Ramsey Rule with Progressive Utility in Long Term Yield Curves Modeling

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Abstract

This paper draws a parallel between the economic and financial points of view in the modeling of long term yield curves and provides new results on asymptotic long rates. The Ramsey rule, which is the reference equation in the economic literature to compute long term discount rates, links endogenous discount rate and marginal utility of aggregate optimal consumption at equilibrium. This paper gives a financial interpretation of the economic discount rate given by the Ramsey rule, using marginal utility indifference prices for non-replicable zero-coupon bonds. For such a long term modeling, the possibility of calibrating the utility to a learning set and adjusting the preferences to new economic information is crucial. This is achieved here by means of consistent progressive utility, which is also a convenient and flexible framework to take into account the heterogeneity of the economic investors. Contrary to the standard backward approach, this forward approach leads to time-coherent optimal processes that do not depend on a fixed time-horizon related to the optimization problem. The dynamics and the long term behavior of the marginal utility yield curve is studied. We also analyse the dependency of the interest rates on the wealth of the economy.

Keywords: Ramsey rule, Yields curves, Long run rates, Marginal indifference pricing, Market-consistent progressive utility of investment and consumption, Forward/backward portfolio optimization.

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Introduction

This paper focuses on the modeling of long term yield curves with different viewpoints. Modeling accurately long term interest rates is a crucial challenge in many financial topics,

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such as the financing of ecological projects, or the pricing of longevity-linked securities or any other investment with long term impact. The bond market is highly illiquid for longer maturities and standard financial interest rates models cannot be easily extended as observed in Hulley and Platen [HP12]. Besides, as mentioned by Piazzesi [Pia10], "in most industrialized countries, the central bank seems to be able to move the short term of the yield curve. What matters for aggregate demand, however, are long-term yields".

Our approach, which is inspired by the economic literature on long-term policy-making, has been nourished by fruitful discussions with members of the French State Planning Commission "France Stratégie", who is in charge of evaluating public policies. Based on the equilibrium theory, an extensive literature has been developed to propose an endogenous definition of the economic discount rate in order to evaluate the future value of an investment by giving a current equivalent value.
The Ramsey rule, introduced in 1928 by Ramsey in his seminal work [Ram28], is the reference equation to compute the discount rate. It has been further discussed by numerous economists such as Gollier [Gol10, Gol12] and Weitzman [Wei98, Wei07]. The issue is addressed at a macroeconomic level, where long run interest rates have not necessarily the same meaning as in financial markets. We call them "economic" interest rates bec ause they are affected mainly by structural characteristics of the economy. The Ramsey rule links the discount rate with the marginal utility of aggregate consumption at the economic equilibrium. Even if this rule is very simple, there is no consensus among economists about the values of the parameters that should be considered. In particular, economic rates are very sensitive to the rate of preference for the present, which can be viewed as the intensity of an independent exponential random horizon. Besides, economists are unanimous in the necessity of a sequential decision scheme that allows to revise the first decisions and preferences in the light of new knowledge and direct experiences. At equilibrium, it is also important to take into account the heterogeneity of the economic agents. Doing so calls for stochastic approaches for modeling agents' aggregate preference. Therefore the utility criterion must be adaptive and adjusted to the information flow, and it obviously must be consistent with respect to a given investment universe. Musiela and Zariphopoulou [MZ07, MZ10] were the first to suggest to use instead of the classic criterion the concept of progressive dynamic utility, that gives an adaptive way to model possible changes over the time of individual preferences of an agent. Progressive utilities of investment and consumption were considered at first by Berrier and Tehranchi [BT11]. The questions of the existence and the characterization of market-consistent progressive utilities has been studied in a general setting in El Karoui and Mrad [EKM13], using a PDE point of view. As pointed out in El Karoui, Hillairet and Mrad [EKHM18], the optimal processes in the standard approach are computed through a backward analysis, emphasizing their dependency on the horizon of the optimization problem, and leading to intertemporality issues; while the progressive approach relies on a calibration viewpoint, given a learning set. The problem is then posed forward, leading to time-coherent optimal processes and putting emphasis on their monotonicity with respect to their initial values.

Whereas the economic framework relies on the theory of general equilibrium, the financial framework is based on a no-arbitrage condition and links yield curves and zero-coupon bonds prices. However the pricing issue in incomplete market is complex. Utility functions are also the cornerstone in the utility indifference pricing method, for the pricing of non-replicable contingent claims. For a small amount of transaction, this pricing methods lead to a linear pricing rule (see Davis [Dav98]) called the *Davis price* or the *marginal utility price*. As the zero-coupon bond market is highly illiquid for long maturities, it is relevant to study utility indifference pricing method for progressive utility with consumption. The economic and financial frameworks are actually very closed: both rely on a similar optimization problem that determines the optimal discounted pricing kernel used to evaluate claims under the historical (also called physical) probability measure. The discounted pricing kernels are the key processes for the yield curve modeling and provide an unifying approach for the economic and financial viewpoints. The main difference is that in the economic framework, this is the spot interest rate r^e (which is the drift term of the optimal discounted pricing kernel) that is determined endogenously by the market clearing condition at the equilibrium, while in the financial framework, r is exogenous and this is the orthogonal volatility of r that is determined at the optimum. Therefore, according to the Ramsey rule, we show that equilibrium interest rates and marginal utility interest rates coincide, being careful that marginal utility prices are robust only for *small* trades. We then study the dynamics of the yield curve and its long term behavior. Special attention is paid on the dependency of the interest rates on the global wealth of the economy, and in the backward setting, on the time-horizon of the underlying optimization problem. In particular, in the case of backward power utilities, we provide a new relation between the orthogonal volatilities of the optimal discounted pricing kernel and the zero-coupon bond price. As a consequence, for non replicable zero-coupon bonds, the time-horizon dependency of the discounted pricing kernel process implies long term yield curves that have a diffusion component and thus that are not necessarily monotonous in time. We illustrate our results with the important example of mixture of power utilities, that corresponds to the aggregation of investors having different Constant Relative Risk Aversion (CRRA). We prove that when the maturity tends to infinity, the asymptotic long aggregate rate is the lowest individual asymptotic rate. The asymptotic limit with respect to the wealth of the economy is also studied: when the wealth tends to infinity the aggregate zero-coupon price converges to the one priced by the less risk averse agent, whereas when the wealth tends to zero, it converges to the one priced by the more risk averse agent.

The paper is organized as follows. Section 1 introduces the economic equilibrium and financial no-arbitrage frameworks, emphasizing their similarities and their differences. The related optimization problem, that determines the discounted pricing kernel, is presented. To help for a clear understanding of the main ideas without too much technicalities, the equilibrium framework is first developed in a simplified model. We then introduce a more general model as well as the financial viewpoint. We discuss the relevance and the flexibility of consistent progressive utilities for the modeling of the representative agent preferences, in this context of long term decision making. Section 2 presents the Ramsey rule, as stated in the seminal paper [Ram28], as well as a pathwise version, written in terms of the optimal discounted pricing kernel. We provide then a financial interpretation of the Ramsey rule and of the economic discount rates, using marginal utility indifference pricing. The yield curve dynamics and its dependency on the wealth of the economy are studied in Section 3. Section 4 is devoted to the long term behavior of the spot forward rate and zero-coupon rates, in the the backward and forward approach, as well as to aggregated rates. Technical details and proofs on utility indifference pricing are postponed in the Appendix.

1 The discounted pricing kernel : an unifying approach for discount rates

This first section is dedicated to basic concepts of the economic equilibrium and financial no-arbitrage frameworks. The purpose here is to compare both the economic and financial points of view, and the differences between them are sometimes quite subtle. Although the results in this section are not completely new, we are convinced that comparing both approaches is instructive because it sheds a new light on concepts that are sometimes posed as evidence.

1.1 The economic and financial view points for interest rates

For the financing of ecological projects reducing global warming and for longevity issues or any other investment with a long term impact, it is necessary to model accurately long run interest rates. The answer is not to be found in financial market, since for longer maturities (30 years and more), the bond market becomes highly illiquid and standard financial interest rates models cannot be easily extended. In general, these issues are addressed at macroeconomic level, where long-run interest rates have not necessarily the same interpretation as in financial market. To avoid confusion, we refer to it as *socially efficient or economic* interest rates, because they would be mainly affected by structural characteristics of the economy, and be low-sensitive to monetary policy. Nevertheless, correct estimates of these rates are useful for long term decisions and understanding their determinants is important.

The economic interest rate process (r_t) is determined endogenously at equilibrium. General macroeconomic models often assume that at equilibrium, the sum of agents' choices is mathematically equivalent to the decision of one individual, called the representative agent. Nevertheless, how preferences of multiple agents aggregates is very complex, and the aggregate utility is unlikely to have a simple expression, unless all agents are identical. In particular, it is shown in El Karoui et al. [EKHM17] that taking a power utility for the representative agent assumes actually that all agents have a power utility with the same risk aversion. Cvitanic, Jouini et al. [CJMN11] propose an equilibrium model dealing with three types of heterogeneity: investors may differ in their beliefs, in their level of risk aversion and in their time-preference rate. Computing the equilibrium requires the resolution of an optimization problem, in which each agent derives his optimal strategy and the optimal Pareto allocation is computed (that is the optimal allocation among each agent, such that there is no possible allocations whose realization would increase the global satisfaction). The theory of the representative agent summarizes this step by assigning to the representative agent a utility and a strategy. Since this utility and this strategy are solution of an underlying optimization problem, they actually should satisfy intrinsic properties, such that a dynamic programming principle. The dynamic programming principle is deeply related to the notion of market-consistency in the context of dynamic utilities (see Musiela and Zariphopoulou [MZ07, MZ10] and also El Karoui et al. [EKHM18] in a consumption framework). As the utility of the representative agent is solution of an optimization problem, his utility at time t can not be constant nor deterministic and should be the value function of the underlying optimization problem, which is an example of market-consistent dynamic utilities. This motivates the use of market-consistent progressive utilities to model the preference of the representative agent at equilibrium.

In modern dynamic macroeconomics, it is standard to represent intertemporal behavior by a time separable intertemporal utility function with a constant relative risk aversion and infinite time horizon. Indeed one can show (see [EKHM18]) that time separable market-consistent utilities are necessarily power utilities, which partly explains the importance of power utilities in the economic modeling. The time component is often taken as a discount rate of the form $e^{-\lambda t}$ where the time preference rate λ is a difficult parameter to calibrate (see the discussion of Lecocq and Hourcade [HL04]). In his seminal paper [Ram28], Ramsey prefers not to discount later enjoyments in comparison with earlier ones, "a practice which is ethically indefensible and arises merely from the weakness of the imagination". Then, to overcome the problem of well-posedness of the underlying optimization problem, he introduces a "maximum obtainable rate of enjoyment or utility" called "Bliss". We will see in this paper that market-consistent dynamic utilities are another way to get rid off the time preference parameter and to provide time-coherent strategies.

1.2 The underlying optimization problems

We draw hereafter some parallels and comparisons between the economic and the financial frameworks for the modeling of interest rates and we present formally some strategic tools that are common to both frameworks. Overall, an agent is concerned with an optimization problem. His choice variables are how much to consume or save at each point in time, how much to invest in each security, under the constraint that no bankruptcy is permitted. His optimization problem is to maximize the expected utility over the class of admissible wealth-consumption process subject to a continuous time budget constraint to be written down.
As usual, a *utility function* u is a strictly concave, increasing, and non-negative function on $\mathbb{R}^+$, with continuous marginal utility u_z, satisfying the Inada conditions, $\lim_{z\mapsto\infty} u_z(z) = 0$ and $\lim_{z\mapsto 0} u_z(z) = +\infty$ to prevent 0 consumption at optimum. The risk aversion is measured by the ratio $\mathrm{R_A}(u)(z) = -u_{zz}(z)/u_z(z)$ and the relative risk aversion by $\mathrm{R^r_A}(u)(z) = z\,\mathrm{R_A}(u)(z)$.

1.2.1 An economic and financial model setup

The economic and financial setups have a lot of similarities. We set here the general framework and notations, following Björk [Bjö20], with a focus on the short rate (r_t) and the concern to emphasize points that are strategic for the paper : namely the impact of the time-horizon T_H, of the initial conditions, and the existence of a representative agent.
We consider here a model setup as in Merton (1969) with consumption and budget constraint (a general model with more details will be introduced in Section 1.4). The financial universe consists in d long-lived securities (also called technology in economics) with prices (S_t), and a riskless security with price (S^0_t). This bank account (S^0_t) yields instantaneously riskless short rate (r_t) $(dS^0_t = S^0_t r_t dt)$. To characterize the portfolio strategies and consumption plan, the trades are assumed to occur continuously in time without any friction: no transaction costs and no taxes, and securities are infinitely divisible.
At time t, the wealth X_t of the investor is the liquidation value of his investment: his strategy consists in investing a proportion π_t of his wealth X_t in the securities S_t (or a physical production technology), the proportion $\pi^0_t := 1 - \sum_{i=1}^d \pi^i_t$ of his remaining wealth being invested in the riskless security S^0_t. The investor also consumes at a non-negative rate c. π

is a d-dimensional process and the dot "." denotes the scalar product between two vectors of same dimension.
The pair (π, c) is called a wealth-consumption plan if the wealth process $X^{\pi,c}$ satisfies the following self-financing dynamics (see [Bjö20, chapter 6])

$$dX_t^{\pi,c} = X_t^{\pi,c}\left(\pi_t.\frac{dS_t}{S_t} + \pi_t^0\frac{dS_t^0}{S_t^0}\right) - c_t dt, \quad X_0 = x \tag{1.1}$$

and if the wealth (starting from $X_0 = x$) remains positive: $X_t^{\pi,c} > 0$. The set of admissible wealth-consumption plans (π, c) is denoted $\mathcal{A}$, also called the opportunity set. $\mathcal{A}$ may have different forms, depending on the framework and the optimization problem that is considered.

The following optimization program has to be solved in both financial and economic frames; in the usual setting it is formulated on a given horizon T_H.

$$\sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\Big(\int_0^{T_H} v(t, c_t)dt + u(T_H, X_{T_H}^{\pi,c})\Big). \tag{1.2}$$

The time horizon T_H is either a fixed date, or the hitting time of a given level of satisfaction (such as the "Bliss" in Ramsey [Ram28]), or a lifetime (as the one of the economy) given by an independent exponential random variable with mean $1/\lambda$, where λ is known as the rate of time preference. In general, in this last situation, it is optimal to have a vanishing wealth at the time horizon T_H and the criterion becomes $\mathbb{E}(\int_0^\infty e^{-\lambda t}v(t, c_t)\lambda dt)$. Thus the dependency of the optimal processes (and consequently of the equilibrium discount rate) on the time horizon T_H is analog to the dependency on the time preference parameter λ.
In the financial point of view, assets, bank account, time-horizon, and utility functions are given exogenously, and the problem is to characterize the optimal wealth-consumption plan. The economic equilibrium point of view relies on the existence of a representative agent, whose optimal investment strategy π^e is given and determines the return of the riskless asset. Then the problem is twofold: firstly finding (if they exist) two utility functions (u, v) and a consumption rate c^e such that the pair (π^e, c^e) is optimal; and secondly verifying that the utility criterium is indeed the one of a representative agent, this point being related to a Pareto optimality criterium.
Similarly to (1.2), at any time $t \leq T_H$, we associate the "value function" $\mathcal{U}(t, x)$ (also called "indirect" utility) given the wealth $X_t = z$ at time t (not to be confused with the initial wealth $X_0 = x$):

$$\mathcal{U}(t, z) = \text{"}\sup_{(\pi,c)\in\mathcal{A}}\text{"}\ \mathbb{E}\Big(u(T_H, X_{T_H}^{\pi,c}(t, z)) + \int_t^{T_H} v(s, c_s)ds | X_t = z\Big),\ a.s. \tag{1.3}$$

with terminal value $\mathcal{U}(T_H, .) = u(T_H, .)$.

HJB Equation To derive the optimal wealth-consumption plan, we need to specify the dynamics of the securities. To ease the presentation, we provide here some results in a one dimensional Markovian setting; higher dimension and a more general setting will be considered in Section 1.4. In early works, the risky security is assumed to satisfy a geometric

Brownian motion driven by a one-dimensional Brownian motion W

$$dS_t = S_t\big(\mu_t dt + \sigma_t dW_t\big)$$

where the deterministic functions σ and μ represent respectively the instantaneous volatility and the security return. Without loss of generality, we normalize the initial condition to $S_0 = 1$. The Markovian framework leads to a differential formulation of the problem, the well-known HJB equation, and the idea is to proceed by verification. Using the wealth dynamic (1.1), the dynamic programming principle implies the following HJB equation

$$\begin{cases} \mathcal{U}_t(t,z) + \sup\limits_{(\pi,c)\in\mathcal{A}} \big\{ v(t,c) + (z\pi_t(\mu_t - r_t) + zr_t - c_t)\mathcal{U}_z(t,z) + \frac{1}{2}\sigma_t^2 z^2 \pi_t^2 \mathcal{U}_{zz}(t,z) \big\} = 0, \\ \mathcal{U}(T_H, z) = u(T_H, z) \end{cases} \tag{1.4}$$

of which a concave solution, if it exists, turns out to be the value function (1.3). By concavity, the natural candidates to be the optimal consumption and optimal portfolio (that realize the supremum in the HJB equation (1.4)) are

$$\pi^*(t,z) = -\frac{(\mu_t - r_t)}{\sigma_t^2}\frac{\mathcal{U}_z(t,z)}{z\mathcal{U}_{zz}(t,z)}, \quad c^*(t,z) = v_c^{-1}(t, \mathcal{U}_z(t,z)). \tag{1.5}$$

The first order condition $v_c(t, c^*(t,z)) = \mathcal{U}_z(t,z)$ says that the marginal utility of current consumption must be equal, at any time, to the marginal utility of the current wealth. Plugging the expression of π^* into (1.4) yields the following HJB equation

$$\mathcal{U}_t(t,z) + v(t, c^*(t,z)) + (r_t z - c^*(t,z))\mathcal{U}_z(t,z) - \frac{1}{2}\mathcal{U}_{zz}(t,z)\left(\frac{(\mu_t - r_t)\mathcal{U}_z(t,z)}{\sigma_t \mathcal{U}_{zz}(t,z)}\right)^2 = 0. \tag{1.6}$$

The optimal investment/consumption strategy (π_t^*, c_t^*) depends on the initial wealth x through the optimal wealth process $X_t^*(x) : \pi_t^*(x) = \pi^*(t, X_t^*(x))$ and $c_t^*(x) = c^*(t, X_t^*(x))$, where $\pi^*(t,z) = -\frac{(\mu_t - r_t)}{\sigma_t^2}\frac{\mathcal{U}_z(t,z)}{z\mathcal{U}_{zz}(t,z)}$, and $c^*(t,z) = v_c^{-1}(t, \mathcal{U}_z(t,z))$.

The optimization problem (1.2) consists in optimizing jointly the consumption rate c and the investment π. The equilibrium market clearing condition for the risk free asset implies that the optimal investment π^* of the representative agent is fixed at equilibrium and equal to 1. This equilibrium constraint ($\pi^* = 1$) determines endogenously the short term risk free rate r^*. This differs from the no-arbitrage financial framework in which the risk free rate is fixed exogenously and the agent optimizes his investment strategy π. In turn, when the market is incomplete, the excesses of return of some less basic assets, such as some bonds, are the one that are endogenously determined in the arbitrage approach.

1.2.2 The discounted pricing kernels and the dual problem

The discounted pricing kernels are the cornerstone of the Ramsey rule, since they are used for evaluating contingent claims under the historical probability measure $\mathbb{P}$. Note that a discounted pricing kernel involves both a discounted factor $\exp(-\int_0^t r_s ds)$ (with the process r that may be stochastic) and a martingale density process corresponding to a change of probability measure. It is also called stochastic discount factor in the economic literature or state price density process in the financial literature. A discounted pricing kernel Y is a dual process of the risky assets (or the production technology) S ; it is orthogonal to any price process P, that is the product PY is a martingale. This martingale property, formulated in terms of conditional expectation, is used for the pricing ; we come back to this point in Section 2.2.1. The drift of Y is necessarily the opposite of the interest rate r (since S_0Y is a martingale) and its diffusion coefficient is the opposite of the risk premium of the risky assets (denoted $\eta^{\mathcal{R}}$ in the sequel) plus an orthogonal component (denoted ν) in the case of incomplete markets. A discounted pricing kernel will be indexed by the orthogonal component ν and denoted by Y^{ν}. In a complete market in which all risks could be hedged, the orthogonal set is trivial and reduced to $\nu = 0$. This is the standard economic framework. In the economic framework, this is the short rate r and thus the drift term of the optimal discounted pricing kernel that is determined at the optimum (at the equilibrium). Whereas in the financial framework, r is exogenous and this is the orthogonal component ν that is determined at the optimum. Both r and the risk premium are crucial for determining discount rates, that is why it is natural to focus on the discounted pricing kernel in this study. The choice of the discounted pricing kernel is achieved by solving the dual optimization problem of (1.2).

The dual problem is based on the Fenchel-Legendre convex conjugate transformation $\tilde{u}(\zeta)$ of a utility function u, where $\tilde{u}(\zeta) = \sup_{z>0} \big(u(z) - \zeta z\big)$. In particular, $\tilde{u}(\zeta) \geq u(z) - \zeta z$ and the maximum is attained at $u_z(z) = \zeta$. Under Inada conditions, $\tilde{u}$ is twice continuously differentiable, strictly convex, strictly decreasing, with $\tilde{u}(0^+) = u(+\infty)$, $\tilde{u}(+\infty) = u(0^+), a.s.$. Moreover, the marginal utility u_z is the inverse of the opposite of the marginal conjugate utility $\tilde{u}_\zeta$; that is $u_z^{-1}(\zeta) = -\tilde{u}_\zeta(\zeta)$; $\tilde{u}(\zeta) = u\big(-\tilde{u}_\zeta(\zeta)\big) + \tilde{u}_\zeta(\zeta)\,\zeta$, and $u(z) = \tilde{u}\big(u_z(z)\big) + z\,u_z(z)$. These strategic relations will also be applied with stochastic utilities U (throughout the paper, we adopt the convention of capital letter for stochastic utility and small letter for deterministic utility).

The conjugate system $(\widetilde{U}(t,\zeta), \tilde{v}(t,\zeta))$ is associated with the following dual optimization problem (starting from $Y_t^{\nu} = \zeta$)

$$\widetilde{\mathcal{U}}(t,\zeta) = \text{"}\inf_{Y^{\nu}}\text{"}\ \mathbb{E}\Big(\tilde{u}(T_H, Y_{T_H}^{\nu}) + \int_t^{T_H} \tilde{v}(s, Y_s^{\nu})ds | Y_t^{\nu} = \zeta\Big),\ a.s. \tag{1.7}$$

The link between the primal wealth process (X_t^*) and the optimal discounted pricing kernel (Y_t^*) is given by the first order relation

$$\mathcal{U}_z(t, X_t^*) = Y_t^*, \quad -\tilde{\mathcal{U}}_\zeta(t, Y_t^*) = X_t^*,$$

which allows to derive the HJB equation of the dual problem from the one of the primal problem.

1.3 Determining the equilibrium spot rate

For evaluating public policies, the economy is usually assumed to be at equilibrium. Nevertheless, it must be borne in mind that this assumption puts strong constraints on the economic framework that could be considered (see He and Leland [HL93] and El Karoui and Mrad [EKM20]). The equilibrium is usually stated in a Markovian setting. It relies on the existence of a representative agent and on the market clearing condition for the risk free asset. That is the investment of the representative agent in the technology π is identical to

1, and thus the coefficient of the risk free rate r cancels in the representative agent's wealth self-financing dynamics (1.1). Then the equilibrium constraint (that is $\pi = 1$ is optimal) determines endogenously the risk free rate r_t^*. In what follows, we assume the existence of an equilibrium, that is the existence of utility processes $\mathcal{U}$ and v that are compatible with the equilibrium constraints. A power utility function, together with a geometric Brownian motion for the discounted pricing kernel Y^*, provides a classic example of such an equilibrium. Let us first recall the definition of an equilibrium (see Dumas [DL17]).

Definition 1.1. *At time t, an equilibrium is an allocation π_t^*, a consumption level c_t^*, a rate of interest r_t^*, such that the representative investor is at the optimum and the market (for the risky security/technology as well as for the riskless security) clears. Market-clearing conditions are as follows:*

- *The supply-equals-demand condition for risky security: $\pi^* = 1$.*
- *The zero-net supply condition for the riskless security.*

The optimal consumption/investment problem (1.2) is solved for the representative agent, either using a dynamic programming principle when the parameters μ and σ are deterministic or through a martingale (dual) approach. Since at equilibrium the optimal investment is fixed (equal to 1), the optimization program consists in optimizing the consumption only. The equilibrium is then expressed in terms of the representative agent's value function $\mathcal{U}(t,z) = \mathbb{E}(\int_t^{T_H} v(s,c_s^*)ds + u(T_H, X_{T_H}^*)|X_t^* = z)$, that is central for the Ramsey rule (see Section 2). The market clearing condition determines endogenously the equilibrium rate, as function of (t,z), by identifying the optimal investment $\pi^*(t,z) = -\frac{(\mu_t - r_t)\mathcal{U}_z(t,z)}{\sigma_t^2 z \mathcal{U}_{zz}(t,z)}$ (see (1.5)) to 1:

$$r_t^*(x) = r(t, X_t^*(x)) \qquad \text{with } r(t,z) = \mu_t + \sigma_t^2 z \frac{\mathcal{U}_{zz}(t,z)}{\mathcal{U}_z(t,z)}. \tag{1.8}$$

Power (CRRA) utility functions $u(z) = \frac{z^{1-\theta}}{1-\theta}$ and deterministic coefficients σ, μ is the standard model used in economy; it is an important case in which computations simplify and the existence of an equilibrium can be stated. It notably implies, using (1.8), that the equilibrium rate does not depend on the wealth process (X_t^*). Nevertheless this case hides some important features on the dependency of the optimal processes and rates on initial conditions, as we will see in what follows.

For a general utility function u that is not necessarily of power type, then the existence of an equilibrium is not guaranteed and the relations given here are conditioned to its existence. Remark that the no-arbitrage assumption implies the existence of a risk premium η, which is given by $\eta_t = \sigma_t^{-1}(\mu_t - r_t)$ in a complete market framework. Thus (1.8) implies that the risk premium is linked to the relative risk aversion of the utility process $\mathcal{U}$:

$$\eta(t, X_t^*) = -\sigma_t \frac{X_t^* \mathcal{U}_{zz}(t, X_t^*)}{\mathcal{U}_z(t, X_t^*)} = \sigma_t \mathrm{R}_\mathrm{A}^\mathrm{r}(\mathcal{U})(t, X_t^*). \tag{1.9}$$

As explained in Section 1.2.2, we will prefer in this work to express the equilibrium in terms of the discounted pricing kernel Y and the dual conjugate functions $\tilde{\mathcal{U}}$ (see (1.7)) and $\tilde{v}$. The relation at time 0 $u_z(x) = y$ propagates into the following dynamic relation between optimal processes $Y_t^* = \mathcal{U}_z(t, X_t^*)$, see for example Björk [Bjö20, Proposition 35.4]. In this dual formulation, the optimal processes are expressed as functions of the dual variable Y: $c_t^* = v_c^{-1}(t, Y_t^*)$, $X_{T_H}^* = u_z^{-1}(T_H, Y_{T_H}^*)$, $X_t^* = (\mathcal{U}_z)^{-1}(t, Y_t^*) = -\tilde{\mathcal{U}}_\zeta(t, Y_t^*)$ where $\tilde{\mathcal{U}}(t,\zeta)$ is the dual conjugate of $\mathcal{U}(t,z)$. Identifying the martingale part of X_t^* and of $-\tilde{\mathcal{U}}_y(t, Y_t^*)$ leads to the equilibrium interest rate

$$r_t^*(y) = \tilde{r}(t, Y_t^*(y)) \quad \text{with } \tilde{r}(t,\zeta) = \mu_t + \sigma_t^2 \frac{\tilde{\mathcal{U}}_\zeta(t,\zeta)}{\zeta \tilde{\mathcal{U}}_{\zeta\zeta}(t,\zeta)}. \tag{1.10}$$

Note that shifting from relation (1.8) to (1.10) is straightforward using the duality relations between the utility function $\mathcal{U}$ and its dual utility $\tilde{\mathcal{U}}$. (1.10) gives the one to one correspondence between the equilibrium rate r^* and the discounted pricing kernel Y^*.

This simple equilibrium model has numerous extensions, as the famous one proposed by Cox-Ingersoll-Ross in [CIR85]. One sought feature of this model was that it yields positive rate (but nowadays the desire of having model with positive rates is not current anymore). Taking into account the presence of a financial market, [CIR85] adopts an equilibrium approach to endogenously determine the term structure of interest rates. In their model, the dynamics of the production process and the utility function depend on an exogenous stochastic factor which in some way influences the economy. At equilibrium, all financial assets are in zero net supply. The risk-free rate and the financial assets prices are determined endogenously such that the representative agent is not better off by trading in the money market, i.e. he is indifferent between an investment in the production opportunity and the risk-free instrument. This is related to the theory of indifference pricing, that we will use in the sequel (see Section 2.2). Then assuming a CIR dynamic for the exogenous stochastic factor implies also a CIR dynamics for the equilibrium short rate.

To summarize, in the equilibrium approach, the interest rate is determined endogenously. Replacing the expression of the equilibrium rate (1.8) into the HJB equation (1.6) yields the following HJB equation[1], that does not involve the interest rate, and which is linear in $\mathcal{U}_z$ and $\mathcal{U}_{zz}$

$$\mathcal{U}_t(t,z) + v(t, c^*(t,z)) + (\mu_t z - c^*(t,z))\mathcal{U}_z(t,z) + \frac{1}{2}\sigma_t^2 z^2 \mathcal{U}_{zz}(t,z) = 0, \quad \mathcal{U}(T_H, z) = u(T_H, z)$$

with the following dynamics $dX_t^* = (\mu_t X_t^* - c_t^*)dt + X_t^* \sigma_t dW_t$ for the equilibrium optimal wealth process, in which the terms in the interest rate r cancel due to the market clearing conditions. In fact, the utility function u at time T_H should not be given and is part of the processes that should be determined at equilibrium. Thus the dependency of the rate on the time-horizon T_H of the optimization problem is artificial. Besides, the expression for the interest rate (1.8), together with the dynamics of the wealth process (X_t^*) shows that the problem is naturally posed forward in the equilibrium setting. This forward approach is typically market-consistent and emphasizes the dependency of the rate on the initial condition x which represents the initial wealth of the economy (or equivalently its dual quantity $y = u_z(x)$). The forward approach is developed hereafter in the financial viewpoint.

[1]When the time horizon T_H is an exponential variable, the terminal condition disappears and is replaced by a linear term of order 0 in the HJB equation

$$\mathcal{U}_t(t,z) + v(t, c^*(t,z)) + (\mu_t z - c^*(t,z))\mathcal{U}_z(t,z) + \frac{1}{2}\sigma_t^2 z^2 \mathcal{U}_{zz}(t,z) - \lambda\,\mathcal{U}(t,z) = 0.$$

1.4 Primal and dual progressive utilities in financial markets

We first describe a general setting of a financial market.

1.4.1 The financial no-arbitrage framework in incomplete markets

The financial viewpoint is based on a no-arbitrage approach with exogenously given interest rate, instead of an equilibrium approach that determines them endogenously. We generalize hereafter the one-dimensional setting presented in Section 1.2. The financial investment universe is now assumed to be an incomplete Itô market, defined on a standard filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ that supports a n-standard Brownian motion W (see for example Karatzas et al. [KLS87], Karatzas and Shreve [KS01] or Skiadas [Ski07]).
The market is characterized by the short rate (r_t), the n-dimensional risk premium vector (η_t), and by the $d \times n$ volatility matrix (σ_t) of the risky assets $(d \leq n)$. As usual in finance, the processes r, η and σ are taken exogenous. We assume that $\int_0^T (|r_t| + \|\eta_t\|^2)dt < \infty$, for any $T > 0$, a.s. We specify here the class of admissible strategies in terms of (κ_t, ρ_t) where[2] $\kappa_t = \sigma_t^{tr}\pi_t$, $c_t = \rho_t X_t$: π_t is $\mathbb{R}^d$-valued and corresponds to the proportion of wealth invested in the risky assets, while ρ_t is the wealth-proportional consumption rate. The incompleteness of the market is expressed by restrictions on the risky portfolios κ_t constrained to live in a given progressive vector space $\mathcal{R}_t$. For example, if the incompleteness follows only from the fact that the number of assets is less than the dimension n of the Brownian motion, then typically $\mathcal{R}_t = \sigma_t^{tr}(\mathbb{R}^d)$.
To avoid technicalities, we assume throughout the paper that all the processes satisfy the necessary measurability and integrability conditions such that the following formal manipulations and statements are meaningful. The following short notations will be used extensively: let $\mathcal{R}$ be a vector subspace of $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$, $x^{\mathcal{R}}$ is the orthogonal projection of the vector x onto $\mathcal{R}$ and $x^{\perp}$ is the orthogonal projection onto $\mathcal{R}^{\perp}$.
In the line with (1.1), the self-financing dynamics of a positive wealth process with risky portfolio κ and wealth consumption rate ρ is given by[3]

$$dX_t^{\kappa,\rho} = X_t^{\kappa,\rho}[(r_t - \rho_t)\,dt + \kappa_t.(dW_t + \eta_t dt)] \qquad X_0^{\kappa,\rho} = x \tag{1.11}$$

where (ρ_t) is a non negative progressive process associated with the consumption process $c_t = \rho_t X_t$, (κ_t) is a n-dimensional vector measuring the multivariate volatility process of the wealth $(X_t^{\kappa,\rho})$. A self-financing strategy (κ, ρ) is admissible (we denote $(\kappa, \rho) \in \mathcal{A}^c$) if the portfolio κ_t lives in a given progressive family of vector spaces $\mathcal{R}_t$ *a.s.*
The set of the admissible wealth processes with admissible (κ_t, ρ_t) is a convex cone denoted by $\mathscr{X}^c$. The existence of a multivariate risk premium η formulates the absence of arbitrage opportunity. Since from (1.11), the impact of the risk premium on the wealth dynamics only appears through the term $\kappa_t.\eta_t$ for $\kappa_t \in \mathcal{R}_t$, there is a "minimal" risk premium $(\eta_t^{\mathcal{R}})$, the projection of η_t on the space $\mathcal{R}_t$ $(\kappa_t.\eta_t = \kappa_t.\eta_t^{\mathcal{R}})$, to which we refer in the sequel.
The class $\mathscr{Y}$ of the discounted pricing kernels Y^ν plays the role of "orthogonal cone" of the cone of admissible wealth processes $\mathscr{X}^c$ in the "martingale" sense. A non-negative Itô semimartingale Y^ν is called an admissible discounted pricing kernel if the current wealth

[2] The upperscript ^{tr} denotes the matrix transpose.
[3] In this paper, the scalar product of two vectors $\mathfrak{X}$ and $\mathfrak{Z}$ (of the same dimension) will be denoted by $\mathfrak{X}.\mathfrak{Z}$ or sometimes by $< \mathfrak{X}, \mathfrak{Z} >$.

plus the cumulative consumption, both discounted by Y^ν, is a local martingale. Then the supermartingale property of this non-negative local martingale $(Y_t^\nu X_t^{\kappa,\rho}+\int_0^t Y_s^\nu c_s ds)$ implies that $\mathbb{E}\left(Y_T^\nu X_T^{\kappa,\rho}+\int_0^T Y_s^\nu c_s ds\right)\le x$. This inequality, also known as the *budget constraint*, provides a necessary condition of admissibility, directly written in terms of the terminal wealth X_T and the consumption process $(c_s)_{s\in[0,T]}$.

Definition 1.2 (Discounted pricing kernel)**.**
A non-negative Itô semimartingale Y^ν is called an admissible discounted pricing kernel if for any admissible consumption plan (κ_t,ρ_t), the process $(Y_t^\nu X_t^{\kappa,\rho}+\int_0^t Y_s^\nu X_s^{\kappa,\rho}\rho_s ds)$ is a local martingale. The dynamics of Y^ν is then given by

$$dY_t^\nu = Y_t^\nu[-r_t dt + (\nu_t-\eta_t^{\mathcal{R}}).dW_t],\quad \nu_t\in\mathcal{R}_t^\perp,\quad Y_0^\nu = y. \tag{1.12}$$

The minimal discounted pricing kernel Y^0 corresponds to $\nu\equiv 0$

$$yY_t^0 = y\exp\big(-\int_0^t r_s ds-\int_0^t \eta_s^{\mathcal{R}}.dW_s-\frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 ds\big). \tag{1.13}$$

Note that $\mathscr{Y}$ does not depend on the presence of the consumption process, and is uniquely characterized by the financial market. The volatility process $\sigma^Y=(\nu-\eta^{\mathcal{R}})$ of Y^ν consists of two components: the minimal risk premium $\eta^{\mathcal{R}}$ that lies in $\mathcal{R}$ and an orthogonal component ν that lies in $\mathcal{R}^\perp$. Observe that any discounted pricing kernel $Y_t^\nu(y)$, starting from y at time 0, is the product of Y_t^0 by the exponential local martingale $L_t^{\perp,\nu}=\exp\big(\int_0^t\nu_s.dW_s-\frac{1}{2}\int_0^t||\nu_s||^2ds\big)$, since $\eta_s^{\mathcal{R}}.\nu_s\equiv 0$.
The inverse of the minimal discounted pricing kernel, $\frac{1}{Y^0}$, is the admissible market numeraire, also called GOP (growth optimal portfolio) (see Geman, El Karoui, Rochet [EKGR95], Heath, Platen [PH06], or Filipovic, Platen [FP09]). The discounted pricing kernels are the cornerstone of the financial interpretation of the Ramsey rule.

1.4.2 Primal and dual progressive utility processes

As explained in Section 1.3, the economic equilibrium framework with a representative agent is well suited to a forward approach, and writes conveniently in a dual formulation. Thus we detail below the dual progressive approach. Progressive utilities are particularly useful for long term decision schemes. Indeed, in the presence of generalized long term uncertainty, the decision scheme must evolve: economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions according to the evolution of the knowledge and to direct experiences, see Lecocq and Hourcade [HL04]. Besides, a sequential decision allows to cope with situations in which it is important to find the core of an agreement between partners having different views or anticipations, in order to give time for solving their controversy. Progressive utilities give also time-coherence in the optimal choices and allow to get rid off the sensitivity of economic discount rate in the pure time preference parameter λ (which can be interpreted as a random horizon T_H).

Since the preference criterion of the representative agent results from an equilibrium (in which each agent optimizes his own preference), we model it as a pair of progressive utilities $(\mathbf{U},\mathbf{V})$, satisfying a dynamic programming principle called market-consistency. $(\mathbf{U},\mathbf{V})$ is defined as a family of stochastic utility processes, that is for any t, $(U(t,z),V(t,c))$ are some utility functions. To express that the adaptive criterion $(\mathbf{U},\mathbf{V})$ is market-consistent, given the investment universe $\mathscr{X}^c$, we introduce the following supermartingale condition.

Definition 1.3 (Consistent primal progressive utility system).
Let $(\mathbf{U},\mathbf{V})$ be a progressive utility system and $\mathscr{X}^c$ a test family of portfolio with consumption. $(\mathbf{U},\mathbf{V})$ is said to be $\mathscr{X}^c$-consistent, if
(i) *for any admissible wealth process $X^{\kappa,\rho}\in\mathscr{X}^c$, with consumption rate $c=\rho X^{\kappa,\rho}$, the value process*

$$\mathcal{G}_t^{\kappa,\rho}=U(t,X_t^{\kappa,\rho})+\textstyle\int_0^t V(s,c_s)ds \text{ is a positive supermartingale.}$$

(ii) *there exists an optimal strategy such that the value process*

$$\mathcal{G}_t^{*}=U(t,X_t^{\kappa^*,\rho^*})+\textstyle\int_0^t V(s,c_s^*)ds \text{ is a martingale.}$$

As the discounted pricing kernels Y^ν have a major role in evaluating discount rate, we rather concentrate on the progressive dual problem. We consider the dual progressive utilities $(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}})$, defined as the family of stochastic dual utility processes such that for any t, $(\widetilde{U}(t,y),\widetilde{V}(t,y))$ are the Fenchel-Transforms of $(U(t,x),V(t,c))$. The market-consistency property on the primal progressive utility system $(\mathbf{U},\mathbf{V})$ translates into a market-consistency property on the dual progressive utilities $(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}})$, given the learning set $\mathscr{Y}$.

Definition 1.4 (Consistent dual progressive utility system).
Let $(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}})$ be a progressive utility system with learning set $\mathscr{Y}$. The dual utility system $(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}})$ is said to be $\mathscr{Y}$- consistent, if
(i) *for any admissible test process $Y^\nu\in\mathscr{Y}$ (with $\nu\in\mathcal{R}^\perp$)*

$$\widetilde{J}_t^\nu=\widetilde{U}(t,Y_t^\nu)+\int_0^t\widetilde{V}(s,Y_s^\nu)ds \text{ is a submartingale.} \tag{1.14}$$

(ii) *there exists an optimal process Y^* in $\mathscr{Y}$ such that*

$$\text{the optimal preference process}\quad \tilde{J}_t^*=\widetilde{U}(t,Y_t^*)+\int_0^t\widetilde{V}(s,Y_s^*)ds \text{ is a martingale.} \tag{1.15}$$

In what follows, the deterministic initial utilities $U(0,.)$ and $V(0,.)$ are denoted $u(.)$ and $v(.)$. The following result is proved in [EKHM18, Corollary 4.9].

Proposition 1.1. *Let $(\mathbf{U},\mathbf{V})$ be a $\mathscr{X}^c$-consistent progressive utility system satisfying regularities conditions. Then the optimal processes are linked by the first order relation*

$$Y_t^*(y)=U_z(t,X_t^*(x))=V_c(t,c_t^*(c_0)) \text{ with } y=u_z(x)=v_c(c_0)$$

or equivalently in terms of the dual utilities

$$X_t^*(x)=-\widetilde{U}_\zeta(t,Y_t^*(y)) \text{ and } c_t^*(c_0)=-\widetilde{V}_\zeta(t,Y_t^*(y)).$$

The dual value function system $(\widetilde{U}(t,\zeta),\tilde{v}(t,\zeta))$ of the classic consumption optimization problem is an example of strongly consistent system (with respect to $\mathscr{Y}$), defined from its terminal condition $\widetilde{U}(T_H,\zeta)=\tilde{u}(\zeta)$. Conversely, given $\mathscr{Y}$, and given an initial utility system $(\tilde{u}(x),\tilde{v}(c))$, a strongly consistent system $(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}})$ is the dual value function system of some investment-consumption problem, with stochastic terminal condition $\widetilde{U}(T_H,y)$ for any time horizon T_H. In this backward approach, the optimal processes are denoted respectively

$(c^{*,H}, X^{*,H})$ and $Y^{*,H}$, the additional symbol H underlining the dependency of the optimal processes on the optimization horizon T_H. In the forward approach, the dual utility process $\widetilde{\mathbf{U}}$ is an horizon unbiased utility function since the value function is the same for any time horizon T_H.

The forward and backward settings differ by their boundary conditions, the terminal utility is given in the standard case and the initial one in the progressive case. Although the $\mathscr{Y}$-consistency constraints are the same, this point induces major differences in the interpretation and in the mathematical treatment of the utility's characterization, apart from the issue of time-coherence. In the standard (backward) framework, the initial value of the value function $\mathcal{U}$ is usually not explicit and is computed through a backward analysis, starting from its given terminal utility (possibly random) $U(z)$ at time T_H. From a "practical" point of view, the Markov property is strategic for the resolution in the backward framework and sometimes additional regularity assumption are needed to use stochastic calculus for the value function. This point is not an issue for a forward utility, that is taken as a "regular"[4] Itô random field with differential decomposition

$$dU(t,z) = \beta(t,z)dt + \gamma(t,z).dW_t.$$

But in return it is not easy to read directly on its local random characteristics β and γ that $\mathbf{U}$ is a utility random field (increasing and concave), in absence of general comparison results for stochastic integrals. This can be done by using similar arguments as the ones that show that the solution of a stochastic differential equation (SDE) is monotone with respect to the initial condition as soon as sufficient regularity assumption are put on the coefficients. Therefore for consistent progressive utilities, the initial value is given and the problem is solved forward, and the emphasis is placed on the monotonicity of optimal processes with respect to the initial condition. We refer to [EKHM18] for discussion about those issues and for explicit regularity conditions and characterization of the consistent pairs of consistent utilities of investment and consumption and the optimal policies.

Compared to equation (1.5) for the deterministic case, the optimal portfolio is then given by

$$z\kappa_t^*(z) = x\sigma_t^{tr}\pi^*(t,z) = -\frac{U_z(t,z)}{U_{zz}(t,z)}\big(\eta_t^{\mathcal{R}} + \frac{\gamma_z^{\mathcal{R}}(t,z)}{U_z(t,z)}\big) \tag{1.16}$$

with the additional term risk premium term $\frac{\gamma_z^{\mathcal{R}}(t,z)}{U_z(t,z)}$ coming from the diffusion term of the progressive utility $\mathbf{U}$. The market-consistency implies the following HJB constraint

$$\beta(t,z) = -U_z(t,z)zr_t + \frac{1}{2}U_{zz}(t,z)\|z\kappa_t^*(z)\|^2 - \tilde{V}(t, U_z(t,z)) \tag{1.17}$$

which is similar to the HJB constraint (1.6).

Power utilities (corresponding to a constant relative risk aversion) are the standard framework in the economic literature. A consistent progressive power utility system is necessarily a pair of power utilities that are time-separable, with the same risk aversion coefficient θ $(0 < \theta < 1)$

$$U^{(\theta)}(t,z) = Z_t^u \frac{z^{1-\theta}}{1-\theta} \quad \text{and} \quad V^{(\theta)}(t,c) = Z_t^v \frac{c^{1-\theta}}{1-\theta}. \tag{1.18}$$

[4]Explicit regularity conditions are given in [EKHM18, Section 4].

The positive processes Z^u and Z^v are linked by the SDE satisfied by Z^u and that is given by the HJB drift constraint (1.17) (see [EKHM18, Section 4.2] for the study of progressive power utilities with consumption). One important feature is that the optimal processes for power utilities are linear with respect of their initial condition. Power utility is the usual framework of the Ramsey rule.

2 The Ramsey rule and its financial interpretation

2.1 The Ramsey rule

The macroeconomics literature typically relates the *economic equilibrium* rate to the time preference rate and to the average rate of productivity growth. A typical example is the Ramsey rule proposed in the seminal paper of Ramsey [Ram28] in 1928 where economic interest rates are linked with the marginal utility of the aggregate consumption at the economic equilibrium. More precisely, the economy is represented by the strategy of a risk-averse representative agent, whose utility function on consumption rate at date t is the function $v(t,c)$. Using an equilibrium point of view, the Ramsey rule at time 0 connects the equilibrium rate for maturity δ with the marginal utility $v_c(t,c)$ of the random optimal consumption rate (c_t^*) by

$$R_0^e(\delta) = -\frac{1}{\delta}\ln\frac{\mathbb{E}[v_c(\delta, c_\delta^*)]}{v_c(0, c_0^*)}. \tag{2.1}$$

Indeed, reducing the consumption at time 0 of a marginal amount ϵ induces a loss equal (at the first order) to $\epsilon v_c(c_0^*)$. This amount is then invested and rises the consumption at time δ to $c_\delta \epsilon e^{\delta R_0^e(\delta)}$. At equilibrium, at the first order, the gain $\epsilon e^{\delta R_0^e(\delta)} v_c(\delta, c_\delta^*)$ should offset the loss at time 0, which implies (2.1).

2.1.1 Historical Ramsey rule

In the optimization problem (1.2), an exponential random horizon T_H with rate λ is similar to the usual setting of a time separable utility function with exponential decay at rate $\lambda > 0$. For example, $\lambda = 0.1$ corresponds to a random horizon T_H with mean of 10 years. λ is the pure time preference parameter, i.e. λ quantifies the agent preference of immediate goods versus future ones. But as mentioned by Lecocq and Hourcade [HL04], it can also be interpreted as a preference of no sacrifice for the present. If we expect to consume more in the future, this parameter gives a lower bound for the Ramsey rule : indeed relation (2.1) applied to a time separable utility function with exponential decay at rate λ and assuming increasing expected consumption implies that $R_0^e(T) \geq \lambda$. Since λ is an intrinsic data, it may be questionable especially in a context of low interest rate, and fixing this value is a strong normative choice. Thus the choice of this pure time preference parameter λ is likely to continue to be a controversial issue.
In the meantime, as explained in Section 1.3, time separable power utility functions (with constant relative risk aversion θ, $0 < \theta < 1$) are often used

$$v(t,c) = Ke^{-\lambda t}\frac{c^{1-\theta}}{1-\theta}.$$

In the seminal paper of Ramsey [Ram28], the optimal consumption is a deterministic function $c_t^* = c_0^* \exp(gt)$ (with g being the growth rate of the economy) and the Ramsey rule (2.1) is written as

$$R_0^e(T) = \lambda + \theta g. \tag{2.2}$$

Although equation (2.2) is very simple, there is no consensus on the parameter values. In the Stern review on the climate change [SS07] in 2006, $\theta = 0.1$, $g = 1.3\%$ $\lambda = 0.1\%$, which leads to a discount rate of 1.4%, whereas the UK-treasury uses a discount rate of 2.5%. Thus 1 million of dollars in 100 years is equivalent today either on 250 000 dollars or 82 000 dollars, depending on which rate is taken.

In order to add some randomness in the future optimal consumption, the consumption process is also typically modeled as a geometric Brownian motion

$$c_t^* = c_0^* \exp(gt + \varphi W_t). \tag{2.3}$$

In fact, assuming power utilities and a geometric Brownian motion for the optimal consumption is quite natural in the light of the previous discussion on the existence of an equilibrium (see Section 1.3). The dynamics (2.3) can be linked to the previous equilibrium setting in the case of constant parameters: $g = \frac{r-\lambda}{\theta} + \varphi^2 \frac{\theta}{2}$ and $\varphi = \frac{\mu - r}{\theta\sigma}$. The Ramsey rule still induces a flat curve

$$R_0^e(T) = \lambda + \theta g - \frac{1}{2}\theta^2\varphi^2. \tag{2.4}$$

The same model holds true at any date in the future, by using conditional expectation. When the parameters of the model are homogeneous, the yields curve at any date in the future is still given by $R_t^e(T-t) = \lambda + \theta g - \frac{1}{2}\theta^2\varphi^2$.

The Ramsey rule is still the reference equation in macroeconomics and it was discussed by numerous economists, such as Gollier [Gol10, Gol12] and Weitzman [Wei98, Wei07]. Relaxing the assumption of a power utility and of modeling the consumption with a geometric Brownian motion as in (2.3), a second order expansion of $v_c(c_t^*)$ in relation (2.1) leads to a decomposition of the yield curve in three terms (impatience effect, wealth effect and precautionary effect) as follows (see Gollier [Gol07])

$$R_0^e(T) = \lambda + \frac{\mathrm{R}_\mathrm{A}^\mathrm{r}(v)(c_0^*)}{T}\Big(\mathbb{E}(\frac{c_T^*}{c_0^*} - 1) - \frac{1}{2}\mathrm{P}_\mathrm{R}(v)(c_0^*)\mathbb{E}((\frac{c_T^*}{c_0^*} - 1)^2)\Big) + o((\mathbb{E}(c_T^*) - c_0^*)^2) \tag{2.5}$$

with $\mathrm{R}_\mathrm{A}^\mathrm{r}(v)(z) = -zv_{zz}(z)/v_z(z)$ the relative risk aversion and $\mathrm{P}_\mathrm{R}(v)(z) = -zv_{zzz}(z)/v_{zz}(z)$ the relative prudence. This leads to an increasing (resp. decreasing) yield curve if the wealth effect becomes more (resp. less) compared to the precautionary effect. Remark that for power utilities, $\mathrm{R}_\mathrm{A}^\mathrm{r}(v)(z) = \theta$, $\mathrm{P}_\mathrm{R}(v)(z) = (\theta + 1)$, which gives the analogy between (2.4) and (2.5). Gollier also investigates in [Gol07] the effects of changes on the consumption on the yield curve computed at time 0.

In what follows, the equilibrium yield curve is computed at any date t in a general setting, using a "pathwise" Ramsey rule, but without questioning the existence of an equilibrium.

2.1.2 A pathwise Ramsey Rule

In the sequel, the upper-script $.^*$ denotes interchangeably optimal process of the forward and backward formulation, keeping in mind that, for the backward formulation, the statements are valid up to time T_H. We focus on the optimality relations given by Proposition 1.1

$$\begin{cases} c_t^*(c_0) & = -\tilde{V}_\zeta(t, Y_t^*(y)) \quad \text{i.e.} \qquad V_c(t, c_t^*(c_0)) = Y_t^*(y),\ t \geq 0 \\ c_0 & = -\tilde{v}_\zeta(y) \qquad \text{i.e.} \qquad v_c(c_0) = y. \end{cases} \tag{2.6}$$

Remark that a parametrization in y is equivalent to a parametrization in the initial wealth x or in the initial consumption rate c_0, based on the one to one correspondence $v_c(c_0) = u_z(x) = y$. The forward point of view emphasizes the key role played by the monotonicity of Y with respect to the initial condition y (under regularity conditions of the progressive utilities). Then as function of y, c_0 is decreasing, and $c_t^*(c_0)$ is an increasing function of c_0. This question of monotonicity is frequently avoided, maybe because with power utility functions $Y_t^*(y)$ is linear in y.

Equation (2.6) may be interpreted as a **pathwise Ramsey rule**, between the marginal utility of the optimal consumption and the optimal discounted pricing kernel:

$$\frac{V_c(t, c_t^*(c_0))}{v_c(c_0)} = \frac{Y_t^*(y)}{y}, \quad t \geq 0 \quad \text{with} \quad v_c(c_0) = y. \tag{2.7}$$

This one to one correspondence between the optimal consumption and the optimal discounted pricing kernel holds at any date t, that is why we called it a "pathwise Ramsey rule". Remark that formulating this pathwise relation (2.7) in terms of the optimal consumption leads to an expression that only involves the utility process $\mathbf{V}$ of the consumption, which contrary to $\mathbf{U}$, is a given process. Formulating the pathwise relation (2.7) in terms of the wealth would have involve the utility $\mathbf{U}$ which is complex to compute, $\mathbf{U}$ being the value function of the optimization problem.

The Ramsey rule leads to a description of the equilibrium yield curve as a function of the optimal discounted pricing kernel Y^*, $R_0^e(T)(y) = -\frac{1}{T}\ln \mathbb{E}[Y_T^*(y)/y]$ which allows to give a financial interpretation in terms of zero-coupon bonds. More dynamically in time, we define for $t < T$ and denoting by $\delta := (T-t)$ the time to maturity

$$R_t^e(\delta)(y) = R_t^e(T-t)(y) := \frac{-1}{T-t}\ln \mathbb{E}\left[\frac{V_c(T, c_T^*(c_0))}{V_c(t, c_t^*(c_0))}\big|\mathcal{F}_t\right] = \frac{-1}{T-t}\ln \mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\right]. \tag{2.8}$$

The Ramsey rule brings us to study the quantity $\mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\right]$. In the context of a financial complete market, it is well-known that this quantity corresponds to the price at date t of zero-coupon bonds (maturing at time T). Nevertheless, its interpretation for incomplete market is less trivial and will be investigated in the next section. Before going on with the financial interpretation of this equilibrium yield curve given in terms of the discounted pricing kernel, we recall that in the equilibrium framework the short term interest rate r_t is endogenous and fixed at equilibrium to satisfy the market clearing condition of the aggregate demands. On the contrary, in the financial no-arbitrage framework, the short rate is exogenous and the discounted pricing kernel is optimized not through its drift r_t but through its volatility ν_t. In the financial no-arbitrage context, the optimization procedure impacts only the form on the yield curve (through the risk premium), and not the beginning of the curve. This helps to understand how yield curve movements of the short end (monitored by a central bank) translate into long-term yield. For this financial interpretation purpose, it is natural to link zero-coupon bonds and the equilibrium yield curve.

2.2 Marginal indifference pricing interpretation of the Ramsey rule

In this section, we investigate the financial interpretation of the Ramsey rule. The financial point of view focuses more on the financial products than the rates, namely in this context on the zero-coupon bonds, which is a contract that pays 1 at a given date T. We thus want to interpret, in terms of price of zero-coupon bonds, the quantities $\mathbb{E}\Big[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\Big]$ for all $t < T$. This question is related to a more general issue in finance, that consists in the pricing of a bounded contingent claim ξ_T, paid at date T ($\xi_T = 1$ in the case of zero-coupon bond). We thus address this pricing issue for replicable and non replicable claims, with both backward (in this case $T \leq T_H$) and forward approaches. When all risks are replicable, then the price is uniquely determined as the value of the replicating portfolio (by no-arbitrage arguments). When some risks remain not replicable, several valuation methodologies exist (such as super-replicating prices or indifference prices), leading to different prices or bid-ask prices ; we refer the interested reader to the Appendix for further discussion. To evaluate small amounts of non replicable claims, we will consider here the marginal utility indifference pricing. This pricing procedure consists in choosing an optimal discount pricing kernel Y^* among the set $\mathcal{Y}$ of all admissible pricing kernels.

2.2.1 Valuation of replicable payoffs

The valuation of a (bounded) contingent claim ξ_T (paid at date T) is done through the choice of a discounted pricing kernel Y^ν, the price at time t being then given by the expectation $\mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T\big|\mathcal{F}_t\big]$. The question that arises is the choice of this discounted pricing kernel Y^ν. As mentioned in Definition 1.2, any discounted pricing kernel Y_t^ν is written as the product of the so-called minimal discounted pricing kernel $Y_t^0(y) = yY_t^0 = y\exp\big(-\int_0^t r_s ds - \int_0^t \eta_s^{\mathcal{R}}.dW_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 ds\big)$ and an orthogonal local martingale $L_t^{\perp,\nu}(y) = \exp\big(\int_0^t \nu_s(y).dW_s - \frac{1}{2}\int_0^t ||\nu_s(y)||^2 ds\big)$. In finance (r_t) and $(\eta_t^{\mathcal{R}})$ are exogenous, while $\nu_t \in \mathcal{R}_t^\perp$ is endogenous and may depend on y. The minimal discounted pricing kernel Y^0 plays a "universal" rule and any Y^ν differs only in the orthogonal part $L^{\perp,\nu}(y)$. Y^0 includes both the short term interest rate r and the risk premium $\eta^{\mathcal{R}}$, it can be decomposed as $Y_t^0 = e^{-\int_0^T r_s ds} L_T^{\mathcal{R}}$ with $L_T^{\mathcal{R}} = \exp\big(-\int_0^t \eta_s^{\mathcal{R}}.dW_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 ds\big)$ an exponential martingale which corresponds to the density process of a change of probability.
If the bounded contingent claim ξ_T is *replicable* by an admissible self-financing portfolio, its market price $p^m(\xi_T)$ (p^m when it is not ambiguous) is the value of the replicating portfolio (by no-arbitrage). Thus p_t^m is a bounded process such that $Y_t^\nu(y)p_t^m$ is a martingale for any discounted pricing kernel $Y^\nu(y)$, and in particular for yY_t^0. This leads to the classic pricing formula of a replicable contingent claim

$$p_t^m(\xi_T) = \mathbb{E}\big[\frac{Y_T^0}{Y_t^0}\xi_T\big|\mathcal{F}_t\big] = \mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T\big|\mathcal{F}_t\big]. \tag{2.9}$$

Therefore, for replicable payoff, the price is uniquely given by $\mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T\big|\mathcal{F}_t\big]$, whatever the discounted pricing kernel Y^ν. In finance, it is interpreted as the risk neutral conditional expectation of the discounted claim between t and T,

$$p_t^m(\xi_T) = \mathbb{E}\big[\frac{Y_T^0}{Y_t^0}\xi_T\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds}\xi_T\big|\mathcal{F}_t\big]$$

where $\mathbb{Q}$ is the minimal risk-neutral probability with density $L_T^{\mathcal{R}}$ with respect to $\mathbb{P}$ (on $\mathcal{F}_T$). Under the risk neutral probability $\mathbb{Q}$, all assets and admissible self-financing portfolios have the same return (r_t). Remark also that in a complete market (which is the natural framework of equilibrium modeling), any contingent claim is replicable, and Y^0 is the only discounted pricing kernel.
In conclusion, for replicable zero-coupon bonds, equilibrium yield curve (2.8) *and market yield curve have the same expression in terms of the discounted pricing kernel.*
However, for long maturities, this replicable assumption seems very strong (even if the payoff of the zero coupon is constant, the short term interest rate and the risk premium are stochastic). If the contingent claim is not replicable, the price is not uniquely determined and different discounted pricing kernel Y^ν may lead to different prices $\mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T\big|\mathcal{F}_t\big]$. What is the financial interpretation of the Ramsey rule in this context? It is important to point out that the Ramsey rule is a marginal linear pricing rule, that is computed for relative small amounts. The following section relates it with the marginal utility indifference pricing. Indeed, similarly to the heuristic of the Ramsey rule recalled in (2.1), the marginal utility indifference price is also a linear price that corresponds to a small perturbation of first order around an equilibrium.

2.2.2 Marginal indifference pricing

When hedging strategies cannot be implemented, the nominal amount of the transaction becomes an important risk factor. One way to evaluate non-replicable claims is the utility based indifference pricing, which is a non-linear pricing rule. The utility *indifference price* is the price at which the investor is indifferent from investing or not in the contingent claim. We consider the two following maximization problems stated at time $t = 0$ to simplify the notations (this can be easily extended to any time $t \le T$). The first one without the claim ξ_T has already been introduced previously

$$\mathcal{U}^T(x) := \sup_{(\kappa,\rho)\in\mathcal{A}^c} \mathbb{E}[U(T, X_T^{\kappa,\rho}(x)) + \int_0^T V(s, c_s)ds]. \tag{2.10}$$

The terminal utility $U(T, .)$ is then perturbed by the random payment $q\xi_T$, leading to the second maximization problem

$$\mathcal{U}^{\xi,T}(x, q) := \sup_{(\kappa,\rho)\in\mathcal{A}^c} \mathbb{E}[U(T, X_T^{\kappa,\rho}(x) - q\,\xi_T) + \int_0^T V(s, c_s)ds]. \tag{2.11}$$

The *utility indifference price*[5] is the cash amount $\widehat{p}^q_{0,T}(x, \xi_T, q)$ determined by the relationship

$$\mathcal{U}^{\xi,T}(x + \widehat{p}^q_{0,T}(x, \xi_T, q), q) = \mathcal{U}^T(x). \tag{2.12}$$

As in (2.1), (2.12) provides the additional initial wealth $\widehat{p}^q$ that offsets the loss of providing a q-quantity of the claim ξ_T at time T.
When the investors are aware of their sensitivity to the non-replicable risk, they can try to transact for only a little amount in the risky contract, which corresponds to the zero marginal rate of substitution p_T^u (u for utility), also called *Davis price* [Dav98] or *marginal*

[5]If $q > 0$ (resp. $q < 0$) it is a selling (resp. a buying) indifference price. For $q > 0$ one should assume that $q\xi_T$ is super replicable at price x.

indifference price. This is a classic pricing approach in economics, less frequently used in option pricing. The *marginal utility indifference price* is determined by the relationship

$$p^u_{0,T}(x,\xi_T) := \lim_{q\to 0} \frac{\partial \hat{p}^q_{0,T}}{\partial q}(x,\xi_T,q). \tag{2.13}$$

The marginal utility price is characterized by the optimal discounted pricing kernel of the consumption optimization problem (2.10).

Proposition 2.1. *Let $(Y^*_t(y))$ be the optimal discounted pricing kernel associated with the (forward or backward) consumption optimization problem. For any non negative contingent claim ξ_T delivered at time T, the marginal utility price is given at any time $t \le T$ by*

$$p^u_{t,T}(x,\xi_T) = \mathbb{E}\big[\xi_T \frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\big], \quad y = \mathcal{U}_z(0,x). \tag{2.14}$$

Proof. We refer to the Appendix for the proof, as well as a discussion on the time-coherence of this pricing rule, in the backward and forward settings (see Proposition 5.1). □

Using the marginal utility indifference pricing, the price of the contingent claim is computed as the expectation under a pricing measure. This corresponds to the "local expectations hypothesis" of Piazzesi [Pia10], in which the transition from the data-generating measure $\mathbb{P}$ to the pricing measure $(Y^*_T.\mathbb{P})$ is tied to preference parameters.‘ The marginal utility price is a linear pricing rule ; this means that there exists a consensus on this price for a small amount, but investors are not sure to have liquidity at this price. Nevertheless, this linear pricing rule may not be well adapted for larger nominal amount of transaction and highly illiquid market. From a financial viewpoint, this linear pricing rule given by the discounted pricing kernel Y^* allows to enrich the financial market with the zero-coupon bonds whose prices become coherent assets under $(Y^*.\mathbb{P})$. In this extended market, the minimal discounted pricing kernel Y^0 is then replaced by Y^*.
From an economic viewpoint, utility indifference pricing relies on the disturbance of a partial equilibrium by adding a new contingent claim/asset that should be financed. A complete market can not be disturbed by a new asset because any contingent claim/asset can be hedged. But in incomplete markets the equilibrium is not perfect and the new claims to be financed have an impact on it. In the case of new claims whose size are small, the disturbance is marginal, leading to a marginal utility indifference price. This indicates similarities between the marginal utility indifference price and the Ramsey rule. We now interpret the previous results on the marginal utility pricing of zero-coupon bonds in terms of the yield curve.

2.3 Marginal utility yield curve

As usual, we use the generic notation $\big(B(t,T),\ t \le T\big)$ for the price at time t of a zero-coupon bond paying one unit of cash at maturity T. In finance, the market yield curve $(\delta \to R_t(\delta))$ is expressed in term of the time to maturity $\delta = T-t$ and is defined through the price of a zero-coupon bonds by $B(t,T) = \exp(-(T-t)R_t(T-t))$. We use the previous results of Sections 2.2.1 and 2.2.2 concerning the pricing of contingent claims: the case of a zero-coupon bond corresponds to a contract delivery 1 at maturity T, i.e. $\xi_T = 1$.
(i) If the zero-coupon bonds are replicable, then there is no ambiguity about their prices,

as any discounted pricing kernel Y leads to the same price (see (2.2.1))

$$B^0(t,T) = \mathbb{E}\big[\frac{Y_T}{Y_t}|\mathcal{F}_t\big] = \mathbb{E}\big[\frac{Y^0_T}{Y^0_t}|\mathcal{F}_t\big].$$

In practice, this pricing rule $B^0(t,T) = \mathbb{E}\big(\frac{Y^0_T}{Y^0_t}|\mathcal{F}_t\big)$, using the minimal pricing kernel Y^0, is often used as a benchmark, even if the bonds are not replicable. In these case, the price does not depend on y (for exogenous $r, \eta^{\mathcal{R}}$).
(ii) For non hedgeable zero-coupon bond, we can apply the marginal indifference pricing rule (with consumption) based on the u-optimal pricing kernel $Y^*_t(y)$. Although it is important to emphasize the dependence of the optimal pricing kernel $Y^*(y)$ on the utility, we avoid this dependence to simplify the notations. Similarly, the marginal utility price at time t of a zero-coupon bond depends on the utility only through the optimal discounted pricing kernel $Y^*(y)$, so we denote it by $B^*(t,T)(y)$ (note that this price depends on y):

$$B^*(t,T)(y) = \mathbb{E}\Big[\frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\Big].$$

Based on the link between optimal discounted pricing kernel and optimal consumption, we see that

$$B^*(t,T)(y) = \mathbb{E}\Big[\frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\Big] = \mathbb{E}\Big[\frac{V_c\big(T, c^*_T(c_0)\big)}{V_c\big(t, c^*_t(c_0)\big)}|\mathcal{F}_t\Big], \quad y = v_c(c_0) = u_z(x) \tag{2.15}$$

where V_c is given by the first order relation (2.7).
According to the Ramsey rule (2.8)*, equilibrium interest rates and marginal utility interest rates are the same, in terms of the discounted pricing kernel. One should keep in mind that in the equilibrium framework the discounted pricing kernel is determined at equilibrium through the spot rate r_t endogenously, while it is optimized through its orthogonal volatility ν_t in the financial setting.* Besides, it is worth emphasizing that the marginal utility prices are only valid for small trades. Indeed for non replicable claims, the size of the transactions is an important source of risk; for larger trades, we cannot use the first order approximation given by the marginal utility price, and we should add a correcting second order term or use indifference pricing (see the Appendix, Theorem 5.2).

3 Yield curves dynamics and their volatilities

The increase of the fixed income market in size and number of products has transformed the way of considering the links between rates of different maturities, leading to leave the economic theory of rational expectation for the principle of no-arbitrage between bonds of different terms. Initiated by Vasicek in 1977, this evolution has matured with Heath-Jarrow-Morton theory [HJM92], and the theory of bond as a numeraire in El Karoui et al. [EKGR95]. Note that this point of view, that follows from the no-arbitrage principle, is relevant for a day by day management of the rate fluctuations, but does not replace the analysis of the economic fundamentals that explain the broad patterns of the fluctuations. This section revisits the previous results on the yield curve, using Heath-Jarrow-Morton (HJM) theory in incomplete market, for both the economic and financial viewpoints, and both the forward and backward frameworks (in the backward approach, the maturity T of

the zero-coupon should be taken smaller than the horizon T_H).
The notion of forward contracts will be used, such as the forward zero-coupon bonds, whose price $B_t(T_0, T)$ is the price at time t of a bond starting at time T_0 and paying one unit of cash at time $T > T_0$. By non arbitrage, $B_t(T_0, T) = B(t, T)/B(t, T_0)$. The family of forward instantaneous rates $\big(f(t, T_0) = -\partial_T \ln\big(B_t(T_0, T)\big)|_{T=T_0}\big)$ takes also a large place in the HJM theory.
Instead of starting with a given dynamic for the short rate r and deducing the zero-coupon bonds and their volatilities (as it is the case for example for the Vasicek model), the Heath-Jarrow-Morton framework adopts a reverse approach based on the prices of zero-coupon bonds and their volatility. It is worth emphasizing that in the HJM approach the spot rate is not given and is deduced from the volatility process, and of the initial conditions of the forward rates $(f(0, T))$. Thus in what follows, we focus on the volatility family of the zero-coupon bonds, that characterizes the dynamics of the yield curve. It is important to highlight that this characteristic is determined directly by the martingale property of the process $(Y_t^*(y)B^*(t, T)(y))_{t\in[0,T]}$, in both the economic and the financial viewpoints. The subtle difference consists of the endogeneity for the economic viewpoint (resp. exogeneity for the financial viewpoint) of the spot rate r, that may depend (or not) on y ; we will discuss this point in Proposition 3.2.

3.1 Heath Jarrow Morton framework for forward rates

Recall that any discounted pricing kernel $Y^*(y)$ is characterized by its volatility process $\sigma^{Y^*}(y) := \nu^*(y) - \eta^{\mathcal{R}}(y)$ (resp. $-\eta^{\mathcal{R}}(y)$ for Y^0), where $\eta^{\mathcal{R}}(y)$ is the minimal risk premium (that lies in $\mathcal{R}$) and $\nu^*(y)$ is the orthogonal component that lies in $\mathcal{R}^\perp$. In the economic framework $\eta^{\mathcal{R}}(y)$ is endogenous, while in the financial setting it is exogenous and usually taken independent of y. $\sigma^{Y^*}(y)$ does not depend on the maturity T, but may depend on the horizon T_H in the backward framework, through the orthogonal component $\nu^*(y)$. The dynamics of the associated bonds $B^*(t, T)(y)$ differ by their volatility vectors, denoted by $\Gamma^*(t, T)(y)$ that are assumed to be progressive processes with the convention $\Gamma^*(t, T)(y) = 0$ a.s. for $t \geq T$. In the sequel, we use the usual short notation for exponential martingale[6], $\mathcal{E}_t(\phi) := \exp\big(\int_0^t \phi_s.dW_s - \frac{1}{2}\int_0^t \|\phi_s\|^2 ds\big)$. The study is based on the martingale property of the process $Y_t^*(y)B^*(t, T)(y)$ (resp. $Y_t^0 B^0(t, T)$), whose volatility $\big(\sigma_t^{Y^*}(y) + \Gamma^*(t, T)(y)\big)$ is the sum of the volatilities of each term, and whose terminal value is $Y_T^*(y)$. Thus, the exponential martingale $Y_t^*(y)B^*(t, T)(y)$ has the following representation:

$$Y_t^*(y)B^*(t, T)(y) = \quad yB^*(0, T)(y)\mathcal{E}_t\big(\sigma^{Y^*}(y) + \Gamma^*(., T)(y)\big). \tag{3.1}$$

The same formula holds for $Y_t^0 B^0(t, T)$ ($\nu^* \equiv 0$). As a byproduct, (3.1) written for $t = T$ provides another formula for the random variable $Y_T^*(y) = y\exp(-\int_0^T r_s ds)\mathcal{E}_T\big(\sigma^{Y^*}(y)\big)$: observing that $B^*(T, T)(y) = 1$,

$$Y_T^*(y) = Y_T^*(y)B^*(T, T)(y) = yB^*(0, T)(y)\mathcal{E}_T\big(\sigma^{Y^*}(y) + \Gamma^*(., T)(y)\big). \tag{3.2}$$

[6] Additional assumptions on ϕ, of Novikov type, are necessary to ensure that this local martingale is a true martingale, see e.g. Novikov [Nov73] or [Kry19].

Identifying the two formulas for the random variable $Y_T^*(y)$ yields, where Cst(y) is a deterministic term

$$\begin{aligned}\int_0^T r_s(y)ds &= \mathrm{Cst(y)} - \int_0^T \Gamma^*(s,T)(y).dW_s + \frac{1}{2}\int_0^T ||\Gamma^*(s,T)(y) + \sigma_s^{Y^*}(y)||^2 - ||\sigma_s^{Y^*}(y)||^2 ds \\ &= \mathrm{Cst(y)} - \int_0^T \Gamma^*(s,T)(y).(dW_s + \sigma_s^{Y^*}(y)ds) + \frac{1}{2}\int_0^T ||\Gamma^*(s,T)(y)||^2 ds. \qquad (3.3)\end{aligned}$$

The spot forward rates are defined by $f^*(t,T)(y) = -\partial_T \ln B^*(t,T)(y)$. They represent the spot rate of the forward zero-coupon bond defined at time t with starting date T. The limit of the spot forward rate, when the maturity T tends to the current date t, is the spot rate r of no-arbitrage:

$$\lim_{T\to t} f^*(t,T)(y) = r_t(y). \qquad (3.4)$$

The spot forward rates are easier to compute than the rates $R_t^*(T-t)(y)$ themselves: indeed they are computed directly from (3.1) by taking the logarithmic derivative of the product $Y_t^*(y)B^*(t,T)(y)$ with respect to the maturity T.

Proposition 3.1. *We recall that $\sigma^{Y^*}(y) = \nu^*(y) - \eta^{\mathcal{R}}(y)$ is the volatility process of $Y^*(y)$. We assume that the volatility vectors $\Gamma^*(t,T)(y)$ are differentiable with respect to T with locally bounded derivative $\gamma^*(t,T)(y) := \partial_T\Gamma^*(t,T)(y)$. Then the spot forward rates satisfy*

$$\begin{cases} f^*(t,T)(y) = f^*(0,T)(y) - \int_0^t \gamma^*(s,T)(y).\big(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s,T)(y))ds\big), & (3.5)\\ df^*(t,T)(y) = -\gamma^*(t,T)(y).\big(dW_t - (\sigma_t^{Y^*}(y) + \Gamma^*(t,T)(y))dt\big). & (3.6)\end{cases}$$

The yield curve $\delta \mapsto R_t^*(\delta)(y)$ is obtained as the primitive of the forward rate curve:

$$R_t^*(\delta)(y) = \frac{1}{\delta}\int_0^\delta f^*(t,t+s)(y)ds. \qquad (3.7)$$

The market practice, that uses the minimal pricing kernel Y^0 (for which $\nu = 0$) as benchmark, induces a spot forward rate $f^0(t,T)(y)$ instead of $f^*(t,T)(y)$. We compute below the dynamics of the difference between the spot forward rates .

Dynamics of the error $f^*(t,T) - f^0(t,T)$.
The difference $\Delta f(t,T)(y) := f^*(t,T)(y) - f^0(t,T)(y)$ ($\Delta f(T,T) = 0$) between the spot forward rates has the following dynamics (with similar notations for $\Delta\gamma$ and $\Delta\Gamma$)

$$\begin{aligned} d_t(\Delta f(t,T))(y) &= d_t f^*(t,T)(y) - d_t f^0(t,T)(y) \\ &= -\Delta\gamma(t,T)(y).(dW_t + \eta_t^{\mathcal{R}}(y)dt) + <\gamma^*(t,T)(y), \nu_t^*(y)> dt \\ &+ <\gamma^*(t,T)(y), \Gamma^*(t,T)(y)> dt - <\gamma^0(t,T)(y), \Gamma^0(t,T)(y)> dt \\ &= -\Delta\gamma(t,T)(y).(dW_t - \sigma_t^{Y^*}(y)dt) + <\Delta\gamma(t,T)(y), \Gamma^*(t,T)(y)> dt \\ &+ <\gamma^0(t,T)(y), \Delta\Gamma(t,T)(y) + \nu_t^*(y)> dt. \end{aligned}$$

The dynamics of the "error" of using the minimal discounted pricing kernel Y^0 instead of $Y^*(y)$ is similar to the dynamics of a forward rate (equation (3.5)), plus the additional source term $<\gamma^0(t,T)(y), \Delta\Gamma(t,T)(y) + \nu_t^*(y)> dt$.

3.2 Exogenous spot rate and wealth dependency

Writing (3.5) in a backward formulation and since $f^*(T,T)(y) = r_T(y)$ from equation (3.4),

$$f^*(t,T)(y) = r_T(y) + \int_t^T \gamma^*(s,T)(y).(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s,T)(y))ds). \tag{3.8}$$

It appears that the spot rate seems to be depending on y (recall that $y = u_x(x)$ where x is the initial wealth), even for an exogenous spot rate. This dependency is conveyed by the orthogonal component $\nu^*(y)$ of the volatility $\sigma^{Y^*}(y)$. Nevertheless, it is usual in the financial modeling to take the spot rate r_t and the minimal risk premium $\eta^{\mathcal{R}}$ independent of the initial parameter y, on the contrary to the economic framework in which they are endogenous and thus naturally depend on y. But this assumption implies a constraint on the initial slope of the spot forward rates. The dynamics of the spot rate and the condition under which r does not depend on y are given in the following proposition.

Proposition 3.2 (Properties of the spot rate)**.** *The spot rate is given by*

$$r_t(y) = f^*(0,t)(y) - \int_0^t \gamma^*(s,t)(y).(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s,t)(y))ds), \tag{3.9}$$

and its dynamics is given by

$$dr_t(y) = \partial_\delta f^*(t,t)(y)dt - \gamma^*(t,t)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big) \tag{3.10}$$

This implies that for exogenous spot rate r that does not depend on y, $\gamma^*(t,t)$ and $\partial_\delta f^*(t,t)(y) + \gamma^*(t,t).\sigma_t^{Y^*}(y)$ do not depend on y.

Remark Since the yield curve $R_t^*(\delta)$ is a more natural market data than the spot forward rate $f^*(t, t+\delta)$, it is interesting to write its initial slope in terms of the initial slope of the yield curve, namely $\partial_\delta f^*(t,t)(y) = 2\partial_\delta R_t^*(0)(y)$.

Proof. Note that Equation (3.9) is a backward formulation of (3.5). Contrary to the differential form (3.6), T is not fixed anymore, instead $T = t + \delta$ with $\delta \to 0$. Therefore, as in Musiela and Rutkowski [MR05], we denote $r(t,\delta)(y) := f^*(t, t+\delta)(y)$. To get its dynamics, we apply Itô's formula to equation (3.5):
$f^*(t,T)(y) = f^*(0,T)(y) - \int_0^t \gamma^*(s,T)(y).\big(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s,T)(y))ds\big)$,
with $T = t + \delta$ which is of finite variation, and thus we get

$$\begin{aligned} dr(t,\delta)(y) &= \partial_\delta r(t,\delta)(y)dt + \gamma^*(t,t+\delta)(y).\Gamma^*(t,t+\delta)(y)dt \\ &- \gamma^*(t,t+\delta)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big). \end{aligned} \tag{3.11}$$

When the time to maturity δ goes to zero, using the relation $r_t(y) = f^*(t,t)(y)$ and the fact that $\Gamma^*(t,t)(y) = 0$, the dynamics of the spot rate is given by

$$dr_t(y) = \partial_\delta f^*(t,t)(y)dt - \gamma^*(t,t)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big).$$

This implies that for exogenous spot rate r, γ^* does not depend on y on the diagonal and $\partial_\delta f^*(t,t)(y) + \gamma^*(t,t)(y).\sigma_t^{Y^*}(y)$ does not depend on y . Besides, the initial slope of the spot forward rate can be interpreted in terms of the initial slope of the yield curve.

Indeed, differentiating (3.7) w.r.t. δ, one gets $\partial_\delta R_t^*(\delta) = -\frac{R_t^*(\delta)-r_t}{\delta} + \frac{f^*(t,t+\delta)-r_t}{\delta}$. Since $r_t = R_t^*(0) = f^*(t,t)$, passing to the limits when $\delta \to 0$, yields $\partial_\delta f^*(t,t)(y) = 2\partial_\delta R_t^*(0)(y)$. Thus, the dynamics of the spot rate can also be written as

$$dr_t(y) = 2\partial_\delta R_t^*(0)(y)dt - \gamma^*(t,t)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big). \qquad \square$$

We now illustrate these constraints of exogenous spot rates in an affine framework with deterministic volatilities.

A Gaussian affine framework The Vasicek model (1977) was the first model for interest rate coming from a financial point of view. It is stated in a complete market and its starting point is the dynamics of the spot rate (r_t) which is assumed to be an Ornstein-Uhlenbeck process. As a consequence, all the rates in the Vasicek model are affine and Gaussian. We provide here a similar affine framework in an incomplete market. We only assume that the spot forward rates $f^*(t,T)(y)$ are affine function of $r_t(y)$

$$f^*(t,T)(y) = \Lambda(t,T)(y)r_t(y) + \Upsilon(t,T)(y), \quad \Lambda(t,T)(y) \text{ and } \Upsilon(t,T)(y) \text{ deterministic}, \quad (3.12)$$

together with the hypothesis of a deterministic diffusion coefficient for the spot rate. Then differentiating this identity with respect to T, and replacing into (3.10) implies an Ornstein-Uhlenbeck dynamics for the spot rate, with $a_t(y) := -\partial_\delta\Lambda(t,t)(y)$ and $b_t(y) := \partial_\delta\Upsilon(t,t)(y)$

$$dr_t(y) = (b_t(y) - a_t(y)r_t(y))\,dt - \gamma^*(t,t)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big).$$

Furthermore, identifying the diffusion coefficient in (3.12) and (3.6) implies that $\gamma^*(t,T)(y) = \Lambda(t,T)(y)\gamma^*(t,t)(y)$. Besides, differentiating $r(t,\delta) = f(t,t+\delta)$ using relation (3.12) and identifying with (3.11) the term in $r_t(y)dt$ implies $\partial_t\Lambda(t,T)(y) - a_t(y)\Lambda(t,T)(y) = 0$, hence $\Lambda(t,T)(y) = e^{-\int_t^T a_u(y)du}$ since $\Lambda(T,T)(y) = 1$. Therefore we have proved that the affine structure (3.12) induces a time-dependent version of the standard Vasicek model with $\gamma^*(t,T)(y) = e^{-\int_t^T a_u(y)du}\gamma^*(t,t)(y)$. If in addition the volatility $\sigma^{Y^*}(y)$ is deterministic then this affine model is also Gaussian.[7]

Illustration of Proposition 3.2

If the spot rate r does not depend on y, then the diffusion coefficient $\gamma^*(s,s)$ is independent of y. Remark also that if r_t does not depend on y, then $\mathbb{E}(r_t(y))$ does not either, and this implies that a is also independent of y. To summarize, in this affine model, if the spot rate r does not depend on the initial condition y then $\gamma^*(t,t)$, a_t and the drift $b_t(y) - a_t r_t + \gamma^*(t,t).\sigma_t^{Y^*}(y)$ do not depend on y. We recover the result of Proposition 3.2, since in this affine framework, $\partial_\delta f^*(t,t)(y) = b_t(y) - a_t r_t$ as a direct consequence of (3.12). This approach can be generalized into a multidimensional affine model, as in Duffie et al. [DFS+03].

4 Asymptotic long run rates

We are interested in the dynamics behavior of the yield curve, when the maturity goes to infinity

$$R_t^*(\infty)(y) := \lim_{T\to+\infty} R_t^*(T)(y). \quad (4.1)$$

[7] In fact, up to a change of probability that depends on $\sigma^{Y^*}(y)$, this affine model is always Gaussian.

Recalling the relation $R_t^*(T)(y) = \frac{1}{T}\int_0^T f^*(t,t+s)(y)ds$, we study hereafter the asymptotic limit of the forward rate $f^*(t,\infty)(y) := \lim_{T\to+\infty} f^*(t,T)(y)$ and by Cesaro's Lemma[8] we deduce the limit of the yield curve from the one of the spot forward rate.
Recalling (3.5)

$$f^*(t,T)(y) = f^*(0,T)(y) - \int_0^t \gamma^*(s,T)(y).\big(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s,T)(y))ds\big),$$

we have to study together the behavior of the stochastic integral $\int_0^t \gamma^*(s,T)(y).dW_s$ and of the finite variation process $\int_0^t \gamma^*(s,T)(y).(\sigma_s^{Y^*}(y)+\Gamma^*(s,T)(y))ds$, for a fixed t and when T is large. [9] A particular attention is paid on the parameters : the initial value y, or the time horizon T_H. Notably the backward and forward frameworks induce different asymptotic behaviors, as detailed hereafter. This extends previous results of Dybvig et al. [DIR96] and El Karoui et al. [EKFG97].

4.1 Asymptotic long run rates with backward utilities

We study the yield curve dynamics for infinite maturity, first in the framework of backward utility, for which the orthogonal component $\nu^{*,H}(y)$ of $\sigma^{Y^{*,H}}(y)$, as well as the volatility $\Gamma^{*,H}(.,T)(y)$, depend on the time-horizon T_H, and consequently impacts the long term behavior of the yield curve. We thus highlight this dependency by using the index H.
To fix the idea, as T tends to infinity, we take $T_H = T$.

Proposition 4.1. *In the backward case, when the maturity T_H tends to infinity, the spot forward rate $f^{*,H}(t,T_H)(y)$ converges uniformly in L^2 (towards a finite limit) if the following limits exist a.s. in $\mathbb{R}$*

$$\begin{cases} k_s(y) := \lim_{T_H\to+\infty} \gamma^{*,H}(s,T_H)(y), & (4.2)\\ \mathfrak{g}_s(y) := \lim_{T_H\to+\infty} \gamma^{*,H}(s,T_H)(y).(\nu_s^{*,H}(y) + \Gamma^{*,H}(s,T_H)(y)). & (4.3)\end{cases}$$

The dynamics of the asymptotic long spot forward rate $f^*(t,\infty)(y)$ is

$$f^*(t,\infty)(y) = f^*(0,\infty)(y) - \int_0^t k_s(y).dW_s + \int_0^t \big(\mathfrak{g}_s(y) - < k_s^{\mathcal{R}}(y), \eta_s^{\mathcal{R}} >\big)\,ds. \tag{4.4}$$

(i) *Having simultaneously the limit $k_.(y)$ not equal to zero $ds\otimes d\mathbb{P}$ a.s. with a finite limit $\mathfrak{g}_.(y)$ is possible only if $\frac{1}{T_H} < k_s^{\perp}(y), \nu_s^{*,H}(y) >= -||k_s(y)||^2 + \mathcal{O}(\frac{1}{T_H})$ $ds\otimes d\mathbb{P}$. Then the spot forward rates for infinite maturity are finite and their dynamics* (4.4) *have a diffusion component.*

(ii) *If $\frac{1}{T_H} < k_s^{\perp}(y), \nu_s^{*,H}(y) >\neq -||k_s(y)||^2 + \mathcal{O}(\frac{1}{T_H})$ $ds\otimes d\mathbb{P}$, the limit $\mathfrak{g}_s(y)$ is finite only if $k_s(y)\equiv 0$ $ds\otimes d\mathbb{P}$ a.s., and the usual form holds for the asymptotic spot forward rates*

$$f^*(t,\infty)(y) = f_0(y) + \int_0^t \mathfrak{g}_s(y)ds.$$

[8] See [Kor13]. Note that this is only a sufficient condition: the two limits of $f^*(t,T)(y)$ and $R_t^*(T)(y)$ are not equivalent from a strict mathematical point of view, but are equal when both of them exist. For the converse result one need for example a monotonicity condition of $u\to f^*(t,u)$ to deduce the infinite limit of f^* from the one of R^*.

[9] We assume sufficient regularity conditions on the coefficients of the SDE satisfied by the process $f^*(t,T)$ (typically $\gamma^*(s,T)(y)$ uniformly bounded (in T) by an L^2-integrable process, as in [EKFG97]) to use convergence results of SDE.

Proof. We have to study the limit of the terms in (3.5), where in the backward case the orthogonal component of $\sigma^{Y^*}(y)$, namely $\nu^{*,H}(y)$, may depend on T_H and has to be taken into account to compute the limit.
First remark that if $\gamma^{*,H}(s,T_H)(y)$ converges (which is equivalent to $\gamma^{*,H,\mathcal{R}}(s,T_H)(y)$ and $\gamma^{*,H,\perp}(s,T_H)(y)$ converge), then the stochastic integral in (3.5) converges. Besides,

$$\begin{aligned}\gamma^{*,H}(s,T_H)(y).(\sigma_s^{Y^*}(y)+\Gamma^{*,H}(s,T_H)(y)) &= -<\gamma^{*,H,\mathcal{R}}(s,T_H)(y),\eta_s^{\mathcal{R}}> \\ &+ <\gamma^{*,H}(s,T_H)(y),\nu_s^{*,H}(y)+\Gamma^{*,H}(s,T_H)(y)> .\end{aligned}$$

Since $\eta_s^{\mathcal{R}}$ does not depend on T_H, (4.2) and (4.3) imply that the right hand side converges a.s. and the dynamics is given by (4.4).
We recall that $\gamma^*(t,T)(y)=\partial_T\Gamma^*(t,T)(y)$. Therefore by Cesaro's Lemma, when $T_H\to+\infty$, $<\gamma^{*,H}(s,T_H)(y),\Gamma^{*,H}(s,T_H)(y)>$ is asymptotically equivalent to $T_H||\gamma^{*,H}(s,T_H)(y)||^2$. Thus, if $k_s(y)=\lim_{T_H\to+\infty}\gamma^{*,H}(s,T_H)(y)$ is not equal to zero a.s., then $<\gamma^{*,H}(s,T_H)(y),\nu_s^{*,H}(y)+\Gamma^{*,H}(s,T_H)(y)>$ converges if and only if $\frac{1}{T_H}<k_s^{\perp}(y),\nu_s^{*,H}(y)>=-||k_s(y)||^2+\mathcal{O}(\frac{1}{T_H})$.
Otherwise, to ensure the limit $\mathfrak{g}_s(y)$ to be finite, one should have $\lim_{T_H\to+\infty}\gamma^{*,H}(s,T_H)(y)=0$ $ds\otimes d\mathbb{P}$ *a.s.*, which implies that there is no stochastic integral in the dynamics (4.4), which is then given by $f_t(y)=f_0(y)+\int_0^t\mathfrak{g}_s(y)ds$. □

By applying again Cesaro's Lemma, this time on the rates $f^{*,H}(t,T_H)(y)$ and $R_t^{*,H}(T_H)(y)=\frac{1}{T_H}\int_0^{T_H}f^{*,H}(t,t+s)(y)ds$, we conclude that $R^*(t,\infty)(y)=f^*(t,\infty)(y)$.

The diffusion component in the dynamics (4.4) of asymptotic long rates is a consequence of the dependency on T_H of the orthogonal volatility $\nu^{*,H}$ of the optimal discounted pricing kernel $Y^{*,H}$. To specify the dynamics (4.4), one need to determine the links between the orthogonal volatilities $\nu^{*,H}$ and $\Gamma^{*,H,\perp}(.,T_H)$, which is not an easy task in full generality. Nevertheless, the computations are tractable for power utilities, which is the natural setting for the Ramsey rule (cf. Section 1.3 and Section 2.1.1).

4.2 Yield curve properties with backward power utilities

The following theorem provides a new and non-asymptotic relation between the orthogonal volatilities of the optimal discounted pricing kernel and the zero-coupon bond price.

Theorem 4.2. *For backward power utilities, the orthogonal volatility $\nu^{*,H}$ of the optimal discounted pricing kernel $Y^{*,H}$ and the orthogonal volatility $\Gamma^{*,H,\perp}(.,T_H)$ of the zero-coupon bond price are linked by the relation*

$$\nu_t^{*,H}=-\Gamma^{*,H,\perp}(t,T_H),\quad 0\le t\le T_H. \tag{4.5}$$

Proof. $\nu^{*,H}$ is the orthogonal volatility of the optimal discounted pricing kernel $Y^{*,H}$, solution of the dual optimization problem. According to Definition 1.4, the dual problem relies on the submartingale/martingale property of the preference process $\left(\tilde{\mathcal{U}}(t,Y_t^\nu)+\int_0^t\tilde{v}(s,Y_s^\nu)ds\right)$, which is sometimes better to write in a multiplicative form. It is then equivalent to study the submartingale/martingale property of $\left(\exp\left(\int_0^t\frac{\tilde{v}(s,Y_s^\nu)}{\tilde{\mathcal{U}}(s,Y_s^\nu)}ds\right)\tilde{\mathcal{U}}(t,Y_t^\nu)\right)$.
In the backward power framework, the terminal dual utility from wealth $\tilde{\mathcal{U}}(T_H,.)$ and the dual utilities from consumption $\tilde{v}(s,.)$ are given: they are dual power utilities, with the same

risk aversion parameter θ, $\tilde{\mathcal{U}}(T_H,y) = Z^{\tilde{u}}_{T_H} y^{\frac{\theta-1}{\theta}}$, and for $s\in[0,T_H]$, $\tilde{v}(s,y) = Z^{\tilde{v}}_s y^{\frac{\theta-1}{\theta}}$, where $Z^{\tilde{v}}_s$ is a given process and $Z^{\tilde{u}}_{T_H}$ is a given $\mathcal{F}_{T_H}$-random variable.
Then, as recalled in (1.18), $\tilde{\mathcal{U}}$ is also time-separable with risk aversion parameter θ. This implies that for $s\in[0,T_H]$, $\frac{\tilde{v}(s,Y^\nu_s)}{\tilde{\mathcal{U}}(s,Y^\nu_s)} = \tilde{Z}_s$ where $\tilde{Z}$ is a progressive process that does not depend[10] on ν. The backward dual optimization problem (1.7) turns out to find $\nu\in\mathcal{R}^\perp$ that minimizes the drift of $\tilde{\mathcal{U}}(T_H,Y^\nu_s)$ that is the drift of $(Y^\nu_{T_H})^{\frac{\theta-1}{\theta}}$

$$(Y^\nu_{T_H})^{\frac{\theta-1}{\theta}} = \exp(-\frac{\theta-1}{\theta}\int_0^{T_H} r_s ds + \frac{\theta-1}{\theta}\int_0^{T_H}\sigma^{Y^\nu}_s.dW_s - \int_0^{T_H}\frac{\theta-1}{2\theta}||\sigma^{Y^\nu}_s||^2 ds).$$

Using relation (3.3) with the discounted pricing kernel $Y^{\nu,H}$ instead of Y^*, leading to zero-coupon prices $B^{\nu,H}(t,T)$ ($B^{\nu,H}(0,T) = B(0,T)$ and $B^{\nu,H}(T,T)=1$) with volatility $\Gamma^{\nu,H}$, we have (where Cst denotes a deterministic constant that does not depend on ν)

$$\begin{aligned}
(Y^\nu_{T_H})^{\frac{\theta-1}{\theta}} &= \mathrm{Cst}\exp\Big(\frac{\theta-1}{\theta}\int_0^{T_H}(\Gamma^{\nu,H}(s,T_H)+\sigma^{Y^\nu}_s).dW_s - \frac{\theta-1}{2\theta}\int_0^{T_H}||\Gamma^{\nu,H}(s,T_H)+\sigma^{Y^\nu}_s||^2 ds\Big)\\
&= \mathrm{Cst}\,\mathcal{E}_{T_H}\left(\frac{\theta-1}{\theta}(\Gamma^{\nu,H}(.,T_H)+\sigma^{Y^\nu})\right)\exp(-\frac{\theta-1}{2\theta}\int_0^{T_H}||\Gamma^{\nu,H}(s,T_H)+\sigma^{Y^\nu}_s||^2 ds)\\
&\qquad \exp(\int_0^{T_H}\frac{1}{2}(\frac{\theta-1}{\theta})^2||\Gamma^{\nu,H}(s,T_H)+\sigma^{Y^\nu}_s||^2 ds)\\
&= \mathrm{Cst}\,\mathcal{E}_{T_H}\left(\frac{\theta-1}{\theta}(\Gamma^{\nu,H}(.,T_H)+\sigma^{Y^\nu})\right)\exp(\frac{1-\theta}{2\theta^2}\int_0^{T_H}||\Gamma^{\nu,H}(s,T_H)+\sigma^{Y^\nu}_s||^2 ds).
\end{aligned}$$

This implies that the minimisation problem is equivalent to minimize (in ν) the quadratic form

$$\frac{1-\theta}{2\theta^2}||\sigma^{Y^\nu}_t + \Gamma^{\nu,H}(t,T_H)||^2 = \frac{1-\theta}{2\theta^2}||\nu^H_t - \eta^{\mathcal{R}}_t + \Gamma^{\nu,H}(t,T_H)||^2$$

which achieves its minimum at $\nu^{*,H}_t = -\Gamma^{*,H,\perp}(t,T_H)$. □

Even in this simple framework of backward power utilities, the backward approach and relation (4.5) imply a diffusion component in the dynamics of asymptotic long rates. Recall that for power utilities, the optimal discounted pricing kernel is linear with respect to its initial condition y, which implies that the interest rates do not depend on y.

Corollary 4.3. *For backward power utilities, the asymptotic long spot forward rate $f^*(t,\infty)$ (that may be infinite) is given by*

$$f^*(t,\infty) = f^*(0,\infty) - \int_0^t k_s.dW_s + \int_0^t \mathfrak{g}_s ds, \tag{4.6}$$

with
$$\begin{cases} k_s := \lim_{T_H\to+\infty}\gamma^{*,H}(s,T_H),\\ \mathfrak{g}_s := \lim_{T_H\to+\infty}\frac{1}{T_H}||\Gamma^{*,H,\mathcal{R}}(s,T_H)||^2.\end{cases}$$

(i) *If $k^{\mathcal{R}}_s$ is not equal to zero $dt\otimes d\mathbb{P}$ a.s., then $f^*(t,\infty)$ is infinite.*
(ii) *Otherwise, $f^*(t,\infty) = f^*(0,\infty) - \int_0^t k^\perp_s.dW_s + \int_0^t \mathfrak{g}_s ds$.*

Proof. Applying Proposition 4.1 with $\nu^{*,H}_t = -\Gamma^{*,H,\perp}(t,T_H)$ and using Cesaro's Lemma

$$\lim_{T_H\to+\infty}\gamma^{*,H}(s,T_H).(\nu^{*,H}_s + \Gamma^{*,H}(s,T_H)) = \lim_{T_H\to+\infty}\frac{1}{T_H}||\Gamma^{*,H,\mathcal{R}}(s,T_H)(y)||^2.$$

[10] This has been proved in [EKHM18, Section 4.2].

If $k_s^{\mathcal{R}}(y) = \lim_{T_H\to+\infty} \gamma^{*,H,\mathcal{R}}(s,T_H) \neq 0 \; dt\otimes d\mathbb{P}$ a.s., then $\lim_{T_H\to+\infty} \frac{1}{T_H}||\Gamma^{*,H,\mathcal{R}}(s,T_H)(y)||^2$ and $f^*(t,\infty)$ are infinite. Otherwise $\quad f^*(t,\infty)(y) = f^*(0,\infty)(y) - \int_0^t k_s^{\perp}.dW_s + \int_0^t \mathfrak{g}_s ds.$ $\square$

Even in this simple framework of backward power utilities, the long run yield curves (if they are not infinite) have a diffusion component and thus are not monotonous in time. This differs from the framework of forward utility for which they are nondecreasing processes, as detailed below.

4.3 Asymptotic long run rates with forward utility

We study the yield curve dynamics for infinite maturity, in the framework of forward utility, for which the orthogonal risk premium $\nu^*(y)$ does not depend on the time-horizon. As a consequence the limit behavior is more straightforward compared to the backward case, and has no diffusion component.

Proposition 4.4. *In the forward case, the asymptotic long spot forward rate $f^*(t,\infty)(y)$ is*
(i) *infinite if $\lim_{T\to+\infty} \gamma^*(t,T)(y)$ exists and is not equal to zero $dt\otimes d\mathbb{P}$ a.s.*
(ii) *Otherwise, $f^*(t,\infty)(y) = f^*(0,\infty)(y) + \int_0^t g_s(y)ds$ with $g_s(y) = \lim_{T\to+\infty} \frac{1}{T}||\Gamma^*(s,T)(y)||^2$.*
So, the asymptotic long forward rate $f^*(t,\infty)(y)$ is a nondecreasing process in time starting from $f^*(0,\infty)(y)$, constant if $g_s(y) \equiv 0 \; ds\otimes d\mathbb{P}$ a.s.

Proof. The proof is based on the following observation (using Cesaro's Lemma)

$$\lim_{T\to+\infty} \frac{1}{T}\Gamma^*(t,T)(y) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \gamma^*(t,u)(y)du = \lim_{T\to+\infty} \gamma^*(t,T)(y).$$

(i) If $\lim_{T\to+\infty} \gamma^*(t,T)(y)$ exists and is not equal to zero $dt\otimes d\mathbb{P}$ a.s. then

$$\lim_{T\to+\infty} \Gamma^*(t,T)(y) = \infty \;\; a.s \text{ and } \;\; l_t(y) \text{ is infinite.}$$

(ii) Otherwise, $\int_0^T \gamma^*(s,T)(y).dW_s$ and $\int_0^T \gamma^*(s,T)(y).\sigma_s^{Y^*}(y)ds$ converge to zero and $l_t(y) = l_0(y) + \int_0^t g_s(y)ds,$ where $g_t(y)$ is the non-negative process
$g_t(y) = \lim_{T\to+\infty}(\gamma^*(t,T)(y).\Gamma^*(t,T)(y)) = \lim_{T\to+\infty} T||\gamma^*(t,T)(y)||^2 = \lim_{T\to+\infty} \frac{1}{T}||\Gamma^*(s,T)(y)||^2.$
$\square$

As a corollary, by Cesaro's Lemma, $R^*(t,\infty)(y) = f^*(t,\infty)(y)$.

Throughout this paper, we have pointed out the key role of the discounted pricing kernel Y^* in the computation of the Ramsey rule and the yield curve, such Y^* being optimal relatively to a given preference criterium. A natural question arising is how to handle the heterogeneity of economic actors, that may have different preferences, and thus different discounted pricing kernel Y^*. To do this, considering N investors characterized by their utility U^{θ_i}, we aggregate the discounted pricing kernels as follows:

$$Y^*(y) := \sum_{i=1}^N Y^{*,\theta_i}(y^{\theta_i}(y)), \quad y = \sum_{i=1}^N y^{\theta_i}(y).$$

We propose to study the impact of aggregation on the yield curve, in particular for infinite maturity, or when the wealth of the economy tends to 0 or ∞.

4.4 Aggregation of utilities

As pointed out in [EKHM17], aggregating discounted pricing kernels corresponds to the aggregation of utilities. We concentrate of aggregating power utilities, since as explained in Section 1.3, power utility functions is an important case of utility functions, in which computations are tractable and the existence of an equilibrium can be stated. Besides, [EKM20] proved that the utility functions that are compatible with an equilibrium can be written as mixtures of power utilities.

Let us consider an economy composed of N investors, with consistent power utilities characterized by (constant) relative risk aversion parameters $\theta_1 < \cdots < \theta_N$. Then, their optimal discounted pricing kernels $Y_t^{*,\theta_i}(y)$ are linear in y with coefficient $\bar{Y}_t^{*,\theta_i}$ and the individual price of zero-coupon bonds with maturity T does not depend on y and is given by $B^{*,\theta_i}(t,T) = \mathbb{E}\big(\frac{\bar{Y}_T^{*,\theta_i}}{\bar{Y}_t^{*,\theta_i}}|\mathcal{F}_t\big)$.

The aggregate indifference zero-coupon bond price $B^*(0,T)(y)$, computed at time 0 for simplicity, is given by

$$B^*(0,T)(y) = \frac{1}{y}\sum_{i=1}^{N} y^{\theta_i}(y) B^{*,\theta_i}(0,T), \qquad \text{with} \qquad y = \sum_{i=1}^{N} y^{\theta_i}(y). \tag{4.7}$$

4.4.1 Asymptotic limit for infinite maturity

For any agent, we define his asymptotic long rate

$$R_0^{*,\theta_i}(\infty) := \lim_{T\to\infty} R_0^{*,\theta_i}(T) = \lim_{T\to\infty}\left(-\frac{1}{T}\ln(B^{*,\theta_i}(0,T))\right).$$

The following proposition shows that when the maturity tends to infinity, the asymptotic long aggregate rate is the one with the lowest asymptotic long rate. This is a similar result to that in Cvitanic et al. [CJMN11, Section 7].

Proposition 4.5. *We consider the aggregation of N heterogeneous agents having CRRA utility functions, and we denote by $R_0^*(T)(y)$ the corresponding aggregate indifference rate. Then the asymptotic long aggregate rate*

$$R_0^*(\infty) := \lim_{T\to\infty} R_0^*(T)(y) = \min_{i\in[\![1;N]\!]} R_0^{*,\theta_i}(\infty) \text{ (possibly infinite).}$$

Proof. First remark that if for any $i \in [\![1;N]\!]$, $R_0^{*,\theta_i}(T)(y)$ have the same limit (infinite or not) then it is straightforward to see that the aggregate yield curve $R_0^*(T)(y)$ converges to this limit. We define $\mathfrak{I} := \underset{i\in[\![1;N]\!]}{\arg\min}\, R_0^{*,\theta_i}(\infty)$, and we choose $i_o \in \mathfrak{I}$. Then

$$\begin{aligned}
R_0^*(T)(y) &= -\frac{1}{T}\ln(B^*(0,T)(y)) \\
&= -\frac{1}{T}\ln\left(\frac{y^{\theta_{i_o}}(y)}{y}B^{*,\theta_{i_o}}(0,T)\right) - \frac{1}{T}\ln\left(1 + \sum_{i\neq i_o}\frac{y^{\theta_i}(y)B^{*,\theta_i}(0,T)}{y^{\theta_{i_o}}(y)B^{*,\theta_{i_o}}(0,T)}\right) \\
&= R_0^{*,\theta_{i_o}}(T) - \frac{1}{T}\ln\left(\frac{y^{\theta_{i_o}}(y)}{y}\right) \\
&\quad -\frac{1}{T}\ln\left(1 + \sum_{i\neq i_o}\frac{y^{\theta_i}(y)}{y^{\theta_{i_o}}(y)}e^{-T\left(R^{*,\theta_i}(0,T) - R^{*,\theta_{i_o}}(0,T)\right)}\right)
\end{aligned} \tag{4.8}$$

If $i \notin \mathfrak{I}$, $e^{-T\left(R^{*,\theta_i}(0,T)-R^{*,\theta_{i_o}}(0,T)\right)} \to 0$ since $\lim_{T\to\infty}(R_0^{*,\theta_i}(T) - R_0^{*,\theta_{i_o}}(T)) > 0$. Thus the factor inside the logarithm is greater than one and for, large T, is smaller than $(N+1)e^{T\max_{i\in\mathfrak{I}}|R_0^{*,\theta_i}(T)-R_0^{*,\theta_{i_o}}(T)|}$. Therefore the last term (4.8) converges to zero since for all $i \in \mathfrak{I}$, $\lim_{T\to\infty}(R_0^{*,\theta_i}(T) - R_0^{*,\theta_{i_o}}(T)) = 0$. We conclude that $R_0^*(\infty) = \lim_{T\to\infty} R_0^*(T)(y) = \lim_{T\to\infty} R_0^{*,\theta_{i_o}}(T) = \min_{i\in[\![1;N]\!]} R_0^{*,\theta_i}(\infty)$. □

4.4.2 Asymptotic limit with respect to the initial wealth

Power utility functions imply equilibrium rates that do not depend on the wealth process of the economy (see Section 1.3), and thus does not allow to capture some important features concerning the impact of the wealth of the economy on the rates. This can be circumvented with aggregation of power utilities, which provides a more flexible preference criterium. Thus we study hereafter the asymptotic behavior of the aggregate zero-coupon bond price $B^*(0,T)(y)$ for small and large wealth $x = u_z^{-1}(y)$, and for any maturity T.
If any investor is endowed at time 0 with a proportion α_i of the initial global wealth x ($\sum_{i=1}^N \alpha_i = 1$), then $y^{\theta_i}(y) = u_z^{\theta_i}(\alpha_i x) = (\alpha^i x)^{-\theta_i}$, $y = \sum_{i=1}^N y^{\theta_i}(y) = u_z(x)$ and

$$B^*(0,T)(y) = \frac{1}{y}\sum_{i=1}^N y^{\theta_i}(y) B^{*,\theta_i}(0,T) = \frac{\sum_{i=1}^N (\alpha_i x)^{-\theta_i} B^{*,\theta_i}(0,T)}{\sum_{i=1}^N (\alpha_i x)^{-\theta_i}} \tag{4.9}$$

Proposition 4.6. *We consider the aggregation of N heterogeneous agents having CRRA utility functions. When the wealth tends to infinity the aggregate zero-coupon price converges to the one priced by the less risk averse agent, whereas when the wealth tends to zero, it converges to the one priced by the more risk averse agent.*

Proof. We use (4.9), and the fact that for power utility u^{θ_i}, $y^{\theta_i}(y) = u_z^{\theta_i}(\alpha_i x) = (\alpha^i x)^{-\theta_i}$. When the wealth tends to infinity (corresponding to $y = u_z(x)$ tends to zero) the discrete random measure $\frac{\sum_{i=1}^N y^{\theta_i}(y)\,\delta_{\theta_i}(\theta)}{y}$ converges towards a Dirac measure that charges the agent with the smallest risk aversion θ_i ; and respectively towards the largest risk aversion θ_i when the wealth tends to zero (corresponding to y tends to infinity):

$$\lim_{y\to 0} B^*(0,T)(y) = B_0^{\theta_1}(T) \text{ and } \lim_{y\to+\infty} B^*(0,T)(y) = B_0^{\theta_N}(T).$$

This is coherent with the result of Cvitanic et al. [CJMN11, Corollary 4.6]. □

This can be generalized into a continuum of heterogeneous investors indexed by θ, with any utility function (not necessarily power) and having different weights in the economy (see [EKHM17, Section 3]).

Conclusion

This paper draws a parallel between financial and economic discount rates and provides a financial interpretation of the Ramsey rule, using consistent pair of progressive utilities of investment and consumption and using marginal utility indifference price (Davis price) for the pricing of non replicable zero-coupon bonds. We have highlighted that forward utilities provide a more flexible framework than standard backward utilities, which induce time dependency on the time horizon ; this difference between forward and backward approaches is particularly relevant in the computation of the infinite maturity yield curve. The case of power utilities is also developed, in order to provide tractable computations and to remain deliberately close to the economic equilibrium setting. Nevertheless power utilities imply that the optimal processes are linear with respect to their initial conditions, and due to this simplification, power utilities are not able to catch the impact of the wealth of the economy on the discount rates. Considering aggregation of power utilities, which is equivalent to an aggregation of discounted pricing kernels, overcomes this issue while keeping tractable formulas. This arises naturally in a context of heterogeneous investors, while being compatible with the existence of an equilibrium. Our approach can also be related to multi-curve modeling, that attracts significant attention since the crisis, see Grbac and Runggaldier [GR15].

In this paper, we have chosen a framework close to the one of the economic equilibrium framework, with a linear pricing rule (given by the marginal utility price), and for illustrative purpose, we have provided explicit examples in Gaussian markets. We would like to point out the limitations of such framework and to suggest some extensions. Indeed, models that are linear with respect to the noise could result to an underestimation of extreme risks, especially for the long-term, and one would like to give more importance to the randomness of the economy.
Alternative models to Gaussian markets for interest rate are affine models and quadratic Gaussian models, for which calculations can be carried out. A short-rate model is affine if it is a linear combination of an affine state space process, whose conditional characteristic function is exponential affine with respect to the initial value. Affine models lead to tractable pricing formula, using Riccati's equations, see for example [EKHM14] in the context of the Ramsey rule. Quadratic Gaussian models are factor models where interest rates are quadratic functions of underlying Gaussian factors, see Beaglehole and Tenney [BT91], Durand and El Karoui [EKD98], or Jamshidian [Jam96], among others. Quadratic Gaussian models allow an extra quadratic term of the state variable in the expression for the short-rate. For these quadratic short-rate models similar properties hold as for the affine models - as well as analytical and computational tractability - in which the zero-coupon price changes to an expression with an extra quadratic term.
Besides, marginal utility price is a linear pricing rule which means that investors agree on this price for a small amount, but they are not sure to have liquidity at this price. For larger nominal amount of transaction and highly illiquid market, the size of the transaction impacts the price. One may use utility indifference pricing, which induces a bid ask spread. Nevertheless, computing utility indifference prices is often a difficult task. An alternative is to use a second order expansion of the Davis price, which is more tractable. This is developed in the Appendix.

5 Appendix

This Appendix provides theoretical details and proofs on utility indifference pricing, on the time-coherence of the marginal utility price in both the forward and backward setting, as well as the derivation of the second order development of the utility indifference price with respect to the amount of claim.

5.1 Utility indifference pricing

When the payoff ξ_T of the claim is not replicable, there are different ways to evaluate the risk coming from the non-replicable part, while taking into account the size of the transaction. A way is the pricing by indifference, that leads to a bid-ask spread. The utility *indifference price* $\widehat{p}^q_{0,T}(x,\xi_T,q)$ is the price at which the investor is indifferent from investing or not in the contingent claim ; it is given by the non-linear relationship

$$\mathcal{U}^{\xi,T}(x+\widehat{p}^q_{0,T}(x,\xi_T,q),q)=\mathcal{U}^T(x). \tag{5.1}$$

where the two maximization problems[11] (with and without the claim ξ_T) have been introduced in Section 2.2

$$\begin{cases} \mathcal{U}^T(x)=\sup\limits_{(\kappa,\rho)\in\mathcal{A}^c}\mathbb{E}[U(T,X^{\kappa,\rho}_T)+\int_0^T V(s,c_s)ds]. & (5.2)\\ \mathcal{U}^{\xi,T}(x,q):=\sup\limits_{(\kappa,\rho)\in\mathcal{A}^c}\mathbb{E}[U(T,X^{\kappa,\rho}_T-q\,\xi_T)+\int_0^T V(s,c_s)ds]. & (5.3)\end{cases}$$

Remark: The formulation of the utility indifference pricing problem is the same for forward and backward utilities, with the appropriate utility process $\mathbf{U}$ that should be considered in the definitions (5.2) and (5.3). In both cases, the utility indifference pricing problem is posed backward, with the natural maturity T which is the date of payment of the claims, and the associated optimal processes depend on T. The literature usually considers the utility indifference pricing problem in the backward framework (that is with $U(T_H,.)$ a given deterministic function, and $T\le T_H$), see for example Davis [Dav98], the survey of Hobson [HH09] or Carmona [CN90]. If $T<T_H$, thanks to the dynamic programming principle, the stochastic utility $U(T,.)$ that should be considered in (5.2) and (5.3) is the value function at time T of the backward optimization problem with utility $U(T_H,.)$ at time T_H. In the forward framework, $U(T,.)$ is the forward utility itself at time T (and T is not restricted to be less than T_H). In what follows, we consider both the forward and backward settings and we comment the differences when needed. We use the index H (such as $Y^{*,H}_{.}$) to emphasize the time horizon dependency in the backward optimization problem.

5.2 Marginal indifference pricing and time-coherence

For a small amount of the claim, one can use marginal indifference price, which corresponds to the zero marginal rate of substitution $p^u_{0,T}(x,\xi_T):=\lim\limits_{q\to 0}\frac{\partial\hat{p}^q_{0,T}}{\partial q}(x,\xi_T,q)$ as defined in (2.13). In this section, we prove Proposition 2.1 that characterizes the marginal indifference price in terms of the optimal discounted pricing kernel Y^*, and we investigate the time-coherence of this linear pricing rule.
Marginal indifference price is defined for any maturity $T\in[0,+\infty[$ in the forward case and for any $T\le T_H$ in the backward case. In the backward case, the value function $U(T,.)$ depends on the horizon T_H. In particular, if the contingent claim ξ_T is delivered at time $T\le T_H$, then ξ_T can be invested between time T and T_H into any admissible portfolio

[11] To ease the notations, we will often write $\mathcal{U}^\xi$ and $\mathcal{U}$ rather than $\mathcal{U}^{\xi,T}$ and $\mathcal{U}^T$.

$X_.(T,\xi_T)$ (martingale under $Y^{*,H}$) and computing the marginal utility price with terminal payoff $\xi_{T_H} = X_{T_H}(T,\xi_T)$ leads to the same price, as explained below.

Proposition 5.1. *Let $(Y^*_t(y))$ be the optimal discounted pricing kernel associated with a (forward or backward) consumption optimization problem. For any non negative contingent claim ξ_T delivered at time T, the marginal utility price is given at any time $t \le T$ by*

$$p^u_{t,T}(x,\xi_T) = \mathbb{E}\big[\xi_T \frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\big], \quad y = \mathcal{U}_z(0,x). \tag{5.4}$$

(i) ***In the forward case****, the pricing rule is time-coherent :*
for all T and T', with $T \le T'$

$$p^u_{t,T'}(x,\xi_{T'}) = p^u_{t,T}(x,\xi_T(t,x)) \text{ with } \xi_T(t,x) = p^u_{T,T'}(X^*_T(t,x),\xi_{T'}). \tag{5.5}$$

(ii) ***In the backward case****, the time-coherence property (5.5) is satisfied*
• *for $T \le T' \le T_H$ with $\xi_T(t,x) = \xi^H_T(t,x) = p^{u,H}_{T,T'}(X^{*,H}_T(t,x),\xi_{T'})$.*
• *for $T \le T'$ with $T' > T_H$ if the utility function $U(T',.)$ at the horizon T' is the consistent progressive utility starting from $u(T_H,.)$ at time T_H.*

Proof. To simplify the notations, the proof is given for $t=0$ (the dynamic version can be proved in the same way) and the indifference price is denoted $\widehat{p}^q := \widehat{p}^q_{0,T}(x,\xi_T,q)$.
Following Davis [Dav98], we compute the marginal indifference price of any contingent claim as follows. Denote by $(X^{*,q}(x), c^{*,q}(x))$ the optimal strategy of the optimization program (5.3) (q-quantity of the claim ξ_T), such that

$$\mathbb{E}\big[U(T,X^{*,q}_T(x) - q\xi_T) + \int_0^T V(s,c^{*,q}_s(x))ds\big] = \mathcal{U}^\xi(0,x,q).$$

Thanks to the envelope theorem we can invert optimization and differentiation along the optimal paths (see Milgrom [MS02]); in our setting, the q-derivative concerns the random variables $U\big(T,X^{\kappa,c}_T(x) - q\xi_T\big) + \int_t^T V(s,c_s(x))ds$. Then

$$\partial_q\,\mathcal{U}^\xi(0,x,q) = -\mathbb{E}\big(U_z(T,X^{*,q}_T(x) - q\xi_T)\xi_T\big).$$

On the other hand, since by definition $\mathcal{U}^\xi(0,x,q) = \mathcal{U}(0,x-\hat{p}^q)$

$$\partial_q\mathcal{U}^\xi(0,x,q) = \partial_q\,\mathcal{U}(0,x-\hat{p}^q) = -\frac{\partial\hat{p}^q}{\partial q}\mathcal{U}_z(0,x-\hat{p}^q)$$

thus, we obtain the q-sensitivity of the indifference price

$$\frac{\partial\hat{p}^q}{\partial q} = \frac{\mathbb{E}\big(U_z(T,X^{*,q}_T(x) - q\xi_T)\xi_T\big)}{\mathcal{U}_z(0,x-\hat{p}^q)}. \tag{5.6}$$

This quantity depends on the optimal process $X^{*,q}_T(x)$ which is not easy to compute, but at the limit in $q=0$, it becomes, since $\lim_{q\to 0} X^{*,q}_T = X^*_T$

$$p^u_{0,T}(x,\xi_T) = \lim_{q\to 0}\frac{\partial\hat{p}^q}{\partial q}(x,\xi_T) = \frac{\mathbb{E}\big[\xi_T U_z(T,X^*_T(x))\big]}{\mathcal{U}_z(0,x)}.$$

The marginal pricing rule is linear, and associated with the pricing kernel

$$\frac{U_z(T, X_T^*(x))}{\mathcal{U}_z(0,x)} = \frac{Y_T^*(\mathcal{U}_z(0,x))}{\mathcal{U}_z(0,x)}.$$

(i) In the forward case, for any maturity T', we have

$$p^u_{0,T'}(x, \xi_{T'}) = \frac{1}{u_z(x)}\mathbb{E}\big[U_z(T', X^*_{T'}(x))\,\xi_{T'}\big] = \mathbb{E}\big[\xi_{T'}\frac{Y^*_{T'}(y)}{y}\big].$$

In particular, for any $T \leq T'$, one can easily prove (5.5):

$$\begin{aligned} p^u_{0,T'}(x,\xi_{T'}) &= \frac{1}{u_z(x)}\mathbb{E}\big[U_z(T', X^*_{T'}(x))\,\xi_{T'}\big] \\ &= \frac{1}{u_z(x)}\mathbb{E}\big[\frac{1}{U_z(T, X^*_T(x))}\mathbb{E}\big[U_z(T', X^*_{T'}(x))\,\xi_{T'}|\mathcal{F}_T\big]U_z(T, X^*_T(x))\big] \\ &= \frac{1}{u_z(x)}\mathbb{E}\big[p^u_{T,T'}(X^*_T(x), \xi_{T'})U_z(T, X^*_T(x))\big] \\ &= p^u_{0,T}(x, p^u_{T,T'}(X^*_T(t,x), \xi_{T'})). \end{aligned}$$

(ii) In the backward case, if the maturity of the claim is $T \leq T_H$, then the amount ξ_T may be invested in any admissible portfolio $X_.(T,\xi_T)$ such that $(X_t(T,\xi_T)\,Y^{*,H}_t(y))_{T\leq t\leq T_H}$ is a martingale and taking $\xi_{T'} = X_{T'}(T,\xi_T),\ T' \in [T, T_H]$. Then the proof of (5.5) in the backward case is identical to the one of the forward case as soon as $T \leq T' \leq T_H$:

$$\begin{aligned} p^{u,H}_{0,T'}(x,\xi_{T'}) &= \mathbb{E}\big[\mathbb{E}(X_{T'}(T,\xi_T)\frac{Y^{*,H}_{T'}(y)}{y}|\mathcal{F}_T\big] \\ &= \mathbb{E}\big[\xi_T\frac{Y^{*,H}_T(y)}{y}\big] = p^{u,H}_{0,T}(x,\xi_T), \qquad y = u_z(x). \end{aligned} \tag{5.7}$$

The backward marginal utility pricing is a well-posed pricing rule only for $T \leq T_H$. Nevertheless, for $T' > T_H$, in order to still have (5.5), the utility function should be extended between T_H and T' in a time-coherent way in order to get the optimal Y^* until T'. □

As mentioned before, the marginal utility indifference pricing rule is not well adapted for larger nominal amount of transaction and highly illiquid market. A correcting term of Davis's price consists in providing a second order development of the utility indifference price, with respect to the number of claim q. In the backward case, this has first been studied by Henderson [Hen02] in the Black and Scholes model for power and exponential utilities, and it has been generalized in a semi-martingale financial model and backward utility function by Kramkov and Sirbu [KS06, Theorem A.1]. Theorem 5.2 provides a more direct proof for forward utility.

5.3 Second order extension of the marginal utility price

The following result provides a second order expansion of the utility indifference price, for small quantity q of the claim ξ_T.

Theorem 5.2. *Suppose the optimal strategy $X^{*,q}(x)$ of the optimization program (5.3) to be continuously differentiable [12] with respect to q. The utility indifference price at time t of*

[12] In the semimartingale framework, this regularity is obtained from that of the SDE coefficients with respect to

a q-quantity of the claim ξ_T delivered at time T admits the following second order expansion in the neighborhood of $q = 0$

$$\begin{aligned}\widehat{p}^q_{t,T}(x,\xi_T) &= q p^u_{t,T}(x,\xi_T)\Big(1 + q\frac{U_{zz}(t,X^*_t(x))}{U_z(t,X^*_t(x))} p^u_{t,T}(x,\xi_T)\Big) \\ &+ q^2 \frac{\mathbb{E}\big(U_{zz}(T,X^*_T(x))(\partial_q X^{*,q}_T(x)|_{q=0} - \xi_T)\xi_T\big)}{U_z(t,x)} + o(q^2) \qquad (5.8)\end{aligned}$$

recalling the Davis price $p^u_{t,T}(x,\xi_T) = \mathbb{E}\big[\xi_T \frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\big], \quad y = u_x(x) = U_z(0,x)$.

Remark that the term $\mathrm{R_A}(u) = -\frac{U_{zz}(t,z)}{U_z(t,z)}$ is the absolute risk aversion coefficient. Besides, the term $\partial_q X^{*,q}_T(x)|_{q=0}$ makes it difficult to compute explicitly this second order term.

Proof. We prove the result at time $t = 0$, the dynamic version is obtained in the same way. From (5.6),

$$U_z(0, x - \hat{p}^q)(\partial_q \hat{p}^q) = \mathbb{E}\big(U_z(T, X^{*,q}_T(x) - q\xi_T)\xi_T\big).$$

Differentiating again with respect to q, it follows under regularity assumptions

$$U_z(0,x-\hat{p}^q)(\partial^2_q \hat{p}^q) - U_{zz}(0,x-\hat{p}^q)(\partial_q\hat{p}^q)^2 = \mathbb{E}\big(U_{zz}(T,X^{*,q}_T(x) - q\xi_T)(\partial_q X^{*,q}_T(x) - \xi_T)\xi_T\big).$$

Then, since $\hat{p}^q \to 0$ and $(\partial_q \hat{p}^q) \to p^u$ when $q \to 0$

$$\partial^2_q \hat{p}^q|_{q=0} = \frac{\mathbb{E}\big(U_{zz}(T,X^*_T(x))(\partial_q X^{*,q}_T(x)|_{q=0} - \xi_T)\xi_T\big) + U_{zz}(0,x)(p^u)^2}{U_z(0,x)}.$$

Therefore the second order expansion of $\hat{p}^q$ in the neighborhood of $q = 0$ is

$$\hat{p}^q = qp^u(1 + qp^u\frac{U_{zz}(0,x)}{U_z(0,x)}) + q^2\frac{\mathbb{E}\big(U_{zz}(T,X^*_T(x))(\partial_q X^{*,q}_T(x)|_{q=0} - \xi_T)\xi_T\big)}{U_z(0,z)} + o(q^2).$$

□

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