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CONTINUOUS FIELDS OF PROPERLY INFINITE C*-ALGEBRAS

ETIENNE BLANCHARD

Abstract. Any unital separable continuous C(X)-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space X has finite topological dimension. We show that this is still true when the compact space X has infinite topological dimension.

1. Introduction

One of the basic C*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C*-algebras through K-theoretical invariants is the Cuntz C*-algebra O∞ generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous C(X)-algebra A the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra D is a trivial C(X)-algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra D tensorially absorbs the Jiang-Su algebra Z ([Win09]). Hence, this C*-algebra D is K₁-injective ([Rør04]) and the C(X)-algebra A satisfies A ∼= D ⊗ C(X) ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product Π₀∞ S₂ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2∞ ([HRW07, Example 4.7]). More recently, M. Dădălă has constructed in [Dăd09, §3] for all pair (Γ₀, Γ₁) of discrete countable torsion groups a unital separable continuous C(X)-algebra A such that:

– the base space X is the compact Hilbert cube X = X of infinite dimension,
– all the fibres Aₓ (x ∈ X) are isomorphic to the strongly self-absorbing Cuntz C*-algebra O₂ generated by two isometries s₁, s₂ satisfying 1O₂ = s₁s₁* + s₂s₂*,
– K₁(A) ∼= C(Y₀, Γ₁) for i = 0, 1, where Y₀ ⊂ [0, 1] is the canonical Cantor set.

These K-theoretical conditions imply that the C(X)-algebra A is not a trivial one. But these arguments does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra O∞ ([Cun77]), in so far as K₀(O∞) = Z is a torsion free group.

We still do not know whether all unital continuous C(X)-algebra with fibres O∞ are trivial. But we show that any unital separable continuous C(X)-algebra with fibres O∞ is at least properly infinite (see [Blan13, §6]).

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We present in this section the main notations which are used in this article. We denote by \( \mathbb{N} = \{0, 1, 2, \ldots \} \) the set of positive integers and we denote by \([S]\) the closed linear span of any subset \( S \) in a Banach space.

**Definition 2.1.** ([Dix69], [Kas88], [Blan97]) Let \( X \) be a compact Hausdorff space and let \( C(X) \) be the \( C^* \)-algebra of continuous function on \( X \).

- A unital \( C(X) \)-algebra is a unital \( C^* \)-algebra \( A \) endowed with a unital morphism of \( C^* \)-algebra from \( C(X) \) to the centre of \( A \).
- For all closed subset \( F \subset X \) and all element \( a \in A \), one denotes by \( a|_F \) the image of \( a \) in the quotient \( A|_F := A/C_0(X \setminus F) \cdot A \). If \( x \in X \) is a point in \( X \), one calls fibre at \( x \) the quotient \( A_x := A|_{\{x\}} \) and one write \( a_x \) for \( a|_{\{x\}} \).
- The \( C(X) \)-algebra \( A \) is said to be continuous if the upper semicontinuous map \( x \in X \mapsto \|a_x\| \in \mathbb{R}_+ \) is continuous for all \( a \in A \).

**Remarks 2.2.**

a) ([Cun81], [BRR08]) For all integer \( n \geq 2 \), the \( C^* \)-algebra \( T_n := T(C_n) \) is the universal unital \( C^* \)-algebra generated by \( n \) isometries \( s_1, \ldots, s_n \) satisfying the relation
\[
s_1s_1^* + \ldots + s_ns_n^* \leq 1. \tag{2.1}
\]
b) A unital \( C^* \)-algebra \( A \) is properly infinite if and only if one the following equivalent conditions holds ([Cun77], [Rør03, Proposition 2.1]):

- the \( C^* \)-algebra \( A \) contains two isometries with mutually orthogonal range projections, \( i.e. \), \( A \) unitally contains a copy of \( T_2 \),
- the \( C^* \)-algebra \( A \) contains a unital copy of the simple Cuntz \( C^* \)-algebra \( O_\infty \) generated by infinitely many isometries with pairwise orthogonal ranges.

### 3. GLOBAL PROPER INFINITENESS

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous \( C(X) \)-algebras with properly infinite fibres.

**Proposition 3.1.** Let \( X \) be a second countable perfect compact Hausdorff space, \( i.e. \) without any isolated point, and let \( A \) be a separable unital continuous \( C(X) \)-algebra with properly infinite fibres.

1) There exist:
   
   (a) a finite integer \( n \geq 1 \),
   
   (b) a covering \( X = \overset{n}{\bigcup} \overset{n}{\bigcup} F_k \) by the interiors of closed balls \( F_1, \ldots, F_n \),
   
   (c) unital embeddings of \( C^* \)-algebra \( \sigma_k : \mathcal{O}_\infty \hookrightarrow A|_{F_k} \ (1 \leq k \leq n) \).

2) The tensor product \( M_p(\mathbb{C}) \otimes A \) is properly infinite for all large enough integers \( p \).

**Proof.** 1) For all point \( x \in X \), the semiprojectivity of the \( C^* \)-subalgebra \( \mathcal{O}_\infty \hookrightarrow A_x \) ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood \( F \subset X \) of the point \( x \) and a unital embedding \( \mathcal{O}_\infty \otimes C(F) \hookrightarrow A|_F \) of \( C(F) \)-algebra. The compactness of the topological space \( X \) enables to conclude.
2) Proposition [BRR08, Proposition 2.7] entails that $M_{2^n-1}(A)$ is properly infinite and [Ror97, Proposition 2.1] implies that $M_p(A)$ for all integer $p \geq 2^n-1$. □

**Remark 3.2.** If $X$ is an ordinary second countable compact Hausdorff space and $A$ is a separable unital continuous $C(X)$-algebra, then $\tilde{X} := X \times [0,1]$ is a perfect compact space, $\tilde{A} := A \otimes C([0,1])$ is a unital continuous $C(\tilde{X})$-algebra and any unital morphism $O_\infty \to A$ induces a unital morphism $O_\infty \to A$ by composition with the projection map $A \to A$ coming from the injection $x \in X \mapsto (x,0) \in \tilde{X}$.

The proper infiniteness of the tensor product $M_p(C) \otimes A$ does not always imply that the $C^*$-algebra $A$ is properly infinite ([HR98]). Indeed, there exists a unital $C^*$-algebra $A$ which is not properly infinite, but such that the tensor product $\tilde{A}$ coming from the injection $A \to \tilde{A}$ is a special case of (a) since $A$ is not properly infinite. Thus, (b) is a special case of (a) since $K_1(\tilde{T}_2 \ast \tilde{T}_2) = \{1\}$ (see e.g. [Blan10, Lemma 4.4]). □

**Corollary 3.3.** Let $j_0,j_1$ denote the two canonical unital embeddings of the Cuntz extension $T_2$ in the full unital free product $T_2 \ast T_2$ and let $\tilde{u} \in \mathcal{U}(T_2 \ast T_2)$ be a $K_1$-trivial unitary satisfying $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$ ([BRR08, Lemma 2.4]).

Then the following conditions are equivalent:

(a) The full unital free product $T_2 \ast T_2$ is $K_1$-injective.

(b) The unitary $\tilde{u}$ belongs to the connected component $U_0(T_2 \ast T_2)$ of $1_{T_2 \ast T_2}$.

(c) Every separable unital continuous $C(X)$-algebra $A$ with properly infinite fibres is a proper infinite $C^*$-algebra.

**Proof.** (a)⇒(b) A unital $C^*$-algebra $A$ is called $K_1$-injective if and only if all $K_1$-trivial unitaries $v \in \mathcal{U}(A)$ are homotopic to the unit $1_A$ in $\mathcal{U}(A)$ (see e.g. [Roh09]). Thus, (b) is a special case of (a) since $K_1(\tilde{T}_2 \ast \tilde{T}_2, \tilde{\sigma}_n) = \{1\}$ (see e.g. [Blan10, Lemma 4.4]).

(b)⇒(c) Let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = \tilde{F}_1 \cup \ldots \cup \tilde{F}_n$ such that there exist unital embeddings $\sigma_k : \tilde{T}_2 \to A|_{F_k} (1 \leq k \leq n)$. Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A|_{G_k}$ such that the two projections $w_k w_k^*$ and $1_{G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A|_{G_k}$.

- If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

- If $k \in \{1,\ldots,n-1\}$ and the isometry $w_k \in A|_{G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital $C^*$-algebra $\pi_k : \tilde{T}_2 \ast \tilde{T}_2 \to A|_{G_k \cap F_{k+1}}$ satisfying:

\[
\begin{align*}
\pi_k(j_0(s_1)) &= w_k|_{G_k \cap F_{k+1}}, \\
\pi_k(j_1(s_1)) &= \sigma_{k+1}(s_1)|_{G_k \cap F_{k+1}} = \pi_k(\tilde{u}) \cdot w_k|_{G_k \cap F_{k+1}}.
\end{align*}
\]  

If the unitary $\tilde{u}$ belongs to $U_0(\tilde{T}_2 \ast \tilde{T}_2)$, then $\pi_k(\tilde{u})$ is homotopic to $1_{A|_{G_k \cap F_{k+1}}} = \pi_k(1_{\tilde{T}_2 \ast \tilde{T}_2})$ in $U(1_{A|_{G_k \cap F_{k+1}}})$, so that $\pi_k(\tilde{u})$ admits a unitary lifting $z_{k+1}$ in $U_0(A|_{F_{k+1}})$ (see e.g. [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A|_{G_{k+1}}$ satisfying the two constraints:

\[
w_{k+1}|_{G_k} = w_k \quad \text{and} \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1)
\]  

3
verifies that the two projections $w_{k+1} w_{k+1}^*$ and $1_{[G_{k+1}]} - w_{k+1} w_{k+1}^*$ are properly infinite and full in $A_{[G_{k+1}]}$.

The proper infiniteness of the projection $w_n w_n^* \in A_{[G_n]} = A$ implies that the unit $1_A = w_n^* w_n = w_n^* \cdot w_n \cdot w_n^* \cdot w_n$ is also a properly infinite projection in $A$, i.e. the C*-algebra $A$ is properly infinite.

(c)$\Rightarrow$(a) The C*-algebra $D := \{ f \in C([0,1], T_2 \ast C_2) : f(0) \in j_0(T_2) \text{ and } f(1) \in j_1(T_2) \}$ is a unital continuous $C([0,1])$-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra $D$ is properly infinite, a statement which is equivalent to the $K_1$-injectivity of $T_2 \ast C_2$ ([Blan10, Proposition 4.2]).

\[ \square \]

**Proposition 3.4.** Let $a \in T_2$ be the sum $a := 1 - s_1 s_1^* + s_1$.

1) The operator $a$ is a unitary in $T_2$.

2) The unitary $u := j_1(a) \cdot j_0(a)^\ast \in T_2 \ast C_2$ belongs to the connected component $U_0(T_2 \ast C_2)$. It also satisfies $j_1(s_1) \cdot j_0(s_1^*) = u \cdot j_0(s_1 s_1^*)$.

3) The C*-algebra $T_2 \ast C_2$ is $K_1$-injective.

**Proof.** 1) Let $H$ be the Hilbert space $H := \ell^2(\mathbb{N})$ with canonical orthonormal basis $\{ e_k : k \in \mathbb{N} \}$ and identify $T_2$ with the C*-algebra generated by the two isometries $s_1, s_2$ given by $s_i \cdot e_k := e_{2k+i}$ for $i = 1, 2$ and $k \in \mathbb{N}$.

$a$ is an isometry since $a^* a = ((1 - s_1^* s_1) + s_1)^* ((1 - s_1^* s_1) + s_1) = (1 - s_1^* s_1) + s_1 s_1^* = 1$.

For all vector $\xi \in H$, we also have

\[
\begin{align*}
    a a^* \xi &= 0 \\
    \iff a^* \xi &= s_1^* \xi + (1 - s_1 s_1^*)^* \xi = 0 \\
    \Rightarrow (1 - s_1 s_1^*)^* \xi &= 0 \text{ and } s_1^* \xi = 0 \\
    \Rightarrow \xi &= s_1 s_1^* \xi \text{ and } s_1^* \xi = 0 \\
    \Rightarrow \xi &= 0 .
\end{align*}
\]

Thus, the coisometry $a^*$ is injective and the operator $a$ is a unitary.

2) The unitary $a$ is homotopic to 1 in $U(T_2)$ (see e.g. [Blan10, Proposition 3.2]). Hence, the product $u = j_1(a) \cdot j_0(s_1^*)$ is homotopic to 1 in $U(T_2 \ast C_2)$.

Besides, $u \cdot j_0(s_1 s_1^*) = j_1(a) \cdot j_0(s_1^*) = j_1(s_1) \cdot j_0(s_1)^*$.

3) This derives for Corollary 3.3. \( \square \)

### 4. The Pimsner-Toeplitz Algebra of a Hilbert $C(X)$-Module

We look in this section at the special case of unital continuous $C(X)$-algebras with fibres $O_\infty$ corresponding to the Pimsner-Toeplitz $C(X)$-algebras of Hilbert $C(X)$-modules with infinite dimension fibres.

**Definition 4.1.** ([Pim95]) Let $X$ be a compact Hausdorff space and $E$ a full Hilbert $C(X)$-module $E$, i.e. without any zero fibre.

a) The full Fock Hilbert $C(X)$-module $F(E)$ of $E$ is the direct sum of Hilbert $C(X)$-module

\[
F(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)})^m},
\]

(4.1)
where $E^{(\otimes_{C(X)})m} := \begin{cases} C(X) & \text{if } m = 0, \\
E \otimes_{C(X)} \ldots \otimes_{C(X)} E & (m \text{ terms}) \text{ if } m \geq 1. \end{cases}$

b) The Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ of $E$ is the unital subalgebra of the $C(X)$-algebra $\mathcal{L}_{C(X)}(\mathcal{F}(E))$ of adjointable $C(X)$-linear operators acting on $\mathcal{F}(E)$ generated by the creation operators $\ell(x) \in \mathcal{T}(E)$, where:

\begin{align*}
- & \quad \ell(\zeta) (f \cdot \hat{1}_{C(X)}) := f \cdot \zeta = \zeta \cdot f & \text{for } f \in C(X) \quad \text{and} \\
- & \quad \ell(\zeta_1 \otimes \ldots \otimes \zeta_k) := \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k & \text{for } \zeta_1, \ldots, \zeta_k \in E \quad \text{if } k \geq 1. \quad (4.2)
\end{align*}

c) Let $(C^*(\mathbb{Z}, \Delta)$ be the compact quantum group generated by a unitary $u$ with spectrum the unit circle and with coproduct $\Delta(u) = u \otimes u$. Then, there is a unique coaction $\alpha$ of the Hopf $C^*$-algebra $(C^*(\mathbb{Z}), \Delta)$ on the Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ such that $\alpha(\ell(\zeta)) = \ell(\zeta) \otimes u$ for all $\zeta \in E$, i.e.

\begin{align*}
\alpha : \quad \mathcal{T}(E) & \to \mathcal{T}(E) \otimes C^*(\mathbb{Z}) = C(T, \mathcal{T}(E)) \\
\ell(\zeta) & \mapsto \ell(\zeta) \otimes u = (z \mapsto \ell(\zeta z)) \quad (4.3)
\end{align*}

The fixed point $C(X)$-subalgebra $\mathcal{T}(E)^{\alpha} = \{ a \in \mathcal{T}(E) ; \alpha(a) = a \otimes 1 \}$ under this coaction is the closed linear span

\begin{align*}
\mathcal{T}(E)^{\alpha} = \left[ C(X) \cdot 1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^* \right]. \quad (4.4)
\end{align*}

Besides, the projection $P \in \mathcal{L}(\mathcal{F}(E))$ onto the submodule $E$ induces a quotient morphism of $C(X)$-algebra $a \in \mathcal{T}(E)^{\alpha} \mapsto q(a) := P \cdot a \cdot P \in K(E) + C(X) \cdot 1 \subset \mathcal{L}(E)$.

**Proposition 4.2.** Let $X$ be a second countable compact Hausdorff perfect space and let $E$ be a separable Hilbert $C(X)$-module with infinite dimensional fibres.

1) There exist a covering $X = F_1 \cup \ldots \cup F_m$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ sections $\zeta_1, \ldots, \zeta_m$ in $E$ such that $\mathcal{T}(E) = C^* < \mathcal{T}(E)^{\alpha}, \ell(\zeta_1), \ldots, \ell(\zeta_m) >$ and $\| \zeta_k \| = 1$ for all $k \in \{ 1, \ldots, m \}$ and $\zeta_k \in F_k$.

2) The Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ is properly infinite.

**Proof.** 1) For all point $x \in X$, there exists a section $\zeta \in E$ satisfying $\| \zeta_x \| = 1$, whence an isomorphism of $C^*$-algebra $\mathcal{T}(E)_x \cong \mathcal{T}(E_x) = C^* < \mathcal{T}(E_x)^{\alpha}, \ell(\zeta_x) >$. The semi-projectivity of the $C^*$-algebra $\mathcal{O}_x \cong \mathcal{T}(E)_x$ and the compactness of the space $X$ imply that there exist a finite covering $X = \hat{F}_1 \cup \ldots \cup \hat{F}_m$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ contractions $\zeta_1, \ldots, \zeta_m$ in $E$ such that $\| \zeta_k \| = 1$ for all index $k \in \{ 1, \ldots, m \}$ and all point $x \in F_k$, so that $\ell(E)|F_k = \ell(E) \cdot \ell(\zeta_k)^* \cdot \ell(\zeta_k)|F_k$ and $\mathcal{T}(E)|F_k = C^* < \mathcal{T}(E)^{\alpha}, \ell(\zeta_k) >|F_k$.

2) The equivalence $(a) \Leftrightarrow (c)$ in Corollary 3.3 implies that $\mathcal{T}(E)$ is properly infinite.  \( \square \)

**Question 4.3.** Is there a unitary $v \in \mathcal{U}((\mathcal{T}_2 \ast \mathcal{T}_2)^{\alpha})$ such that $j_1(s_1 s_1^*) = v \cdot j_0(s_1 s_1^*) \cdot v^*?\)
References


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