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Non-linear eigenvalue problems arising from growth maximization of positive linear dynamical systems

Vincent Calvez\textsuperscript{1} Pierre Gabriel\textsuperscript{2} Stéphane Gaubert\textsuperscript{3}

Abstract—We study a growth maximization problem for a continuous time positive linear system with switches. This is motivated by a problem of mathematical biology (modeling growth-fragmentation processes and the PMCA protocol). We show that the growth rate is determined by the non-linear eigenvalue of a max-plus analogue of the Ruelle-Perron-Frobenius operator, or equivalently, by the ergodic constant of a Hamilton-Jacobi (HJ) partial differential equation, the solutions or subsolutions of which yield Barabanov and extremal project-team Inria NUMED, 46, allée d’Italie 69364 Lyon Cedex 07, France.

We investigate in this note the optimal control of time continuous positive linear dynamical systems in infinite horizon. We wish to compute the maximal growth rate that can be obtained from infinitesimal combinations of a set of nonnegative matrices.

More precisely, we consider a compact set \( \mathcal{M} \subset \mathcal{M}_n(\mathbb{R}) \) of irreducible Metzler matrices. That is to say, we assume that for all \( m \in \mathcal{M} \) and for all \( i \neq j \), \( m_{ij} \geq 0 \). In addition for every partition of indices \( \{1 \ldots n\} = I \cup J \) one can pick \( i \in I \) and \( j \in J \) such that \( m_{ij} > 0 \). A direct consequence of compactness is uniform irreducibility: there exists a constant \( \nu > 0 \) such that for all \( m \in \mathcal{M} \), and every partition of indices one can pick \( i \in I \) and \( j \in J \) such that \( m_{ij} \geq \nu \).

Let \( K \) be the nonnegative orthant in \( \mathbb{R}^n \), \( K_+ \) the positive orthant, and \( K_0 = K \setminus \{0\} \). For \( t > 0 \), \( x \in K \) and a measurable control function \( M : [0, t] \to \mathcal{M} \), we define \( x_M \in W^{1,\infty}([0, t], \mathbb{R}^n) \) as the solution of the following linear problem with control \( M \):

\[
\begin{cases}
\dot{x}_M(s) = M(s)x_M(s), \\
x_M(0) = x. 
\end{cases}
\]

We also denote \( x_M(s) = R(s,M)x \), where \( R \) is the resolvent. Finally we denote in short \( L^\infty(0, t) \) the set of measurable (bounded by assumption) control functions \( M : [0, t] \to \mathcal{M} \). We are interested in control functions maximizing the growth rate

\[
\limsup_{t \to \infty} \frac{1}{t} \log \|x_M(t)\|.
\]

We assume w.l.o.g. that \( \mathcal{M} \) is convex. The results presented here are still valid for nonconvex sets \( \mathcal{M} \), provided the controls are replaced by relaxed controls which take values in the closed convex hull \( \overline{\mathcal{M}}(\mathcal{M}) \).

For a constant control \( M(s) \equiv m \) we have \( R(t, m) = e^{tm} \). It is an immediate consequence of the Perron-Frobenius theorem that, being \( \phi_m \in K_+ \) a left Perron-Frobenius (PF) eigenvector of \( m \), the linear function \( \pi(x) = \langle \phi_m, x \rangle \) satisfies the following identity,

\[
(\forall t \in \mathbb{R}_+) \ (\forall x \in K) \ e^{\lambda(m)t}\pi(x) = \pi(x_m(t)),
\]

where \( \lambda(m) \in \mathbb{R} \) is the dominant eigenvalue of \( m \). The following result can be thought of as a non-linear extension of the Perron-Frobenius theorem.

Theorem 1: Under previous assumptions there exist a real \( \lambda(M) \) and a function \( \pi : K \to \mathbb{R}_+ \), homogeneous of degree 1, positive on \( K_0 \), globally Lipschitz continuous, which satisfy the following identity

\[
(\forall t \in \mathbb{R}_+) \ (\forall x \in K) \ e^{\lambda(M)t}\pi(x) = \sup_{M \in \mathcal{M}} \pi(x_M(t)),
\]

The scalar \( \lambda(M) \) is unique as soon as \( \pi \) belongs to the class of homogeneous functions of degree 1 which are locally bounded on \( K \), and it determines the optimal growth rate (2). Moreover, \( \pi = \log \tau \) is characterized as a viscosity solution of an ergodic Hamilton-Jacobi PDE:

\[
-\lambda(M) + H(D_y \tau(y), y) = 0, \quad y \in S,
\]

where \( S \) is the standard simplex.

The Hamiltonian \( H \) will be given in Section II.

Corollary 2 (Ergodicity): Let \( v_0 : K \to \mathbb{R}_+ \) be a continuous function, homogeneous of degree 1, positive on \( K_0 \). Define \( v(t,x) = \sup_{M \in \mathcal{M}} v_M(x(t)) \). Then we have the following ergodicity result,

\[
(\forall x \in K_0) \ \lim_{t \to +\infty} \frac{1}{t} \log(v(t,x)) = \lambda(M).
\]

Moreover the convergence is locally uniform on \( K_0 \).

Theorem 1 is closely related to results belonging to the theory of stability of linear inclusions. There, matrices are not necessarily assumed to be Metzler matrices. The non-linear eigenvalue \( \lambda(M) \) coincides with the joint spectral radius [32]. In his seminal paper [4], Barabanov proved

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the existence of extremal norms in $\mathbb{R}^n$ which saturates (3),
under a different irreducibility condition. Later the same
author investigated the behaviour of extremal trajectories in
the three-dimensional case $n = 3$, first when $\mathcal{M}$ has the
specific structure of a segment with a rank one matrix
for direction [5], secondly under a uniqueness condition
for extremal trajectories verifying the Pontrjagin Maximum
Principle (PMP) [6] (see also the recent improvement by
Gaye et al [22]). We also refer to [35] for an alternative
proof of the existence of Barabanov extremal norms, and to
the work of Chitour, Mason and Sigalotti [13] for the analysis
of situations in which there are obstructions to the existence
of such norms.

Several authors have analyzed specially the stability of
positive linear systems. Very recently, Mason and Wirth [28]
have established the existence of an extremal norm, that
is, a viscosity subsolution of the spectral problem (3) (the
equality relation being replaced by $\geq$), corresponding to a
critical subsolution of the ergodic Hamilton-Jacobi equation.
They use an irreducibility condition which is milder than our,
but which does not guarantee the existence of a viscosity
solution. Conditions for the existence of subsolutions are
typically less restrictive. It is an interesting issue to see
whether the assumptions of Theorem 1 could be relaxed.

We emphasize that we take advantage of an illuminat-
ing connection between problem (3) and the weak KAM
theory in Lagrangian dynamics [20]. In particular long-time
dynamics of optimal trajectories appear to be encoded in the
so-called Aubry sets. Such eigenproblems have been widely
studied in ergodic control, and also by dynamicians in the
setting of the weak KAM theory, where the eigenfunction
is known as a weak KAM solution. However, basic existence
results for eigenvectors rely on controllability conditions
which are not satisfied in our setting.

We exploit tools from the theory of Hamilton-Jacobi PDE
to prove Theorem 1, combined with techniques from Perron-
Frobenius theory. In particular, we use the Birkhoff-Hopf
theorem in a crucial way. The latter states that a linear
map leaving invariant the interior of a closed, convex and
pointed cone is a strict contraction in Hilbert’s projective
metric. The contraction of the controlled flow turns out to
take the existence of the eigenvector. We note that tools
from Lagrangian dynamics (Mather sets) have been recently
applied by Morris to study joint spectral radii [30]. This
deserves to be further studied in the present setting.

The same type of equations has been studied in the context
of infinite dimensional max-plus spectral theory. In particu-
lar, the existence of continuous eigenfunctions for max-plus
operators with a continuous kernel is established in [24].
More general conditions, exploiting quasi-compactness tech-
niques, can be found in [26]. It would be interesting to see
whether such techniques to the present problems.

A natural question that arises in the literature is whether
the knowledge of $\{\lambda(m)\}_{m \in \mathcal{M}}$, say $\forall m \in \mathcal{M}$ $\lambda(m) < 0$
guarantees the stability of the differential inclusion (1). A
positive answer has been given in [23] in dimension $n = 2$. A
negative answer has been given in (possibly) high dimension
in the same work. Soon after, Fainshil et al give a counter-
example in dimension $n = 3$ [18]. It is a pair of matrices
such that every convex combination has a negative spectral
radius but the associated joint spectral radius is positive.

We address similar questions in the present note, namely
whether $\lambda(\mathcal{M}) = \max_m \lambda(m)$ or $\lambda(\mathcal{M}) > \max_m \lambda(m)$. We
give a new and self-contained proof of the positive answer in
dimension $n = 2$. We also give three dimensional numerical
examples with positive and negative answers. The case where
$\lambda(\mathcal{M}) > \max_m \lambda(m)$ is of particular interest. To find such
a numerical example we restrict to the case where $\mathcal{M}$ is a
segment, and the maximum of $\lambda(m)$ is attained at an interior
point. We investigate periodic perturbations of the optimal
constant control in the spirit of [15], [14]. More precisely
we compute the second order directional derivative of the
Floquet eigenvalue. We derive a criterion about the local
optimality of the constant control with respect to periodic
perturbations. We exhibit a numerical example for which
this condition is satisfied. Numerical simulations of the full
optimal control problem clearly shows the convergence of the
optimal trajectory towards a limit cycle, suggesting that
the optimal control in infinite horizon is indeed a BANG-
BANG periodic control. It is worth noticing that the criterion
that we derive is the exact opposite of a so-called Legendre
condition in geometric optimal control theory [1], [8]. The
latter condition ensures the local optimality of the extremal
trajectory (here the trajectory corresponding to the maximal
Perron eigenvalue) for short times.

II. TECHNIQUES OF PROOF OF THEOREM 1

We present in this section the main elements of the proof
of Theorem 1. In this Section we write in short $\lambda = \lambda(\mathcal{M})$.

Step #1. Homogeneity and projection of the dynamics onto the simplex. The infinitesimal version of (3) writes as a
Hamilton-Jacobi equation in the viscosity sense,

$$
\lambda \bar{\nu}(x) = \max_{m \in \mathcal{M}} \langle D_\nu \bar{\nu}(x), mx \rangle .
$$

(5)

Using the homogeneity of the function $\bar{\nu}$ we can project (5)
onto the simplex $\mathcal{S} = \{x \in K : \langle 1, x \rangle = 1\}$. We write

$$
\bar{\nu}(x) = \langle 1, x \rangle \hat{\nu} \left( \frac{x}{\langle 1, x \rangle} \right),
$$

where $\hat{\nu}$ is defined on $\mathcal{S}$. Then problem (5) is equivalent to finding $(\lambda, \hat{\nu})$ such that

$$
\lambda \hat{\nu}(y) = \max_{m \in \mathcal{M}} \left( L(y, m) \hat{\nu}(y) + \langle D_\nu \hat{\nu}(y), b(y, m) \rangle \right),
$$

(6)

where the pay-off $L$ and the vector fields $b$ are given by

$$
L(y, m) = \langle 1, m y \rangle, \quad b(y, m) = m y - L(y, m) \cdot y.
$$

Note that each vector field $b(\cdot, m)$ is tangent to the simplex
$\mathcal{S}$. It gives indeed the projected dynamics on the simplex: if
$x_M$ is solution to (1) then $y_M = \frac{x_M}{\langle 1, x_M \rangle}$ is solution to the
non-linear ODE

$$
\dot{y}_M(s) = b(y_M(s), M(s)) .
$$

(7)
Step #2. Computation of a Lipschitz constant with respect to Hilbert’s projective metric

Recall that Hilbert’s (projective) metric is defined, for all $x, y \in K_+$, by

$$d(x, y) = \log \max_{1 \leq i, j \leq n} \frac{x_i y_j}{x_j y_i}. \quad (8)$$

It is a metric in the set of half-lines included in the interior of $K$. In particular, $d(x, y) = 0$ if and only if $x$ and $y$ are proportional.

It is known to be a weak Finsler structure [31], obtained by thinking of $K_+$ as a manifold with the seminorm $\|h\|_x = \max_i h_i x_i^{-1} - \min_j h_j x_j^{-1}$ in the tangent space at point $x$.

Then,

$$d(x, y) = \inf \int_0^1 \|\gamma(t)\|_{\gamma(t)} ds$$

where the infimum is taken over all differentiable paths $\gamma$ contained in the interior of $K$, such that $\gamma(0) = x$ and $\gamma(1) = y$.

Let $q \in K_+$ and $m \in \mathcal{P}$. We aim to compute the Lipschitz constant of the function $l(x) = \langle q, m/x \rangle$, when the source set is endowed with the Hilbert metric. For a given matrix $m = (m_{ij})$ we denote $|m| = |(m_{ij})|$ the matrix obtained by taking absolute values of the coefficients pointwise.

The following lemma is established by exploiting the Finsler’s nature of Hilbert projective metric, along the lines of [31]. See also [21].

**Lemma 3:** Let $l : (K, d) \to (\mathbb{R}, |\cdot|)$ defined as $l(x) = \langle \frac{q, m}{q, x} \rangle$. It is Lipschitz continuous with the following bound on the Lipschitz constant,

$$\text{Lip} l \leq \sup_{x \in K_+} \inf_{a \in \mathbb{R}} \frac{\langle q, m-a, 1/d \rangle}{\langle q, x \rangle}.$$

Step #3. Exponential contraction of the flow (after some time).

A key technical ingredient is the following lemma, which shows that for a fixed time $\tau > 0$, the flow maps the closed cone to its interior.

**Lemma 4:** Let $\tau > 0$. Define the cone $K_\tau \subset K$ as the convex closure of images of $K$ by the flow after a time step $\tau > 0$,

$$K_\tau = \overline{\bigcup_{M \in C(\tau)} R(\tau, M)K}.\quad$$

It satisfies the following properties,

- $K_\tau$ is stable with respect to every flow $R(s, M), s \geq 0$, $M \in L^{\infty}(0, s)$.
- $K_\tau$ is included in the interior of the cone, closed, and bounded in Hilbert’s projective metric.

A classical result of Birkhoff and Hopf shows that a linear map sending a (closed, convex, and pointed) cone to its interior is a strict contraction in Hilbert’s projective metric, see for instance [25] for more information. We deduce from the Birkhoff-Hopf theorem and from Lemma 4 the following contraction result for the flow.

**Lemma 5:** There exist a time $T > 0$ and a positive rate $\mu > 0$ such that the flow $R(t, M)$ is uniformly exponentially contractive for $t \geq T$:

$$(\forall t \geq T) \ (\forall M \in L^{\infty}(0, t)) \ (\forall (x, y) \in K_+ \times K_+) \ d(R(t, M)x, R(t, M)y) \leq e^{-\mu t} d(x, y). \quad (9)$$

**Remark 6:** If $\inf_{m \in \mathcal{M}} \min_{i \neq j} m_{ij} > 0$, one can choose $T = 0$ in the Lemma 5, and accordingly,

$$\mu = \inf_{m \in \mathcal{M}} \left( \min_{i \neq j} \left( 2(m_{ij} m_{ji})^{1/2} \right) \right) > 0. \quad (10)$$

See also [21].

Step #4. Weak KAM Theorem. As suggested by the expected exponential growth, we make a logarithmic transformation. Let introduce $\bar{u} = \log \hat{u}$. The original problem (3) writes equivalently: find a real $\lambda$ and a function $\bar{u}$, defined on the simplex $S$, such that

$$\lambda + \bar{u}(y) = \sup_{M \in L^{\infty}(0, t)} \left\{ \int_0^t L(y_M(s), M(s)) ds + \bar{u}(y_M(t)) \right\},$$

for all $t \geq 0$, or in its infinitesimal setting: find a real $\lambda$ and a function $\bar{u}$ such that $\bar{u}$ is the viscosity solution of the stationary Hamilton-Jacobi equation

$$-\lambda + H(D_y \bar{u}(y), y) = 0, \quad y \in S, \quad (12)$$

where the Hamiltonian is defined as $H(p, y) = \max_m (L(y, m) + \langle p, b(y, m) \rangle)$.

The existence of a solution $(\lambda, u)$ is known as a weak KAM Theorem in the context of dynamical systems, see the work by Fathi [19], [20]. Here, we follow the now classical argument of Lions-Papanicolaou-Varadhan to prove the existence of such a pair $(\lambda, \bar{u})$, the vector $\bar{u}$ being obtained as a rescaled limit of the solution $u_\epsilon$ of a Hamilton-Jacobi PDE with discount rate $\epsilon > 0$. In doing so, we make use of the contraction property of Lemma 5 with respect to Hilbert’s projective metric.

Step #5. Calibrated trajectories. Before we proceed with the end of the proof (boundedness of $\bar{u}$ and uniqueness of $\lambda$), we recall some definitions from [20] adapted to our context.

**Definition 7 (Calibrated trajectories):** A Lipschitz curve $\gamma : I \to S_+$ defined on the interval $I \subset \mathbb{R}$, associated to some control $M \in L^{\infty}(I)$, $\gamma = y_M$, is calibrated if for every $t \leq t' \in I$, we have

$$\bar{u}(\gamma(t')) - \bar{u}(\gamma(t)) = \int_t^{t'} (L(y_M(s), M(s)) - \lambda) \ ds$$

Along the lines of [20], we show that calibrated trajectories do exist.

Step #6. Regularity of $\bar{u}$ up to the boundary $\partial S$ and uniqueness of $\lambda$. First of all we deduce from the fixed point formulation (11) that $\bar{u}$ is Lipschitz continuous on the whole $S$ with the respect to the $\ell^1$ norm $|\cdot|_1$. Notice that the previous argument only yields local Lipschitz continuity due to the singularity of the Hilbert metric at the boundary $\partial S$.

From the fixed point formulation (11) we have in particular,

$$\lambda + \bar{u}(y) = \sup_{M \in L^{\infty}(0, 1)} \left\{ \int_0^1 L(y_M(s), M(s)) ds + \bar{u}(y_M(1)) \right\}. \quad (13)$$

It suffices to observe that for all $M \in L^{\infty}(0, 1)$, $y_M(1) = R(1, M)y \in K_1$ which is a compact subset of $S$ with respect to the Hilbert metric. Thus $K_1$ is at uniform positive distance
from the boundary $\partial S$ and there exists a constant $C(K_1)$ such that for all $(x, y) \in K_1 \times K_1$, $d(x, y) \leq C(K_1)|x - y|_1$. Finally we observe that (13) is a supremum of Lipschitz functions as it is the case for $\tilde{u}(y_M(1))$:

$$|\tilde{u}(y_M(1)) - \tilde{u}(x_M(1))|$$

$$\leq (\text{Lip } \tilde{u}_{\alpha}) d(R(1, M)y, R(1, M)x)$$

$$\leq (\text{Lip } \tilde{u}_{\alpha}) C(K_1)|R(1, M)y - R(1, M)x|_1$$

$$\leq (\text{Lip } \tilde{u}_{\alpha}) C(K_1) \left( \sup_{M \in L^\infty((0,1))} \|R(1, M)\|_1 \right) |y - x|_1.$$
where \( \gamma_i(t) \) is the unique \( T \)-periodic solution of the relaxation ODE

\[
\frac{\gamma_i(t)}{\lambda_i - \lambda_i^*} + \gamma_i(t) = \gamma(t).
\]

The idea of computing directional derivatives has been used in a similar context in [29] for optimizing the Perron eigenvalue in a continuous model for cell division. See also [15] for a more general discussion on the comparison between Perron and Floquet eigenvalues.

Taking \( \gamma \equiv 1 \) in Equation (15), we get the second derivative of the Perron eigenvalue at \( \alpha^* \),

\[
\frac{d^2 \lambda_F}{d\alpha^2}(\alpha^*) = 2 \sum_{i=2}^{n} \left( \phi_i^* F e_i^* \right) \left( \phi_i^* F e_i^* \right)
\]

which is nonpositive since \( \alpha^* \) is a maximum point. Therefore we are led to the following question: is it possible to construct counter-examples such that the sum in (15) is positive for some periodic control \( \gamma \), whereas the sum (16) is nonpositive? This is clearly not possible in dimension \( n = 2 \) because the sum in (15) is reduced to a single nonpositive term by (15). For \( n \geq 3 \), considering periodic perturbations \( \gamma(t) = \cos(\omega t) \), we get the formula

\[
\frac{d^2 \lambda_F}{d\alpha^2}(\alpha^* + \varepsilon \gamma) = \sum_{i=2}^{n} \frac{\lambda_i - \lambda_i^*}{\omega^2 + (\lambda_i - \lambda_i^*)^2} \left( \phi_i^* F e_i^* \right) \left( \phi_i^* F e_i^* \right).
\]

An asymptotic expansion when \( \omega \to +\infty \) indicates that if the condition

\[
\sum_{i=2}^{n} (\lambda_i - \lambda_i^*) (\phi_i^* F e_i^*) (\phi_i^* F e_i^*) > 0
\]

is satisfied, then (15) is positive for some \( \omega \) large enough.

C. Legendre type condition for local optimality on short times

Within the framework described in the previous section, we introduce the endpoint mapping

\[
F_T : \{ L^\infty(0, T), \alpha(\cdot) \to x(T), \}
\]

which maps a control \( \alpha \in L^\infty(0, T) \) to the terminal value \( x(T) \) of the corresponding trajectory, i.e. the solution of the ODE

\[
\dot{x}(s) = (G + \alpha(s)F)x(s).
\]

We analyse in the following the behaviour of this mapping in the neighbourhood of the best constant control \( \alpha^* \) and its associated trajectory \( x(t) = e^{\lambda t} e_1^* \). Moreover we make the link with the computations on the Floquet eigenvalue in the previous section. Consider a variation \( \alpha(\cdot) = \alpha^* + \epsilon \gamma(\cdot) \in L^\infty(0, T) \) (not necessarily periodic) and define the quadratic form

\[
Q(\gamma) := \phi_1^* (D^2_F \alpha^* \cdot F_T)(\gamma, \gamma).
\]

A straightforward computation gives the expression

\[
Q(\gamma) = 2 \epsilon \lambda_1 T \int_0^T \gamma(t) \gamma(s) \phi_1^* F e_i^* (G + \alpha^* F - \lambda_i^* I)(t-s) Fe_i^* d\alpha dt
\]

Looking at the leading terms in \( Q \) when \( T \) is small, we get a sufficient condition for the control \( \alpha^* \) to be locally optimal for small times on the hyperplane

\[
L_0^\infty := \{ \gamma \in L^\infty(0, T), \int_0^T \gamma(t) dt = 0 \}.
\]

Proposition 11: If the condition

\[
\phi_1^* F (G + \alpha^* F - \lambda_i^* I) Fe_i^* > 0.
\]

is satisfied, then the quadratic form \( Q \) restricted to \( L_0^\infty \) is negative definite for short times with respect to the negative Sobolev space \( H^{-1}(0, T) \), that is to say

\[
\exists \delta > 0, \exists \epsilon > 0, \forall T \in (0, \epsilon), \| Q|_{L^\infty_0}(\gamma) \leq -\delta \| \gamma \|_{H^{-1}}^2.
\]

We skip the proof of this result.

Condition (18) is the so-called generalized Legendre condition of our problem. The generalized Legendre condition appears in the study of optimality for totally singular extremals, i.e. when the second derivative of the Hamiltonian is identically zero along the trajectory. A typical example is provided by the single-input affine control systems, namely, \( \dot{x}(t) = f_0(x(t)) + \alpha(t) f_1(x(t)) \), where \( \alpha(t) \in \mathbb{R} \), and \( f_0, f_1 \) are smooth vector fields. In this case the generalized Legendre condition writes

\[
\langle p(\cdot) | [f_1, [f_1, f_1]](x(\cdot)) \rangle > 0
\]

where \([\cdot, \cdot]\) is the Lie bracket of vector fields. Our linear control system belongs to this class of problems, and straightforward computations show that for \( q(t) = e^{\lambda t} e_1^* \) and \( p(t) = e^{-2 \lambda t} \phi_1^* \), we have \( p(\cdot) | [F, [G, F]]| x(\cdot) = \phi_1^* F (G + \alpha^* F - \lambda_i^* I)| Fe_i^* \). The generalized Legendre condition ensures that the quadratic form \( Q|_{KerD_{F^*} \cdot F_T} \) is definite negative. Then it allows to deduce that the trajectory \( x(\cdot) \) is locally optimal for short final times \( T \) in the \( C^0 \) topology (we refer to [1] and [8] for details).

Here we proved the negativity of \( Q \) on \( L_0^\infty \) instead of \( KerD_{F^*} \cdot F_T \) under the condition (11). Our aim is to make clearer the link with the computation of the second derivative of the Floquet eigenvalue. It follows from the previous section that the second derivative of \( \lambda_F \) is positive for periodic controls \( \cos(\omega t) \) when \( \omega \) is large if condition (17) is satisfied. Considering \( T \) small and \( \omega = k \frac{T}{2} \) with \( k \geq 1 \), we have that \( \cos(\omega t) \in L^\infty_0 \) and \( \omega \to +\infty \) when \( k \to +\infty \). The following proposition points out the consistency between conditions (18) and (17).

Proposition 12: We have

\[
\phi_1^* F (G + \alpha^* F - \lambda_i^* I)| Fe_i^* = - \sum_{i=2}^{n} (\lambda_i - \lambda_i^*) (\phi_i^* F e_i^*) (\phi_i^* F e_i^*).
\]

This relation is instructive since it emphasizes the relation between a condition for optimality for small times and a condition for optimality with high frequencies.

D. Lack of controllability/coercivity and the ergodic set

Classical arguments for proving ergodicity results such as Theorem 1 or Corollary 2 rely on short time dynamics of the system. This is the case for instance of the ergodicity result
in Capuzzo-Dolcetta and Lions [11], and of the Weak KAM Theorem of Fathi [19]. In the former the authors assume a uniform controllability condition,

\[(\exists r > 0) \quad (\forall y \in S) \quad B(0, r) \subset \operatorname{co}\{b(y, m) \mid m \in \mathcal{M}\} \, ,\]

The latter relies on a regularizing property of the Lax-Oleinik semi-group which holds true for Tonelli Lagrangians. Both cases imply that the Hamiltonian is coercive, i.e., \(\lim_{|p| \to +\infty} H(p, y) = +\infty\), a property which is not satisfied in our case. One noticeable exception can be found in [7, Section VII.1.2], where the controllability condition is replaced by a dissipativity condition which is somehow similar to our uniform contraction estimate in Lemma 5.

The lack of controllability is clear from Lemma 4, where some strict subsets of the simplex are positively invariant by all the flows. In a couple of papers, Arisawa made clear the equivalence between ergodicity (in the sense of Corollary 2) and the existence of a so-called ergodic set when controllability is lacking. The ergodic set satisfies the following properties: it is non empty, closed and positively invariant by the flows; it is attractant; it is controllable. We refer to [2], [3] for the precise meaning of this statement, and to Section V for illustrations of ergodic sets in low-dimensional examples.

IV. ILLUSTRATION IN DIMENSION 2

We first illustrate our results on a simple two-dimensional example. Let \(a \in (0, \frac{1}{2})\). We consider the one-parameter family of matrices \(\mathcal{M} = \left\{ \begin{pmatrix} 0 & 1 - \alpha \\ \alpha & 0 \end{pmatrix} \right\}, \quad \alpha \in [a, 1 - a] \) .

The dominant eigenvalue is \(\sqrt{\alpha(1 - \alpha)}\), with a maximum attained at \(\alpha = \frac{1}{2}\). We identify \([0, 1]\) and \(\mathcal{S}\) under the parameterization

\[\mathcal{S} = \left\{ \begin{pmatrix} 1 - \theta \\ \theta \end{pmatrix} \right\}, \quad \theta \in [0, 1] \) .

We have \(L(\theta, \alpha) = \alpha(1 - \theta) + (1 - \alpha)\theta\), and \(b(\theta, \alpha) = \alpha(1 - \theta)^2 - (1 - \alpha)\theta^2\). We look for a solution \((\lambda, \hat{u})\) to the Hamilton-Jacobi equation (12). For computing \(\hat{u}\) we have to combine two branches, corresponding to the choice \(\alpha = a\) or \(\alpha = 1 - a\). We realize that the first branch has a vertical asymptote at \(\theta = z_a\) since \(b(\cdot, a)\) vanishes at this point, whereas the second branch has a vertical asymptote at \(\theta = z_{1-a}\). Therefore there is only one possible combination which gives a bounded viscosity solution on \(\mathcal{S}\).

In this example the ergodic set is the segment \([z_a, z_{1-a}]\).

V. APPLICATION TO THE OPTIMIZATION OF GROWTH-FRAGMENTATION PROCESSES IN DIMENSION 3

We consider a toy model for a stage-structured linear polymerization-fragmentation process. It is inspired from a nonlinear discrete polymerization-fragmentation introduced in [27] for the dynamics of prion proliferation. We do not take into account nonlinear saturation effects, and we further reduce the size of the system to \(n = 3\). Polymers can have three states relative to their lengths: small (monomers), medium (oligomers), large (polymers). We denote by \(x_i\), \(i = 1, 2, 3\) the density of polymers in each compartment. Transition rates due to growth in size of polymers from smaller to larger compartments (polymerization) are denoted by \(\tau_i\), \(i = 1, 2\). Transition rates due to fragmentation from larger to smaller compartments are denoted by \(\beta_i\), \(i = 2, 3\). The corresponding matrices are

\[G = \begin{pmatrix} -\tau_1 & 0 & 0 \\ \tau_1 & -\tau_2 & 0 \\ 0 & \tau_2 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 2\beta_2 & \beta_3 \\ -\beta_2 & 0 & \beta_3 \\ 0 & 0 & -\beta_3 \end{pmatrix} \, .\]

Denoting by \(q = (1 \ 2 \ 3)^T\) the vector of relative sizes of polymers, we have the following properties: \(1^T G = 0\) (conservation of the number of polymers by growth) and \(q^T F = 0\) (conservation of the total size of polymers by fragmentation).

Optimal control issues come up in the development of efficient diagnosis tools for early detection of prion diseases from blood samples. The protocol PMCA (Protein Misfolding Cyclic Amplification) has been introduced by Soto and co-authors [34], [12] as very powerful method to achieve this goal. It aims at quickly generating in vitro detectable quantities of PrPsc being given minute quantities of it. PMCA consists in successive switching between incubation phases (where aggregates are expected to grow following a seeding-nucleation scenario alimented by purified PrPc) and sonication phases (where breaking of polymers is expected to increase the number of nucleation sites). This clear distinction between two phases with a control parameter which is the intensity of sonication makes the framework of (1) very well adapted to model PMCA.

The minimal model for PMCA goes as follows: introduce \(\alpha : [0, t] \to [a, A]\) the intensity of sonication (i.e. fragmentation). The goal is to maximize the total size of polymers \(\langle q, x(\alpha(t))\rangle\) following the linear growth-fragmentation process:

\[
\begin{aligned}
x(\alpha(s)) &= (G + \alpha(s)F)x(\alpha(s)), \\
x(\alpha(0)) &= x.
\end{aligned}
\]
A generalization of this model, which includes an incidence of the sonication on the growth process, is investigated in [16]. For problem (20), corollary 2 implies that \( \langle q, x_\alpha(t) \rangle \) has an exponential growth with exponent \( \lambda(M) \). When \( \tau_1 = \tau_2 \) it is clearly better to sonicate as much as possible (\( \alpha(s) = A \)) because smaller monomers are equally efficient at growing in size than intermediate oligomers, but they are more numerous for a given size. However there are some biological evidence that polymerization rate is size-dependent: polymerization of intermediate aggregates have been postulated to be the most efficient [33] (see also [10] for a continuous PDE model and a discussion of this phenomenon). Mathematically speaking we have a precise description of the variations of \( \lambda(\alpha) \) as stated in the following proposition.

**Proposition 13:** The Perron eigenvalue \( \lambda(\alpha) \) of \( G + \alpha F \) reaches a maximum value for some \( \alpha^* \in (0, +\infty) \) if and only if \( \tau_2 > 2\tau_1 \). Furthermore we face the following alternative:

- either \( \tau_2 \leq 2\tau_1 \) and \( \lambda(\alpha) \) increases from 0 to \( \tau_1 \),
- or \( \tau_2 > 2\tau_1 \) and \( \lambda(\alpha) \) increases from 0 to \( \lambda(\alpha^*) \) and then decreases from \( \lambda(\alpha^*) \) to \( \tau_1 \).

We refer to [9] for the details of the proof of Proposition 13.

Qualitative analysis of the optimal control and associated optimal trajectories rely on the description of relevant sets in the simplex. The ergodic set introduced by Arisawa [2], [3] can be characterized as the set enclosed by two remarkable trajectories: each starting from one of the two extremal Perron eigenvectors (resp. \( e_1(\alpha) \) and \( e_1(A) \)) and evolving with constant control (resp. \( A \) and \( a \)). This set is of particular interest since it attracts all trajectories, not necessarily optimal ones. However it does not give any insight about the fate of optimal trajectories inside the ergodic set.

So far we have only access to local second-order conditions (16) to test the optimality of the best constant control. Numerical tests suggest that we always have \( \lambda(M) = \max_{m} \lambda(m) \) in the case of (19), that is to say the optimal trajectory converges towards the optimal Perron eigenvector in the simplex. This is confirmed by finite-volume numerical simulations of the ergodic Hamilton-Jacobi equation (4), see Figure 2. We computed the optimal control \( \alpha^* \) as a function of the position in the simplex. We observed a clear separation between two connected regions of the simplex (result not shown), corresponding to the extremal choices \( \alpha = \alpha \) or \( \alpha = A \). This gives an optimal vector field which drives optimal trajectories. Outcomes of the numerical simulations are consistent with the presumable stability of the best constant control against periodic perturbations.

**A three-dimensional example with an optimal limit cycle**

Proposition 9 rules out the existence of optimal limit cycles in dimension \( n = 2 \). Although the previous example of the three-dimensional growth-fragmentation process did not exhibit limit cycles apparently, we were able to find another three-dimensional example by testing random choices of matrices with respect to the stability criterion (16):

\[
G = \begin{pmatrix}
0 & 0.245 & 0.007 \\
0 & 0 & 0.141 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
F = \begin{pmatrix}
-0.245 & 0 & 0 \\
0.272 & -0.499 & 0 \\
0.645 & 0.026 & -0.035
\end{pmatrix}.
\]

We assume as in the previous example that the control \( \alpha \) takes values in \([a, A]\). The maximal Perron eigenvalue is obtained for \( \alpha^* \approx 0.415 \). The stability criterion (16) has been checked numerically: the optimal constant control is not stable with respect to periodic perturbations at high frequency. Therefore we expect limit cycles to attract optimal trajectories in the simplex in the spirit of the Poincaré-Bendixson theory. This has been checked using finite-volume numerical simulations of the ergodic Hamilton-Jacobi equation (4), see Figure 3. It is worth mentioning that a similar counter-example has been proposed in [18] to answer a question raised in [23]. Our quantitative approach based on second-order conditions provides another example. Furthermore it illustrates the rich possible dynamics of optimal trajectories. The connexion with the Poincaré-Bendixson theory seems
appealing. However, proving that limit cycles are the only alternative to pointwise convergence seems out of reach at the moment, due to the complexity of the Poincaré-Bendixson theory in the case of discontinuous vector fields [17].

REFERENCES


