conditions, the equation (3.1.5) must be written at  $J = 2(N + M - 4)$  collocation points only. At the four remaining points, the values of  $\omega = \omega_0$  and  $\omega_i$  are determined so that (3.2.4) and (3.2.5) are satisfied. Here, we have chosen these special points  $S$  adjacent to the corners (Fig. 1), that is to say in the regions where the density of collocation points is high.

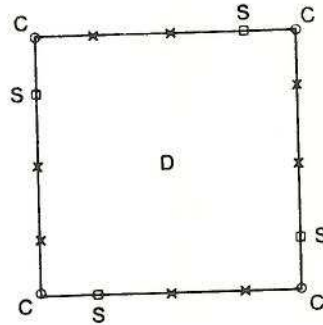


Fig. 1

### 3.3 Difficulty associated with the Tau-Method

The use of the Tau-Method to solve problem B for  $\psi_i$  leads to a difficulty arising because the last weighted residuals, corresponding to the higher modes, are not considered (they are replaced by the boundary conditions). So it appears that there exists a linear combination  $\bar{\omega}$  of the  $\omega_i$  such that the only non-zero coefficients have indices  $n = N - 1$  or  $N$ , or  $m = M - 1$  or  $M$ , precisely those coefficients which are not taken into account in the calculation of the corresponding stream function. Consequently we have a numerical solution  $\bar{\psi} = 0$  and the matrix  $A$  is not invertible. This difficulty can be surmounted if the whole spectrum of the vorticity  $\omega_i$  is considered when solving the equation for  $\psi_i$ . This is done by increasing the number of polynomials in the expansion (2.11) for  $\psi$ , which goes up to  $N + 2, M + 2$  now.

We are thus led to the problem of expressing  $\partial\psi/\partial x$  and  $\partial\psi/\partial y$ , defined by  $(N + 3)(M + 3)$  coefficients, at the usual  $(N + 1)(M + 1)$  collocation points  $(\cos n\pi/N, \cos m\pi/M)$ . The evaluation of  $G$  is done by simply neglecting the last two coefficients in either direction. On the boundary, two variants have been considered:

Method 1, 2: all the coefficients of  $\partial\psi/\partial v$  are taken into account.

Method 1', 2': the two last modes of  $\partial\psi/\partial v$  are neglected.

The numerical results have shown that 1 is much more accurate than 1', and that 2' is slightly better than 2.

## 4. NUMERICAL RESULTS

All the calculations reported here have been made with  $N = M$ .

## 4.1 Exact unsteady solution

We consider the solution

$$\psi_{ex} = \frac{2\pi - 1 + \sin 2\pi t}{2\pi} \quad \psi_0, \psi_0 = (1-x)^2 (1-y)^2 e^{x(y-1)} \quad (4.1.1)$$

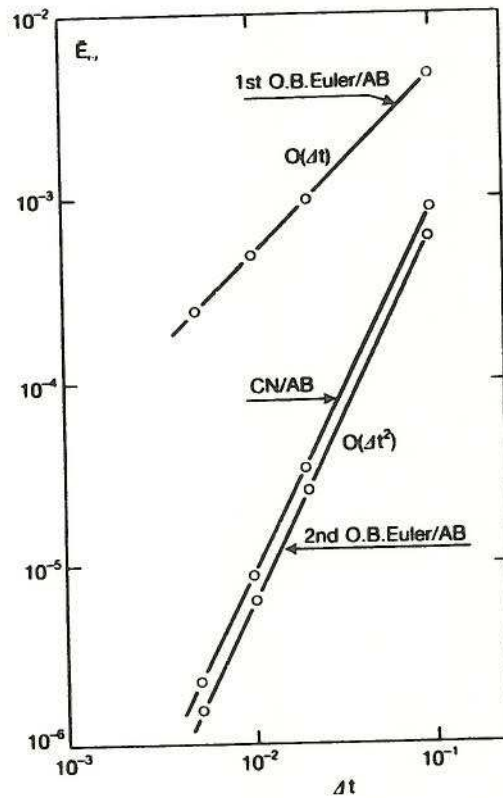
which defines  $f$ ,  $g$  and  $h$ . Here we compare the accuracy of various time-discretization schemes:

- (1) 1<sup>st</sup>-order Backward Euler/Adams-Bashforth
- (2) 2<sup>nd</sup>-order Backward Euler/Adams-Bashforth type [Eqs. (2.6) - (2.7)].
- (3) Crank-Nicolson/Adams-Bashforth.

The calculations have been made with  $Re = 1$ ,  $N = 20$  and  $0.1 \leq \Delta t \leq 0.005$ . With these values of  $N$  and  $\Delta t$ , the leading part of the error comes from the time-discretization. Fig. 2 shows the normalized mean quadratic error:

$$\bar{E}_\omega = \max_t \{ \|\omega - \omega_{ex}\|_{L2} / \|\omega_{ex}\|_{L2} \} . \quad (4.1.2)$$

The presence of three levels in time in the schemes necessitates starting-up procedures with modified schemes which, in the present calculations, are first order only. Hence, the maximum in (4.1.2) is taken when the effect of the initialization has disappeared: the error  $\bar{E}_\omega$  is then periodic in time. From Fig. 2, it can be seen that scheme (2) is slightly more accurate than scheme (3), at least for the special solution computed here.

Fig. 2 Error  $\bar{E}_w$ 

#### 4.2 Exact steady solution

In order to compare the accuracy of methods 1 and 2' we have considered the steady solution  $\psi_{ex} = \psi_0$ , using the scheme (2.6) - (2.7) with  $Re = 1$ . From Table 1, it is seen that method 2' is more accurate than method 1 for small  $N$ . The accuracy is comparable with some advantage for Method 1 when  $N$  is larger.

| N         | 10                     | 12                     | 16                      | 20                      |
|-----------|------------------------|------------------------|-------------------------|-------------------------|
| Method 1  | $1.223 \times 10^{-4}$ | $5.828 \times 10^{-6}$ | $9.179 \times 10^{-11}$ | $4.045 \times 10^{-11}$ |
| Method 2' | $2.928 \times 10^{-5}$ | $3.507 \times 10^{-7}$ | $1.723 \times 10^{-10}$ | $3.573 \times 10^{-10}$ |

Table 1: Error  $\bar{E}_w$

#### 4.3 Steady Cavity flow

Finally, the method has been used to calculate the steady solution of (2.1) - (2.2) with  $f = 0$ ,  $Re = 50$  and with the boundary conditions:  $\psi = 0$  on the whole boundary;  $\partial\psi/\partial y = -(1-x^2)^2$  at  $y = 1$ ,  $\partial\psi/\partial n = 0$  elsewhere; and with the initial condition  $\vec{V}^0 = 0$  and therefore  $\omega^0 = 0$ .

Fig. 3, which shows the critical time-step, makes clear the good stability properties of the method.

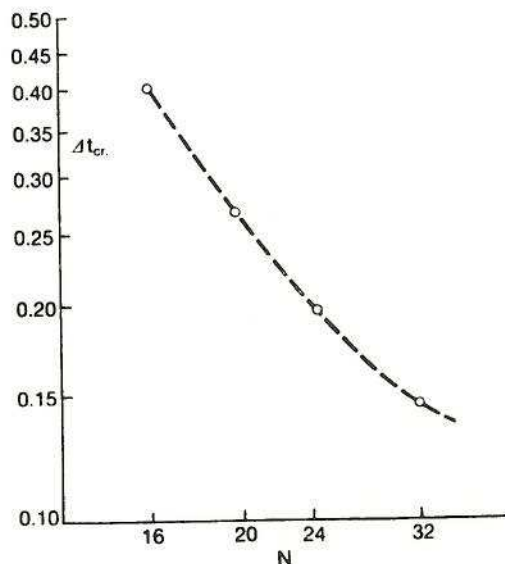


Fig. 3 Critical time-step

Table 2 gives some results. Here, the variables have been redefined by using the side  $L$  of the square as reference length, so that  $0 \leq X, Y \leq 1$ ,  $\Psi = \psi/2$ .  $\Omega = 2\omega$  and  $Re_L = 100$ .

refer to (Peyret and Taylor, 1983) for results given by other methods. Because of the strong variation of  $\Omega$  on the upper side  $Y = 1$ , the consideration of the maximum on collocation points only gives a bad idea of the convergence of the method with respect to  $N$ . Since the spectral approximation gives the solution at every  $X \in [0, 1]$ , we have also listed in the last column of Table 2 the maximum value of  $\Omega(X, 1)$  on 201 equally spaced points.

Table 2. Results for cavity flow,  $Re_L = 100$ 

| Method                   | N  | $Max \psi ^{(e)}$ | $Max \Omega(x,1) ^{(e)}$ | $Max \Omega(x,1) ^{(f)}$ |
|--------------------------|----|-------------------|--------------------------|--------------------------|
| pseudo-spectral<br>1     | 16 | 0.083139          | 13.4235                  | 13.5108                  |
|                          | 20 | 0.082692          | 13.1519                  | 13.4357                  |
|                          | 24 | 0.083315          | 13.4149                  | 13.4357                  |
|                          | 32 | 0.083403          | 13.3418                  | 13.4462                  |
| pseudo-spectral<br>2'    | 16 | 0.083158          | 13.3475                  | 13.4527                  |
|                          | 20 | 0.082695          | 13.1790                  | 13.4506                  |
|                          | 24 | 0.083315          | 13.4260                  | 13.4490                  |
|                          | 32 | 0.083403          | 13.3441                  | 13.4441                  |
| pseudo-spectral<br>(a)   | 16 | 0.083057          | 13.3574                  | 13.4485                  |
|                          | 20 | 0.082545          | 13.1810                  | 13.4453                  |
|                          | 24 | 0.083201          | 13.4315                  | 13.4522                  |
|                          | 32 | 0.083265          | 13.3496                  | 13.4517                  |
| Collocation (V,p)<br>(b) | 16 | 0.083686          | 13.3574                  | 13.4428                  |
|                          | 32 | 0.083685          | 13.3422                  | 13.4445                  |
| Hermitian (c)            | 20 | 0.0835            | 13.31                    | -                        |
| Finite-Diff. (d)         | 20 | 0.0829            | 13.14                    | -                        |

(a) Elie et al. (1983),

(b) Quazzani and Peyret (1983),

(c) 4th-order, Bontoux et al. (1978),

(d) 2nd-order, Peyret and Taylor (1983), p. 205, run 16,

(e) taken on collocation points,

(f) taken on 201 equally spaced points and obtained at  $x=0.62$ .

Fig. 4 shows the evolution of  $\Omega(x,1)$ . On the scale of the figure the difference between  $N = 16$  and  $N = 32$  cannot be distinguished. Fig. 5 is a blow-up of a very small region near  $x = 0$ . This provides an illustration of the high accuracy of Chebyshev approximation near the boundaries. Also, it is interesting to note the good agreement of the present results with those obtained by a Collocation - Chebyshev - Artificial Compressibility method (Ouazzani and Peyret, 1983) for the velocity-pressure equations, the latter method needing no special treatment at a corner.

The CPU time (CRAY 1S) is 0.03s/time-step for  $N = 20$  and 0.078s/time-step for  $N = 32$ . The preprocessing is roughly equivalent to  $4N$  time-steps.

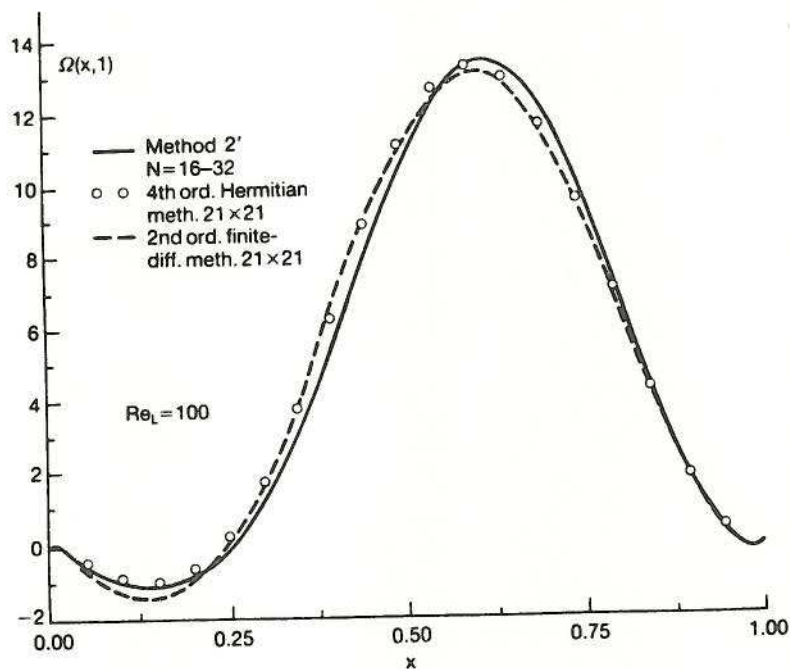


Fig. 4 Vorticity  $\Omega(x, 1)$

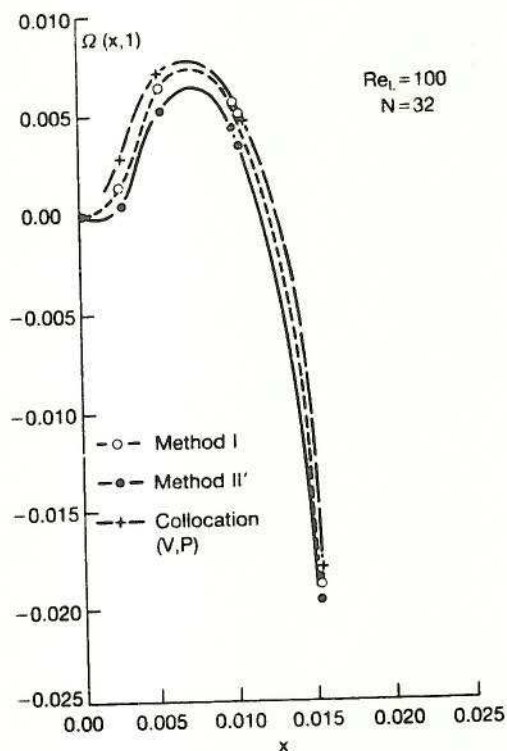


Fig. 5 Vorticity  $\Omega(x, 1)$  near  $x = 0$

#### 5. ACKNOWLEDGEMENTS

The computations were performed on the CRAY 1S computer of "Centre de Calcul Vectoriel pour la Recherche".

#### 6. REFERENCES

- Bontoux, P., Forestier, B. and Roux, B., (1978) "Analysis of higher order methods for the numerical simulation of confined flows", Lecture Notes in Phys., vol. 90, pp. 94-102. Springer-Verlag, New York.
- Dennis, S.C.R. and Quartapelle, L., (1983) "Direction solution of the vorticity-stream function ordinary differential equations by a Chebyshev approximation", *J. Comp. Phys.*, 22, 448-468.

- Elie, F., Chikhaoui, A., Randriamampianina, A., Bontoux, P. and Roux, B., (1983). "Spectral approximation for Boussinesq double diffusion", Proc. 5th GAMM Conf. Numer. Methods in Fluid Mech., (M. Pandolfi, R. Piva, eds). pp. 57-64, Vieweg, Braunschweig.
- Gottlieb, D. and Orszag, S.A., (1977) Numerical Analysis of Spectral Methods. Monograph No. 26, SIAM, Philadelphia.
- Haidvogel, D.B. and Zang, T., (1979) "The accurate solution of Poisson's equation by expansion in Chebyshev polynomials". *J. Comp. Phys.*, 30, 167-180.
- Kleiser, L. and Schumann, U., (1980) "Treatment of incompressibility and boundary conditions in 3-D numerical simulations of plane channel flows", Proc. 3th GAMM Conf. Numer. Methods in Fluid Mech., (E.H. Hirschel ed.), pp. 165-173, Vieweg, Braunschweig.
- Le Quéré, P., and Alziary de Roquefort, T., (1982) "Sur une méthode spectrale semi-implicite pour la résolution des équations de Navier-Stokes d'un écoulement bidimensionnel visqueux d'un fluide incompressible". *C.R. Acad. Sci. Paris*, 294, 11. pp. 941-943.
- Ouazzani, J. and Peyret, R., (1983) "A pseudo-spectral solution of binary gas-mixture flows", Proc. 5th GAMM Conf. Numer. Methods in Fluid Mech., (M. Pandolfi, R. Piva, eds.), pp. 275-282, Vieweg, Braunschweig.
- Peyret, R. and Taylor, T.D., (1983) Computational Methods for Fluid Flow. Springer-Verlag, New York.
- Tuckerman, L., (1983) "Formation of Taylor vortices in spherical Couette flow", Ph.D. Thesis, M.I.T., Cambridge, Mass.
- Zakaria, A., (1985) "Etude de divers schémas pseudo-spectraux de type collocation pour la résolution des équations aux dérivées partielles. Application aux équations de Navier-Stokes". Thèse 3ième cycle, Départ. Math., Université de Nice.