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Hadamard states for the linearized Yang-Mills equation on curved spacetime

C. Gérard and M. Wrochna

Abstract. We construct Hadamard states for the Yang-Mills equation linearized around a smooth, space-compact background solution. We assume the spacetime is globally hyperbolic and its Cauchy surface is compact or equal $\mathbb{R}^d$.

We first consider the case when the spacetime is ultra-static, but the background solution depends on time. By methods of pseudodifferential calculus we construct a parametrix for the associated vectorial Klein-Gordon equation. We then obtain Hadamard two-point functions in the gauge theory, acting on Cauchy data. A key role is played by classes of pseudodifferential operators that contain microlocal or spectral type low-energy cutoffs.

The general problem is reduced to the ultra-static spacetime case using an extension of the deformation argument of Fulling, Narcowich and Wald.

As an aside, we derive a correspondence between Hadamard states and parametrices for the Cauchy problem in ordinary quantum field theory.

1. Introduction

The construction of a sufficiently explicit parametrix for the Klein-Gordon is essential in Quantum Field Theory on curved spacetime, where two-point functions of physically admissible states (*Hadamard states*) are required to be distributions with a specified wave front set. By using methods of pseudodifferential calculus it is possible to control at the same time the propagation of singularities and the additional properties of the parametrix, which are needed to treat physical conditions such as positivity (or purity) of states. As shown in the scalar case in [J, GW] for a large class of spacetimes, this allows to construct a large class of Hadamard states.

The generalization to gauge theories poses difficulties which are due to two main obstacles.

First of all, the equations of motions are given by a *non-hyperbolic* differential operator P. This is usually coped with by identifying the space of solutions of P with a quotient $\mathcal{V}_P$ of subspaces of solutions of some hyperbolic operator D_1. Although one is essentially reduced to constructing two-point functions for D_1, one has to make sure that their restriction to $\mathcal{V}_P$ is well defined. This entails a compatibility condition that will be termed *gauge-invariance*.

Secondly, the hyperbolic operator D_1 is formally self-adjoint w.r.t. a hermitian product which is typically *non-positive* on fibers. This results in a conflict between the Hadamard condition and positivity of states for D_1. Although one can still expect positivity to hold on the subspace $\mathcal{V}_P$, it is not obvious how this can be controlled.

An additional difficulty are *infrared problems*, which are inherent to any massless theory, but have also their special incarnations in the context of gauge-invariance and positivity on $\mathcal{V}_P$.

In the present paper we study those issues in the case of the Yang-Mills equation, linearized around a (possibly non-vanishing) background solution $\bar{A}$.

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Framework for gauge theories. We work (when possible) in the abstract framework for gauge theories proposed recently by Hack & Schenkel [HS]. More precisely, we consider its simplified version, in which the classical theory is determined by:

(1) two vector bundles V_0, V_1 over a globally hyperbolic manifold (M, g), both equipped with a hermitian structure,
(2) a formally self-adjoint operator $P \in \mathrm{Diff}(M; V_1)$, which accounts for the equations of motion,
(3) a non-zero operator $K \in \mathrm{Diff}(M; V_0, V_1)$ s.t. $PK = 0$, which accounts for gauge transformations $u \to u + Kf$.

We then assume $D_1 := P + KK^*$ is hyperbolic and define the physical space by identifying solutions of P with those solutions of D_1 which satisfy the additional constraint $K^*u = 0$ (cf. Sect. 2 for precise definitions). The latter is often called *subsidiary condition* in the physics literature, we will thus term this approach the *subsidiary condition framework*[1]. The version we consider applies to the Maxwell and Yang-Mills equations, K being then the covariant differential $\bar{d}$ (note however that for other gauge theories one would have to use the more extended version from [HS]).

Hadamard two-point functions. In our framework, a pair of operators $\lambda_1^{\pm} : \Gamma_{\mathrm{c}}(M; V_1) \to \Gamma(M; V_1)$ induces two-point functions[2] of a Hadamard state on the phase space of P if it satisfies

$$D_1\lambda_1^{\pm} = \lambda_1^{\pm}D_1 = 0, \quad \lambda_1^{+} - \lambda_1^{-} = \mathrm{i}G_1, \tag{1.1}$$

where G_1 is the causal propagator of D_1 and if moreover:

$$\begin{aligned} &(\mu\mathrm{sc}) \quad \mathrm{WF}'(\lambda_1^{\pm}) \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}, \\ &(\mathrm{g.i.}) \quad (\lambda_1^{\pm})^* = \lambda_1^{\pm} \ \text{ and } \ \lambda_1^{\pm} : \mathrm{Ran}\, K \to \mathrm{Ran}\, K, \\ &(\mathrm{pos}) \quad \lambda_1^{\pm} \geq 0 \ \text{ on } \ \mathrm{Ker}\, K^*. \end{aligned}$$

Condition (μsc) is just the same as the Hadamard condition in ordinary (i.e., hyperbolic) field theory. What differs is the non-trivial requirement of gauge-invariance (g.i.). Moreover, positivity (pos) is no longer required to hold on all test sections, but on a specified subspace instead.

Main results. Our main result is the construction of Hadamard states for the Yang-Mills equation linearized around a smooth background solution $\bar{A}$, under various assumptions on $\bar{A}$ and the spacetime (M, g). Let us first formulate some hypotheses.

1.0.1. *Spacetimes.*

Hypothesis 1.1. *(M, g) is a globally hyperbolic spacetime with a Cauchy surface Σ diffeomorphic either to $\mathbb{R}^d$ for $d \geq 3$, or to a compact, parallelizable manifold.*

Hypothesis 1.2. *If $\Sigma = \mathbb{R}^d$, $h_{ij}(x)dx^i dx^j$ is a smooth Riemannian metric on Σ such that:*

$$c^{-1}\mathbf{1} \leq [h_{ij}(x)] \leq c\mathbf{1}, \ c > 0, \ |\partial_x^{\alpha} h_{ij}(x)| \leq C_{\alpha}, \ \forall \alpha \in \mathbb{N}^d, \ x \in \mathbb{R}^d.$$

[1] Because we are working in a purely algebraic setting, the terminology is rather ambiguous. We refer the interested reader to [Der] for a review on the flat case that explains the terminology used in the physics literature.
[2] We work with complex fields rather than with real ones, therefore it is natural to speak of a *pair* of two-point functions, cf. [Hol, GW, W2]. It should be noted that the real and complex approaches are equivalent, see for instance [GW] for the bosonic case.

1.0.2. *Background Yang Mills connections.*

Hypothesis 1.3. *G is a linear Lie group with compact Lie algebra $\mathfrak{g}$.*

We consider the trivial principal bundle $(M \times G, M, G)$ and the associated trivial vector bundle $(M \times \mathfrak{g}, M, \mathfrak{g})$. Using the horizontal connection on $M \times G$, a connection on $M \times \mathfrak{g}$ can be identified with a section $\bar{A}$ of the bundle $T^*M \times \mathfrak{g}$, i.e. with a Lie algebra valued $1-$form $\bar{A}$.

Hypothesis 1.4. *If $\Sigma = \mathbb{R}^d$, $\bar{A}$ is a smooth global solution of the non-linear Yang-Mills equation (2.14) on $\mathbb{R}_t \times \Sigma$ such that*

i) $\bar{A}$ *is in the* temporal gauge *i.e.* $\bar{A}_t = 0$,

ii) $|\partial_x^\alpha \bar{A}_\Sigma(t,x)| \leq C_\alpha$, *locally uniformly in* t,

iii) $|\partial_x^\alpha \bar{\delta}_\Sigma \bar{F}_\Sigma(0,x)| \leq C_\alpha \langle x\rangle^{-1}$, $|\partial_x^\alpha \bar{F}_t(0,x)| \leq C_\alpha \langle x\rangle^{-2}$ $\alpha \in \mathbb{N}^d$, $x \in \mathbb{R}^d$,

where the components $\bar{A}_\Sigma$, $\bar{A}_t$, $\bar{F}_\Sigma$, $\bar{F}_t$ of $\bar{A}$ and the curvature $\bar{F} = \bar{d}\bar{A}$ are defined in 4.4.1.

Our first theorem deals with ultra-static background metrics and background solutions $\bar{A}$ satisfying conditions near infinity in the case $\Sigma = \mathbb{R}^d$.

Theorem 1.1. *Let us assume Hypotheses 1.1, 1.3 and if $\Sigma = \mathbb{R}^d$ also Hypotheses 1.2, 1.4. Let $g = -dt^2 + h_{ij}(x)dx^i dx^j$ on $M = \mathbb{R}_t \times \Sigma$. Then there exist quasi-free Hadamard states for the linearized Yang Mills equation on (M, g) around $\bar{A}$.*

Our next theorem covers the general case, with a *space-compact* background solution $\bar{A}$. We will deduce it from Thm. 1.1 by a deformation argument explained in Subsect. 3.5. This deformation relies on the global solvability of the *non-linear* Yang-Mills equation, which requires that $\dim M \leq 4$.

Theorem 1.2. *Let us assume Hypotheses 1.1, 1.3 and* $\dim M \leq 4$.
Let $\bar{A} \in \mathcal{E}^1_{\mathrm{sc}}(M) \otimes \mathfrak{g}$ a smooth, space-compact solution of the non-linear Yang-Mills equation (2.14) on (M, g). Then there exist quasi-free Hadamard states for the linearized Yang Mills equation around $\bar{A}$.

Let us emphasize that the case $\bar{A} \neq 0$ differs substantially from the case of a vanishing background solution (or of an abelian gauge group), as was so far assumed in other works on Hadamard states. Indeed, if $\bar{A} \neq 0$ then the deformation argument cannot be used to reduce the problem to the situation when (M, g) is ultra-static *and* the coefficients of D_1, P do not depend on time.

As further explained in Subsect. 3.5, the difficulty comes from the fact that the background $\bar{A}$ must be a solution of the non-linear Yang-Mills equation and therefore cannot be arbitrarily deformed. This is our main motivation for considering the case of a *time-dependent* Klein-Gordon operator D_1 on an ultra-static spacetime.

Known results. In the literature, other constructions were already considered in the special case of the Maxwell equations or Yang-Mills linearized around $\bar{A} = 0$.

In these cases the deformation argument yields a time-independent problem, and it is possible to use arguments from spectral theory at least if the Cauchy surface Σ has special properties that make the infrared problems less serious. For the Maxwell equations, this strategy was employed in [FP] for Σ compact with vanishing first cohomology group (extending some earlier results of [Fur]), and in [FS] for Σ subject to an 'absence of zero resonances' condition for the Laplace-Beltrami operator on 1-forms. This condition appears to be more general but similar in nature to our assumptions, as it involves the behaviour of Σ at infinity[3]. The Yang-Mills

[3]The two methods are difficult to compare: in [FS] the infrared problem amounts to an obstruction to invertibility of the Laplacian, whereas in our approach the Laplacian is effectively replaced by an invertible operator and an infrared problem occurs in attempts of restoring gauge-invariance.

equation with $\bar{A} = 0$ was considered in [Hol2] (in the BRST framework) for Σ compact with vanishing first cohomology group.

Another approach was studied in [DS] on asymptotically flat spacetimes, where the use of spectral theory arguments is made possible by considering a characteristic Cauchy problem.

Summary of the construction. Let us summarize the strategy adopted in the paper.

The construction of the parametrix by pseudodifferential calculus is a generalization of the arguments used in [GW] in the scalar case. As an output, we obtain Hadamard two-point functions $\lambda_1^{\pm}$ that satisfy (g.i.) only 'modulo smooth terms'. Moreover, they are positive on some subspace (the space of 'purely spatial' 1-forms on M) that needs not to coincide with $\mathrm{Ker}K^*$.

To solve this, we work with quantities on a fixed Cauchy surface Σ. We define a Cauchy-surface analogue K_Σ of the operator K, and deduce that the Cauchy-surface version of the phase space for P can be expressed as a quotient $\mathrm{Ker}K_\Sigma^\dagger/\mathrm{Ran}K_\Sigma$ (where † is the *symplectic adjoint*, defined in (2.9)).

Next, we argue that gauge-invariance can be obtained by modifying $\lambda_1^{\pm}$ with the help of a projection Π that maps to a complement of $\mathrm{Ran}K_\Sigma$. The whole task that remains then is to show that:

- The range of Π is a space on which $\lambda_1^{\pm}$ is positive (after restricting to the phase space of P).
- The modification of $\lambda_1^{\pm}$ does not affect (μsc).

Both tasks are unfortunately made difficult by infrared problems. For example, the projection Π can contain terms such as $(\bar{\delta}_\Sigma \bar{d}_\Sigma)^{-1}\bar{\delta}_\Sigma$ (see Subsect.8.2), whose definition is already ambiguous, not to mention boundedness between Sobolev spaces of appropriate order.

One way we deal with such problems is to use a *Hardy's inequality* on $\mathbb{R}^d$ for the Hodge Laplacian on 0-forms.

The essential novelty is the systematic use of two classes of pseudodifferential operators

$$\Psi^p_{\mathrm{as}}(\Sigma; V_\alpha, V_\beta), \quad \Psi^p_{\mathrm{reg}}(\Sigma; V_\alpha, V_\beta),$$

that contain infrared regularizations of different type — either a simple 'microlocal' cutoff in the low frequencies (for the Ψ^p_{as} class), or in addition to that a 'spectral' cutoff (the Ψ^p_{reg} class), defined using (functions of) some elliptic self-adjoint operators. Moreover, the norm of the regularization is controlled by a parameter R that can be chosen arbitrarily large. This allows to obtain *exact inverses* in situations where standard pseudodifferential calculus gives only inverses modulo regularizing remainders. Using this method, we first construct a reference projection, establish its boundedness as an operator between appropriate (weighted) Sobolev spaces, and then perturb it in order to finally get the positivity.

Auxiliary results. Beside of what is of direct interest for Maxwell and Yang-Mills fields, let us mention some auxiliary results obtained in the present work.

First of all, in the context of ordinary field theory (without gauge), we derive a direct relation between (bosonic) Hadamard two-point functions and parametrices that satisfy certain special properties (Subsect. 3.3). This allows to generalize and simplify results in [GW] that tell how to obtain more Hadamard states out of an already given one.

We also derive a number of results for the classical Yang-Mills theory linearized around a non-vanishing background, for instance our formula for the phase space of P in terms of Cauchy data appears to be new (see 2.4.1).

Outlook. An evident limitation of our method is that we have to assume that the Cauchy surface Σ is either compact or equal $\mathbb{R}^d$, as the construction is based on standard pseudodifferential operator classes. We also use Hardy's inequality in the case $\Sigma = \mathbb{R}^d$. We expect, however, that

it would be possible to extend our results to other Cauchy surfaces by considering extensions of the standard pseudodifferential calculus on classes of non-compact manifolds on which a generalized form of Hardy's inequality still holds true.

Let us also stress that all our results are formulated in the subsidiary condition framework to gauge theories. Especially for applications in perturbative Quantum Field Theory, a different approach — the *BRST framework*, is commonly believed to be more efficient [Hol2]. We do not consider it here, although it seems plausible that one can transport Hadamard states from one framework to the other, as illustrated in [FS, Appendix B]. Another assumption that we implicitly make is that $\bar{A}$ is a connection on a *trivial* principal bundle and one can ask whether the methods of this paper can be applied to the non-trivial case. We plan to address these issues in a future work.

Structure of the paper. The paper is structured as follows.

Sect. 2 concerns the classical theory. We first recall well-known facts on ordinary field theories, then in Subsect. 2.4 review gauge theories on curved spacetime in the (simplified) subsidiary condition framework. We introduce the corresponding quantities on a Cauchy surface in 2.4.1 and then in Subsect. 2.5 we show how the linearized Yang-Mills equation fits into this framework.

Sect. 3 discusses Hadamard states for both ordinary field theories and for gauge theories in the subsidiary framework in general terms. We introduce in Subsect. 3.2 the definition of Hadamard states that we use for ordinary field theories. We then set up in Subsect. 3.3 a correspondence between Hadamard states and parametrices subject to special conditions. Next, we discuss in Subsect. 3.4 two-point functions in gauge theory, and formulate the conditions (μsc), (g.i.), (pos) and the Cauchy surface analogues of the latter two. In the same subsection we outline our method to cope with (g.i.) and (pos), and discuss the main technical obstructions. The section ends with an extended version of the Fulling, Narcowich & Wald argument in Subsect. 3.5, which allows us to reduce the construction of Hadamard states for the Yang-Mills equation to a situation where the spacetime is static, but the equations of motions still depend on the time coordinate.

Sect. 4 reviews the vector and scalar Klein-Gordon equations on ultra-static spacetimes.

In Sect. 5 we give a detailed construction of the parametrix for the vector Klein-Gordon equations considered in Sect. 4, generalizing results from [GW].

In Sect. 6, using the results of Sect. 5 we obtain two-point functions for the vector and scalar Klein-Gordon equations on an ultra-static spacetime and study their properties. At this point, the properties (g.i.) and (pos) are not satisfied and only their weaker versions are available.

As a byproduct of our constructions, we prove that for vector Klein-Gordon equations, where the natural hermitian product is not positive-definite on the fibers, there *does not exist Hadamard states*, but only Hadamard *pseudo-states*.

In Sect. 7, we study the relationship between the two-point functions constructed in Sect. 6 in the vector and scalar case. In particular Thm. 7.3 will be important later on.

In Sect. 8 we prove Thm. 1.1 by the method described in Subsect. 3.4. This is the most technical part of the paper.

In Appendix A we introduce the necessary background on pseudodifferential calculus. It includes amongst other a version of Egorov's theorem adapted to the case of matrices of pseudodifferential operators.

Appendix B gathers independent results, used in several parts of the main text. In B.1 we prove a version of Hardy's inequality adapted to our applications for the Yang-Mills equation. In B.2 we recall the transition to the temporal gauge for the non-linear Yang-Mills equation. In B.3 we discuss the constraint equations on Cauchy data for the non-linear Yang-Mills equation

and show how to construct examples of solutions satisfying our hypotheses. In B.4 we sketch the proof of Prop. 3.19.

2. Classical gauge field theory

2.1. **Notation.** Let V be a finite rank vector bundle over a smooth manifold M. We denote by $\Gamma(M;V)$, resp. $\Gamma_{\mathrm{c}}(M;V)$, $\Gamma_{\mathrm{sc}}(M;V)$ the space of smooth, resp. smooth with compact, space-compact support sections of V, the later notation requiring that M is equipped with some causal structure.

If V_1, V_2 are two vector bundles, the set of differential operators (of order m) $\Gamma(M;V_1)\to\Gamma(M;V_2)$ is denoted $\mathrm{Diff}(M;V_1,V_2)$ ($\mathrm{Diff}^m(M;V_1,V_2)$), we also use the notation $\mathrm{Diff}(M;V)=\mathrm{Diff}(M;V,V)$.

By a *bundle with hermitian structure* we will mean a vector bundle V equipped with a fiber wise non-degenerate hermitian form (in the literature the name 'hermitian bundle' is usually reserved for positive definite hermitian structures).

Suppose now that (M,g) is a pseudo-Riemannian oriented manifold. If V is a bundle on M with hermitian structure, we denote V^* the anti-dual bundle. The hermitian structure on V and the volume form on M allow to embed $\Gamma(M;V)$ into $\Gamma'_{\mathrm{c}}(M;V)$, using the non-degenerate hermitian form on $\Gamma_{\mathrm{c}}(M;V)$

$$(u|v)_V := \int_M (u(x)|v(x))_V \, d\mathrm{Vol}_g,\ u,v\in\Gamma_{\mathrm{c}}(M;V). \tag{2.1}$$

Therefore, we have a well-defined notion of the formal adjoint $A^*:\Gamma_{\mathrm{c}}(M;W)\to\Gamma(M;V)$ of an operator $A:\Gamma_{\mathrm{c}}(M;V)\to\Gamma(M;W)$.

If E,F are vector spaces, the space of linear operators is denoted $L(E,F)$. If E,F are additionally endowed with some topology, we write $A:E\to F$ if $A\in L(E,F)$ is continuous.

To distinguish between the same operator A acting on different spaces of functions and distributions, for instance $A:\Gamma_{\mathrm{c}}(M;V)\to\Gamma'_{\mathrm{c}}(M;W)$ and $A:\Gamma(M;V)\to\Gamma(M;W)$, we use the notation $A|_{\Gamma_{\mathrm{c}}}$ and $A|_{\Gamma}$.

2.2. **Quotient spaces.** In the sequel we will frequently encounter operators and sesquilinear forms on quotients of linear spaces, we recall thus the relevant basic facts.

2.2.1. *Operators on quotient spaces.* Let $F_i\subset E_i$, $i=1,2$ be vector spaces and let $A\in L(E_1,E_2)$. Then the induced map

$$[A]\in L(E_1/F_1,E_2/F_2),$$

defined in the obvious way, is

- well-defined if $AE_1\subset E_2$ and $AF_1\subset F_2$;
- injective iff $A^{-1}F_2=F_1$;
- surjective iff $E_2=AE_1+F_2$.

2.2.2. *Sesquilinear forms on quotients.* Let now $E\subset F$ be vector spaces and let $C\in L(E,E^*)$, where E^* is the anti-dual space of E. Then the induced map

$$[C]\in L(E/F,(E/F)^*),$$

defined as before, is

- well-defined if $CE\subset F^\circ$ (where $F^\circ\subset E^*$ denotes the annihilator of F) and $F\subset\operatorname{Ker}C$;
- non-degenerate iff $F=\operatorname{Ker}C$.

If C is hermitian or anti-hermitian (which will usually be the case in our examples) then the condition $F\subset\operatorname{Ker}C$ implies the other one $CE\subset F^\circ$ (and vice versa).

2.3. **Ordinary classical field theory.** We recall now some standard results, see eg [BGP, HS]. Let (M, g) be a globally hyperbolic spacetime (we use the convention $(-, +, \dots, +)$ for the Lorentzian signature). If V is a vector bundle over M, we denote $\Gamma_{\rm sc}(M; V)$ the space of space-compact sections, i.e. sections in $\Gamma(M; V)$ such that their restriction to a Cauchy surface has compact support.

One says that $D \in \mathrm{Diff}(M; V)$ is *Green hyperbolic* if D and D^* possess retarded and advanced propagators — the ones for D will be denoted respectively G^+ and G^- (for the definition, see [BGP]). The *causal propagator* (or Pauli-Jordan commutator function) of D is then by definition $G := G^+ - G^-$. Normally hyperbolic and prenormally hyperbolic operators (defined below) are Green hyperbolic.

Definition 2.1. (1) *An operator $D \in \mathrm{Diff}(M; V)$ is* normally hyperbolic *if its principal symbol equals $-\xi_\mu \xi^\mu \mathbf{1}_V$.*

(2) *An operator $D \in \mathrm{Diff}(M; V)$ is* prenormally hyperbolic *if there exists $\widetilde{D} \in \mathrm{Diff}(M; V)$ s.t. $D\widetilde{D}$ is normally hyperbolic.*

This terminology is slightly more general than the one used in e.g. [Müh], cf. [W, W2] for examples.

Proposition 2.2. *If $D, \widetilde{D} \in \mathrm{Diff}(M; V)$ are such that $D\widetilde{D}$ is Green hyperbolic then D is Green hyperbolic and their retarded/advanced propagators $G^\pm$ and $G^\pm_{D\widetilde{D}}$ are related by*

$$G^\pm = \widetilde{D} G^\pm_{D\widetilde{D}}.$$

The proof of Prop. 2.2 is a straightforward generalization of the arguments of Dimock [Dim, Müh].

Before discussing gauge theories, let us recall the basic data that define an ordinary classical field theory (i.e., with no gauge freedom built in) on a globally hyperbolic manifold (M, g).

Hypothesis 2.1. *Suppose that we are given:*

(1) *a bundle V over M with hermitian structure;*
(2) *a Green hyperbolic operator $D \in \mathrm{Diff}(M; V)$ s.t. $D^* = D$.*

Proposition 2.3. *As a consequence of Hypothesis 2.1,*

(1) *the induced map*

$$[G] : \frac{\Gamma_{\rm c}(M; V)}{\mathrm{Ran}\, D|_{\Gamma_{\rm c}}} \longrightarrow \mathrm{Ker}\, D|_{\Gamma_{\rm sc}}$$

is well defined and bijective.

(2) *$(G^\pm)^* = G^\mp$ and consequently $G^* = -G$;*

To fix some terminology, by a phase space we mean a pair $(\mathcal{V}, q)$ consisting of a complex vector space $\mathcal{V}$ and a sesquilinear form q on $\mathcal{V}$. Actual physical meaning can be associated to $(\mathcal{V}, q)$ if q is hermitian. The classical phase space associated to D is $(\mathcal{V}, q)$, where

$$\mathcal{V} := \frac{\Gamma_{\rm c}(M; V)}{\mathrm{Ran}\, D|_{\Gamma_{\rm c}}}, \quad \overline{u}\, q v := \mathrm{i}(u|[G]v)_V. \tag{2.2}$$

By (2) of Prop. 2.3 the sesquilinear form q is hermitian, and it is not difficult to show that it is non-degenerate. As a rule, we will work with hermitian forms rather than with real symplectic ones, but it should be kept in mind that the two approaches are equivalent.

2.3.1. *Phase space on Cauchy surface.* Let us fix a Cauchy surface Σ of (M,g). Consider a Green hyperbolic operator $D \in \mathrm{Diff}^m(M;V)$. Let V_ρ be a vector bundle over Σ and $\rho : \Gamma_{\mathrm{sc}}(M;V) \to \Gamma_{\mathrm{c}}(\Sigma;V_\rho)$ an operator which is the composition of a differential operator of order $\leq m$ with the pullback $\imath^*$ of the embedding $\imath : \Sigma \hookrightarrow M$.

We equip V_ρ with a hermitian structure $(\cdot|\cdot)_{V_\rho}$, which extends to $\Gamma_{\mathrm{c}}(\Sigma;V_\rho)$ as in (2.1), using the volume form on Σ induced by g. It is convenient to assume that this hermitian structure is *positive definite*. The adjoint map:

$$\rho^* : \Gamma_{\mathrm{c}}(\Sigma;V_\rho) \to \Gamma'(M;V)$$

is defined using the two hermitian structures $(\cdot|\cdot)_V$ and $(\cdot|\cdot)_{V_\rho}$.

Hypothesis 2.2. *Assume that for each initial datum $\varphi \in \Gamma_{\mathrm{c}}(\Sigma;V_\rho)$, the Cauchy problem*

$$\begin{cases} Df = 0, \quad f \in \Gamma_{\mathrm{sc}}(M;V) \\ \rho f = \varphi, \end{cases} \tag{2.3}$$

has a unique solution.

In other words, the map $\rho : \operatorname{Ker} D|_{\Gamma_{\mathrm{sc}}} \to \Gamma_{\mathrm{c}}(\Sigma;V_\rho)$ is a bijection. If D satisfies Hypothesis 2.2, we will say that it is *Cauchy hyperbolic* (for the map ρ). It can be proved that if D is Green hyperbolic then there exists ρ s.t. D is Cauchy hyperbolic[4], cf. the reasoning in [K, Sec. 4.3].

By Hypothesis 2.2, assuming additionally $D = D^*$ and using (1) of Prop. 2.3 we deduce that the phase space $(\mathcal{V},q)$ is isomorphic to $(\mathcal{V}_\Sigma,q_\Sigma)$, defined in the following way:

$$\mathcal{V}_\Sigma := \Gamma_{\mathrm{c}}(\Sigma;V_\rho), \quad \overline{u}\, q_\Sigma v := \mathrm{i}(u|G_\Sigma v)_{V_\rho}, \tag{2.4}$$

where G_Σ is uniquely defined by

$$G =: (\rho G)^* G_\Sigma (\rho G).$$

(Let us stress again that the stars refer to formal adjoints using the hermitian structures of V and V_ρ, the latter can be chosen quite arbitrarily.) As a consequence of this definition,

$$\mathbf{1} = G^*\rho^* G_\Sigma \rho \ \text{ on } \ \operatorname{Ker} D|_{\Gamma_{\mathrm{sc}}}. \tag{2.5}$$

This also implies $\rho = \rho G^* \rho^* G_\Sigma \rho$ on $\operatorname{Ker} D|_{\Gamma_{\mathrm{sc}}}$, hence

$$\mathbf{1} = \rho G^* \rho^* G_\Sigma \ \text{ on } \ \Gamma_{\mathrm{c}}(\Sigma;V_\rho). \tag{2.6}$$

It is useful to introduce the *Cauchy evolution operator*:

$$U := G^*\rho^* G_\Sigma. \tag{2.7}$$

By (2.5) and (2.6), it satisfies $\rho U = \mathbf{1}$ and $U\rho = \mathbf{1}$ (on space-compact solutions of D). Moreover, since $G^* = -G$ we get $DU = 0$. Applying both sides of (2.5) to f we obtain a formula for the solution of the Cauchy problem (2.3).

Proposition 2.4. *Assume D is Cauchy hyperbolic for ρ and $D = D^*$. Then the unique solution of the Cauchy problem (2.3) equals*

$$f = U\varphi = G^*\rho^* G_\Sigma \varphi = -G\rho^* G_\Sigma \varphi.$$

[4]Of course one has to choose ρ sensibly, cf. the example in [BG, Sec. 2.7].

2.4. **Gauge theory in subsidiary condition formalism.** The following data is used to define a classical linearized gauge field theory on a globally hyperbolic manifold (M, g). This is a special case of the setting proposed by Hack and Schenkel in [HS], well suited for the case of Yang-Mills fields.

Hypothesis 2.3. *Suppose that we are given:*

(1) *bundles with hermitian structures V_0, V_1 over M;*
(2) *a formally self-adjoint operator $P \in \mathrm{Diff}(M; V_1)$;*
(3) *an operator $K \in \mathrm{Diff}(M; V_0, V_1)$, such that $K \neq 0$ and*
   (a) *$PK = 0$,*
   (b) *$D_0 := K^*K \in \mathrm{Diff}(M; V_0)$ is Green hyperbolic,*
   (c) *$D_1 := P + KK^* \in \mathrm{Diff}(M; V_1)$ is Green hyperbolic.*

The operator P accounts for the equations of motion, linearized around a background solution. The operator K defines the linear gauge transformation $f \mapsto f + Kg$, and the condition $PK = 0$ states that P is invariant under this transformation, which entails that P is not hyperbolic. Making use of the assumption on D_0, the non-hyperbolic equation $Pf = 0$ can be reduced by gauge transformations to the subspace $K^*f = 0$ of solutions of the hyperbolic problem $D_1 f = 0$. The equation $K^*f = 0$ is traditionally called *subsidiary condition* and can be thought as a covariant fixing of gauge.

The canonical example is the Maxwell theory, in which case K is the differential d acting on 0-forms on M and $P = \delta d$, where δ is the codifferential. The subsidiary condition $K^*f = 0$ is then simply the Lorenz gauge. This example will be further discussed in Subsect. 2.5 as a special case of Yang-Mills theory.

Let us first observe that the differential operators from Hypothesis 2.3 satisfy the algebraic relations

$$K^*D_1 = D_0K^*, \quad D_1K = KD_0.$$

These have the following consequences on the level of propagators and spaces of solutions, proved in [HS].

Proposition 2.5. *As a consequence of Hypothesis 2.3,*

(1) *$K^*G_1^\pm = G_0^\pm K^*$ on $\Gamma_{\mathrm{c}}(M; V_1)$ and $KG_0^\pm = G_1^\pm K$ on $\Gamma_{\mathrm{c}}(M; V_0)$;*
(2) *For all $\psi \in \Gamma_{\mathrm{sc}}(M; V_1)$ there exists $h \in \Gamma_{\mathrm{sc}}(M; V_0)$ s.t. $\psi - Kh \in \mathrm{Ker}\, K^*|_{\Gamma_{\mathrm{sc}}}$. If moreover $\psi \in \mathrm{Ker}\, P|_{\Gamma_{\mathrm{sc}}}$ then $\psi - Kh \in \mathrm{Ker}\, P|_{\Gamma_{\mathrm{sc}}} \cap \mathrm{Ker}\, K^*|_{\Gamma_{\mathrm{sc}}}$;*
(3) *We have*

$$\mathrm{Ker}\, P|_{\Gamma_{\mathrm{sc}}} \cap \mathrm{Ker}\, K^*|_{\Gamma_{\mathrm{sc}}} \subset G_1 \mathrm{Ker}\, K^*|_{\Gamma_{\mathrm{c}}} + G_1 \mathrm{Ran}\, K|_{\Gamma_{\mathrm{c}}};$$

(4) $\mathrm{Ran}\, P|_{\Gamma_{\mathrm{c}}} = \mathrm{Ker}\, K^*|_{\Gamma_{\mathrm{c}}} \cap G_1^{-1} \mathrm{Ran}\, K|_{\Gamma_{\mathrm{sc}}}$.

Since the auxiliary operators D_1, D_0 are Green hyperbolic, we can associate to them phase spaces $(\mathcal{V}_1, q_1)$, $(\mathcal{V}_0, q_0)$ as in the previous subsection.

In the 'subsidiary condition' framework, the physical phase space associated to P, denoted $(\mathcal{V}_P, q_P)$, is defined by

$$\mathcal{V}_P := \frac{\mathrm{Ker}\, K^*|_{\Gamma_{\mathrm{c}}}}{\mathrm{Ran}\, P|_{\Gamma_{\mathrm{c}}}}, \quad \overline{u}\, q_P v := \mathrm{i}(u|[G_1]v)_{V_1}.$$

The first thing to check is that the propagator G_1 of D_1 induces a well-defined linear map on the quotient space above.

Proposition 2.6. *The sesquilinear form q_P is well defined on $\mathcal{V}_P$.*

Proof. We need to show that $(u|G_1 v)_{V_1} = 0$ if $u \in \mathrm{Ker}\, K^*|_{\Gamma_c}$ and $v = Pf$ for some $f \in \Gamma_c(M; V_1)$. We have in such case

$$G_1 P f = -G_1 K K^* f = -K G_0 K^* f,$$

hence $(u|G_1 P f)_{V_1} = -(K^* u|G_0 K^* f)_{V_0} = 0$. □

The definition of the phase space $\mathcal{V}_P$ agrees with the one considered in [Dim2, FP, P, HS] and is arguably the most natural one. Other possible definitions are discussed in [DHK, HS, B]. Let us also mention that the form q_P needs not be non-degenerate on $\mathcal{V}_P$, cf. examples and further discussion in [DHK, HS, B].

It is possible to give different generalizations of Prop. 2.3, (1) (claim a) below is proved in [HS]).

Proposition 2.7. *The induced maps*

$$\text{a)} \quad [G_1] : \frac{\mathrm{Ker}\, K^*|_{\Gamma_c}}{\mathrm{Ran}\, P|_{\Gamma_c}} \longrightarrow \frac{\mathrm{Ker}\, P|_{\Gamma_{sc}}}{\mathrm{Ran}\, K|_{\Gamma_{sc}}},$$

$$\text{b)} \quad [G_1] : \frac{\mathrm{Ker}\, K^*|_{\Gamma_c}}{\mathrm{Ran}\, P|_{\Gamma_c}} \longrightarrow \frac{\mathrm{Ker}\, D_1|_{\Gamma_{sc}} \cap \mathrm{Ker}\, K^*|_{\Gamma_{sc}}}{\mathrm{Ran}\, G_1 K|_{\Gamma_c}},$$

are well defined and bijective.

Proof. b): For well-definiteness we check that $G_1 \mathrm{Ker}\, K^*|_{\Gamma_c} \subset \mathrm{Ker} D_1$ which is easy, and $G_1 \mathrm{Ker}\, K^*|_{\Gamma_c} \subset \mathrm{Ker} K^*$, which follows from $K^* G_1 = G_0 K^*$. We need also to check that $G_1 \mathrm{Ran} P \subset \mathrm{Ran} G_1 K$ which follows from Hypothesis 2.3 (c).

For injectivity we see that if $K^* u = 0$ and $G_1 u = G_1 K v$, then $u - Kv = D_1 f$ for $f \in \Gamma_c(M; V_1)$, hence $D_0(v + K^* f) = 0$, which implies that $v + K^* f = 0$ and hence $u = Pf$.

Surjectivity amounts to showing

$$\mathrm{Ker}\, D_1|_{\Gamma_{sc}} \cap \mathrm{Ker}\, K^*|_{\Gamma_{sc}} = G_1 \mathrm{Ker} K^*|_{\Gamma_c} + G_1 \mathrm{Ran}\, K|_{\Gamma_c}.$$

The inclusion '$\supset$' is easy, the other one follows from Prop. 2.5, (3). □

Finally, let us quote another useful result, shown in the present context in [HS], and often called the *time-slice property* (or time-slice axiom). Below, $J^+(O)$ (resp. $J^-(O)$) denotes the causal future (resp. causal past) of $O \subset M$.

Proposition 2.8. *Let Σ_+, Σ_- be two Cauchy surfaces s.t. $J^-(\Sigma_+) \cap J^+(\Sigma_-)$ contains properly a Cauchy surface. Then for all $[f] \in \mathrm{Ker}\, K^*|_{\Gamma_c} / \mathrm{Ran}\, P|_{\Gamma_c}$ there exists $\tilde{f} \in \mathrm{Ker}\, K^*|_{\Gamma_c}$ s.t.*

$$[f] = [\tilde{f}], \quad \mathrm{supp}\, \tilde{f} \subset J^-(\Sigma_+) \cap J^+(\Sigma_-).$$

2.4.1. *Phase spaces on a Cauchy surface.* Let us now discuss the corresponding phase spaces on a fixed Cauchy surface $\Sigma \subset M$. Recall that in Hypothesis 2.3 we have required that the operators D_1 and D_0 are Green hyperbolic, and thus Cauchy hyperbolic. The corresponding maps will be denoted

$$\rho_1 : \ \Gamma(M; V_1) \to \Gamma_c(\Sigma; V_{\rho_1}),$$
$$\rho_0 : \ \Gamma(M; V_0) \to \Gamma_c(\Sigma; V_{\rho_0}).$$

We also recall that we have defined operators $G_{i\Sigma}$ such that $G_i = (\rho_i G_i)^* G_{i\Sigma} (\rho_i G_i)$ and Cauchy evolution operators U_i for $i = 0, 1$.

To the operator K we associate an operator $K_\Sigma \in \mathrm{Diff}(\Sigma; V_{\rho_0}, V_{\rho_1})$:

$$K_\Sigma := \rho_1 K U_0. \tag{2.8}$$

It is useful to introduce the adjoint $K_\Sigma^\dagger \in \mathrm{Diff}(\Sigma; V_{\rho_1}, V_{\rho_0})$ w.r.t. the hermitian forms $q_{1\Sigma}$ and $q_{0\Sigma}$ (the so-called *symplectic adjoint*), i.e.

$$G_{0\Sigma} K_\Sigma^\dagger := K_\Sigma^* G_{1\Sigma}. \tag{2.9}$$

The notation † is used to avoid confusion with the formal adjoint * w.r.t. the hermitian structures on the bundles V_{ρ_0}, V_{ρ_1}, appearing for instance in the LHS of the above equation.

Lemma 2.9. *As a consequence of Hypothesis 2.3,*

(1) $KU_0 = U_1K_\Sigma$ *and* $K^*U_1 = U_0K_\Sigma^\dagger$*;*
(2) $\rho_1 K = K_\Sigma\rho_0$ *on* $\mathrm{Ker}\, D_0|_{\Gamma_{\rm sc}}$ *and* $\rho_0K^* = K_\Sigma^\dagger\rho_1$ *on* $\mathrm{Ker}\, D_1|_{\Gamma_{\rm sc}}$*;*
(3) $\mathrm{Ker}\, K_\Sigma^\dagger|_{\Gamma_{\rm c}} = \rho_1 G_1^*\mathrm{Ker}\, K^*|_{\Gamma_{\rm c}}$*;*
(4) $\mathrm{Ran}\, K_\Sigma|_{\Gamma_{\rm c}} = \rho_1 G_1^*\mathrm{Ran}\, K|_{\Gamma_{\rm c}}$*;*
(5) $K_\Sigma^\dagger K_\Sigma = 0$.

Proof. (1): Let us prove the second assertion (the first one is trivial). By (2.9) and Prop. 2.5, (1),

$$\begin{aligned} U_0K_\Sigma^\dagger &= G_0^*\rho_0^*G_{0\Sigma}K_\Sigma^\dagger = G_0^*\rho_0^*K_\Sigma^*G_{1\Sigma} = G_0^*\rho_0^*U_0^*K^*\rho_1^*G_{1\Sigma} \\ &= G_0^*K^*\rho_1^*G_{1\Sigma} = K^*G_1^*\rho_1^*G_{1\Sigma} = K^*U_1. \end{aligned}$$

(2): By (1) we have $\rho_0K^* = \rho_0K^*U_1\rho_1 = \rho_0U_0K_\Sigma^\dagger\rho_1 = K_\Sigma^\dagger\rho_1$. The other assertion is trivial.

(3): If $u = \rho_1G_1^*f$ with $f \in \mathrm{Ker}\, K^*|_{\Gamma_{\rm c}}$ then $K_\Sigma^\dagger u = \rho_0K^*G_1^*f = \rho_0G_0^*K^*f = 0$. Conversely, if $u \in \mathrm{Ker}\, K_\Sigma^\dagger|_{\Gamma_{\rm c}}$ then using that $\mathbf{1} = \rho_1G_1^*\rho_1^*G_{1\Sigma}$ we get $u = \rho_1G_1^*f$ with $f = \rho_1^*G_{1\Sigma}u$ and

$$K^*f = K^*\rho_1^*G_{1\Sigma}u = \rho_0^*K_\Sigma^*G_{1\Sigma}u = \rho_0^*G_{0\Sigma}K_\Sigma^\dagger u = 0.$$

(4): If $u = \rho_1G_1^*Kf$ then $u = \rho_1KG_1f = K_\Sigma\rho_0G_0f$. Conversely, if $u = K_\Sigma h$ then using that $\mathbf{1} = \rho_1G_1^*\rho_1^*G_{1\Sigma}$ we get

$$u = \rho_1G_1^*\rho_1^*G_{1\Sigma}K_\Sigma h = \rho_1G_1^*K\rho_0^*G_{0\Sigma}h.$$

(5): By (1), $K_\Sigma^\dagger K_\Sigma = \rho_0U_0K_\Sigma^\dagger K_\Sigma = \rho_0K^*U_1K_\Sigma = \rho_0K^*KU_0 = 0$. □

Proposition 2.10. *The induced map*

$$[\rho_1]: \ \frac{\mathrm{Ker}\, D_1|_{\Gamma_{\rm sc}} \cap \mathrm{Ker}\, K^*|_{\Gamma_{\rm sc}}}{\mathrm{Ran}\, G_1K|_{\Gamma_{\rm c}}} \longrightarrow \frac{\mathrm{Ker}\, K_\Sigma^\dagger|_{\Gamma_{\rm c}}}{\mathrm{Ran}\, K_\Sigma|_{\Gamma_{\rm c}}}$$

is well defined and bijective.

Proof. Recall that we proved $\mathrm{Ker}\, D_1|_{\Gamma_{\rm sc}} \cap \mathrm{Ker}\, K^*|_{\Gamma_{\rm sc}} = G_1\mathrm{Ker}K^*|_{\Gamma_{\rm c}} + G_1\mathrm{Ran}\, K|_{\Gamma_{\rm c}}$.

For well-definiteness and surjectivity of $[\rho_1]$ it is thus sufficient to check that

$$\rho_1(G_1\mathrm{Ker}K^*|_{\Gamma_{\rm c}} + G_1\mathrm{Ran}\, K|_{\Gamma_{\rm c}}) = \mathrm{Ker}\, K_\Sigma^\dagger|_{\Gamma_{\rm c}},$$

which follows directly from (2) and (3) of Lemma 2.9 (using $G_1^* = -G_1$).

For injectivity we need to show that if $u \in G_1\mathrm{Ker}K^*|_{\Gamma_{\rm c}} + G_1\mathrm{Ran}\, K|_{\Gamma_{\rm c}}$ and $\rho_1u \in \mathrm{Ran}\, K_\Sigma|_{\Gamma_{\rm c}}$ then $u \in \mathrm{Ran}\, G_1K|_{\Gamma_{\rm c}}$. This follows from (4) of Lemma 2.9. □

We deduce from Prop. 2.7 and Prop. 2.10 that the map ρ_1G_1 induces an isomorphism between the phase space $(\mathcal{V}_P, q_P)$ and the phase space $(\mathcal{V}_{P\Sigma}, q_{P\Sigma})$, defined in the following way:

$$\mathcal{V}_{P\Sigma} := \frac{\mathrm{Ker}\, K_\Sigma^\dagger|_{\Gamma_{\rm c}}}{\mathrm{Ran}\, K_\Sigma|_{\Gamma_{\rm c}}}, \quad \overline{u}\, q_{P\Sigma} v := \mathrm{i}(u|[G_{1\Sigma}]v)_{V_{\rho_1}}.$$

2.5. **Linearized Yang-Mills.** We now recall how the formalism of Subsect. 2.4 applies to Yang-Mills equations linearized around a background solution $\bar{A}$. We follow [MM, HS].

Let $\mathfrak{g}$ be a real compact Lie algebra as in Hypothesis 1.3. We still denote by $\mathfrak{g}$ its complexification. The complexification of the Killing form yields a sesquilinear form

$$\mathfrak{k} \in L(\mathfrak{g}, \mathfrak{g}^*), \quad \mathfrak{k} > 0.$$

For simplicity we will work in a geometrically trivial situation[5].

[5]Otherwise one has to use the language of principal bundles, some indications can be found in [MM, Z].

As in [HS] we take V_0 to be the trivial bundle

$$V_0 := M \times \mathfrak{g},$$

equipped with the hermitian structure induced by $\mathcal{k}$, and V_1 the corresponding 1-form bundle

$$V_1 := T^*M \times \mathfrak{g}.$$

We equip V_1 with the hermitian structure given by the tensor product of the canonical hermitian structure on T^*M with $\mathcal{k}$.

Note that under Hypothesis 1.1 this bundle is trivial since Σ and hence M is then parallelizable.

Let us denote by $\mathcal{E}^p(M)$ the space of smooth p-forms on M and by $\mathcal{E}^\oplus(M) = \bigoplus_p \mathcal{E}^p(M)$ the space of smooth forms on M. As explained in 1.3, the spaces of sections $\Gamma(M; V_i)$ $i = 0, 1$ can be identified respectively with $\mathcal{E}^0(M) \otimes \mathfrak{g}$ and $\mathcal{E}^1(M) \otimes \mathfrak{g}$. The exterior product on $\mathcal{E}^\oplus(M) \otimes \mathfrak{g}$ is defined by

$$(\alpha \otimes a) \wedge (\beta \otimes b) := (\alpha \wedge \beta) \otimes [a, b] \quad a, b \in \mathfrak{g},\ \alpha, \beta \in \mathcal{E}^\oplus(M),$$

(note that in the physics literature a bracket notation is sometimes used instead). The interior product is defined by

$$(\alpha \otimes a) \lrcorner (\beta \otimes b) := (\alpha \lrcorner \beta) \otimes [b, a], \quad a, b \in \mathfrak{g},\ \alpha, \beta \in \mathcal{E}^\oplus(M).$$

We also define

$$A \llcorner \cdot :\ \mathcal{E}^\oplus(M) \otimes \mathfrak{g} \ni B \mapsto B \lrcorner A \in \mathcal{E}^\oplus(M) \otimes \mathfrak{g}.$$

It holds that

$$(B \wedge \cdot)^* = \overline{B} \lrcorner \cdot, \quad B \in \mathcal{E}^p(M) \otimes \mathfrak{g}$$

where the bar stands for ordinary complex conjugation. Note also that for 0-forms the interior product reduces to

$$f \lrcorner \cdot = -f \wedge \cdot, \quad f \in \mathcal{E}^0(M) \otimes \mathfrak{g}. \tag{2.10}$$

Let $d : \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)$ be the ordinary differential and let $\bar{A} \in \mathcal{E}^1(M) \otimes \mathfrak{g}$ (the thick bar is designed to distinguish $\bar{A}$ from dynamical variables A, it should not to be confused with complex conjugation $\overline{A}$). The *covariant differential* $\bar{d} : \mathcal{E}^p(M) \otimes \mathfrak{g} \to \mathcal{E}^{p+1}(M) \otimes \mathfrak{g}$ respective to $\bar{A}$ is defined by

$$\bar{d}f := df + \bar{A} \wedge f, \quad f \in \mathcal{E}^p(M) \otimes \mathfrak{g}.$$

Despite its name, it is in general not a differential in the sense that $\bar{d}\bar{d}$ would vanish, instead it holds that

$$\bar{d}\bar{d} = \bar{F} \wedge \cdot, \tag{2.11}$$

where $\bar{F} := d\bar{A} + \bar{A} \wedge \bar{A} \in \mathcal{E}^2(M) \otimes \mathfrak{g}$ is the *curvature* of $\bar{A}$. The covariant co-differential $\bar{\delta} : \mathcal{E}^{p+1}(M) \otimes \mathfrak{g} \to \mathcal{E}^p(M) \otimes \mathfrak{g}$ is by definition the formal adjoint $\bar{d}^*$ of $\bar{d}$. The covariant differential satisfies

$$\bar{d}(A \wedge B) = (\bar{d}A) \wedge B + (-1)^p A \wedge (\bar{d}B), \quad A \in \mathcal{E}^p(M) \otimes \mathfrak{g},\ B \in \mathcal{E}^q(M) \otimes \mathfrak{g}.$$

This can be written as an identity for operators and by taking their adjoints, one gets

$$A \lrcorner \bar{\delta}B = (\bar{d}A) \lrcorner B + (-1)^p \bar{\delta}(A \lrcorner B), \quad A \in \mathcal{E}^p(M) \otimes \mathfrak{g},\ B \in \mathcal{E}^q(M) \otimes \mathfrak{g}. \tag{2.12}$$

A consequence of the definition $\bar{F} = \bar{d}\bar{A}$ is the *Bianchi identity*

$$\bar{d}\bar{F} = 0. \tag{2.13}$$

The *non-linear Yang-Mills equation* for $\bar{A}$ reads

$$\bar{\delta}\bar{d}\bar{A}\ (= \bar{\delta}\bar{F}) = 0. \tag{2.14}$$

This system can be linearized as follows. We fix a real-valued section $\bar{A} \in \mathcal{E}^1(M) \otimes \mathfrak{g}$ and assume it is *on-shell*, i.e. satisfies the Yang-Mills equation (2.14). The *linearized Yang-Mills operator* is

$$P := \bar{\delta}\bar{d} + \bar{F} \llcorner \in \mathrm{Diff}^2(M; V_1), \tag{2.15}$$

where $\bar{d}$, $\bar{\delta}$ and $\bar{F}$ refer to the background solution $\bar{A}$. The linearized Yang-Mills equation is

$$PA = 0. \tag{2.16}$$

Gauge transformations are described in this linearized setting by the differential operator

$$K := \bar{d} \in \mathrm{Diff}^1(M; V_0, V_1).$$

It is not difficult to see that Hypothesis 2.3 is satisfied by P and K. More precisely, the operators $D_0 = K^*K$ and $D_1 = P + KK^*$ equal

$$D_0 = \bar{\delta}\bar{d} \in \mathrm{Diff}^2(M; V_0),$$
$$D_1 = \bar{d}\bar{\delta} + \bar{\delta}\bar{d} + \bar{F} \llcorner \in \mathrm{Diff}^2(M; V_1).$$

To show $PK = 0$, we compute using (2.11), (2.12) and (2.10)

$$\begin{aligned} PKf &= \bar{\delta}\bar{d}\bar{d}f + \bar{F} \llcorner (\bar{d}f) = \bar{\delta}(\bar{F} \wedge f) + (\bar{d}f) \lrcorner \bar{F} \\ &= \bar{\delta}(f \lrcorner \bar{F}) + (\bar{d}f) \lrcorner \bar{F} = f \lrcorner (\bar{\delta}\bar{F}) \quad \forall f \in \mathcal{E}^0(M) \otimes \mathfrak{g}. \end{aligned}$$

By the assumption that $\bar{A}$ is on-shell (2.14) this vanishes.

2.5.1. *Adapted Cauchy data.* Let us denote by n the future directed unit normal vector field to a Cauchy surface Σ.

Since D_1, D_0 are normally hyperbolic, they are Cauchy hyperbolic for the maps ρ_1, ρ_0 defined by taking the restriction to Σ of a given section and of its first derivative along n.

For many purposes it will however be more convenient to consider different maps ρ_1^{F}, ρ_0^{F}, which appear to be due to Furlani [Fur2] (cf. also [P]), and which are defined as follows[6].

We equip $\mathcal{E}^p_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g}$ with their standard (positive) hermitian scalar products, obtained from $\mathfrak{k}$ and the Riemannian metric h induced by g on Σ. We also recall that $\imath^* : \mathcal{E}^p_{\mathrm{sc}}(M) \otimes \mathfrak{g} \to \mathcal{E}^p_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g}$ is the pullback map induced by the embedding $\imath : \Sigma \to M$.

Definition 2.11. *If $\zeta \in \mathcal{E}^1_{\mathrm{sc}}(M) \otimes \mathfrak{g}$, we set:*

$$g^0_t := \imath^* n \lrcorner \zeta \in \mathcal{E}^0_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g},$$

$$g^0_\Sigma := \imath^* \zeta \in \mathcal{E}^1_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g},$$

$$g^1_t := \mathrm{i}^{-1} \imath^* \bar{\delta}\zeta \in \mathcal{E}^0_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g},$$

$$g^1_\Sigma := \mathrm{i}^{-1} \imath^* n \lrcorner \bar{d}\zeta \in \mathcal{E}^1_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g}.$$

For $g^i := (g^i_t, g^i_\Sigma) \in \mathcal{E}^0_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g} \oplus \mathcal{E}^1_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g}$ we set:

$$g := (g^0, g^1) =: \rho_1^{\mathrm{F}} \zeta.$$

Analogously, if $\zeta \in \mathcal{E}^0_{\mathrm{sc}}(M) \otimes \mathfrak{g}$, we set

$$g^0 := \imath^* \zeta \in \mathcal{E}^0_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g},$$

$$g^1 := \mathrm{i}^{-1} \imath^* n \lrcorner \bar{d}\zeta \in \mathcal{E}^0_{\mathrm{c}}(\Sigma) \otimes \mathfrak{g},$$

[6]To be precise, reference [Fur2] uses Cauchy data which are denoted $(A_{(n)}, A_{(0)}, A_{(\delta)}, A_{(d)})$ therein and are related to ours by $g^0_t = A_{(n)}$, $g^0_\Sigma = A_{(0)}$, $g^1_t = \mathrm{i}^{-1} A_{(\delta)}$, $g^1_\Sigma = \mathrm{i}^{-1} A_{(d)}$.

and

$$g := (g^0, g^1) =: \rho_0^{\mathrm{F}} \zeta.$$

In the terminology of Sect. 2.4.1, $\rho_i^{\mathrm{F}} : \Gamma_{\mathrm{c}}(M; V_i) \to \Gamma_{\mathrm{c}}(\Sigma; V_{\rho_i^{\mathrm{F}}})$ where the bundles

$$V_{\rho_1^{\mathrm{F}}} = (T^*\Sigma \oplus T^*\Sigma) \times \mathfrak{g}, \quad V_{\rho_0^{\mathrm{F}}} = (\Sigma \oplus \Sigma) \times \mathfrak{g}$$

are equipped with their canonical hermitian structures inherited from the inverse Riemannian metric on Σ and the Killing form $\mathfrak{k}$.

As in [Fur2, P], it can be checked that the corresponding Cauchy problems are well-posed and that the operators $G_{i\Sigma}$ (defined using the ρ_i^{F} data) can be written as

$$G_{1\Sigma} = \mathrm{i}^{-1} \begin{pmatrix} 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ -\mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix}, \quad G_{0\Sigma} = \mathrm{i}^{-1} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{2.17}$$

We denote by $\bar{d}_\Sigma, \bar{\delta}_\Sigma$ the covariant differential and co-differential on Σ respective to $\bar{A}_\Sigma := \imath^* \bar{A}$, i.e.

$$\bar{d}_\Sigma := d_\Sigma + \bar{A}_\Sigma \wedge \cdot : \ \mathcal{E}_{\mathrm{c}}^p(\Sigma) \otimes \mathfrak{g} \to \mathcal{E}_{\mathrm{c}}^{p+1}(\Sigma) \otimes \mathfrak{g},$$
$$\bar{\delta}_\Sigma := \bar{d}_\Sigma^* : \ \mathcal{E}_{\mathrm{c}}^p(\Sigma) \otimes \mathfrak{g} \to \mathcal{E}_{\mathrm{c}}^{p-1}(\Sigma) \otimes \mathfrak{g},$$

where now the adjoint is computed using the inverse metric on Σ and the Killing form $\mathfrak{k}$.

The ρ_i^{F} Cauchy data are particularly useful to express the operators K_Σ and $K_\Sigma^\dagger$. In the lemma below we still denote by $K_\Sigma^{(\dagger)}$ the maps $R_{1\mathrm{F}} \circ K_\Sigma^{(\dagger)} \circ R_{0\mathrm{F}}^{-1}$, where $R_{i\mathrm{F}} = \rho_i^{\mathrm{F}} \circ \rho_i^{-1}$ for $i = 0, 1$ are the transition maps from standard to adapted Cauchy data. The map $R_{1\mathrm{F}}$ will be simply denoted by R_{F} later on. Its concrete expression is given in Lemma 4.1 below .

Lemma 2.12. *We have:*

$$K_\Sigma = \begin{pmatrix} 0 & \mathrm{i} \\ \bar{d}_\Sigma & 0 \\ 0 & 0 \\ \mathrm{i}^{-1}\mathrm{a} & 0 \end{pmatrix}, \quad K_\Sigma^\dagger = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i}\,\mathrm{a}^* & 0 & \bar{\delta}_\Sigma \end{pmatrix},$$

where $\mathrm{a} := \imath^*(n \lrcorner \bar{F}) \wedge \cdot$.

Proof. The formula for K_Σ is a routine computation. To obtain the formula for $K_\Sigma^\dagger$ we use (2.17) and (2.9). □

Using Lemma 2.12 and the identity $K_\Sigma^\dagger K_\Sigma = 0$ (Lemma 2.9, (5)), we obtain the following important identity:

$$\bar{\delta}_\Sigma \circ \mathrm{a} = \mathrm{a}^* \circ \bar{d}_\Sigma \text{ in } L(\mathcal{E}^0(\Sigma) \otimes \mathfrak{g}). \tag{2.18}$$

3. Hadamard states

In this section we discuss Hadamard states both in ordinary field theory and the subsidiary condition framework. In Subsect. 3.1 we recall basic facts on quasi-free states on complex symplectic spaces. The Hadamard condition in ordinary field theory is recalled in Subsect. 3.2. Subsect. 3.3 contains a streamlined version of the arguments in [GW], dealing with the correspondence between Hadamard states and parametrices for the Cauchy problem in the ordinary framework. In Subsect. 3.4 we consider the subsidiary gauge framework. We explain there in detail the strategy we will follow in later sections to construct Hadamard states in this case, thereby proving Thm. 1.1.

Finally in Subsect. 3.5 we explain the version of the Fulling-Narcowich-Wald deformation argument adapted to the Yang-Mills case, which we use to deduce Thm. 1.2 from Thm. 1.1.

3.1. **Quasi-free states.** Let $\mathcal{V}$ a complex vector space, $\mathcal{V}^*$ its anti-dual and $L_{\rm h}(\mathcal{V},\mathcal{V}^*)$ the space of hermitian sesquilinear forms on $\mathcal{V}$. If $q\in L_{\rm h}(\mathcal{V},\mathcal{V}^*)$ then we can define the *polynomial CCR $*$-algebra* $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$ (see eg [DG, Sect. 8.3.1]) [7]. The (complex) field operators $\mathcal{V}\ni v\mapsto\psi(v),\psi^*(v)$, which generate $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$ are anti-linear, resp. linear in v and satisfy the canonical commutation relations

$$[\psi(v),\psi(w)]=[\psi^*(v),\psi^*(w)]=0,\quad [\psi(v),\psi^*(w)]=\overline{v}qw\mathbf{1},\quad v,w\in\mathcal{V}.$$

The *complex covariances* $\Lambda^\pm\in L(\mathcal{V},\mathcal{V}^*)$ of a (gauge-invariant[8]) state ω on $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$ are defined in terms of the abstract field operators by

$$\overline{v}\Lambda^+w:=\omega\big(\psi(v)\psi^*(w)\big),\quad \overline{v}\Lambda^-w:=\omega\big(\psi^*(w)\psi(v)\big),\quad v,w\in\mathcal{V}$$

By the canonical commutation relations, one has $\Lambda^+-\Lambda^-=q$.

In what follows we will consider only *quasi-free states*, which means that they are uniquely determined by their covariances $\Lambda^\pm$ (since $\Lambda^+-\Lambda^-=q$ it suffices to know one of them).

Definition 3.1. *A pair $\Lambda^\pm$ of hermitian forms on $\mathcal{V}$ such that $\Lambda^+-\Lambda^-=q$ will be called a pair of* pseudo-covariances.

Let us recall the following characterization of covariances of quasi-free states on $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$ (cf. [AS, GW]).

Proposition 3.2. *Pseudo-covariances $\Lambda^\pm\in L_{\rm h}(\mathcal{V},\mathcal{V}^*)$ are covariances of a (bosonic, gauge-invariant) quasi-free state on $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$ iff*

$$\Lambda^\pm\geq 0. \tag{3.1}$$

If q is non-degenerate then this is equivalent to $\pm qc^\pm\geq 0$, where $c^\pm:=\pm q^{-1}\Lambda^\pm$. If moreover, $(c^+)^2=c^+$ on the completion of $\mathcal{V}$ w.r.t. $\Lambda^++\Lambda^-$, then the associated state is pure.

Hence a pair of (pseudo-)covariances $\Lambda^\pm\in L_{\rm h}(\mathcal{V},\mathcal{V}^*)$ uniquely define a (pseudo-)state on $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$, where by pseudo-state we mean a $*-$invariant linear functional on $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$.

Definition 3.3. *A (bosonic) charge reversal on $(\mathcal{V},q)$ is an anti-linear operator κ on $\mathcal{V}$ such that $\kappa^2=\pm\mathbf{1}$ and $\kappa^*q\kappa=-\overline{q}$, where the bar stands for ordinary complex conjugation. A quasi-free state on $\mathrm{CCR}^{\rm pol}(\mathcal{V},q)$ with two-point function Λ^+ is said to be invariant under charge reversal if $\Lambda^-=-\kappa^*\overline{\Lambda^+}\kappa$. If q is non-degenerate then this is equivalent to $c^-=-\kappa c^+\kappa$.*

Clearly, if Λ^+ is a covariance of a quasi-free state invariant under charge conjugation then one of the two conditions in (3.1) implies the other. Note that one can always obtain a state invariant under charge reversal by taking $\frac{1}{2}(\Lambda^+-\kappa^*\overline{\Lambda^-}\kappa)$ instead of Λ^+. For this reason, we will disregard this issue and consider states that need not be invariant under a charge reversal (contrarily to most of the literature on Hadamard states).

3.2. **Hadamard two-point functions.**

[7] See [GW, W2] for remarks on the transition between real and complex vector space terminology.

[8] Here by gauge invariance we mean invariance w.r.t. transformations generated by the complex structure. We always consider states that are gauge-invariant in this sense and not mention it anymore in order to avoid confusion with other possible meanings of gauge invariance.

3.2.1. *Two-point functions.* Let $D \in \mathrm{Diff}^m(M;V)$ be prenormally hyperbolic and formally selfadjoint for $(\cdot|\cdot)_V$. Let us introduce the assumptions:

$$\begin{aligned} &i) \quad \lambda^\pm : \Gamma_{\mathrm{c}}(M;V) \to \Gamma(M;V) \\ &ii) \quad \lambda^\pm = \lambda^{\pm *} \text{ for } (\cdot|\cdot)_V \text{ on } \Gamma_{\mathrm{c}}(M;V), \\ &iii) \quad \lambda^+ - \lambda^- = \mathrm{i}G, \\ &iv) \quad D\lambda^\pm = \lambda^\pm D = 0, \end{aligned} \tag{3.2}$$

$$\lambda^\pm \geq 0 \text{ for } (\cdot|\cdot)_V \text{ on } \Gamma_{\mathrm{c}}(M;V). \tag{3.3}$$

Note that (3.2) implies that $\lambda^\pm : \Gamma'(M;V) \to \Gamma'_{\mathrm{c}}(M;V)$. Let us set

$$\overline{u}\Lambda^\pm v := (u|\lambda^\pm v)_V, \ u, v \in \Gamma_{\mathrm{c}}(M;V).$$

If (3.2) hold, then $\Lambda^\pm$ define a pair of complex pseudo-covariances on the phase space $(\mathcal{V}, q)$ defined in (2.2), hence define a unique quasi-free pseudo-state on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{V}, q)$. If additionally (3.3) holds, they are (true) covariances, and define a unique quasi-free state on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{V}, q)$.

Definition 3.4. *A pair of maps $\lambda^\pm : \Gamma_{\mathrm{c}}(M;V) \to \Gamma(M;V)$ satisfying (3.2) will be called a pair of* spacetime two-point functions*.*

3.2.2. *Hadamard condition.* The (primed) wave front set of $\lambda^\pm$ is by definition the (primed) wave front set of its Schwartz kernel. For $x \in M$, we denote $V_x^{\pm *}$ the *positive/negative energy cones*, dual future/past light cones and set

$$\mathcal{N}^\pm := \{(x,\xi) \in T^*_x M \setminus \{0\} : \ g^{\mu\nu}(x)\xi_\mu\xi_\nu = 0, \ \xi \in V_x^{\pm *}\}, \ \mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-.$$

Definition 3.5. *A pair of two-point functions $\lambda^\pm$ satisfying (3.2) is* Hadamard *if*

$$\mathrm{WF}'(\lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm. \tag{Had}$$

This form of the Hadamard condition is taken from [SV, Hol], see also [W2] for a review on the equivalent formulations.

Remark 3.6. *Assume that there exists an anti-linear operator $\kappa : \Gamma(M;V) \to \Gamma(M;V)$ with $\kappa^2 = \pm\mathbf{1}$ and $D\kappa = \kappa D$. It follows that κ induces a charge reversal on $(\mathcal{V}, q)$ defined in (2.2). If moreover κ has the property that*

$$\kappa(fu) = \overline{f}\kappa u, \ f \in \Gamma(M), \ u \in \Gamma(M;V)$$

then it is easy to see that

$$\mathrm{WF}(\kappa u) = \overline{\mathrm{WF}(u)}, \ u \in \Gamma'_{\mathrm{c}}(M;V)$$

where

$$\overline{\Gamma} := \{(x,-\xi) : \ (x,\xi) \in \Gamma\}, \ \textit{for } \Gamma \subset T^*M.$$

If $\lambda^\pm$ are the two-point functions of a (pseudo-)state ω invariant under the charge reversal κ, then the relation between λ^+ and λ^- shows that the two conditions in (Had) are equivalent. Most of the literature on Hadamard states deals only with the charge-reversal invariant case, see however [Hol, W2].

3.3. **Correspondence between Hadamard states and parametrices.** One of the methods to impose (μsc) is to construct a sufficiently explicit parametrix for the Cauchy problem on a given Cauchy surface Σ, as was done in [GW] for the scalar Klein-Gordon equation. In the present subsection, we will derive the precise relation between two-point functions of Hadamard states in ordinary field theory and parametrices.

3.3.1. *Two-point functions on a Cauchy surface.* Let $D \in \mathrm{Diff}^m(M;V)$ be prenormally hyperbolic, formally selfadjoint on $\Gamma_{\mathrm{c}}(M;V)$ and Cauchy hyperbolic for some map ρ as in 2.3.1.

Lemma 3.7. *The operator ρG extends continuously to a surjection*

$$\rho G : \Gamma'(M;V) \to \Gamma'(\Sigma;V_\rho)$$

with $\mathrm{Ker}\rho G|_{\Gamma'} = \mathrm{Ran}D|_{\Gamma'}$.

Proof. To show that $\rho G : \Gamma'(M;V) \to \Gamma'_{\mathrm{c}}(\Sigma;V_\rho)$ is well-defined and continuous, it suffices to use the well-known fact that

$$\mathrm{WF}'(G) \subset \mathcal{N} \times \mathcal{N} \tag{3.4}$$

and the rules for composition of distributional kernels in terms of the wavefront set (see [Hör]). The fact that $\rho G : \Gamma'(M;V) \to \Gamma'(\Sigma;V_\rho)$ follows then from the support properties of G. To prove the surjectivity it suffices to show that the identity

$$\mathbf{1} = -\rho G \rho^* G_\Sigma \text{ valid on } \Gamma_{\mathrm{c}}(\Sigma;V_\rho)$$

extends to $\Gamma'(\Sigma;V_\rho)$. This is indeed the case because G_Σ is a differential operator (this is usually shown using Green's formula) and consequently acts continuously from Γ' to Γ', hence $\rho^* G_\Sigma : \Gamma'(\Sigma;V_\rho) \to \Gamma'(M;V)$.

The fact that $\mathrm{Ker}\rho G|_{\Gamma'} = \mathrm{Ker}G|_{\Gamma'} = \mathrm{Ran}D|_{\Gamma'}$ follows by the same proof as before. □

Let us introduce the assumptions:

$$\begin{aligned} &i) \quad \lambda^\pm_\Sigma : \Gamma_{\mathrm{c}}(\Sigma;V_\rho) \to \Gamma(\Sigma;V_\rho), \\ &ii) \quad \lambda^\pm_\Sigma = (\lambda^\pm_\Sigma)^* \text{ for } (\cdot|\cdot)_{V_\rho}, \\ &iii) \quad \lambda^+_\Sigma - \lambda^-_\Sigma = \mathrm{i}G_\Sigma. \end{aligned} \tag{3.5}$$

Definition 3.8. *A pair of maps $\lambda^\pm_\Sigma$ satisfying (3.5) will be called a pair of* Cauchy surface two-point functions.

In the proposition below we recall a well known bijection between spacetime and Cauchy surface two-point functions.

Proposition 3.9. *The maps:*

$$\lambda^\pm_\Sigma \mapsto \lambda^\pm := (\rho G)^* \lambda^\pm_\Sigma (\rho G), \tag{3.6}$$

and

$$\lambda^\pm \mapsto \lambda^\pm_\Sigma := (\rho^* G_\Sigma)^* \lambda^\pm (\rho^* G_\Sigma) \tag{3.7}$$

are bijective and inverse from one another. Moreover, $\lambda^\pm$ are the two-point functions of a quasi-free state iff

$$\lambda^\pm_\Sigma \geq 0 \text{ for } (\cdot|\cdot)_{V_\rho}.$$

Proof. (1): let $\lambda^\pm_\Sigma$ satisfy (3.5). Clearly $\lambda^\pm$ is well defined as a map from $\Gamma_{\mathrm{c}}(M;V)$ to $\Gamma'_{\mathrm{c}}(M;V)$. If $u \in \Gamma_{\mathrm{c}}(M;V)$, then $f^\pm := \lambda^\pm_\Sigma \rho G u \in \Gamma(\Sigma;V_\rho)$, hence $\mathrm{WF}(\rho^* f^\pm) \subset N^*_\Sigma M$, the *conormal bundle* to Σ in M. We use now (3.4), the fact that Σ is non-characteristic i.e. $N^*_\Sigma M \cap \mathcal{N} = \emptyset$ and standard arguments with wave front sets (see [Hör]) to obtain that $\lambda^\pm u = -G\rho^* f^\pm \in \Gamma(M;V)$. The other conditions in (3.2) are straightforward.

(2): let $\lambda^\pm$ satisfies (3.2). Since $\lambda^\pm D = 0$, we have $\mathrm{WF}'(\lambda^\pm) \subset T^*M \times \mathcal{N}$ which implies that $\lambda^\pm(\rho^* G_\Sigma) : \Gamma_{\mathrm{c}}(\Sigma;V_\rho) \to \Gamma(M;V)$. Next we use that G_Σ is a differential operator hence $G_\Sigma : \Gamma(\Sigma;V_\rho) \to \Gamma(\Sigma;V_\rho)$ to obtain that $\lambda^\pm_\Sigma : \Gamma_{\mathrm{c}}(\Sigma;V_\rho) \to \Gamma(\Sigma;V_\rho)$. The other conditions in (3.5) are straightforward.

The fact that the two maps are inverse from each other follows from $\rho U = \rho G^* \rho^* G_\Sigma = \mathbf{1}$. The last statement about positivity is obvious. □

Prop. 3.9 leads to the following definition:

Definition 3.10. *A pair $\lambda^\pm_\Sigma$ of Cauchy surface two-point functions is* Hadamard *if the associated spacetime two-point functions $\lambda^\pm$ are Hadamard.*

3.3.2. *Hadamard two-point functions and parametrices.* Let us now discuss the link between Hadamard two-point functions and parametrices for the Cauchy problem. Let $\lambda^\pm$ be the two-point functions of a state. We set[9]

$$H^0(\Sigma;V_\rho) := \big(\Gamma_{\rm c}(\Sigma;V_\rho)\big)^{\rm cpl} \tag{3.8}$$

where the completion is taken w.r.t. $(\cdot|(\lambda^+_\Sigma+\lambda^-_\Sigma)\cdot)_{V_\rho}$.

Theorem 3.11. *Let $D\in\mathrm{Diff}^m(M;V)$ be prenormally hyperbolic, formally self-adjoint and Cauchy hyperbolic. Let $\lambda^\pm$ be the two-point functions of a quasi-free Hadamard state and define*

$$U^\pm := Uc^\pm : \Gamma'(\Sigma;V_\rho)\to\Gamma'_{\rm c}(M;V),$$

where $c^\pm=\pm(\mathrm{i}G_\Sigma)^{-1}\lambda^\pm_\Sigma$. Then

(1) $U^+ + U^- = U$.
(2a) *The spaces* $\mathrm{Ker}\,U^+|_{H^0}$ *and* $\mathrm{Ker}\,U^-|_{H^0}$ *are orthogonal for q_Σ.*
(2b) *if the state is pure then*

$$H^0(\Sigma;V_\rho)=\mathrm{Ker}\,U^+|_{H^0}\oplus\mathrm{Ker}\,U^-|_{H^0}.$$

(3) $\pm\mathrm{i}G_\Sigma$ *is positive on* $\mathrm{Ker}\,U^\pm|_{H^0}$ *for $(\cdot|\cdot)_{V_\rho}$.*
(4) $\mathrm{WF}(U^\pm f)\subset\mathcal{N}^\pm$ *for all $f\in\Gamma'(\Sigma;V_\rho)$.*

Proof. (1) follows from $c^+ + c^- = \mathbf{1}$. To prove (2a) we note that for $u^\pm\in\mathrm{Ker}c^\mp$ and q_Σ defined in (2.4) one has:

$$\overline{(u^+ + zu^-)}q_\Sigma u^+ = \overline{(u^+ + zu^-)}q_\Sigma c^+(u^+ + zu^-)\in\mathbb{R},\ \forall\, z\in\mathbb{C},$$

which implies that $\overline{u^-}q_\Sigma u^+ = 0$. (2b) follows from the fact that $c^\pm$ are bounded projections on H^0 if the state ω is pure, (3) follows from the conditions $\lambda^\pm_\Sigma\geq 0$. To show (4), observe that for all $u\in\Gamma'(M;V)$

$$\lambda^+ u = (\rho G)^*\lambda^+_\Sigma\rho G u = U^+\rho G u.$$

Thus, the Hadamard condition entails that $\mathrm{WF}(U^+\rho G u)\subset\mathcal{N}^+$. Since ρG is surjective this means $\mathrm{WF}(U^+ f)\subset\mathcal{N}^+$ for all $f\in\Gamma'(\Sigma;V_\rho)$. The proof for U^- is analogous. □

To obtain a converse statement, we need spaces that can replace the space $H^0(\Sigma;V_\rho)$, and that will allow to compose operators.

To this end, suppose $\mathcal{H}(\Sigma;V_\rho)$ is a topological vector space s.t.

$$\Gamma_{\rm c}(\Sigma;V_\rho)\subset\mathcal{H}(\Sigma;V_\rho)\subset\Gamma(\Sigma;V_\rho),$$

with continuous and dense embedings. Examples of such spaces are (intersections of) scales of Sobolev spaces associated to a positive, elliptic pseudodifferential operator. The dual space of $\mathcal{H}(\Sigma;V_\rho)$, denoted $\mathcal{H}'(\Sigma;V_\rho)$, satisfies

$$\Gamma'(\Sigma;V_\rho)\subset\mathcal{H}'(\Sigma;V_\rho)\subset\Gamma'_{\rm c}(\Sigma;V_\rho).$$

We will denote $B^{-\infty}(\Sigma;V_\rho)$ the class of operators that map $\mathcal{H}'(\Sigma;V_\rho)$ into $\Gamma(\Sigma;V_\rho)$.

We assume that

$$G_\Sigma,\ G_\Sigma^{-1} : \mathcal{H}(\Sigma;V_\rho)\to\mathcal{H}(\Sigma;V_\rho), \tag{3.9}$$

[9]For instance, if $\lambda^\pm$ are the two-point functions of the vacuum for the scalar Klein-Gordon equation on Minkowski space then $H^0(\Sigma;V_\rho)=H^{\frac12}(\mathbb{R}^d)\oplus H^{-\frac12}(\mathbb{R}^d)$, where $H^m(\mathbb{R}^d)$ are the usual Sobolev spaces.

which since $\mathrm{i}G_\Sigma$ is selfadjoint for $(\cdot|\cdot)_{V_\rho}$ implies of course

$$G_\Sigma,\ G_\Sigma^{-1} : \mathcal{H}'(\Sigma; V_\rho) \to \mathcal{H}'(\Sigma; V_\rho),$$

The corresponding natural assumption for a pair of Cauchy surface two-point functions $\lambda_\Sigma^\pm$ is

$$\begin{aligned} \lambda_\Sigma^\pm &: \mathcal{H}(\Sigma; V_\rho) \to \mathcal{H}(\Sigma; V_\rho), \\ \lambda_\Sigma^\pm &: \mathcal{H}'(\Sigma; V_\rho) \to \mathcal{H}'(\Sigma; V_\rho), \end{aligned} \tag{3.10}$$

where as before one of the above conditions implies the other.

Theorem 3.12. *Assume that there exist operators $U^\pm : \mathcal{H}'(\Sigma; V_\rho) \to \Gamma'_{\mathrm{c}}(M; V)$ such that $U^\pm : \mathcal{H}(\Sigma; V_\rho) \to \Gamma(M; V)$ and*

$$DU^\pm = 0, \quad U^+ + U^- = U,$$

up to remainders that map $\mathcal{H}'(\Sigma; V_\rho) \to \Gamma(M; V)$.

Assume moreover that

(1) *the spaces $\operatorname{Ker} U^+|_{\mathcal{H}}$ and $\operatorname{Ker} U^-|_{\mathcal{H}}$ are orthogonal for q_Σ and*

$$\mathcal{H}(\Sigma; V_\rho) = \operatorname{Ker} U^+|_{\mathcal{H}} \oplus \operatorname{Ker} U^-|_{\mathcal{H}}.$$

(2) $\mathrm{WF}(U^\pm f) \subset \mathcal{N}^\pm$ *for all $f \in \Gamma'(\Sigma; V_\rho)$.*

Let $c^\pm : \mathcal{H}(\Sigma; V_\rho) \to \mathcal{H}(\Sigma; V_\rho)$ be the projection s.t.

$$\operatorname{Ran} c^\pm = \operatorname{Ker} U^\mp|_{\mathcal{H}}, \quad \operatorname{Ker} c^\pm = \operatorname{Ker} U^\pm|_{\mathcal{H}}.$$

Then $\lambda_\Sigma^\pm := \pm\mathrm{i}G_\Sigma c^\pm$ are Hadamard Cauchy surface two-point functions. If moreover

(3) $\pm\mathrm{i}G_\Sigma c^\pm \geq 0$ *for $(\cdot|\cdot)_{V_\rho}$,*

then $\lambda_\Sigma^\pm$ are the Cauchy surface two-point functions of a Hadamard state.

Proof. (1) implies $c^+ + c^- = \mathbf{1}$. By duality, $c^\pm : \mathcal{H}'(\Sigma, V_\rho) \to \mathcal{H}'(\Sigma, V_\rho)$. Next, for all $f \in \Gamma'(\Sigma; V_\rho)$ we have:

$$Uc^\pm f = (U^+ + U^-)c^\pm f = U^\pm c^\pm f = U^\pm(\mathbf{1} - c^\mp)f = U^\pm f \mod C^\infty.$$

Therefore,

$$\lambda^\pm u = \pm\mathrm{i}Uc^\pm\rho G u = \pm\mathrm{i}U^\pm\rho G u \mod C^\infty, \quad u \in \Gamma'(M; V).$$

Let $a^\pm$ be a properly supported pseudodifferential operator, non-characteristic on $\mathcal{N}^\pm$ and with essential support disjoint from $\mathcal{N}^\mp$. From (2) and the relation above it follows that $a^\pm\lambda^\pm$ is smoothing, hence $\mathrm{WF}'(\lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}$. Since $\lambda^\pm = (\lambda^\pm)^*$ this implies $\mathrm{WF}'(\lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$. This proves the first statement of the proposition. The second statement is obvious. □

Thm. 3.12 allows to simplify the construction of Hadamard states for the scalar Klein-Gordon equation given in [GW] — it is in fact not difficult to check properties (1)-(3) directly from the construction of the parametrix therein. The space $\mathcal{H}(\Sigma; V_\rho)$ is taken there to be the intersection of usual Sobolev spaces on $\mathbb{R}^d$. The next proposition is an abstract version of a result from [GW].

Proposition 3.13. *Assume that $\lambda_\Sigma^\pm, \tilde{\lambda}_\Sigma^\pm$ satisfy (3.10) and are the Cauchy surface two-point functions of two quasi-free states, and suppose the first of them is pure and Hadamard. Then the other one is Hadamard iff*

$$c^-\tilde{c}^+c^-,\ c^+\tilde{c}^+c^-,\ c^+\tilde{c}^-c^+ \in B^{-\infty}(\Sigma; V_\rho) \tag{3.11}$$

or, equivalently, iff

$$\tilde{c}^\pm - c^\pm \in B^{-\infty}(\Sigma; V_\rho) \tag{3.12}$$

Proof. $\Leftarrow$: if (3.11) or (3.12) holds then

$$U\tilde{c}^{\pm} - Uc^{\pm}\tilde{c}^{\pm}c^{\pm} : \ \mathcal{H}'(\Sigma; V_\rho) \to \Gamma(M; V).$$

By Thm. 3.11, it follows that $\mathrm{WF}(U\tilde{c}^{\pm} f) \subset \mathcal{N}^{\pm}$ for all $f \in \Gamma'(\Sigma; V_\rho)$ and consequently $\tilde{\lambda}$ is Hadamard by Thm. 3.12.

$\Rightarrow$: for all $f \in \Gamma'(\Sigma; V_\rho)$,

$$Uc^{-}\tilde{c}^{+}c^{\pm}f = U\tilde{c}^{+}c^{\pm}f - Uc^{+}\tilde{c}^{+}c^{\pm}f.$$

By Thm. 3.11, the wave front set of the LHS is contained in $\mathcal{N}^{-}$, and the wave front set of the RHS is contained in $\mathcal{N}^{+}$. This shows that the operators $Uc^{-}\tilde{c}^{+}c^{\pm}$ are smoothing, therefore $c^{-}\tilde{c}^{+}c^{\pm} = \rho Uc^{-}\tilde{c}^{+}c^{\pm}$ are smoothing. The assertion $c^{+}\tilde{c}^{-}c^{+} \in B^{-\infty}(\Sigma; V_\rho)$ is shown similarly.

Moreover, (3.11) entails that

$$\begin{aligned}\tilde{c}^{+} - c^{+} &= (c^{+} + c^{-})\tilde{c}^{+}(c^{+} + c^{-}) - c^{+} = c^{+}\tilde{c}^{+}c^{+} - c^{+} \\ &= c^{+}(\tilde{c}^{+} - \mathbf{1})c^{+} = -c^{+}\tilde{c}^{-}c^{+} \mod B^{-\infty}(\Sigma; V_\rho),\end{aligned}$$

where the last term belongs to $B^{-\infty}(\Sigma; V_\rho)$. This proves (3.12). □

Corollary 3.14. *If $\lambda_{\Sigma}^{\pm}$ satisfying (3.10) are Hadamard Cauchy surface two-point functions then so are $v^{*}\lambda_{\Sigma}^{\pm}v$ for any $v \in \mathbf{1} + B^{-\infty}(\Sigma; V_\rho)$ s.t. $v^{*}G_{\Sigma}v = G_{\Sigma}$.*

3.4. **Hadamard states in the subsidiary condition formalism.**

3.4.1. *Hadamard states in the subsidiary condition formalism.* Definition 3.5 can be generalized to gauge theories in the 'subsidiary condition' framework. Recall that to a given non-hyperbolic operator P we have assigned a hyperbolic operator D_1 and introduced phase spaces $\mathcal{V}_P = \mathrm{Ker}K^{*}/\mathrm{Ran}P$, $\mathcal{V}_1 = \Gamma_{\mathrm{c}}/\mathrm{Ran}D$. We consider the following definition, which generalizes the one used by [FP, FS].

Definition 3.15. *A quasi-free state ω on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{V}_P, q_P)$ is* Hadamard *if there exists Hadamard two-point functions $\lambda_1^{\pm}$ on $\Gamma_{\mathrm{c}}(M; V_1)$ such that the complex covariances of ω are given by:*

$$\overline{[u]}\Lambda^{\pm}[v] = (u|\lambda_1^{\pm}v)_V, \ u, v \in \mathrm{Ker}K^{*}|_{\Gamma_{\mathrm{c}}},$$

where $\mathrm{Ker}K^{*}|_{\Gamma_{\mathrm{c}}} \ni u \mapsto [u] \in \mathrm{Ker}K^{*}/\mathrm{Ran}P$ is the canonical map.

We say that $\lambda_1^{\pm}$ are the two-point functions of the Hadamard state ω on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{V}_P, q_P)$. The following lemma is straightforward.

Lemma 3.16. *$\lambda_1^{\pm} : \Gamma_{\mathrm{c}}(M; V_1) \to \Gamma(M; V_1)$ are the two-point functions of a Hadamard state on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{V}_P, q_P)$ if:*

$$\begin{aligned}&(\mu\mathrm{sc}) \quad D_1\lambda_1^{\pm} = \lambda_1^{\pm}D_1 = 0, \quad \mathrm{WF}'(\lambda_1^{\pm}) \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}, \\ &(\mathrm{g.i.}) \quad (\lambda_1^{\pm})^{*} = \lambda_1^{\pm} \ \textit{and} \ \lambda_1^{\pm} : \ \mathrm{Ran}\, K|_{\Gamma_{\mathrm{c}}} \to \mathrm{Ran}\, K|_{\Gamma'_{\mathrm{c}}}, \\ &(\mathrm{pos}) \quad \lambda_1^{\pm} \geq 0 \ \textit{on} \ \mathrm{Ker}\, K^{*}|_{\Gamma_{\mathrm{c}}}.\end{aligned} \tag{3.13}$$

It is worth mentioning that in perturbative interacting Quantum Field Theory, some constructions seem to survive if one replaces gauge-invariance (g.i.) by a condition 'modulo smooth terms' [Rej]. Nevertheless, (μsc) and positivity (pos) are still essential (cf. [DF] and [Hol2, Sec. 4.1.2] for discussion on the latter), and gauge-invariance (g.i.) is needed to have a reasonable non-interacting theory, we will thus aim at solving all of them when possible.

We now discuss gauge-invariance and positivity on the level of Cauchy surface two-point functions $\lambda_{1\Sigma}^{\pm}$. We explain the main steps of the construction of Hadamard states for the linearized Yang-Mills equations, leading to a proof of Thm. 1.1, which will be completed in Sect. 8.

The construction is somewhat complicated by the need to justify that various operators can be composed. These technical points can be bypassed on the first reading.

We fix spaces $\mathcal{H}(\Sigma;V_{\rho_i})$, $i=0,1$ as in Subsect. 3.3 and assume that $G_{i\Sigma}$ satisfy (3.9). The corresponding assumption on K_Σ is:

$$\begin{aligned} &K_\Sigma : \mathcal{H}(\Sigma;V_{\rho_0}) \to \mathcal{H}(\Sigma;V_{\rho_1}),\\ &K_\Sigma : \mathcal{H}'(\Sigma;V_{\rho_0}) \to \mathcal{H}'(\Sigma;V_{\rho_1}). \end{aligned} \tag{3.14}$$

The operator $K_\Sigma^\dagger$ has then the same properties as K_Σ.

3.4.2. *Cauchy surface two-point functions.* Assume that we are given Cauchy surface two-point functions $\lambda_{i\Sigma}^\pm$ for $i=0,1$ satisfying (3.5) and (3.10) for $V=V_i$.

To $\lambda_{i\Sigma}^\pm$ we associate as before operators $c_i^\pm := \pm \mathrm{i} G_{i\Sigma}^{-1}\lambda_{i\Sigma}^\pm$ which by the above assumptions satisfy:

$$\begin{aligned} &i) \quad c_i^\pm : \mathcal{H}(\Sigma;V_{\rho_i}) \to \mathcal{H}(\Sigma;V_{\rho_i}),\\ &ii) \quad c_i^\pm : \mathcal{H}'(\Sigma;V_{\rho_i}) \to \mathcal{H}'(\Sigma;V_{\rho_i}),\\ &iii) \quad c_i^+ + c_i^- = \mathbf{1}. \end{aligned} \tag{3.15}$$

Conditions (pos), (g.i.) on $\lambda_1^\pm$ in (3.13) can be rewritten as

$$\begin{aligned} &\text{(pos)} \quad \lambda_{1\Sigma}^\pm = \pm \mathrm{i} G_{1\Sigma} c_1^\pm \geq 0 \text{ for } (\cdot|\cdot)_{V_{\rho_1}} \text{ on } \mathrm{Ker} K_\Sigma^\dagger,\\ &\text{(g.i.)} \quad (c_1^\pm)^\dagger = c_1^\pm, \quad c_1^\pm : \mathrm{Ran} K_\Sigma \to \mathrm{Ran} K_\Sigma. \end{aligned}$$

Note that the last condition can be rewritten as:

$$\text{(g.i.)} \quad (c_1^\pm)^\dagger = c_1^\pm, \quad c_1^\pm : \mathrm{Ker} K_\Sigma^\dagger \to \mathrm{Ker} K_\Sigma^\dagger.$$

Let us now set:

$$c_1^\pm K_\Sigma - K_\Sigma c_0^\pm =: \pm R_{-\infty}. \tag{3.16}$$

Condition (g.i.) is clearly satisfied if $R_{-\infty}=0$.

The operators $c_i^\pm$ are obtained from parametrices $U_i^\pm$ for the Cauchy problems for D_i as in Thm. 3.12, in order to enforce the Hadamard condition for $\lambda_1^\pm$. The construction of parametrices done in Sect. 5 relies on pseudodifferential calculus, from which we will only be able to obtain that $R_{-\infty}$ is smoothing.

Nevertheless, it is possible to ensure (g.i.) by subtracting to $c_1^\pm$ a term $c_{1\,\mathrm{reg}}^\pm$, which is expected to be smoothing, and hence will not invalidate the Hadamard property.

The method works as follows.

3.4.3. *Construction of a projection.* Let Π be a projection s.t.

$$\begin{aligned} &\mathrm{Ker}\,\Pi = \mathrm{Ran}\, K_\Sigma,\\ &\Pi : \mathcal{H}(\Sigma;V_{\rho_1}) \to \mathcal{H}(\Sigma;V_{\rho_1}),\\ &\Pi : \mathcal{H}'(\Sigma;V_{\rho_1}) \to \mathcal{H}'(\Sigma;V_{\rho_1}). \end{aligned} \tag{3.17}$$

Clearly $\Pi^\dagger$ has the same mapping properties as Π. Moreover one has:

$$\mathrm{Ran}\Pi^\dagger = \mathrm{Ker} K_\Sigma^\dagger, \quad \mathrm{Ran}(\mathbf{1}-\Pi) = \mathrm{Ran} K_\Sigma, \quad \mathrm{Ker}(\mathbf{1}-\Pi^\dagger) = \mathrm{Ker} K_\Sigma^\dagger. \tag{3.18}$$

Since $\mathrm{Ran} K_\Sigma \subset \mathrm{Ker} K_\Sigma^\dagger$ we also have:

$$\Pi^\dagger K_\Sigma = K_\Sigma, \ K_\Sigma^\dagger \Pi = K_\Sigma^\dagger. \tag{3.19}$$

3.4.4. *Construction of a right inverse to K_Σ.* Let also $B : \Gamma_{\rm c}(\Sigma; V_{\rho_1}) \to \Gamma(\Sigma; V_{\rho_0})$ an operator such that

$$K_\Sigma B = \mathbf{1} - \Pi, \text{ and hence } B^\dagger K_\Sigma^\dagger = \mathbf{1} - \Pi^\dagger. \tag{3.20}$$

The operator B is typically *unbounded* from $\mathcal{H}(\Sigma; V_{\rho_1})$ to $\mathcal{H}(\Sigma; V_{\rho_0})$, because of infrared problems. To control its unboundedness, we introduce a smooth positive function $\langle x\rangle : \Sigma \to \mathbb{R}$ and still denote by $\langle x\rangle$ the operator of multiplication by $\langle x\rangle$, acting on $\Gamma(\Sigma; V_{\rho_i})$. If Σ is compact the weight is unnecessary and one can take $\langle x\rangle = \mathbf{1}$.

We assume that:

$$\begin{aligned} &i) \quad \langle x\rangle G_{i\Sigma}\langle x\rangle^{-1} : \mathcal{H}(\Sigma; V_{\rho_i}) \to \mathcal{H}(\Sigma; V_{\rho_i}),\ i = 0, 1,\\ &ii) \quad \langle x\rangle^{-1} K_\Sigma \langle x\rangle : \mathcal{H}(\Sigma; V_{\rho_0}) \to \mathcal{H}(\Sigma; V_{\rho_1}),\\ &iii) \quad \langle x\rangle^{-1} c_0^\pm \langle x\rangle : \mathcal{H}(\Sigma; V_{\rho_0}) \to \mathcal{H}(\Sigma; V_{\rho_0}), \end{aligned} \tag{3.21}$$

Concerning the operator B we assume that:

$$\begin{aligned} &B : \mathcal{H}(\Sigma; V_{\rho_1}) \to \langle x\rangle \mathcal{H}(\Sigma; V_{\rho_0}),\\ &B : \mathcal{H}'(\Sigma; V_{\rho_1}) \to \langle x\rangle \mathcal{H}'(\Sigma; V_{\rho_0}), \end{aligned} \tag{3.22}$$

Theorem 3.17. *Let $c_i^\pm$, Π, B be as above. Let us set:*

$$\tilde{c}_1^\pm := \Pi^\dagger c_1^\pm \Pi + B^\dagger c_0^\pm K_\Sigma^\dagger + K_\Sigma c_0^\pm B,$$

$$c_{1\,\rm reg}^\pm := \pm(B^\dagger R_{-\infty}^\dagger + \Pi^\dagger R_{-\infty} B),$$

$$\tilde{\lambda}_{1\Sigma}^\pm := \pm {\rm i} G_{1\Sigma} \tilde{c}_1^\pm.$$

Then:

(1) $\tilde{c}_1^\pm : \langle x\rangle^{-1}\mathcal{H}(\Sigma; V_{\rho_1}) \to \langle x\rangle \mathcal{H}(\Sigma; V_{\rho_1})$, *hence* $\tilde{c}_1^\pm : \Gamma_{\rm c}(\Sigma; V_{\rho_1}) \to \Gamma(\Sigma; V_{\rho_1})$.

(2) *One has:*

$$\begin{aligned} &i) \quad (\tilde{c}_1^\pm)^\dagger = \tilde{c}_1^\pm,\\ &ii) \quad \tilde{c}_1^+ + \tilde{c}_1^- = \mathbf{1},\\ &iii) \quad \tilde{c}_1^\pm : {\rm Ker} K_\Sigma^\dagger \to {\rm Ker} K_\Sigma^\dagger,\\ &iv) \quad (f|\tilde{\lambda}_{1\Sigma}^\pm g)_{V_{\rho_1}} = (\Pi f|\lambda_{1\Sigma}^\pm \Pi g)_{V_{\rho_1}},\ f, g \in {\rm Ker} K_\Sigma^\dagger,\\ &v) \quad c_1^\pm = \tilde{c}_1^\pm + c_{1\,\rm reg}^\pm, \end{aligned}$$

in particular $\tilde{\lambda}_{1\Sigma}^\pm$ satisfy (g.i.).

(3) *If the projection Π is such that*

$$\lambda_{1\Sigma}^\pm \geq 0 \text{ on } \Pi {\rm Ker} K_\Sigma^\dagger, \tag{3.23}$$

then $\tilde{\lambda}_{1\Sigma}^\pm$ satisfy also (pos).

(4) *If moreover*

$$c_{1\,\rm reg}^\pm : \Gamma_{\rm c}'(\Sigma; V_{\rho_1}) \to \Gamma(\Sigma; V_{\rho_1})$$

and $\lambda_{1\Sigma}^\pm$ are Hadamard, then $\tilde{\lambda}_{1\Sigma}^\pm$ are Hadamard.

Proof. Let us first prove (1). Clearly $\Pi^\dagger c_1^\pm \Pi : \mathcal{H}(\Sigma; V_{\rho_1}) \to \mathcal{H}(\Sigma; V_{\rho_1})$, by (3.15), (3.17). Next we obtain that $K_\Sigma c_0^\pm B : \mathcal{H}(\Sigma; V_{\rho_1}) \to \langle x\rangle \mathcal{H}(\Sigma; V_{\rho_1})$, by (3.22), (3.21). Using the same assumptions and duality we obtain that $B^\dagger c_0^\pm K_\Sigma^\dagger : \langle x\rangle^{-1}\mathcal{H}(\Sigma; V_{\rho_1}) \to \mathcal{H}(\Sigma; V_{\rho_1})$.

Let us now prove (2). *i*) is easy. To prove *ii*) we write

$$\begin{aligned}\tilde{c}_1^+ + \tilde{c}_1^- &= \Pi^\dagger\Pi + B^\dagger K_\Sigma^\dagger + K_\Sigma B\\ &= \Pi^\dagger\Pi + B^\dagger K_\Sigma^\dagger\Pi + K_\Sigma B\\ &= \Pi^\dagger\Pi + (\mathbf{1}-\Pi^\dagger)\Pi + (\mathbf{1}-\Pi) = \mathbf{1},\end{aligned}$$

using successively $c_i^+ + c_i^- = \mathbf{1}$, (3.19), and (3.20). *iii*) follows from $\mathrm{Ran}\Pi^\dagger = \mathrm{Ker}K_\Sigma^\dagger$ (see (3.18)), and $\mathrm{Ran}K_\Sigma \subset \mathrm{Ker}K_\Sigma^\dagger$. *iv*) follows from the definition of $\tilde{\lambda}_{1\Sigma}^\pm$. To prove *v*) we write:

$$\begin{aligned}\tilde{c}_1^\pm &= \Pi^\dagger c_1^\pm\Pi + B^\dagger c_0^\pm K_\Sigma^\dagger + K_\Sigma c_0^\pm B\\ &= \Pi^\dagger c_1^\pm\Pi + B^\dagger c_0^\pm K_\Sigma^\dagger + \Pi^\dagger K_\Sigma c_0^\pm B\\ &= \Pi^\dagger c_1^\pm\Pi + B^\dagger K_\Sigma^\dagger c_1^\pm + \Pi^\dagger c_1^\pm K_\Sigma B \mp B^\dagger R_{-\infty}^\dagger \mp \Pi^\dagger R_{-\infty} B\\ &= \Pi^\dagger c_1^\pm\Pi + (\mathbf{1}-\Pi)^\dagger c_1^\pm + \Pi^\dagger c_1^\pm(\mathbf{1}-\Pi) \mp B^\dagger R_{-\infty}^\dagger \mp \Pi^\dagger R_{-\infty} B\\ &= c_1^\pm - c_{1\,\mathrm{reg}}^\pm.\end{aligned}$$

(3) follows from the fact that $(\cdot|\tilde{\lambda}_{1\Sigma}^\pm\cdot)_{V_{\rho_1}} = (\cdot|\lambda_{1\Sigma}^\pm\cdot)_{V_{\rho_1}}$ on $\mathrm{Ker}K_\Sigma^\dagger$.

Under the hypotheses of (4) $\lambda_{1\Sigma}^\pm - \tilde{\lambda}_{1\Sigma}$ is smoothing, hence so is $\lambda_1^\pm - \tilde{\lambda}_1^\pm$. This completes the proof of the theorem. □

Remark 3.18. *If B satisfies additionally $BK_\Sigma = \mathbf{1}$ (as will be the case in Sect. 8), then $\tilde{c}_1^\pm$ satisfies a stronger version of gauge-invariance, namely*

$$\tilde{c}_1^\pm K_\Sigma = K_\Sigma c_0^\pm. \tag{3.24}$$

Such property is needed to construct two-point functions in the BRST framework, cf. [Hol2] *for discussion in the case of Yang-Mills fields with flat background connection and* [WZ] *for generalization and computations on the Cauchy surface.*

3.5. **Reduction to ultra-static spacetimes by deformation.** A well-known argument due to Fulling, Narcowich and Wald [FNW] allows one to reduce the construction of Hadamard states for the Klein-Gordon equation to the special case of an ultra-static spacetime, and an extension of this method can be used for the Maxwell equations [FP] and Yang-Mills linearized around $\bar{A} = 0$ [Hol2].

Let us first recall the FNW deformation argument for ordinary field theory: let g, g' be Lorentzian metrics on M such that (M, g) and (M, g') are globally hyperbolic and $\Sigma \subset M$ a Cauchy surface for (M, g) and (M, g'). Assume that $g = g'$ on a causal neighborhood $O(\Sigma)$ of Σ. Assume also that $D, D' \in \mathrm{Diff}^m(M; V)$ are normally hyperbolic operators satisfying the assumptions in Subsect. 2.3 such that $D = D'$ on $O(\Sigma)$. Then by the time-slice property and Hörmander's propagation of singularities theorem, the restriction of a Hadamard state for D' to $O(\Sigma)$ yields a Hadamard state for D.

In the subsidiary condition formalism, one has to assume the existence of operators P, K, P', K' as in Hypothesis 2.3 such that $P = P'$, $K = K'$ on $O(\Sigma)$. The same argument using the gauge invariant version of the time slice property, i.e. Prop. 2.8, shows that the restriction of a Hadamard state for $(P,' K')$ to $O(\Sigma)$ yields a Hadamard state for (P, K).

In the ordinary case one fixes an ultra-static metric g_{us}, a normally hyperbolic operator D_{us}, an interpolating metric g' sharing a Cauchy surface Σ with g and a Cauchy surface Σ_{us} with g_{us}, and finally a normally hyperbolic operator D' with $D' = D$ near $O(\Sigma)$ and $D' = D_{\mathrm{us}}$ near $O(\Sigma_{\mathrm{us}})$. Applying twice the above argument, one obtains a one-to-one correspondence between

Hadamard states for D and Hadamard states for $D_{\rm us}$. The construction of Hadamard states for $D_{\rm us}$ is easier since $D_{\rm us}$ can be chosen in such way that its coefficients are independent on the time coordinate and then it admits a natural vacuum state which can be shown to be Hadamard.

3.5.1. *Deformation argument for Yang-Mills.* In the subsidiary condition formalism, it is not obvious how to find interpolating operators P', K' equal to P, K near $O(\Sigma)$ and satisfying Hypothesis 2.3 *globally* on M. Moreover even if (M, g') is ultra-static on some $O(\Sigma_{\rm us})$, this does not imply in general that P', K' will be independent on the time coordinate on $O(\Sigma_{\rm us})$.

For linearized Yang-Mills equations, it is possible to find interpolating operators P', K' if we can find a $1-$form $\bar{A}'$ on (M, g') such that $\bar{\delta}'\bar{F}' = 0$ and $\bar{A}' = \bar{A}$ near $O(\Sigma)$. This will follow in turn from a result of *global existence* of *smooth* solutions of the non-linear Yang-Mills equation, on the spacetime (M, g'), with smooth Cauchy data on Σ.

Assuming this problem is solved, there is another issue that we need to consider:

by the deformation argument explained above, to prove the existence of Hadamard states for the linearized Yang-Mills equations on (M, g), we may assume that (M, g) is ultra-static, i.e. $g = g_{\rm us} = -dt^2 + h_{ij}(x)dx^i dx^j$ on $M = \mathbb{R}_t \times \Sigma_x$.

Recall that we assume that Σ is either a compact manifold or $\Sigma = \mathbb{R}^d$. The Riemannian metric $h_{ij}(x)dx^i dx^j$ on Σ can be chosen as we wish, in particular if $\Sigma = \mathbb{R}^d$ is not compact, we may assume that it satisfies Hypothesis 1.2. However if $\Sigma = \mathbb{R}^d$, we need also to ensure Hypothesis 1.4 on the (non necessarily time-independent) background solution $\bar{A}_{\rm us}$ (recall that this is a decay condition at spatial infinity). Moreover we have to assume that $\bar{A}_{\rm us}$ is in the *temporal gauge*, i.e. that $\bar{A}_{{\rm us},t} \equiv 0$.

If our model problem is obtained from the above deformation argument, $\bar{A}_{\rm us}$ is obtained by solving two Cauchy problems for non-linear Yang-Mills equations:

in the first step one has to solve it on (M, g'), from a Cauchy surface Σ in the future (where $g' = g$) to a Cauchy surface $\Sigma_{\rm us}$ in the past (where $g' = g_{\rm us}$). In a second step one has to solve it globally on $(M, g_{\rm us})$ with the Cauchy data on $\Sigma_{\rm us}$ obtained in the first step.

Clearly if the Cauchy problem for the Yang-Mills equation (2.14) on a globally hyperbolic spacetime (M, g) can be globally solved in the space of smooth *space-compact* solutions, then all the intermediate background fields $\bar{A}'$ and $\bar{A}_{\rm us}$ will be space compact, and hence $\bar{A}_{\rm us}$ will satisfy the decay condition (1.4). As a consequence the FNW deformation argument can be applied, giving the existence of Hadamard states if the background field $\bar{A}$ is space-compact.

Fortunately it is not very difficult to deduce the result we need in dimensions lower than 4, from the existing literature, in particular from the work by Chruściel & Shatah [CS, Thm. 1.1]. The proof of the following proposition will be sketched in Appendix B.4.

Proposition 3.19. *Assume that* $\dim M \leq 4$ *and* (M, g) *is globally hyperbolic. Let* $\bar{A} \in \mathcal{E}^1_{\rm sc}(M; \mathfrak{g})$ *a local solution of the Yang-Mills equation (2.14) near some Cauchy surface* Σ. *Then there exists* $\bar{A}' \in \mathcal{E}^1_{\rm sc}(M; \mathfrak{g})$ *such that:*

(1) $\bar{A}' \sim \bar{A}$ *near* Σ, *where* $\sim$ *denotes gauge equivalence,*
(2) $\bar{A}'_t \equiv 0$, *ie* $\overline{A}'$ *is in the temporal gauge,*
(3) $\bar{A}'$ *is a global solution of (2.14).*

Combining Prop. 3.19 with the above discussion, we see that Thm. 1.2 follows from Thm. 1.1.

4. Vector and scalar Klein-Gordon equations on ultra-static spacetimes

In this section we consider a general framework containing the operators $D_0 = \overline{\delta}\overline{d}$ and $D_1 = \overline{d}\overline{\delta} + \overline{\delta}\overline{d} + \bar{F}\llcorner$ associated to the Yang-Mills equation (defined in Subsect. 2.5) on ultra-static spacetimes. This will provide a basis for the construction of the parametrix in Sect. 5.

4.1. **Preparations.** The operator D_1, (resp. D_0) acts on $\mathcal{E}^1(M)\otimes\mathfrak{g}$ (resp. $\mathcal{E}^0(M)\otimes\mathfrak{g}$). Since by Hypothesis 1.1 $M = \mathbb{R}_t\times\Sigma$ is parallelizable, we fix a global trivialization of T^*M and identify $\mathcal{E}^1(M)\otimes\mathfrak{g}$ (resp. $\mathcal{E}^0(M)\otimes\mathfrak{g}$) with $C^\infty(M;W)$ for

$$W := V\otimes\mathfrak{g} \quad \text{and} \quad V = \mathbb{C}^{1+d} \text{ (resp. } V = \mathbb{C}\text{)}. \tag{4.1}$$

We refer to the two cases as the *vector case* (resp. *scalar case*).

The background metric is ultra-static:

$$g = -dt^2 + h_{ij}(x)dx^i dx^j,$$

on $M = \mathbb{R}\times\Sigma$, with either $\Sigma = \mathbb{R}^d$ or Σ a compact manifold. We obtain a splitting

$$V = V_t\oplus V_\Sigma,\ W_{t,\Sigma} := V_{t,\Sigma}\otimes\mathfrak{g},\ W = W_t\oplus W_\Sigma, \tag{4.2}$$

by writing a $1-$form as $A = A_t dt + A_\Sigma dx$, and we identify V_t with $\mathbb{C}$. In the scalar case we take $V_t = \{0\}$, $V_\Sigma = \mathbb{C}$. Defining $J\in L(V)$ by

$$J := \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}\end{pmatrix} \text{ if } V = \mathbb{C}^{1+d},\ J := 1 \text{ if } V = \mathbb{C}, \tag{4.3}$$

we see that $V_t = \mathrm{Ker}(J+\mathbf{1}),\ V_\Sigma = \mathrm{Ker}(J-\mathbf{1})$.

We denote by $(\cdot|\cdot)$ the canonical positive definite scalar product on $C_0^\infty(M;W)$. In the scalar case we set:

$$(u|v) := \int_M \overline{u}(t,x)\not{k}v(t,x)|h|^{\frac{1}{2}}dtdx,$$

in the vector case we set:

$$(u|v) := \int_M \overline{u}(t,x)Jg^{-1}(x)\otimes\not{k}v(t,x)|h|^{\frac{1}{2}}dtdx,$$

To avoid introducing too much notation, we also denote by $(\cdot|\cdot)$ the analogous scalar product on $C_0^\infty(\Sigma;W)$, i.e.:

$$\begin{aligned}(u|v) &:= \int_\Sigma \overline{u}(x)\otimes\not{k}v(x)|h|^{\frac{1}{2}}dx, \text{ resp.}\\ (u|v) &:= \int_\Sigma \overline{u}(x)Jg^{-1}(x)\otimes\not{k}v(x)|h|^{\frac{1}{2}}dx,\end{aligned} \tag{4.4}$$

which is also positive definite.

We denote by $\Gamma_a\in C^\infty(\Sigma;L(V))$ the coefficients of the Levi-Civita connection for (M,g). Since this connection is metric for g^{-1}, we have:

$$\partial_a g^{-1} = \Gamma_a^* g^{-1} + g^{-1}\Gamma_a. \tag{4.5}$$

Since the metric is ultra-static we have moreover $\Gamma_0 = 0$, and Γ_i are the Levi-Civita connection coefficients for $(\Sigma; h_{ij}dx^i dx^j)$.

We denote by $M_a = \mathrm{ad}_{\overline{A}_a}\in C^\infty(\mathbb{R}\times\Sigma;L(\mathfrak{g}))$ the connection coefficients for the algebra degrees of freedom. They can also depend on x^0 because the background Yang-Mills solution is in general time-dependent. We have of course $M_a^*\not{k} + \not{k}M_a = 0$.

In the vector case we set

$$T_a := \Gamma_a\otimes\mathbf{1}_\mathfrak{g} + \mathbf{1}_V\otimes M_a\in C^\infty(\mathbb{R}\times\Sigma;L(W)),$$

and $T_a := M_a$ in the scalar case.

In the vector case we also fix a map $\rho \in C^\infty(\mathbb{R}\times\Sigma; L(W))$ representing the term $F_\llcorner$ such that

$$\rho^*(g^{-1}\otimes \not{k}) = (g^{-1}\otimes \not{k})\rho,$$

in the scalar case we take $\rho = 0$. We set:

$$\nabla_a^T := \partial_a + T_a,\ D := -|g|^{-\frac{1}{2}}\nabla_a^T|g|^{\frac{1}{2}}g^{ab}\nabla_b^T + \rho. \tag{4.6}$$

The *charge* q defined in (2.2) equals:

$$\overline{\zeta}q\zeta := \int_{\{t\}\times\Sigma} \overline{\mathrm{i}^{-1}\nabla_0^T\zeta}\cdot g^{-1}\otimes \not{k}\,\zeta + \overline{\zeta}\cdot g^{-1}\otimes \not{k}\,\mathrm{i}^{-1}\nabla_0^T\zeta|h|^{\frac{1}{2}}dx, \tag{4.7}$$

in the vector case and

$$\overline{\zeta}q\zeta := \int_{\{t\}\times\Sigma} \overline{\mathrm{i}^{-1}\nabla_0^T\zeta}\cdot \not{k}\,\zeta + \overline{\zeta}\cdot \not{k}\,\mathrm{i}^{-1}\nabla_0^T\zeta|h|^{\frac{1}{2}}dx, \tag{4.8}$$

in the scalar case.

4.2. **Temporal gauge.** The temporal gauge is $\bar{A}_0(t,x)\equiv 0$, which since $M_a = \mathrm{ad}_{\bar{A}_a}$ implies that $T_0 = 0$, i.e. $\nabla_0^T = \partial_t$. It is well known that one can always assume that one is in the temporal gauge, cf. Appendix B.2.

In this case the operator D takes the form:

$$D = \partial_t^2 + a(t,x,D_x),\quad a(t,x,D_x) = -|h|^{-\frac{1}{2}}\nabla_i^T h^{ij}(x)|h|^{\frac{1}{2}}\nabla_j^T + \rho(t,x). \tag{4.9}$$

Denoting by a^* the formal adjoint of a for the positive scalar product $(\cdot|\cdot)$, we deduce from the fact that q defined in (4.7), (4.8) is independent on t that:

$$a^*J = Ja, \tag{4.10}$$

for J defined in (4.3). In other terms, D is self-adjoint for $(\cdot|\cdot)_V := (\cdot|J\cdot)$. In the next sections we will use primarily the product $(\cdot|\cdot)$.

4.3. **Cauchy problem.** The standard Cauchy problem for the operator D is

$$\begin{cases} D\zeta = 0,\\ \rho\zeta = f, \end{cases} \tag{4.11}$$

for $\rho\zeta(x) = (\zeta(0,x), \mathrm{i}^{-1}\partial_t\zeta(0,x))$, $f = (f^0, f^1)$. We denote by $\zeta = Uf$ the solution of (4.11). We will denote by f_t^i, f_Σ^i, $i = 0,1$ the time and space components of f^i, according to the decomposition $W = W_t\oplus W_\Sigma$.

Denoting still by q the charge expressed in terms of Cauchy data we obtain that in the vector case:

$$\begin{aligned} \overline{f}qf &= (f^1|Jf^0) + (f^0|Jf^1)\\ &= (f_\Sigma^1|f_\Sigma^0) + (f_\Sigma^0|f_\Sigma^1) - (f_t^1|f_t^0) - (f_t^0|f_t^1). \end{aligned} \tag{4.12}$$

In the first line above the positive scalar product $(\cdot|\cdot)$ is defined in (4.4), the positive scalar products in the second line are equal to

$$(f_\Sigma|f_\Sigma) := \int_\Sigma \overline{f_\Sigma}h^{-1}\otimes \not{k}f_\Sigma|h|^{\frac{1}{2}}dx,\ (f_t|f_t) := \int_\Sigma \overline{f_t}\cdot \not{k}f_t|h|^{\frac{1}{2}}dx. \tag{4.13}$$

In the scalar case we have instead

$$\overline{f}qf = (f^1|f^0) + (f^0|f^1),\ \text{for } (u|v) = \int_\Sigma \overline{u}\cdot \not{k}v|h|^{\frac{1}{2}}dx.$$

4.4. **Adapted Cauchy data.** The above choice of Cauchy data is the usual one for an operator obtained from a metric connection. In the vector case, however, it will often be more convenient to work with the adapted Cauchy data ρ_i^{F} defined in Sect. 2.5.1. In this subsection we discuss the transition from one choice of Cauchy data to the other.

4.4.1. *Identifications.* The space $\mathcal{E}^1_{\mathrm{sc}}(M)\otimes\mathfrak{g}$ equals $C^\infty_{\mathrm{sc}}(M;W)$.

For $A\in\mathcal{E}^1_{\mathrm{sc}}(M)\otimes\mathfrak{g}$ we set:

$$A =: A_t dt + A_\Sigma, \tag{4.14}$$

for $A_t\in C^\infty(\mathbb{R},\mathcal{E}^0_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g})$, $A_\Sigma\in C^\infty(\mathbb{R},\mathcal{E}^1_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g})$, which corresponds to the decomposition $\zeta=\zeta_t\oplus\zeta_\Sigma$, using (4.2). We will use the corresponding identifications for restrictions to Σ, i.e.:

$$C^\infty_0(\Sigma;W)\sim C^\infty_0(\Sigma;W_t)\oplus C^\infty_0(\Sigma;W_\Sigma)\sim(\mathcal{E}^0_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g})\oplus(\mathcal{E}^1_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g}). \tag{4.15}$$

We have also corresponding decompositions for $2-$forms. Namely, if $F\in\mathcal{E}^2_{\mathrm{sc}}(M)\otimes\mathfrak{g}$ we set:

$$F =: dt\wedge F_t + F_\Sigma, \tag{4.16}$$

for $F_t\in C^\infty(\mathbb{R},\mathcal{E}^1_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g})$, $F_\Sigma\in C^\infty(\mathbb{R},\mathcal{E}^2_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g})$.

We recall that $\bar{A}\in\mathcal{E}^1_{\mathrm{sc}}(M)\otimes\mathfrak{g}$ is the background connection, which we assume to be in the temporal gauge. We introduce the derivative and co-derivative on Σ:

$$\bar{d}_\Sigma := d_\Sigma + \bar{A}_\Sigma\wedge\cdot\,:\ \mathcal{E}^p_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g}\to\mathcal{E}^{p+1}_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g},$$
$$\bar{\delta}_\Sigma := \bar{d}^*_\Sigma:\ \mathcal{E}^p_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g}\to\mathcal{E}^{p-1}_{\mathrm{c}}(\Sigma)\otimes\mathfrak{g},$$

and one has $\bar{d}_\Sigma\bar{d}_\Sigma=\bar{F}_\Sigma\wedge\cdot$ using the notation in (4.16). An easy computation using that $\bar{A}_t\equiv 0$ shows that:

$$\begin{aligned}
&\bar{d}u = \partial_t u dt + \bar{d}_\Sigma u,\ u\in\mathcal{E}^0_{\mathrm{sc}}(M)\otimes\mathfrak{g},\\
&\bar{d}A = dt\wedge(\partial_t A_\Sigma - \bar{d}_\Sigma A_t) + \bar{d}_\Sigma A_\Sigma,\quad A\in\mathcal{E}^1_{\mathrm{sc}}(M)\otimes\mathfrak{g},\\
&\bar{\delta}A = \partial_t A_t + \bar{\delta}_\Sigma A_\Sigma,\ A\in\mathcal{E}^1_{\mathrm{sc}}(M)\otimes\mathfrak{g},\\
&\bar{\delta}F = -(\bar{\delta}_\Sigma F_t)dt + \partial_t F_t + \bar{\delta}_\Sigma F_\Sigma,\quad F\in\mathcal{E}^2_{\mathrm{sc}}(M)\otimes\mathfrak{g}.
\end{aligned} \tag{4.17}$$

Using (4.17), we see that

$$\bar{F}_t = \partial_t\bar{A}_\Sigma,\quad \bar{F}_\Sigma = \bar{d}_\Sigma\bar{A}_\Sigma,$$

and that the Yang-Mills equation $\bar{\delta}\bar{F}=0$ is equivalent to:

$$\bar{\delta}_\Sigma\bar{F}_t = 0,\quad \partial_t\bar{F}_t + \bar{\delta}_\Sigma\bar{F}_\Sigma = 0, \tag{4.18}$$

where of course (4.18) holds for all $t\in\mathbb{R}$.

4.4.2. *Transition to adapted Cauchy data.* The adapted Cauchy data were defined in Sect. 2.5.1. Using (4.17) we obtain the following relation between the standard Cauchy data ρ_1 and the adapted ones ρ_1^{F}.

Lemma 4.1. *Let $R_{\mathrm{F}} := \rho_1^{\mathrm{F}}\circ\rho_1^{-1}$. Then:*

(1)

$$R_{\mathrm{F}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0\\ 0 & \mathbf{1} & 0 & 0\\ 0 & -\mathrm{i}\bar{\delta}_\Sigma & \mathbf{1} & 0\\ \mathrm{i}\bar{d}_\Sigma & 0 & 0 & \mathbf{1}\end{pmatrix},\ R_{\mathrm{F}}^{-1} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0\\ 0 & \mathbf{1} & 0 & 0\\ 0 & \mathrm{i}\bar{\delta}_\Sigma & \mathbf{1} & 0\\ -\mathrm{i}\bar{d}_\Sigma & 0 & 0 & \mathbf{1}\end{pmatrix}.$$

(2) *We have:*

$$R_{\mathrm{F}}^* q R_{\mathrm{F}} = q,$$

i.e. R_{F} is symplectic.

Note that the precise form of $R_{\rm F}$ relies on the assumption that the spacetime is ultra-static. It enjoys some good properties particular to that case, like for instance $JR_{\rm F} = R_{\rm F}J$, which is used implicitly in some computations in Sect. 8.

5. Parametrices for the Cauchy problem

In this section we give a construction of the parametrix for the Cauchy problem (4.11), by adapting arguments in [GW] to vector-valued Klein-Gordon equations. In the rest of the paper, the principal part of the operator $a(t, x, D_x)$ below is time-independent, since the background metric is ultra-static. In this section however we treat the more general case where the principal part is time-dependent, which corresponds to the case when the riemannian metric $h_{ij}(t, x)dx^i dx^j$ is time-dependent. The completely general situation of a metric $-\beta(t, x)dt^2 + h_{ij}(t, x)dx^i dx^j$ could be treated as well by our methods.

The construction of a parametrix for the Cauchy problem given later on will rely heavily on pseudodifferential calculus. For the necessary basic facts and definitions we refer the reader to Appendix A.

5.1. Setup and notation.

We consider an operator

$$D = \partial_t^2 + a(t, x, D_x),\ a(t, x, D_x) = -|h|^{-\frac{1}{2}}\nabla_i^T h^{ij}(t, x)|h|^{\frac{1}{2}}\nabla_j^T + \rho(t, x),$$

where T, ρ etc. are as in Sect. 4.

We assume that the metric $h_{ij}(t, x)dx^i dx^j$ satisfies Hypothesis 1.2, locally uniformly in t, and that the background Yang-Mills solution $\bar{A}$ satisfies Hypothesis 1.4 *ii)*.

In the sequel we denote $a(t, x, D_x)$ simply by $a(t) \in C^\infty(\mathbb{R}, \Psi^2(\Sigma; W))$ (see Appendix A for the definition of pseudodifferential operators classes Ψ^m, $\Psi^m_{\rm scal}$). One has:

$$\sigma_{\rm pr}(a(t)) = k_i h^{ij}(t, x)k_j \otimes \mathbf{1}_W, \tag{5.1}$$

hence $a(t)$ has a scalar principal part. For V a finite dimensional vector space, we set

$$\mathcal{H}(\Sigma; V) := \bigcap_{m\in\mathbb{Z}} H^m(\Sigma; V),\ \mathcal{H}'(\Sigma; V) := \bigcup_{m\in\mathbb{Z}} H^m(\Sigma; V), \tag{5.2}$$

equipped with their natural topologies, where $H^m(\Sigma; V)$ are the Sobolev spaces, which are canonically defined since Σ is equal either to $\mathbb{R}^d$ or to a compact manifold. We set also

$$L^2(\Sigma; W) = H^0(\Sigma; W),$$

where in the situation considered in Sect. 4, $L^2(\Sigma; W)$ is equipped with the scalar product (4.4).

5.2. Some classes of pseudodifferential operators.

In this subsection we introduce some special classes of pseudodifferential operators which will play an important role later on.

5.2.1. *High momenta localization.* A first problem that we have to face is the need to construct *exact* inverses to some elliptic operators, not only inverses modulo smoothing errors. Let us explain the well-known way to solve this problem on a simple scalar example:

if $r \in \Psi^{-1}(\mathbb{R}^d)$, the operator $\mathbf{1} + r$ is not necessarily invertible on $L^2(\mathbb{R}^d)$. However if we fix some cutoff function $\chi \in C^\infty(\mathbb{R})$, with $\chi(s) \equiv 0$ for $|s| < 1$, $\chi(s) \equiv 1$ for $|s| \geq 2$ and set

$$r_R(x, k) := \chi(R^{-1}|k|)r(x, k),\ r_R := r_R(x, D_x), \tag{5.3}$$

then $r - r_R \in \Psi^{-\infty}(\mathbb{R}^d)$ and $r_R \to 0$ in $\Psi^0(\mathbb{R}^d)$ as $R \to +\infty$. It follows that

$$\mathbf{1} + r_R \text{ is invertible on } L^2(\mathbb{R}^d) \text{ for } R \gg 1,\ (\mathbf{1} + r_R)^{-1} \in \mathbf{1} + \Psi^{-1}(\mathbb{R}^d). \tag{5.4}$$

We formalize this method by introducing the following definition.

Definition 5.1. *Let V_1, V_2 be finite dimensional hermitian spaces. We denote by $\Psi^p_{\rm as}(\Sigma; V_1, V_2)$ the space of $R-$dependent pseudodifferential operators c_R such that:*
i) c_R is uniformly bounded in $\Psi^p(\Sigma; V_1, V_2)$,
ii) $c_R \to 0$ in $\Psi^{p+\varepsilon}(\Sigma; V_1, V_2)$ when $R \to +\infty$ for some (and hence for all) $\varepsilon > 0$.
The space $\Psi^p_{\rm as}(\Sigma; V, V)$ will be simply denoted by $\Psi^p_{\rm as}(\Sigma; V)$.

We now collect some easy properties of the above classes (the meaning of statement (2) below is explained in the proof).

Lemma 5.2. (1) $\big(\Psi^p_{\rm as}(\Sigma; V_1, V_2)\big)^* = \Psi^p_{\rm as}(\Sigma; V_2, V_1)$,
(2) $\Psi^p(\Sigma; V_1, V_2) \subset \Psi^p_{\rm as}(\Sigma; V_1, V_2) + \Psi^{-\infty}(\Sigma; V_1, V_2)$,
(3) *let $c_R \in \Psi^{-\varepsilon}_{\rm as}(\Sigma; V)$ for $\varepsilon > 0$ and let $\alpha \in \mathbb{R}$. Then for $R \geq R_0$ we have:*

$$(\mathbf{1} + c_R)^\alpha \in \mathbf{1} + \Psi^{-\varepsilon}_{\rm as}(\Sigma; V).$$

Proof. (1) follows from the definition. If $c \in S^p(\Sigma; V_1, V_2)$ we set $c_R(x,k) = \chi(R^{-1}|k|)c(x,k)$, for χ as in (5.3), and obtain that $c_R(x, D_x) \in \Psi^p_{\rm as}(\Sigma; V_1, V_2)$, $c(x, D_x) - c_R(x, D_x) \in \Psi^{-\infty}(\Sigma; V_1, V_2)$, which proves (2). Let us now prove (3). We obtain that $c_R \to 0$ in $\Psi^0(\Sigma; V)$, hence in $B(L^2(\Sigma; V))$. It follows that for $R \geq R_0$ $(\mathbf{1} + c_R)^\alpha$ is well defined by the holomorphic functional calculus of bounded operators. The map $c_R \mapsto (\mathbf{1} + c_R)^\alpha - \mathbf{1}$ is then continuous on $\Psi^{-\varepsilon}(\Sigma; V)$ for all $\varepsilon > 0$, from which we deduce that $(\mathbf{1} + c_R)^\alpha \in \mathbf{1} + \Psi^{-\varepsilon}_{\rm as}(\Sigma; V)$. □

5.2.2. *Infrared cutoffs.* Some operators will need to contain additional low energy (infrared) cutoffs, defined using some selfadjoint operators. These cutoffs will play an important role in Sect. 8.

In the rest of the paper we denote by $\chi_<, \chi_> \in C^\infty(\mathbb{R})$ two cutoff functions with

$$\chi_< + \chi_> = 1, \quad \mathrm{supp}\chi_> \subset]-\infty, -1] \cup [1, +\infty[, \quad \mathrm{supp}\chi_< \subset [-2, 2]. \tag{5.5}$$

Definition 5.3. *Let V_1, V_2 be finite dimensional hermitian spaces and $h_i \in \mathrm{Diff}^2(\Sigma; V_i)$ be elliptic, selfadjoint and bounded from below. We denote by $\Psi^p_{\rm reg}(\Sigma; V_1, V_2)$ the space of $R-$dependent pseudodifferential operators c_R such that:*
i) $c_R \in \Psi^p_{\rm as}(\Sigma; V_1, V_2)$,
ii) $c_R = \chi_>(h_2) c_R \chi_>(h_1)$ for some $\chi_>$ as in (5.5).
The space $\Psi^p_{\rm reg}(\Sigma; V, V)$ will be simply denoted by $\Psi^p_{\rm reg}(\Sigma; V)$.

Lemma 5.4. (1) $\big(\Psi^p_{\rm reg}(\Sigma; V_1, V_2)\big)^* = \Psi^p_{\rm reg}(\Sigma; V_2, V_1)$,
(2) $\Psi^p(\Sigma; V_1, V_2) \subset \Psi^p_{\rm reg}(\Sigma; V_1, V_2) + \Psi^{-\infty}(\Sigma; V_1, V_2)$,
(3) *let $c_R \in \Psi^{-\varepsilon}_{\rm reg}(\Sigma; V)$ for $\varepsilon > 0$ and let $\alpha \in \mathbb{R}$. Then for $R \geq R_0$ we have:*

$$(\mathbf{1} + c_R)^\alpha \in \mathbf{1} + \Psi^{-\varepsilon}_{\rm reg}(\Sigma; V).$$

Proof. (1) follows from the definition. (2) follows from Lemma 5.2 (2) and the fact that $\chi_<(h_i) \in \Psi^{-\infty}(\Sigma; V_i)$, since h_i is elliptic and bounded below. Next $(1 + c_R)^\alpha$ is well defined for R large enough by Lemma 5.2. For $f(\lambda) = (1 + \lambda)^\alpha$ we have (denoting $\chi_>(h)$ simply by $\chi_>$):

$$f(c_R) = f(\chi_> c_R \chi_>) = \mathbf{1} + f'(0)\chi_> c_R \chi_> + \chi_> c_R \chi_> g(\chi_> c_R \chi_>) \chi_> c_R \chi_>,$$

for $g(\lambda) = \lambda^{-2}(f(\lambda) - 1 - f'(0)\lambda)$. Since g is analytic near 0, we obtain that $g(\chi_> c_R \chi_>) \in \Psi^0(\Sigma; V)$ and moreover that $g(\chi_> c_R \chi_>)$ is uniformly bounded in $\Psi^0(\Sigma; V)$. This implies (3). □

We will use the above operators classes for $V = W_t$, W_Σ, W or $W \oplus W$. We start by defining the operators h that will be used in our case.

Definition 5.5. *We set:*

$$h_t := \bar\delta_\Sigma \bar d_\Sigma :\ \mathcal{E}^0_{\rm c}(\Sigma)\otimes\mathfrak{g}\to\mathcal{E}^0_{\rm c}(\Sigma)\otimes\mathfrak{g},$$

$$h_\Sigma := \bar\delta_\Sigma \bar d_\Sigma + \bar d_\Sigma\bar\delta_\Sigma + \bar F_\Sigma\llcorner\cdot :\ \mathcal{E}^1_{\rm c}(\Sigma)\otimes\mathfrak{g}\to\mathcal{E}^1_{\rm c}(\Sigma)\otimes\mathfrak{g},$$

and denote still by h_t, h_Σ their selfadjoint extensions, with domains $H^2(\Sigma;W_t)$, $H^2(\Sigma;W_\Sigma)$. We set:

$$h := h_t\oplus h_\Sigma \text{ acting on } L^2(\Sigma;W).$$

Note that from Hypothesis 1.2 we obtain that h_t, resp. h_Σ belong to $\Psi^2(\Sigma;V_t)$, resp. $\Psi^2(\Sigma;V_\Sigma)$ with principal symbol equal to $h^{ij}(0,x)k_ik_j$. It is well-known that this implies that their closures are selfadjoint with domains equal to $H^2(\Sigma;W_t)$, resp. $H^2(\Sigma;W_\Sigma)$.

We equip then the spaces W_t, W_Σ, W and $W\oplus W$ with the elliptic operators h_t, h_Σ, h and $h\oplus h$ and define the various spaces $\Psi^p_{\rm reg}$ using the above operators.

Finally we choose a number $C\gg 1$ such that $h+C\mathbf{1}\geq\mathbf{1}$ and set:

$$\epsilon := (h+C\mathbf{1})^{\frac12} = \epsilon_t\oplus\epsilon_\Sigma, \tag{5.6}$$

where $\epsilon_t := (h_t+C\mathbf{1})^{\frac12}$, $\epsilon_\Sigma := (h_\Sigma+C\mathbf{1})^{\frac12}$. Let us collect some useful properties of the above operators.

Lemma 5.6. (1) *$h\in\mathrm{Diff}^2(\Sigma;W)$ is an elliptic differential operator with principal symbol*

$$\sigma_{\rm pr}(h)(x,k) = k_ih^{ij}(0,x)k_j\otimes\mathbf{1}_W.$$

(2) *$\epsilon\in\Psi^1(\Sigma;W)$ is an elliptic pseudodifferential operator with principal symbol:*

$$\sigma_{\rm pr}(\epsilon)(x,k) = (k_ih^{ij}(0,x)k_j)^{\frac12}\otimes\mathbf{1}_W.$$

(3)

$$i)\quad h=h^*,\ \epsilon=\epsilon^*,\ [h,J]=[\epsilon,J]=0,$$

$$ii)\quad h_\Sigma\bar d_\Sigma = \bar d_\Sigma h_t + \bar\delta_\Sigma\bar F_\Sigma\wedge\cdot,\ \bar\delta_\Sigma h_\Sigma = h_t\bar\delta_\Sigma + \bar\delta_\Sigma\bar F_\Sigma\lrcorner\cdot\ .$$

Proof. (1) and (3) *i)* are straightfoward. (2) follows from Prop. A.1. (3) *ii)* follows from the Riemannian version of the computations at the end of Subsect. 2.5. □

5.3. **Construction of generators.** In this subsection we construct the two generators for the parametrix of the Cauchy problem, by modifying arguments from [GW].

We first introduce a convenient family $\mathbb{R}\ni t\mapsto\epsilon(t)\in C^\infty(\mathbb{R},\Psi^1(\Sigma;W))$ with the properties below. The operators $\epsilon(t)$ will serve as elliptic 'weight' operators.

$$\begin{aligned} &i)\quad \sigma_{\rm pr}(\epsilon(t)) = (k_ih^{ij}(t,x)k_j)^{\frac12}\otimes\mathbf{1}_W,\\ &ii)\quad \epsilon(t)\text{ is selfadjoint on } L^2(\Sigma;W) \text{ with domain } H^1(\Sigma;W),\\ &iii)\quad \epsilon(t)\geq\mathbf{1},\ \epsilon(t)J=J\epsilon(t),\ \epsilon(0)=\epsilon, \end{aligned} \tag{5.7}$$

where ϵ is defined in (5.6). It is easy to construct such a family $\epsilon(t)$, one way being to introduce the operator $h(t)$ as in Def. 5.5 using the metric $h^{ij}(t,x)dx^idx^j$ and connection coefficients $T_a(t,x)$ at time t instead of at time 0, and to set $\epsilon(t)=(h(t)+C(t))^{\frac12}$ for some $C(t)\gg 1$.

Proposition 5.7. *There exists for $R \geq 1$ a family $\mathbb{R} \ni t \mapsto b_R(t)$ such that:*

$$
\begin{aligned}
&ia) \quad b_R(t) = \epsilon(t) + C^\infty(\mathbb{R}, \Psi^0(\Sigma; W)),\\
&ib) \quad \mathrm{i}\partial_t b_R(t) - b_R^2(t) + a(t) = r_{-\infty}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W)),\\
&ic) \quad \begin{cases} b_R(t) + Jb_R^*(t)J : H^1(\Sigma; W) \to L^2(\Sigma; W) \text{ is invertible},\\ b_R(t) + Jb_R^*(t)J = \epsilon(t)^{\frac{1}{2}}(2\mathbf{1} + C^\infty(\mathbb{R}, \Psi^{-1}(\Sigma; W)))\epsilon(t)^{\frac{1}{2}}, \end{cases}\\
&id) \quad (b_R(t) + Jb_R^*(t)J)^{-\frac{1}{2}} = \epsilon(t)^{-\frac{1}{2}}(\tfrac{1}{2}\mathbf{1} + C^\infty(\mathbb{R}, \Psi^{-1}(\Sigma; W)))\epsilon(t)^{-\frac{1}{2}}.
\end{aligned}
$$

Moreover we have:

$$
\begin{aligned}
&ii) \quad b_R(0) + Jb_R^*(0)J = \epsilon^{\frac{1}{2}}(2\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W))\epsilon^{\frac{1}{2}},\\
&iii) \quad (b_R(0) + Jb_R^*(0)J)^{-1} = \epsilon^{-\frac{1}{2}}(\tfrac{1}{2}\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W))\epsilon^{-\frac{1}{2}}\\
&iv) \quad b_R(0) = \epsilon^{\frac{1}{2}}(\mathbf{1} + r_{-1,R})\epsilon^{\frac{1}{2}},\ r_{-1,R} \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W).
\end{aligned}
$$

Remark 5.8. *It is easy to see that the equation*

$$\mathrm{i}\partial_t b(t) - b^2(t) + a(t) = r_{-\infty}(t)$$

is equivalent to

$$(\partial_t + \mathrm{i}b(t)) \circ (\partial_t - \mathrm{i}b(t)) = \partial_t^2 + a(t) - r_{-\infty}(t) \tag{5.8}$$

The idea of factorizing the Klein-Gordon operator modulo a smoothing error term was already used in [J] *to construct Hadamard states in the scalar case. However, in contrast to* [J]*, instead of solving (5.8) on the level of symbols we work with the operators and supplement arguments from microlocal analysis by Hilbert space techniques (cf.* [GW] *for the scalar case).*

Proof. *Step 1*: in Step 1 the parameter R will be absent, so we suppress the subscript R to simplify notation. We look for $b(t)$ of the form:

$$b(t) =: \epsilon(t) + b_0(t),\ b_0(t) \in C^\infty(\mathbb{R}, \Psi^0(\Sigma; W)). \tag{5.9}$$

Using that $a(t) = \epsilon^2(t) + r_1(t)$ by (5.7) *i)*, we obtain that $b_0(t)$ should solve:

$$
\begin{aligned}
b_0 &= (2\epsilon)^{-1}\mathrm{i}\partial_t\epsilon + (2\epsilon)^{-1}(r_1(t) - \mathbf{1}) + (2\epsilon)^{-1}(\mathrm{i}\partial_t b_0 - b_0^2 + [\epsilon, b_0])\\
&= (2\epsilon)^{-1}(\mathrm{i}\partial_t\epsilon + r_1 - \mathbf{1}) + F(b_0),
\end{aligned} \tag{5.10}
$$

Since $\epsilon(t)$ as a scalar principal symbol, we have $\epsilon(t) \in C^\infty(\mathbb{R}, \Psi^1_{\mathrm{scal}}(\Sigma; W)) + C^\infty(\mathbb{R}, \Psi^0(\Sigma; W))$. Therefore we obtain that $[\epsilon, c] \in C^\infty(\mathbb{R}, \Psi^m(\Sigma; W))$ for any operator $c \in C^\infty(\mathbb{R}, \Psi^m(\Sigma; W))$. It follows that we can apply [GW, Lemma A.1] and find $b(t) = \epsilon(t) + b_0(t)$, unique modulo $C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W))$ such that

$$\mathrm{i}\partial_t b(t) - b^2(t) + a(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W)), \tag{5.11}$$

hence we have satisfied conditions *ia), ib).*

Step 2: in Step 2 we modify $b(t)$ by subtracting an $R-$dependent term in $\Psi^{-\infty}(W)$ to ensure the remaining conditions. We first write $b(t)$ as

$$b(t) = \epsilon(t)^{\frac{1}{2}}(\mathbf{1} + r_{-1}(t))\epsilon(t)^{\frac{1}{2}},\ r_{-1}(t) \in C^\infty(\mathbb{R}, \Psi^{-1}(\Sigma; W)).$$

We fix a cutoff function $\chi \in C^\infty(\mathbb{R})$, with $\chi(\lambda) = 0$ for $|\lambda| \leq 1$, $\chi(\lambda) = 1$ for $|\lambda| \geq 2$, and set for $R \geq 1$ and a function $\lambda \in C^\infty(\mathbb{R})$ to be determined later:

$$r_{-1,R}(t) = \chi\left(\frac{\epsilon(t)}{R\lambda(t)}\right) r_{-1}(t) \chi\left(\frac{\epsilon(t)}{R\lambda(t)}\right)$$

We know that for fixed t we have $\chi(\frac{\epsilon(t)}{\lambda})r_{-1}(t)\chi(\frac{\epsilon(t)}{\lambda}) \to 0$ in $\Psi^0(\Sigma;W)$, when $\lambda \to +\infty$. Therefore we can find a smooth function $\mathbb{R} \ni t \mapsto \lambda(t) \in \mathbb{R}$ such that

$$\|r_{-1,R}(t)\|_{B(L^2(\Sigma;W))} \leq \frac{1}{2},\ \forall\, t \in \mathbb{R},\ R \geq 1. \tag{5.12}$$

Moreover we have

$$r_{-1}(t) - r_{-1,R}(t) = r_{-\infty,R}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma;W)).$$

Finally we set

$$b_R(t) = \epsilon(t)^{\frac{1}{2}}(\mathbf{1} + r_{-1,R}(t))\epsilon(t)^{\frac{1}{2}},$$

so that $b_R(t) = b(t) + C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma;W))$, hence $b_R(t)$ still satisfies *ia)*, *ib)*.

To verify the remaining conditions, we write:

$$b_R(t) + Jb_R^*(t)J = \epsilon(t)^{\frac{1}{2}}(2\mathbf{1} + r_{-1,R}(t) + Jr_{-1,R}^*(t)J)\epsilon(t)^{\frac{1}{2}},$$

since $\epsilon(t)$ is selfadjoint and $[J, \epsilon(t)] = 0$, by (5.7). Since $\|r_{-1,R}(t)\| + \|Jr_{-1,R}^*(t)J\| \leq 1$ by (5.12), we have

$$\begin{aligned}(b_R(t) + Jb_R^*(t)J)^{-\frac{1}{2}} &= \epsilon(t)^{-\frac{1}{2}}(2\mathbf{1} + r_{-1,R}(t) + Jr_{-1,R}^*(t)J)^{-1}\epsilon(t)^{-\frac{1}{2}} \\ &= \epsilon(t)^{-\frac{1}{2}}(\frac{1}{2}\mathbf{1} + C^\infty(\mathbb{R}, \Psi^{-1}(\Sigma;W)))\epsilon(t)^{-\frac{1}{2}},\end{aligned}$$

by Prop. A.1. This proves conditions *ic)*, *id)*.

It remains to check *ii)*, *iii)*, *iv)*. This follows from the fact that $\epsilon(0) = \epsilon$, hence $r_{-1,R}(0) \in \Psi^{-1}_{\mathrm{reg}}(\Sigma;W)$. It suffices then to apply the properties of the space $\Psi^{-1}_{\mathrm{reg}}(\Sigma;W)$ recalled in Lemma 5.4. □

5.4. **Parametrices for the Cauchy problem.** It is well known that if $f \in \mathcal{H}(\Sigma; W \oplus W)$, then the Cauchy problem (4.11) has a unique solution $\zeta = U(t)f \in C^\infty(\mathbb{R}, \mathcal{H}(\Sigma;W))$. In this subsection we give a representation of $U(t)$ by generalizing to vector-valued wave equations the constructions in [GW, Sect. 5] for the scalar case.

Theorem 5.9. *Let $b(t) = b_R(t) \in C^\infty(\mathbb{R}, \Psi^1(\Sigma;W))$ be the operator constructed in Prop. 5.7 and let us set:*

$$b^+(t) := b(t),\ b^-(t) := -Jb^*(t)J,$$

$$u^\pm(t) := \mathrm{Texp}(\mathrm{i}\textstyle\int_0^t b^\pm(\sigma)d\sigma)$$

$$r^{0\pm} := \mp(b^+(0) - b^-(0))^{-1}b^\mp(0) \in \Psi^0(\Sigma;W),$$

$$r^{1\pm} := \pm(b^+(0) - b^-(0))^{-1} \in \Psi^{-1}(\Sigma;W),$$

and

$$r^\pm f := r^{0\pm}f^0 + r^{1\pm}f^1,\ f \in \mathcal{H}(\Sigma; W \oplus W). \tag{5.13}$$

Then

$$U(t) = u^+(t)r^+ + u^-(t)r^- + r_{-\infty}(t),\ r_{-\infty}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W \oplus W, W)).$$

Proof. It is convenient to generalize slightly the situation and to denote by $U(t,s)$ the Cauchy evolution operator for initial data at time s, so that $U(t) = U(t,0)$. We set also

$$T(t,s) := \begin{pmatrix} U(t,s) \\ \mathrm{i}^{-1}\partial_t U(t,s) \end{pmatrix} : \mathcal{H}(\Sigma, W \oplus W) \to \mathcal{H}(\Sigma; W \oplus W),$$

so that

$$\mathrm{i}^{-1}\partial_t T(t,s) = A(t)T(t,s),\ A(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & 0 \end{pmatrix}. \tag{5.14}$$

Note that the operators $r^{\pm}(t)$, defined as in (5.13) with $b^{\pm}(0)$ replaced by $b^{\pm}(t)$ are well defined, by Prop. 5.7. Similarly we set $u^{\pm}(t,s) = \mathrm{Texp}(\mathrm{i}\int_s^t b^{\pm}(\sigma)d\sigma)$, and

$$\tilde{U}(t,s) := u^+(t,s)r^+(s) + u^-(t,s)r^-(s),$$

$$\tilde{T}(t,s) := \begin{pmatrix} \tilde{U}(t,s) \\ \mathrm{i}^{-1}\partial_t\tilde{U}(t,s) \end{pmatrix} = \begin{pmatrix} u^+(t,s)r^+(s) + u^-(t,s)r^-(s) \\ b^+(t)u^+(t,s)r^+(s) + b^-(t)u^-(t,s)r^-(s) \end{pmatrix}.$$

An easy computation shows that

$$\tilde{T}(s,s) = \mathbf{1},\ u^{\pm}(t,s)r^{\pm}(s) = r^{\pm}(t)\tilde{T}(t,s), \tag{5.15}$$

which implies that $(t,s) \mapsto \tilde{T}(t,s)$ is a two-parameter group. From Prop. 5.7

$$\mathrm{i}^{-1}\partial_t\left(b^{\pm}(t)u^{\pm}(t)\right) = \mathrm{i}^{-1}\partial_t b^{\pm}(t) + b^{\pm 2}(t) = a(t) - r^{\pm}_{-\infty}(t),$$

for $r^{\pm}_{-\infty}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W))$. Using then (5.15) we obtain that

$$\mathrm{i}^{-1}\partial_t\tilde{T}(t,s) = \tilde{A}(t)\tilde{T}(t,s),$$

for

$$\tilde{A}(t) = A(t) - R_{-\infty}(t),$$

$$R_{-\infty}(t) = \begin{pmatrix} 0 \\ r^+_{-\infty}(t) \end{pmatrix} \circ r^+(t) + \begin{pmatrix} 0 \\ r^-_{-\infty}(t) \end{pmatrix} \circ r^-(t) \in C^\infty(\mathbb{R}^2, \Psi^{-\infty}(\Sigma; W \oplus W)).$$

We can then express $T(t,s)$ in terms of $\tilde{T}(t,s)$ by setting

$$T(t,s) =: \tilde{T}(t,s) \circ R(t,s), \tag{5.16}$$

where $R(t,s)$ solves the equation

$$\begin{cases} \mathrm{i}^{-1}\partial_t R(t,s) - \tilde{T}(s,t)R_{-\infty}(t)\tilde{T}(t,s) \circ R(t,s) = 0, \\ R(s,s) = \mathbf{1}. \end{cases} \tag{5.17}$$

By Lemma A.5 we first obtain that

$$\tilde{R}_{-\infty}(t,s) := \tilde{T}(s,t)R_{-\infty}(t)\tilde{T}(t,s) \in C^\infty(\mathbb{R}^2, \Psi^{-\infty}(\Sigma; W \oplus W)).$$

The solution of (5.17) is then given by

$$R(t,s) = \mathrm{Texp}(\mathrm{i}\textstyle\int_s^t \tilde{R}_{-\infty}(\sigma,s)d\sigma) = \mathbf{1} + \mathrm{i}\displaystyle\int_s^t \tilde{R}_{-\infty}(\sigma,s)R(\sigma,s)d\sigma. \tag{5.18}$$

By the argument in the proof of Prop. A.3 (see the properties of $m(t,s)$ in the proof), we first obtain that $R(t,s) \in C^\infty(\mathbb{R}^2, \Psi^0(\Sigma; W \oplus W))$. (5.18) then implies that

$$R(t,s) = \mathbf{1} + C^\infty(\mathbb{R}^2, \Psi^{-\infty}(\Sigma; W \oplus W)).$$

By Lemma A.5 we obtain finally that

$$T(t,s) = \tilde{T}(t,s) + C^\infty(\mathbb{R}^2, \Psi^{-\infty}(\Sigma; W \oplus W)),$$

hence

$$U(t,s) = \tilde{U}(t,s) + C^\infty(\mathbb{R}^2, \Psi^{-\infty}(\Sigma; W \oplus W, W)).$$

Setting $s = 0$ completes the proof of the theorem. □

At this point, we could set

$$U^{\pm} := U(t)r^{\pm} = u^{\pm}(t)r^{\pm} + C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W)),$$

and prove directly that these are parametrices that satisfy the properties listed in Thm. 3.12, with the exception of positivity (positivity w.r.t. the product $(\cdot|\cdot)_V$ does not hold if $J \neq \mathbf{1}$, instead one gets positivity w.r.t. the 'non-physical' inner product $(\cdot|\cdot)$). Thus, we could associate to them (non-positive) pseudo-covariances $\lambda^{\pm}$ in an abstract manner as in Thm. 3.12. However, we prefer to construct them in a more systematic way in Sect. 6 in order to derive additional information needed to cope later on with the conditions (g.i.) and (pos) in gauge theory.

6. Hadamard two-point functions

6.1. **Preparations.** In the present section, we continue with the setup of Sect. 5 and deduce expressions for Hadamard two-point functions from the construction of the parametrix. This is done in a similar way as in [GW], i.e. we construct an operator T_R that diagonalizes the symplectic form and separates Cauchy data that propagate with positive and negative energies in the wave front set. We also show in Subsect. 6.3 that Hadamard states do not exist for vector Klein-Gordon equations if the scalar product is not positive-definite on the fibers.

In the sequel, if $b_R(t)$ is the operator constructed in Prop. 5.7 we denote $b_R(0)$ simply by b_R.

Lemma 6.1. *There exists $Z_R \in \Psi^{\frac{1}{2}}(\Sigma; W)$ such that:*

$$b_R J + J b_R^* = Z_R^* J Z_R, \tag{6.1}$$

and additionally:

$$Z_R = (\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W))(2\epsilon)^{\frac{1}{2}},\ Z_R^{-1} = (2\epsilon)^{-\frac{1}{2}}(\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W)).$$

Proof. By Prop. 5.7 we have

$$J b_R + b_R^* J = (2\epsilon)^{\frac{1}{2}}(J + J c_R + c_R^* J)(2\epsilon)^{\frac{1}{2}},\ c_R \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W).$$

We look for Z_R in the lemma under the form $Z_R = S_R(2\epsilon)^{\frac{1}{2}}$ for

$$S_R = \mathbf{1} + d_R,\ d_R \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W). \tag{6.2}$$

The identity (6.1) is satisfied if

$$S_R^* J S_R = J + J c_R + c_R^* J. \tag{6.3}$$

Using $W = W_t \oplus W_\Sigma$ (see (4.2)), we can write:

$$S_R = \begin{pmatrix} s_{tt,R} & s_{t\Sigma,R} \\ s_{\Sigma t,R} & s_{\Sigma\Sigma,R} \end{pmatrix},\ c_R = \begin{pmatrix} c_{tt,R} & c_{t\Sigma,R} \\ c_{\Sigma t,R} & c_{\Sigma\Sigma,R} \end{pmatrix}.$$

Let us now formulate the property that $c_R \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W)$ in terms of the components of c_R.

If α, β are any of the symbols t or Σ, then since $h = h_t \oplus h_\Sigma$, we obtain that $c_{\alpha\beta,R} \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W_\alpha, W_\beta)$. We are looking for $s_{\alpha\beta,R}$ such that

$$s_{\alpha\beta,R} - \delta_{\alpha\beta} \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W_\alpha, W_\beta)$$

Let us now suppress the index R to simplify notation. The equation (6.3) is satisfied iff:

$$\begin{cases} -s_{tt}^* s_{tt} + s_{\Sigma t}^* s_{\Sigma t} = \mathbf{1} - c_{tt}^* - c_{tt}, \\ -s_{tt}^* s_{t\Sigma} + s_{\Sigma t}^* s_{\Sigma\Sigma} = -c_{t\Sigma} + c_{\Sigma t}^*, \\ -s_{t\Sigma}^* s_{tt} + s_{\Sigma\Sigma}^* s_{\Sigma t} = c_{\Sigma t} - c_{t\Sigma}^*, \\ s_{\Sigma\Sigma}^* s_{\Sigma\Sigma} - s_{t\Sigma}^* s_{t\Sigma} = \mathbf{1} + c_{\Sigma\Sigma} + c_{\Sigma\Sigma}^*. \end{cases} \tag{6.4}$$

To solve this system we first set $s_{t\Sigma} = 0$. The last equation of (6.4) can then be solved for R large enough by

$$s_{\Sigma\Sigma} = s^*_{\Sigma\Sigma} = (\mathbf{1} + c_{\Sigma\Sigma} + c^*_{\Sigma\Sigma})^{\frac{1}{2}} \in \mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W_\Sigma, W_\Sigma),$$

using Lemma 5.4 (3). The second and third equations are then solved by

$$s_{\Sigma t} = s^{-1}_{\Sigma\Sigma}(c_{\Sigma t} - c^*_{t\Sigma}) \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W_\Sigma, W_t),$$

again by Lemma 5.4. Finally we solve the first equation by

$$s_{tt} = s^*_{tt} = (\mathbf{1} + c_{tt} + c^*_{tt} + s^*_{\Sigma t} s_{\Sigma t})^{\frac{1}{2}} \in \mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W_t, W_t).$$

This completes the proof of the lemma. □

We now set

$$T_R := Z_R(b^+_R - b^-_R)^{-1} \otimes \mathbf{1}_{\mathbb{C}^2} \circ \begin{pmatrix} -b^-_R & \mathbf{1} \\ b^+_R & -\mathbf{1} \end{pmatrix} \in \Psi^\infty(\Sigma; W \oplus W), \tag{6.5}$$

so that $T_R f = \begin{pmatrix} Z_R r^+_R f \\ Z_R r^-_R f \end{pmatrix}$, where $r^\pm_R$ are defined in (5.13). We have:

$$T^{-1}_R = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ b^+_R & b^-_R \end{pmatrix} \circ Z^{-1}_R \otimes \mathbf{1}_{\mathbb{C}^2}. \tag{6.6}$$

Proposition 6.2. *We have:*

$$(T^{-1}_R)^* \circ q \circ T^{-1}_R = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \tag{6.7}$$

$$T_R = \frac{1}{\sqrt{2}}(\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)) \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{pmatrix}. \tag{6.8}$$

Proof. Let us suppress again the subscript R and denote $b^\pm_R$ simply by $b^\pm$. Set $f^\pm = r^\pm f$, so that

$$f^0 = f^+ + f^-,\ f^1 = b^+ f^+ + b^- f^-.$$

An easy computation using that $b^+ = b$, $b^- = -Jb^*J$ yields:

$$\overline{f} q f = (f^+ | (Jb + b^* J) f^+) - (f^- | (Jb + b^* J) f^-).$$

By Lemma 6.1 we have $Jb + b^*J = Z^*_R J Z_R$. This implies (6.7) by the definition of T_R.

Let us now prove (6.8). From Lemma 6.1 and Prop. 5.7 we have

$$\begin{aligned} &Z_R = (\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W))(2\epsilon)^{\frac{1}{2}}, \\ &(b^+_R - b^-_R)^{-1} = (b_R + J b^*_R J)^{-1} = (2\epsilon)^{-\frac{1}{2}}(\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W))(2\epsilon)^{-\frac{1}{2}}. \end{aligned}$$

Similarly we have

$$\begin{aligned} &\begin{pmatrix} -b^-_R & \mathbf{1} \\ b^+_R & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \epsilon^{-\frac{1}{2}} & 0 \\ 0 & \epsilon^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} J b^*_R J \epsilon^{-\frac{1}{2}} & \epsilon^{\frac{1}{2}} \\ b\epsilon^{-\frac{1}{2}} & -\epsilon^{\frac{1}{2}} \end{pmatrix} \\ = \ &\epsilon^{\frac{1}{2}}(\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)) \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}. \end{aligned}$$

Then (6.8) follows by applying formula (6.5). □

6.2. **Hadamard two-point functions.** In this subsection we construct pairs of Hadamard two-point functions.

Proposition 6.3. *Let us define* $c^{\pm} : \mathcal{H}(\Sigma; W \oplus W) \to \mathcal{H}(\Sigma; W \oplus W)$ *by:*

$$c^{+} := T_R^{-1} \circ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \circ T_R, \quad c^{-} := T_R^{-1} \circ \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \circ T_R, \tag{6.9}$$

Then the following holds:
(1) *One has*

$$c^{\pm} f = \begin{pmatrix} r^{\pm} f \\ b^{\pm} r^{\pm} f \end{pmatrix}, \ f \in \mathcal{H}(\Sigma; W \oplus W),$$

(2)

$$\begin{aligned} &i) \quad c^{+} + c^{-} = \mathbf{1}, \ (c^{\pm})^2 = c^{\pm}, \\ &ii) \quad (c^{\pm})^{\dagger} = c^{\pm}, \\ &iii) \quad r^{\pm} \circ c^{\pm} = r^{\pm}. \end{aligned}$$

Proof. (1) is a routine computation using (6.5), (6.6). (2) follows from (6.7). □

Theorem 6.4. *Let* $c^{\pm}$ *be defined by (6.9) and set*

$$\lambda_{\Sigma}^{\pm} := \pm q \circ c^{\pm} \in B(\mathcal{H}(\Sigma; W \oplus W), \mathcal{H}'(\Sigma; W \oplus W)). \tag{6.10}$$

Then
(1) $\lambda_{\Sigma}^{\pm}$ *is a pair of Hadamard Cauchy surface two-point functions;*
(2) *one has:*

$$\lambda_{\Sigma}^{+} = T_R^{*} \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} T_R, \quad \lambda_{\Sigma}^{-} = T_R^{*} \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} T_R. \tag{6.11}$$

Proof. The proof of (1) is identical to the proof of [GW, Thm. 7.1]. Note that only the proof of the implication $\Rightarrow$ in [GW, Thm. 7.1] needs to be copied. (2) follows from (6.7), (6.9). □

Remark 6.5. *Statement (1) of Thm. 6.4 still holds if we replace* $c^{\pm}$ *by* $c^{\pm} \pm r_{-\infty}$*, for* $r_{-\infty} \in \Psi^{-\infty}(\Sigma; W \oplus W)$.

6.3. **Non-existence of Hadamard states for vector Klein-Gordon equations.** In this subsection we consider a vector Klein-Gordon operator D as above, assuming that $J \neq \mathbf{1}$, i.e. that the hermitian form on W is not positive definite. We show that under a mild additional condition on its two-point functions, there *does not exist* any Hadamard state, but only Hadamard *pseudo-states*.

Theorem 6.6. *Assume that* $J \neq \mathbf{1}$*. Then there does not exist spacetime two-point functions* $\tilde{\lambda}^{\pm}$ *for* D *satisfying* (μsc) *and* (pos) *such that additionally the Cauchy surface two-point functions* $\tilde{\lambda}_{\Sigma}^{\pm}$ *map continuously* $\mathcal{H}(\Sigma; W \oplus W)$ *into itself.*

Proof. Let $\tilde{\lambda}_{\Sigma}^{\pm}$ the Cauchy surface two-point functions of the state ω. Since by assumption $\tilde{\lambda}_{\Sigma}^{\pm}$ preserve $\mathcal{H}(\Sigma; W \oplus W)$ we can apply [GW, Thm. 7.1], which generalizes directly to the vector case. We obtain that if (μsc) holds then $\tilde{\lambda}_{\Sigma}^{\pm} - \lambda_{\Sigma}^{\pm}$ is smoothing. Let us set

$$\tilde{A} := (T_R^{*})^{-1} \left(\tilde{\lambda}_{\Sigma}^{+} + \tilde{\lambda}_{\Sigma}^{-} \right) T_R^{-1}, \quad A := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}.$$

By (6.11) we obtain that $\tilde{A} = A + R_{\infty}$ where R_{∞} is smoothing. We may choose a sequence $f_n \in L^2(\Sigma; W \oplus W)$ with $\|f_n\| = 1$, $(f_n | A f_n) = -1$, $\mathrm{w}-\lim f_n = 0$, with support in some fixed compact $K \subset \Sigma$. Let us denote $\mathbb{1}_K$ the characteristic function of K, understood as a multiplication operator. Since $\mathbb{1}_K R_{\infty} \mathbb{1}_K$ is compact we obtain that $\lim_{n \to \infty} (f_n | \tilde{A} f_n) = -1$. But this contradicts the positivity condition (pos), which implies that $\tilde{A} \geq 0$. □

6.4. **Positivity of Hadamard two-point functions on subspaces.** We saw in Thm. 6.6 that it is impossible to construct Hadamard two-point functions for D_1, since in this case $J \neq \mathbf{1}$. However there exist subspaces of $\mathcal{H}(\Sigma; W \oplus W)$ on which $\lambda^{\pm}_{1\Sigma}$ are positive. This will follow from the fact that J is positive on $W_\Sigma = (\mathrm{Ker}(J - \mathbf{1})) \otimes \mathfrak{g}$.

Proposition 6.7. *Let $\lambda^{1\pm}$ be defined in (6.10), for $D = D_1$. Then there exists $r_{-1,R} \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)$ such that:*

$$\lambda^{\pm}_1 \geq 0 \text{ on } (\mathbf{1} + r_{-1,R})\mathcal{H}(\Sigma; W_\Sigma \oplus W_\Sigma).$$

Proof. From (6.8) we obtain that

$$T_R = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{pmatrix} (\mathbf{1} + \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)). \tag{6.12}$$

This implies, using also Lemma 5.4 (3) that for R large enough there exists $r_{-1,R} \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)$ such that

$$T_R = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{pmatrix} (\mathbf{1} + r_{-1,R})^{-1}.$$

We note next that $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$ are positive on $\mathcal{H}(\Sigma; W_\Sigma \oplus W_\Sigma)$, since J is positive on W_Σ. The operators $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}$ and $\begin{pmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & -\epsilon^{\frac{1}{2}} \end{pmatrix}$ preserve the space $\mathcal{H}(\Sigma; W_\Sigma \oplus W_\Sigma)$, since $\epsilon = \epsilon_t \oplus \epsilon_\Sigma$. The proposition follows then from (6.11) and (6.12). □

7. Pair of Hadamard pseudo-covariances

In this section we consider the pair of operators $D_0 = \bar{\delta}\bar{d}$, $D_1 = \bar{d}\bar{\delta} + \bar{\delta}\bar{d} + \bar{F}\llcorner$ as in Subsect. 2.5. After going to the temporal gauge, we may assume that both operators fit into the framework of Sect. 5, i.e. that:

$$D_i = \partial_t^2 + a_i(t, x, D_x),$$

where $a_i(t) \in C^\infty(\mathbb{R}; \Psi^2(\Sigma; W_i))$ for $W_1 = V_1 \otimes \mathfrak{g}$, and $W_0 = \mathfrak{g}$. The operator $K = \bar{d}$ becomes in this framework:

$$K = K_0(t)\partial_t + K_1(t), \tag{7.1}$$

where $K_j(t) \in C^\infty(\mathbb{R}, \mathrm{Diff}^j(\Sigma; W_0, W_1))$ is a differential operator in x, such that

$$(\partial_t^2 + a_1(t)) \circ K = K \circ (\partial_t^2 + a_0(t)). \tag{7.2}$$

It is easy to check that

$$K_0(t, x) \in L(W_0, W_1) \neq 0, \ \forall\ (t, x) \in \mathbb{R} \times \Sigma. \tag{7.3}$$

We recall that

$$K_\Sigma := \rho_1 \circ K \circ U_0 \in \mathrm{Diff}(W_0 \oplus W_0, W_1 \oplus W_1),$$

where ρ_i, U_i are the trace and Cauchy evolution operators.

7.1. **Some preparations.** Let us denote by $u_i^{\pm}(t)$, $i = 0, 1$ the operators constructed in Thm. 5.9.

Lemma 7.1. *There exist* $m_1^{\pm} \in \Psi^1(\Sigma; W_0, W_1)$ *and* $r_{-\infty}^{\pm}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W_0, W_1))$ *such that:*

$$K \circ u_0^{\pm}(t) = u_1^{\pm}(t) m_1^{\pm} + r_{-\infty}^{\pm}(t).$$

Proof. We consider only the + case and suppress the + superscripts to simplify notation. We also denote by $r_{-\infty}(t)$ a generic operator in $C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; V_1, V_2))$ for appropriate V_1, V_2. We will use repeatedly the following consequence of Prop. A.3: the map

$$m(t) \mapsto u_i(t) m(t) u_j(t)^{-1} \text{ is bijective on } C^\infty(\mathbb{R}, \Psi^p(\Sigma; W_j, W_i)). \tag{7.4}$$

This follows from the fact that $b_i(t)$ have a scalar principal symbol equal to $(k_i h^{ij}(t,x) k_j)^{\frac{1}{2}}$.

We recall the following equivalent identities from Prop. 5.7:

$$\begin{aligned} &i) \quad \mathrm{i}\partial_t b_i(t) - b_i^2(t) + a_i(t) + r_{-\infty}(t) = 0, \\ &ii) \quad (\partial_t + \mathrm{i}b_i(t)) \circ (\partial_t - \mathrm{i}b_i(t)) = \partial_t^2 + a_i(t) + r_{-\infty}(t), \ i = 0, 1. \end{aligned} \tag{7.5}$$

Since $u_0(t) = \mathrm{Texp}(\mathrm{i}\int_0^t b_0(s)ds)$, we obtain from (7.1) that:

$$K \circ u_0(t) = (\mathrm{i}K_0 b_0(t) + K_1) \circ u_0(t).$$

Composing this identity to the left with $\partial_t - \mathrm{i}b_1$ and using (7.5) *i)* we obtain:

$$\begin{aligned} &(\partial_t - \mathrm{i}b_1) \circ K \circ u_0(t) \\ &= (-K_0(a_0 + r_{-\infty,0}) + \partial_t K_1 + \mathrm{i}(\partial_t K_0 + K_1) b_0 + b_1(K_0 b_0 - \mathrm{i}K_1)) \circ u_0(t) \\ &= m_2(t) \circ u_0(t), \text{ for } m_2(t) \in C^\infty(\mathbb{R}, \Psi^2(\Sigma; W_0, W_1)). \end{aligned} \tag{7.6}$$

By (7.4) we obtain that:

$$m_2(t) \circ u_0(t) = u_1(t) \circ \tilde{m}_2(t), \text{ where } \tilde{m}_2(t) \in C^\infty(\mathbb{R}, \Psi^2(\Sigma; W_0, W_1)). \tag{7.7}$$

Combining (7.6) and (7.7), we obtain that:

$$(\partial_t - \mathrm{i}b_1) \circ K \circ u_0(t) = u_1(t) \circ \tilde{m}_2(t).$$

We compose the above identity with $\partial_t + \mathrm{i}b_1(t)$, using again (7.5) and obtain:

$$\begin{aligned} (\partial_t^2 + a_1) \circ K \circ u_0(t) =& (\partial_t + \mathrm{i}b_1) \circ u_1(t) \circ \tilde{m}_2(t) + r_{-\infty}(t) K u_0(t) \\ &= 2\mathrm{i}b_1 \circ u_1(t) \circ \tilde{m}_2(t) + u_1(t) \partial_t \tilde{m}_2(t) + r_{-\infty}(t) K u_0(t) \\ &= u_1(t) \circ \left(\partial_t \tilde{m}_2(t) - \mathrm{i}\tilde{b}_1(t) \tilde{m}_2(t) + r_{-\infty}(t) \right), \end{aligned}$$

where in the last line we use (7.4), and $\tilde{b}_1(t) \in C^\infty(\mathbb{R}, \Psi^1(\Sigma; W_1))$ is again elliptic with a real principal symbol.

On the other hand since $(\partial_t^2 + a_1) \circ K = K \circ (\partial_t^2 + a_0)$, we have by (7.5)

$$(\partial_t^2 + a_1) \circ K \circ u_0(t) = K \circ r_{-\infty}(t) u_0(t) = u_1(t) \circ r_{-\infty}(t),$$

again by (7.4). Summarizing we obtain that $\tilde{m}_2(t)$ solves

$$\partial_t \tilde{m}_2(t) + \mathrm{i}\tilde{b}_1(t) \tilde{m}_2(t) = r_{-\infty}(t),$$

hence

$$\begin{aligned}\tilde{m}_2(t) &= \mathrm{Texp}(\mathrm{i}\int_0^t \tilde{b}_1(\sigma)d\sigma)\tilde{m}_2(0) + \mathrm{Texp}(\mathrm{i}\int_s^t \tilde{b}_1(\sigma)d\sigma)r_{-\infty}(s)ds \\ &= \mathrm{Texp}(\mathrm{i}\int_0^t \tilde{b}_1(\sigma)d\sigma)\tilde{m}_2(0) + r_{-\infty}(t).\end{aligned} \tag{7.8}$$

By Lemma 7.2 below this implies that $\tilde{m}_2(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W_0, W_1))$, hence by Lemma A.5 that $m_2(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W_0, W_1))$. The identity (7.6) becomes

$$(\partial_t - \mathrm{i}b_1) \circ K \circ u_0(t) = r_{-\infty}(t).$$

As in (7.8) this implies that

$$K \circ u_0(t) = u_1(t) \circ (K \circ u_0)(0) + r_{-\infty}(t),$$

and completes the proof of the lemma. □

Lemma 7.2. *Let $b_1(t) \in C^\infty(\mathbb{R}, \Psi^1(\Sigma; W_1))$ satisfying the assumptions of Prop. A.3 and $m(t) \in C^\infty(\mathbb{R}, \Psi^p(\Sigma; W_0, W_1))$, $p \in \mathbb{R}$ such that:*

$$m(t) = \mathrm{Texp}(\mathrm{i}\textstyle\int_0^t b_1(s)ds)m(0) + r_{-\infty}(t),\ r_{-\infty}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W_0, W_1)).$$

Then $m(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W_0, W_1))$.

Proof. We have $\partial_t m(t) - \mathrm{i}b_1(t)m(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty})$. By induction we obtain

$$\partial_t^k m(t) - p_k(t)m(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty}), k \in \mathbb{N},$$

where $p_k(t) \in C^\infty(\mathbb{R}, \Psi^k(\Sigma; W_1))$, $\sigma_{\mathrm{pr}}(p_k) = (\mathrm{i}\sigma_{\mathrm{pr}}(b_1))^k$. Note that b_1 is elliptic in $\Psi^1(\Sigma; W_1)$ hence p_k is elliptic in $\Psi^k(\Sigma; W_1)$ and since $\partial_t^k m(t)$ belongs to $C^\infty(\mathbb{R}, \Psi^p(\Sigma; W_0, W_1))$ by assumption we obtain that $m(t) \in C^\infty(\mathbb{R}, \Psi^{p-k}(\Sigma; W_0, W_1))$. This completes the proof. □

7.2. **Compatibility of Hadamard pseudo-covariances.** We prove now the main result of this section, which will be important later on.

Theorem 7.3. *Let $c_i^\pm \in B(\mathcal{H}(\Sigma; W_i \oplus W_i))$, $i = 0, 1$ be as in Prop. 6.3. Then*

$$c_1^\pm K_\Sigma - K_\Sigma c_0^\pm \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1 \oplus W_1).$$

Proof. Since $c_i^+ + c_i^- = \mathbf{1}$, it suffices to prove the + case, which amounts to show that

$$c_1^- K_\Sigma c_0^+ \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1 \oplus W_1). \tag{7.9}$$

In the sequel we denote simply by $r_{-\infty}(t)$ an error term in $C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; V_1, V_2))$ for appropriate V_1, V_2. We recall from Thm. 5.9 and Prop. 6.3 that:

$$U_i(t) = u_i^+(t)r_i^+ + u_i^-(t)r_i^- + r_{-\infty}(t),\ r_i^\pm c_i^\pm = r_i^\pm.$$

Using Lemma 7.1 this gives first:

$$U_1(t)K_\Sigma c_0^+ = KU_0(t)c_0^+ = Ku_0^+(t)r_0^+ + r_{-\infty}(t) = u_1^+(t)m_1^+ r_0^+ + r_{-\infty}(t)$$

for some $m_1^+ \in \Psi^1(\Sigma; W_0, W_1)$. On the other hand:

$$U_1(t)K_\Sigma c_0^+ = u_1^+(t)r_1^+ c_1^+ K_\Sigma c_0^+ + u_1^-(t)r_1^- c_1^- K_\Sigma c_0^+ + r_{-\infty}(t).$$

It follows that

$$u_1^-(t)r_1^- c_1^- K_\Sigma c_0^+ = u_1^+(t) \circ (m_1^+ r_0^+ - r_1^+ c_1^+ K_\Sigma c_0^+) + r_{-\infty}(t). \tag{7.10}$$

We claim that if $n_1^\pm \in \Psi^p(\Sigma; W_0 \oplus W_0, W_1)$ satisfy

$$u_1^+(t)n_1^+ - u_1^-(t)n_1^- \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)),$$

then $n_1^\pm \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)$. Taking first $t = 0$ we obtain that $n_1^+ - n_1^- \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)$. Next taking derivatives in t at $t = 0$ we obtain that $(b_1^+(0) - b_1^-(0))n_1^+ \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)$, hence $n_1^+ \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)$ by the ellipticity of $b_1^+(0) - b_1^-(0)$. This also implies that $n_1^- \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)$.

Applying this remark to (7.10) we obtain that $r_1^- c_1^- K_\Sigma c_0^+ \in \Psi^{-\infty}(\Sigma; W_0 \oplus W_0, W_1)$. This implies (7.9) since from Prop. 6.3 and $r_1^+ c_1^- = 0$ we obtain:

$$c_1^- = \begin{pmatrix} r_1^- \\ b_1^-(0) r_1^- \end{pmatrix} \circ c_1^-.$$

This completes the proof of the theorem. □

8. Proof of Thm. 1.1

As before, Σ is assumed to be compact or equal to $\mathbb{R}^d$. If $\Sigma = \mathbb{R}^d$ we assume Hypothesis 1.4. In this case it follows from Prop. B.1 that h_t satisfies a Hardy inequality:

$$h_t = \bar{\delta}_\Sigma \bar{d}_\Sigma \geq C\langle x\rangle^{-2}, \tag{8.1}$$

which will be very important in the sequel.

Our goal in this section is to construct a projection Π acting on Cauchy data with the following two properties:

i) $\mathrm{Ker}\Pi = \mathrm{Ran}K_\Sigma$

ii) $\lambda_{1\Sigma}^\pm$ are positive on $\mathrm{Ran}\Pi \cap \mathrm{Ker}K_\Sigma^\dagger$.

We will ensure *ii)* by choosing Π in such a way that

$$\mathrm{Ran}\Pi \cap \mathrm{Ker}K_\Sigma^\dagger \subset (\mathbf{1} + r_{-1,R})\mathcal{H}(\Sigma; W_\Sigma \oplus W_\Sigma), \tag{8.2}$$

where the operator $r_{-1,R}$ appears in Prop. 6.7.

8.1. **Notations.** - As before, if E, F are two topological vector spaces, we write $A : E \to F$ if A is linear continuous from E to F. We write $A : E \xrightarrow{\sim} F$ if additionally A is bijective and both A^{-1} is linear continuous.

- We denote $\langle x\rangle H^m(\Sigma; V)$ the Sobolev space of order m with weight $\langle x\rangle = (1 + |x|)^{\frac{1}{2}}$ (of course this is just the same as $H^m(\Sigma; V)$ if Σ is compact) and $\langle x\rangle L^2(\Sigma; V) = \langle x\rangle H^0(\Sigma; V)$ the weighted L^2 space.

- We will denote $B^{-\infty}(\Sigma; V_1, V_2)$ the space of operators that are bounded from $H^{-m}(\Sigma; V_1)$ to $H^m(\Sigma; V_2)$ for any $m \in \mathbb{R}$.

8.2. **The reference projection for $\Sigma = \mathbb{R}^d$.** In this subsection we assume that $\Sigma = \mathbb{R}^d$. We define a reference projection Π_0, which will be used to construct the projection Π. We first state an easy consequence of the Hardy inequality.

Lemma 8.1. *The following operators are bounded:*

i) $h_t^{-\frac{1}{2}} \bar{\delta}_\Sigma : L^2(\Sigma; W_\Sigma) \to L^2(\Sigma; W_t)$,

ii) $\bar{d}_\Sigma h_t^{-\frac{1}{2}} : L^2(\Sigma; W_t) \to L^2(\Sigma; W_\Sigma)$,

iii) $h_t^{-\frac{1}{2}} \langle x\rangle^{-1} : L^2(\Sigma; W_t) \to L^2(\Sigma; W_t)$

Proof. *i)* and *ii)* follow from the definition of h_t. To prove *iii)* we use the Hardy inequality (8.1) and the Kato-Heinz theorem which yield $h_t^{-1} \leq C\langle x\rangle^{-2}$. □

Definition 8.2. *We set:*

$$\pi := \bar{d}_\Sigma h_t^{-1} \bar{\delta}_\Sigma : L^2(\Sigma; W_\Sigma) \to L^2(\Sigma; W_\Sigma),$$
$$b := h_t^{-1} \bar{\delta}_\Sigma : \ L^2(\Sigma; W_\Sigma) \to \langle x \rangle L^2(\Sigma; W_t),$$
$$a := \bar{F}_t \wedge \cdot : \ \langle x \rangle L^2(\Sigma; W_t) \to L^2(\Sigma; W_\Sigma).$$

The above operators are well defined by Lemma 8.1 and Hypothesis 1.4.

Clearly π is the orthogonal projection on $\mathrm{Ran}\bar{d}_\Sigma$, where $\bar{d}_\Sigma$ is considered as a closed operator on $L^2(\Sigma; W_t)$ with domain $H^1(\Sigma; W_t)$. Moreover one has:

$$\bar{d}_\Sigma \circ b = \pi, \ b \circ \bar{d}_\Sigma = \mathbf{1}. \tag{8.3}$$

We will construct Π by modifying a reference projection Π_0. We denote by Π_0 the operator defined in the adapted Cauchy data by the matrix:

$$\Pi_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} - \pi & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathrm{i}a \circ b & 0 & \mathbf{1} \end{pmatrix}. \tag{8.4}$$

Since $a\langle x \rangle : L^2(\Sigma; W_t) \to L^2(\Sigma; W_\Sigma)$ by Hypothesis 1.4 we see that

$$\Pi_0 : L^2(\Sigma; W \oplus W) \to L^2(\Sigma; W \oplus W).$$

Let us consider the operator K_Σ given in Lemma 2.12 as an unbounded operator

$$K_\Sigma : L^2(\Sigma; W_t \oplus W_t) \to L^2(\Sigma; W_\Sigma \oplus W_\Sigma),$$
$$\mathrm{Dom} K_\Sigma = H^1(\Sigma; W_t) \oplus L^2(\Sigma; W_t).$$

Lemma 8.3. Π_0 *is a bounded projection on* $L^2(\Sigma; W \oplus W)$ *with* $\mathrm{Ker}\Pi_0 = \mathrm{Ran}K_\Sigma$.

Proof. The fact that Π_0 is a projection is a routine computation, using that $b(\mathbf{1} - \pi) = 0$. Since ab is bounded by Lemma 8.1 and Hypothesis 1.4 we see that Π_0 is bounded. To prove the second statement we note first that $\Pi_0 K_\Sigma = 0$, using (8.3). This implies that $\mathrm{Ran}K_\Sigma \subset \mathrm{Ker}\Pi_0$. Conversely let $g \in \mathrm{Ker}\Pi_0$, i.e.

$$g^0_\Sigma = \pi g^0_\Sigma, \ g^1_t = 0, \ g^1_\Sigma = -\mathrm{i}abg^0_\Sigma.$$

From the first equation we get $g^0_\Sigma = \bar{d}_\Sigma u^0$ for $u^0 = bg^0_\Sigma \in H^1(\Sigma; \mathfrak{g})$, and hence $g^1_\Sigma = -\mathrm{i}au^0$, i.e. $g = K_\Sigma u$, for $u = (u^0, \mathrm{i}^{-1} g^0_t)$. □

We end this subsection by constructing an operator B_0 such that $(\mathbf{1} - \Pi_0) = K_\Sigma B_0$ (see the discussion at the end of Subsect. 3.4).

Lemma 8.4. *Let* $B_0 : L^2(\Sigma; W \oplus W) \to \langle x \rangle L^2(\Sigma; W_t) \oplus L^2(\Sigma; W_t)$ *be given by:*

$$B_0 := \begin{pmatrix} 0 & b & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \tag{8.5}$$

Then one has

$$(\mathbf{1} - \Pi_0) = K_\Sigma B_0, \quad B_0 K_\Sigma = \mathbf{1}.$$

Proof. The proof is a direct computation that uses $\bar{d}_\Sigma b = \pi$. □

8.3. **The reference projection for Σ compact.** In this subsection, we assume that Σ is compact. This implies that $\mathrm{Ker}h_t = \mathrm{Ker}\bar{d}_\Sigma$ is not necessarily trivial. Therefore we need to change the definition of π, b and Π_0. We set now:

Definition 8.5.

$$\pi := \bar{d}_\Sigma h_t^{-1} \mathbb{1}_{\mathbb{R}\setminus\{0\}}(h_t)\bar{\delta}_\Sigma : L^2(\Sigma; W_\Sigma) \to L^2(\Sigma; W_\Sigma),$$
$$b := h_t^{-1}\mathbb{1}_{\mathbb{R}\setminus\{0\}}(h_t)\bar{\delta}_\Sigma : \ L^2(\Sigma; W_\Sigma) \to L^2(\Sigma; W_t),$$
$$\mathrm{a} := \bar{F}_t \wedge \cdot : \ L^2(\Sigma; W_t) \to L^2(\Sigma; W_\Sigma),$$

where $\mathbb{1}_{\mathbb{R}\setminus\{0\}}$ stands for the characteristic function of $\mathbb{R}\setminus\{0\}$.

Note that since h_t has compact resolvent, we know that

$$\pi \in \Psi^0(\Sigma; W_\Sigma), \ b \in \Psi^{-1}(\Sigma; W_\Sigma, W_t), \ a \in \Psi^0(\Sigma; W_t, W_\Sigma). \tag{8.6}$$

We also denote by $\pi_1 : L^2(\Sigma; W_\Sigma) \to L^2(\Sigma; W_\Sigma)$ a bounded projection with

$$\mathrm{Ker}\,\pi_1 = \mathrm{a}(\mathrm{Ker}h_t), \tag{8.7}$$

like for example the orthogonal projection for the natural Hilbertian scalar product on $L^2(\Sigma; W_\Sigma)$ along $\mathrm{a}\mathrm{Ker}h_t$. By the ellipticity of h_t, we know that $\mathrm{Ker}h_t \subset C^\infty(\Sigma; W_t)$, hence $\mathrm{a}\mathrm{Ker}h_t \subset C^\infty(\Sigma; W_\Sigma)$ and these two spaces are finite dimensional.

This implies first that there exists a right inverse $\mathrm{a}^{-1} \in L(\mathrm{Ker}\,\pi_1, \mathrm{Ker}h_t)$ such that

$$\mathrm{a} \circ \mathrm{a}^{-1} = \mathbf{1} \text{ on } \mathrm{Ker}\,\pi_1. \tag{8.8}$$

Moreover since $\mathrm{Ker}\,\pi_1$ is a finite dimensional subspace of $C^\infty(\Sigma; W_\Sigma)$ we have:

$$\pi_1 \in \mathbf{1} + \Psi^{-\infty}(\Sigma; W_\Sigma), \ \mathrm{a}^{-1}(\mathbf{1} - \pi_1) \in \Psi^{-\infty}(\Sigma; W_\Sigma, W_t). \tag{8.9}$$

We set now:

$$\Pi_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} - \pi & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathrm{i}\pi_1 \mathrm{a} \circ b & 0 & \pi_1 \end{pmatrix}. \tag{8.10}$$

Lemma 8.6. *Π_0 is a bounded projection on $L^2(\Sigma; W \oplus W)$ with $\mathrm{Ker}\,\Pi_0 = \mathrm{Ran}K_\Sigma$. Moreover $\Pi_0 \in \Psi^0(\Sigma; W \oplus W)$.*

Proof. The fact that Π_0 is bounded follows from the properties of π, a, b stated in Def. 8.5 and from (8.9). Again the fact that Π_0 is a projection follows from $b(\mathbf{1} - \pi) = 0$. Let us now prove that $\Pi_0 K_\Sigma = 0$ hence $\mathrm{Ran}K_\Sigma \subset \mathrm{Ker}\,\Pi_0$. By a routine computation this amounts to show that $(\mathbf{1} - \pi)d_\Sigma = 0$ and that $\pi_1 \mathrm{a}(b\bar{d}_\Sigma - \mathbf{1}) = 0$. The first identity is immediate. To prove the second, we use that $b\bar{d}_\Sigma - \mathbf{1} = \mathbb{1}_{\{0\}}(h_t)$. Then $\pi_1 \mathrm{a}\mathbb{1}_{\{0\}}(h_t) = 0$ since $\mathrm{Ker}\,\pi_1 = \mathrm{a}(\mathrm{Ker}h_t)$.

Let us now prove that $\mathrm{Ker}\,\Pi_0 \subset \mathrm{Ran}K_\Sigma$. Let $g \in \mathrm{Ker}\,\Pi_0$ i.e.

$$g^0_\Sigma = \pi g^0_\Sigma, \ g^1_t = 0, \ \pi_1(g^1_\Sigma + \mathrm{i}\mathrm{a}b g^0_\Sigma) = 0.$$

Then $g = K_\Sigma u$ for $u = (u^0, u^1)$ if

$$\mathrm{i}u^1 = g^0_t, \ \bar{d}_\Sigma u^0 = g^0_\Sigma, \ -\mathrm{i}\mathrm{a}u^0 = g^1_\Sigma. \tag{8.11}$$

We take $u^1 = \mathrm{i}^{-1} g^0_t$ and $u^0 = b g^0_\Sigma + v^0$ for $v^0 \in \mathrm{Ker}h_t$, so that $\bar{d}_\Sigma u^0 = \bar{d}_\Sigma b g^0_\Sigma = \pi g^0_\Sigma = g^0_\Sigma$. It remains to satisfy the third identity in (8.11), which yields $-\mathrm{i}\mathrm{a}v^0 = g^1_\Sigma + \mathrm{i}\mathrm{a}b g^0_\Sigma$. Since $\pi_1(g^1_\Sigma + \mathrm{i}\mathrm{a}b g^0_\Sigma) = 0$, we can find $v^0 \in \mathrm{Ker}h_t$ satisfying the above condition, using that $\mathrm{Ker}\,\pi_1 = \mathrm{a}\mathrm{Ker}h_t$. The fact that $\Pi_0 \in \Psi^0$ follows from (8.6) and (8.9). □

We need the analog of Lemma 8.4 in the compact case.

Lemma 8.7. *Let $B_0 : L^2(\Sigma; W \oplus W) \to L^2(\Sigma; W_t) \oplus L^2(\Sigma; W_t)$ be given by:*

$$B_0 := \begin{pmatrix} 0 & b - a^{-1}(\mathbf{1} - \pi_1)ab & 0 & \mathrm{i}a^{-1}(\mathbf{1} - \pi_1) \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \tag{8.12}$$

where $a^{-1} : \mathrm{Ker}\,\pi_1 \to \mathrm{Ker}h_t$ is defined in (8.8). Then one has

$$(\mathbf{1} - \Pi_0) = K_\Sigma B_0, \quad B_0 K_\Sigma = \mathbf{1}. \tag{8.13}$$

Moreover $B_0 \in \Psi^\infty(\Sigma; W \oplus W, W_t \oplus W_t)$.

Proof. Again the first property of B_0 is a direct computation, the fact that $B_0 \in \Psi^\infty$ follows from (8.6), (8.9). □

8.4. **Change of Cauchy data.** In this section we systematically work with the adapted Cauchy data, in which the operators K_Σ and $K_\Sigma^\dagger$ take simple forms. Therefore the operator $r_{-1,R} \in \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)$ appearing in Prop. 6.7 is replaced by $R_{\mathrm{F}} \circ r_{-1,R} \circ R_{\mathrm{F}}^{-1}$.

Moreover it is convenient to perform another change of Cauchy data, corresponding to putting different weights on the two components f^0, f^1 or g^0, g^1 of a set of Cauchy data. The need for these weights is already apparent from the presence of the matrix

$$S := \begin{pmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{pmatrix}, \tag{8.14}$$

in the expression of the operator T_R in Prop. 6.2. It can also be seen from the fact that the natural space of Cauchy data appearing for example in the quantization of the scalar Klein-Gordon equation is $H^{\frac{1}{2}}(\Sigma) \oplus H^{-\frac{1}{2}}(\Sigma)$. It is convenient to treat the two components of the Cauchy data as follows: If $f \in \mathcal{H}(\Sigma; W \oplus W)$ and $g = R_{\mathrm{F}} f$ we will set

$$\tilde{f} := Sf, \quad \tilde{g} := Sg. \tag{8.15}$$

Note that S maps $H^{\frac{1}{2}}(\Sigma; W) \oplus H^{-\frac{1}{2}}(\Sigma; W)$ into $L^2(\Sigma; W \oplus W)$. Let us now collect a few properties of S. Clearly

$$S^* q_1 S = q_1,$$

i.e. S is symplectic. Moreover:

$$\begin{aligned} S\Psi^p_{\mathrm{as}}(\Sigma; W \oplus W)S^{-1} &= \Psi^p_{\mathrm{as}}(\Sigma; W \oplus W), \\ S\Psi^p_{\mathrm{reg}}(\Sigma; W \oplus W)S^{-1} &= \Psi^p_{\mathrm{reg}}(\Sigma; W \oplus W). \end{aligned} \tag{8.16}$$

If $\tilde{f}, \tilde{g}$ are as in (8.15), then $\tilde{g} = \tilde{R}_{\mathrm{F}} \tilde{f}$ for

$$\tilde{R}_{\mathrm{F}} := S R_{\mathrm{F}} S^{-1} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & -\mathrm{i}\tilde{\delta}_\Sigma & \mathbf{1} & 0 \\ \mathrm{i}\tilde{d}_\Sigma & 0 & 0 & \mathbf{1} \end{pmatrix} \in \Psi^0(\Sigma; W \oplus W), \tag{8.17}$$

and

$$\tilde{\delta}_\Sigma := \epsilon_t^{-\frac{1}{2}} \bar{\delta}_\Sigma \epsilon_\Sigma^{-\frac{1}{2}}, \ \tilde{d}_\Sigma := \epsilon_\Sigma^{-\frac{1}{2}} \bar{d}_\Sigma \epsilon_t^{-\frac{1}{2}}. \tag{8.18}$$

Finally let us express the transformed reference projection. If $\Sigma = \mathbb{R}^d$ then:

$$\tilde{\Pi}_0 := S\Pi_0 S^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} - \epsilon_\Sigma^{\frac{1}{2}} \pi \epsilon_\Sigma^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathrm{i}\epsilon_\Sigma^{-\frac{1}{2}} a \circ b \epsilon_\Sigma^{-\frac{1}{2}} & 0 & \mathbf{1} \end{pmatrix}, \tag{8.19}$$

and if Σ is compact:

$$\tilde{\Pi}_0 := S\Pi_0 S^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} - \epsilon_\Sigma^{\frac{1}{2}} \pi \epsilon_\Sigma^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathrm{i}\epsilon_\Sigma^{-\frac{1}{2}} \pi_1 a \circ b \epsilon_\Sigma^{-\frac{1}{2}} & 0 & \epsilon_\Sigma^{-\frac{1}{2}} \pi_1 \epsilon_\Sigma^{\frac{1}{2}} \end{pmatrix}. \tag{8.20}$$

8.5. **Operator classes for adapted Cauchy data.** It follows from the above discussion that after going to the adapted Cauchy data and conjugating by S, the class $\Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)$ appearing in Sect. 5 should be replaced by $\tilde{R}_{\mathrm{F}} \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W) \tilde{R}_{\mathrm{F}}^{-1}$, which is different from $\Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W)$. In this subsection we introduce classes of pseudodifferential operators in which the operator equation $\tilde{\delta}_\Sigma \circ v = r$ can be solved in v (see Lemma 8.10) and which contain the class $\tilde{R}_{\mathrm{F}} \Psi^{-1}_{\mathrm{reg}}(\Sigma; W \oplus W) \tilde{R}_{\mathrm{F}}^{-1}$. We first introduce some notation.

In the sequel i, j are indices equal to either 0 or 1, and α, β are indices equal to either t or Σ. If $\alpha = t$, resp. Σ, we set $\overline{\alpha} = \Sigma$, resp. t and:

$$s_\alpha = \begin{cases} \tilde{d}_\Sigma, & \text{if } \alpha = t, \\ \tilde{\delta}_\Sigma, & \text{if } \alpha = \Sigma, \end{cases}$$

so that $s_\alpha \in \Psi^0(\Sigma; W_\alpha, W_{\overline{\alpha}})$.

If $c \in \Psi^p(\Sigma; W \oplus W)$ we denote by $c_{i\alpha, j\beta}$ its matrix entries according to the decomposition

$$W \oplus W = (W_t \oplus W) \oplus (W_t \oplus W_\Sigma) = W_{0t} \oplus W_{0\Sigma} \oplus W_{1t} \oplus W_{1\Sigma}.$$

Recall also that $\chi_>$ denotes a cutoff function as in (5.5).

Definition 8.8. *Let $p \in \mathbb{R}$.*

(1) *We set*

$$\begin{aligned} \widetilde{\Psi}^p_{\mathrm{reg,r}}(\Sigma; W_\beta, W_\alpha) &:= \Psi^p_{\mathrm{as}}(\Sigma; W_\beta, W_\alpha) \chi_>(h_\beta) + \Psi^p_{\mathrm{as}}(\Sigma; W_{\overline{\beta}}, W_\alpha) s_\beta, \\ \widetilde{\Psi}^p_{\mathrm{reg,l}}(\Sigma; W_\beta, W_\alpha) &:= \chi_>(h_\alpha) \Psi^p_{\mathrm{as}}(\Sigma; W_\beta, W_\alpha) + s_{\overline{\alpha}} \Psi^p_{\mathrm{as}}(\Sigma; W_\beta, W_{\overline{\alpha}}), \\ \widetilde{\Psi}^p_{\mathrm{reg}}(\Sigma; W_\beta, W_\alpha) &:= \chi_>(h_\alpha) \Psi^p_{\mathrm{as}}(\Sigma; W_\beta, W_\alpha) \chi_>(h_\beta) + s_{\overline{\alpha}} \Psi^p_{\mathrm{as}}(\Sigma; W_\beta, W_{\overline{\alpha}}) \chi_>(h_\beta) \\ &\quad + s_{\overline{\alpha}} \Psi^p_{\mathrm{as}}(\Sigma; W_\beta, W_{\overline{\alpha}}) \chi_>(h_\beta) + s_{\overline{\alpha}} \Psi^p_{\mathrm{as}}(\Sigma; W_{\overline{\beta}}, W_{\overline{\alpha}}) s_\alpha. \end{aligned}$$

(2) *We say that $c \in \widetilde{\Psi}^p_{\mathrm{reg},\sharp}(\Sigma; W \oplus W)$ for $\sharp = \mathrm{l}, \mathrm{r}$, if $c_{i\alpha, j\beta} \in \widetilde{\Psi}^p_{\mathrm{reg},\sharp}(\Sigma; W_\alpha, W_\beta)$ for all i, α, j, β.*

The next lemma shows that the above classes have similar properties to $\Psi^p_{\mathrm{reg}}(\Sigma; W \oplus W)$.

Lemma 8.9. *The following properties hold:*

(1) $\tilde{R}_{\mathrm{F}} \Psi^p_{\mathrm{as}}(\Sigma; W \oplus W) \tilde{R}_{\mathrm{F}}^{-1} = \Psi^p_{\mathrm{as}}(\Sigma; W \oplus W)$,

(2) $\tilde{R}_{\mathrm{F}} \Psi^p_{\mathrm{reg}}(\Sigma; W \oplus W) \tilde{R}_{\mathrm{F}}^{-1} \subset \widetilde{\Psi}^p_{\mathrm{reg}}(\Sigma; W \oplus W) \subset \Psi^p_{\mathrm{as}}(\Sigma; W \oplus W)$,

(3) *Let $c_R \in \Psi^{-\varepsilon}_{\mathrm{reg},\sharp}(\Sigma; W \oplus W)$ for $\varepsilon > 0$ and let $\alpha \in \mathbb{R}$. Then for $R \geq R_0$ we have*

$$(\mathbf{1} + c_R)^\alpha \in \mathbf{1} + \Psi^{-\varepsilon}_{\mathrm{reg},\sharp}(\Sigma; W \oplus W).$$

Proof. (1) follows from the fact that the class Ψ^p_{as} is invariant under left or right composition with elements of Ψ^0. (2) is a routine computation, introducing the matrix entries of some $c \in \Psi^p_{\mathrm{reg}}(\Sigma; W \oplus W)$ and using (8.17). To prove (3) we use the identity $(\mathbf{1} - a)^{-1} = \mathbf{1} + a + a(\mathbf{1} - a)^{-1} a$ and the following easy observations:

$$\Psi^0_{\mathrm{as}} \widetilde{\Psi}^{-\varepsilon}_{\mathrm{reg,r}} \subset \widetilde{\Psi}^{-\varepsilon}_{\mathrm{reg,r}}, \ \widetilde{\Psi}^{-\varepsilon}_{\mathrm{reg,l}} \Psi^0_{\mathrm{as}} \subset \widetilde{\Psi}^{-\varepsilon}_{\mathrm{reg,l}}, \ \widetilde{\Psi}^{-\varepsilon}_{\mathrm{reg,l}} \widetilde{\Psi}^{-\varepsilon}_{\mathrm{reg,r}} \subset \widetilde{\Psi}^{-2\varepsilon}_{\mathrm{reg}}. \ \square$$

We end this subsection with another technical lemma, which will motivate the introduction of the above operator classes.

Lemma 8.10. *Let $r \in \widetilde{\Psi}^p_{\mathrm{reg}}(\Sigma; W_\alpha, W_t)$ for $\alpha = t, \Sigma$. Then there exists $v \in \widetilde{\Psi}^p_{\mathrm{reg,r}}(\Sigma; W_\alpha, W_\Sigma)$ such that*

$$\tilde{\delta}_\Sigma \circ v = r.$$

Proof. Since $r \in \widetilde{\Psi}^p_{\mathrm{reg}}(\Sigma; W_\alpha, W_t)$ we can write

$$r = \chi_>(h_t)m_1 + \tilde{\delta}_\Sigma m_2,\ m_1 \in \widetilde{\Psi}^p_{\mathrm{reg,r}}(\Sigma; W_\alpha, W_t),\ m_2 \in \widetilde{\Psi}^p_{\mathrm{reg,r}}(\Sigma; W_\alpha, W_\Sigma).$$

If follows that

$$v = \epsilon_\Sigma^{\frac{1}{2}} \bar{d}_\Sigma h_t^{-\frac{1}{2}} \epsilon_t^{\frac{1}{2}} \chi_>(h_t)m_1 + m_2 \in \widetilde{\Psi}^p_{\mathrm{reg,r}}(\Sigma; W_\alpha, W_\Sigma)$$

solves $\tilde{\delta}_\Sigma \circ v = r$. □

8.6. **Technical estimates for $\Sigma = \mathbb{R}^d$.** In this subsection we collect some delicate technical estimates on the operators π, b in the case $\Sigma = \mathbb{R}^d$. It is convenient to introduce some notation related to Hypothesis 1.4: if V is a finite dimensional vector space we set:

$$S^m_0(\Sigma; V) := \{f \in C^\infty(\Sigma; V) :\ \partial_x^\alpha f(x) \in O(\langle x\rangle^m),\ \alpha \in \mathbb{N}^d\}.$$

Abusing notation we see that Hypothesis 1.4 implies that

$$\bar{A}_\Sigma \in S^0_0,\ \bar{\delta}_\Sigma \bar{F}_\Sigma \in S^{-1}_0,\ \bar{F}_t \in S^{-2}_0.$$

Recall that $B^{-\infty}(\Sigma; V_1, V_2)$ denotes the space of operators that map $H^{-m}(\Sigma; V_1) \to H^m(\Sigma; V_2)$ for all m.

Lemma 8.11. *Assume that $\Sigma = \mathbb{R}^d$. Then:*
(1) $\bar{d}_\Sigma \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma \in B^{-\infty}(\Sigma; W_\Sigma)$,
(2) $\langle x\rangle^{-1} \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma \in B^{-\infty}(\Sigma; W_\Sigma)$
(3) $\pi \in \Psi^0(\Sigma; W_\Sigma) + B^{-\infty}(\Sigma; W_\Sigma)$,
(4) $b \in \Psi^{-1}(\Sigma; W_\Sigma, W_t) + \langle x\rangle B^{-\infty}(\Sigma; W_\Sigma, W_t)$,
(5) $\chi_>(h_\Sigma)\pi \in \Psi^0(\Sigma; W_\Sigma) + \langle x\rangle^{-1} B^{-\infty}(\Sigma; W_\Sigma)$,
(6) $a \circ b \in \langle x\rangle^{-1}\Psi^{-1}(\Sigma; W_\Sigma) + \langle x\rangle^{-1} B^{-\infty}(\Sigma; W_\Sigma)$.

Proof. (1): let $A = \bar{d}_\Sigma \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma$. We need to prove that

$$(h_\Sigma^n + \mathrm{i})A(h_\Sigma^n + \mathrm{i}) : L^2 \to L^2,\ \forall n \in \mathbb{N},$$

which will follow from

$$i):\ A : L^2 \to L^2, \qquad ii):\ Ah_\Sigma : H^{-n} \to L^2,$$
$$iii):\ h_\Sigma A : L^2 \to H^n, \quad iv):\ h_\Sigma A h_\Sigma : H^{-n} \to H^n.$$

i) is straightforward by Lemma 8.1. Let us now prove *ii)*. By Lemma 5.6 (3), we have:

$$Ah_\Sigma = \bar{d}_\Sigma \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma h_\Sigma = \bar{d}_\Sigma \chi_<(h_t) \bar{\delta}_\Sigma + \bar{d}_\Sigma \chi_<(h_t) h_t^{-1} R,$$

for $R = \bar{\delta}_\Sigma \bar{F}_\Sigma \lrcorner \cdot$. The first term on the right belongs to $\Psi^{-\infty}$. We write the second term as $\bar{d}_\Sigma h_t^{-1} \langle x\rangle^{-1} \circ \langle x\rangle \chi_<(h_t) R$. The first factor is bounded on L^2 by Lemma 8.1, the second belongs to $\Psi^{-\infty}$, since $\bar{\delta}_\Sigma \bar{F}_\Sigma \in S^{-1}_0$. This implies *ii)* and hence *iii)* by duality. To prove *iv)* we write

$$\begin{aligned} h_\Sigma A h_\Sigma = &\quad h_\Sigma \bar{d}_\Sigma \chi_<(h_t) \bar{\delta}_\Sigma + h_\Sigma \bar{d}_\Sigma \chi_<(h_t) h_t^{-1} R \\ = &\quad h_\Sigma \bar{d}_\Sigma \chi_<(h_t) \bar{\delta}_\Sigma + \bar{d}_\Sigma \chi_<(h_t) R + R^* \chi_<(h_t) h_t^{-1} R. \end{aligned}$$

The first two terms belong to $\Psi^{-\infty}$. We factor the third term as:

$$R^* \chi_<(h_t) \langle x\rangle \circ \langle x\rangle^{-1} h_t^{-1} \langle x\rangle^{-1} \circ \langle x\rangle \tilde{\chi}_<(h_t) R,$$

for some cutoff function $\tilde{\chi}_<$ with the same properties as $\chi_<$ and $\tilde{\chi}_<\chi_< = \chi_<$. The first and last factor belong to $\Psi^{-\infty}$, the middle one is bounded on L^2 by Lemma 8.1. This proves *iv)* and completes the proof of (1).

(2): the proof of (2) is completely analogous to the proof of (1) and left to the reader.

(3): we write

$$\pi = \bar{d}_\Sigma \chi_>(h_t) h_t^{-1} \bar{\delta}_\Sigma + \bar{d}_\Sigma \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma.$$

The first term belongs to Ψ^0, the second to $B^{-\infty}$ by (1). This proves (3).

(4): we write

$$b = \chi_>(h_t) h_t^{-1} \bar{\delta}_\Sigma + \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma,$$

the first term belongs to Ψ^{-1}, the second to $\langle x\rangle B^{-\infty}$, by (2).

(5): We write as before:

$$\chi_>(h_\Sigma)\pi = \chi_>(h_\Sigma)\bar{d}_\Sigma \chi_>(h_t) h_t^{-1} \bar{\delta}_\Sigma + \chi_>(h_\Sigma)\bar{d}_\Sigma \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma.$$

The first term belongs to Ψ^0. We write the second term as

$$\chi_>(h_\Sigma) h_\Sigma^{-1} h_\Sigma \bar{d}_\Sigma \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma = \chi_>(h_\Sigma) h_\Sigma^{-1} \bar{d}_\Sigma \chi_<(h_t) \bar{\delta}_\Sigma + \chi_>(h_\Sigma) h_\Sigma^{-1} R^* \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma.$$

The first term belongs to $\Psi^{-\infty}$. We factor the second term as:

$$\langle x\rangle^{-1} \circ \langle x\rangle \chi_>(h_\Sigma) h_\Sigma^{-1} R^* \langle x\rangle \circ \langle x\rangle^{-1} \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma.$$

Now $\langle x\rangle \chi_>(h_\Sigma) h_\Sigma^{-1} R^* \langle x\rangle \in \Psi^0$ since $\bar{\delta}_\Sigma \bar{F}_\Sigma \in S_0^{-2}$ and $\langle x\rangle^{-1} \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma \in B^{-\infty}$ by (2). This proves that the second term belongs to $\langle x\rangle^{-1} B^{-\infty}$ and completes the proof of (5).

(6): we write once again:

$$a \circ b = a\chi_>(h_t) h_t^{-1} \bar{\delta}_\Sigma + a \circ \chi_<(h_t) h_t^{-1} \bar{\delta}_\Sigma.$$

The first term belongs to $\langle x\rangle^{-1}\Psi^{-1}$, since $\bar{F}_t \in S_0^{-1}$. The second term belongs to $\langle x\rangle^{-1} B^{-\infty}$, using (2) and the fact that $\bar{F}_t \in S_0^{-2}$. □

8.7. **Construction of the projection Π.** In this subsection we construct the projection Π. The first step consists in determining its range.

Proposition 8.12. *There exists $s_{-1,R} \in \Psi^{-1}_{\rm as}(\Sigma; W \oplus W)$ such that:*

$$(\mathbf{1} + s_{-1,R}) \mathrm{Ran}\Pi_0 \cap \mathrm{Ker} K_\Sigma^\dagger \subset (\mathbf{1} + r_{-1,R}) L^2(\Sigma; W_\Sigma \oplus W_\Sigma),$$

where $r_{-1,R} \in \Psi^{-1}_{\rm reg}(\Sigma; W \oplus W)$ is the operator in Prop. 6.7.

Proof. We set $g = R_{\rm F} f$. It is easy to check that for Π_0 given either by (8.4) or (8.10):

$$\begin{aligned} f \in \mathrm{Ker} K_\Sigma^\dagger &\Rightarrow\ g_t^1 = 0, \\ f \in \mathcal{H}'(\Sigma; W_\Sigma \oplus W_\Sigma) &\Leftrightarrow\ g_t^0 = 0,\ g_t^1 + \mathrm{i}\bar{\delta}_\Sigma g_\Sigma^0 = 0, \\ f \in \mathrm{Ran}\Pi_0 &\Rightarrow\ g_t^0 = 0,\ \bar{\delta}_\Sigma g_\Sigma^0 = 0. \end{aligned} \tag{8.21}$$

As explained in Subsect. 8.5 it is convenient to work with $\tilde{g} = Sg$, which amounts to replace $r_{-1,R}$ by $\tilde{R}_{\rm F} r_{-1,R} \tilde{R}_{\rm F}^{-1} =: \tilde{r}$, and $s_{-1,R}$ by $\tilde{R}_{\rm F} r_{-1,R} \tilde{R}_{\rm F}^{-1} =: \tilde{s}$.

By Lemma 8.9 we know that $\tilde{r} \in \widetilde{\Psi}^{-1}_{\rm reg}(\Sigma; W \oplus W)$, and we will look for $\tilde{s} \in \widetilde{\Psi}^{-1}_{\rm reg,r}(\Sigma; W \oplus W)$. Again by Lemma 8.9 it will follow that $s \in \Psi^{-1}_{\rm as}(\Sigma; W \oplus W)$.

Expressed in terms of $\tilde{g}$, the statements in (8.21) become:

$$\begin{aligned} f \in \mathrm{Ker} K_\Sigma^\dagger &\Rightarrow\ \tilde{g}_t^1 = 0, \\ f \in \mathcal{H}'(\Sigma; W_\Sigma \oplus W_\Sigma) &\Leftrightarrow\ \tilde{g}_t^0 = 0,\ \tilde{g}_t^1 + \mathrm{i}\tilde{\delta}_\Sigma \tilde{g}_\Sigma^0 = 0, \\ f \in \mathrm{Ran}\Pi_0 &\Rightarrow\ \tilde{g}_t^0 = 0,\ \tilde{\delta}_\Sigma \tilde{g}_\Sigma^0 = 0, \end{aligned} \tag{8.22}$$

where $\tilde{\delta}_\Sigma = \epsilon_t^{-\frac{1}{2}} \overline{\delta}_\Sigma \epsilon_\Sigma^{-\frac{1}{2}}$ was defined in (8.18). We set:

$$A_1 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \tilde{\delta}_\Sigma & \mathrm{i}^{-1} & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \tilde{\delta}_\Sigma & 0 & 0 \end{pmatrix},$$

so that

$$\begin{aligned} f \in (\mathbf{1}+r)\mathcal{H}'(\Sigma; W_\Sigma \oplus W_\Sigma) \quad &\Leftrightarrow \; \tilde{g} \in \mathrm{Ker}\big(A_1 \circ (\mathbf{1}+\tilde{r})^{-1}\big), \\ f \in (\mathbf{1}+s)\mathrm{Ran}\Pi_0 \quad &\Rightarrow \; \tilde{g} \in \mathrm{Ker}\big(A_2 \circ (\mathbf{1}+\tilde{s})^{-1}\big). \end{aligned} \tag{8.23}$$

To prove the proposition it suffices to find $\tilde{s} \in \Psi^{-1}_{\mathrm{reg,r}}(\Sigma; W \oplus W)$ such that

$$\tilde{g} \in \mathrm{Ker}\big(A_2 \circ (\mathbf{1}+\tilde{s})^{-1}\big), \; \tilde{g}^1_t = 0 \Rightarrow \; \tilde{g} \in \mathrm{Ker}\big(A_1 \circ (\mathbf{1}+\tilde{r})^{-1}\big). \tag{8.24}$$

Again by Lemma 8.9 (3), we know that for R large enough $(\mathbf{1}+\tilde{r})^{-1} = \mathbf{1}+\hat{r}$ for $\hat{r} \in \widetilde{\Psi}^{-1}_{\mathrm{reg}}$. Let assume that we have found $\hat{s} \in \widetilde{\Psi}^{-1}_{\mathrm{reg,r}}$ such that

$$\tilde{g} \in \mathrm{Ker}\big(A_2 \circ (\mathbf{1}+\hat{s})\big), \; \tilde{g}^1_t = 0 \Rightarrow \; \tilde{g} \in \mathrm{Ker}\big(A_1 \circ (\mathbf{1}+\hat{r})\big). \tag{8.25}$$

Then setting $\mathbf{1}+\tilde{s} := (\mathbf{1}+\hat{s})^{-1}$, we know that $\tilde{s} \in \widetilde{\Psi}^{-1}_{\mathrm{reg,r}}$ by Lemma 8.9 and that $\tilde{s}$ solves (8.24). Hence to complete the proof of the proposition, it remains to solve (8.25).

We have

$$A_1 = A_2 + A_3 \text{ for } A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}^{-1} & 0 \end{pmatrix}.$$

Therefore we look for $\hat{s} = \hat{r} + \hat{v}$ and need to find $\hat{v} \in \widetilde{\Psi}^{-1}_{\mathrm{reg,r}}$ such that:

$$A_2\hat{v} = A_3(\mathbf{1}+\hat{r}) \text{ on } \{\tilde{g}^1_t = 0\}.$$

Since $A_3 = 0$ on $\{\tilde{g}^1_t = 0\}$, we finally need to find $\hat{v}$ such that

$$A_2\hat{v} = A_3\hat{r} \text{ on } \{\tilde{g}^1_t = 0\}.$$

A routine computation yields the following equations for the entries of $\hat{v}$:

$$\begin{aligned} &\hat{v}_{0t,j\beta} = 0, \; \forall \; {}_{j\beta}, \\ &\tilde{\delta}_\Sigma \hat{v}_{0\Sigma,j\beta} = \mathrm{i}^{-1}\hat{r}_{1t,j\beta} \text{ for } {}_{j\beta=0t,\; 0\Sigma,\; 1\Sigma}. \end{aligned} \tag{8.26}$$

We can set all the other entries of $\hat{v}$ to 0. It remains to solve the equations in the second line of (8.26). This can be done by applying Lemma 8.10. This completes the proof of the proposition. □

In the proof of Prop. 8.12, we use the assumption that (M, g) is ultra-static: otherwise the expression in the second line of (8.22) becomes more complicated and it is not clear how to choose the reference projection Π_0.

If $\Sigma = \mathbb{R}^d$ we will need some further properties of the operator $s_{-1,R}$ constructed in Prop. 8.12.

Proposition 8.13. *Assume that $\Sigma = \mathbb{R}^d$. Then there exists R_0 such that for $R \geq R_0$ and for any $m \in \mathbb{R}$:*

i) $\mathbf{1} + s_{-1,R}\Pi_0 : \; H^{m+\frac{1}{2}}(\Sigma; W) \oplus H^{m-\frac{1}{2}}(\Sigma; W) \xrightarrow{\sim} H^{m+\frac{1}{2}}(\Sigma; W) \oplus H^{m-\frac{1}{2}}(\Sigma; W)$,

ii) $\langle x \rangle (\mathbf{1} + s_{-1,R}\Pi_0)\langle x \rangle^{-1} : \; H^{m+\frac{1}{2}}(\Sigma; W) \oplus H^{m-\frac{1}{2}}(\Sigma; W) \xrightarrow{\sim} H^{m+\frac{1}{2}}(\Sigma; W) \oplus H^{m-\frac{1}{2}}(\Sigma; W)$.

Proof. As before we conjugate all operators by $\tilde{R}_{\mathrm{F}}$, which amounts to replace $s_{-1,R}$ by $\tilde{s}_{-1,R} = \tilde{R}_{\mathrm{F}} s_{-1,R} \tilde{R}_{\mathrm{F}}^{-1}$, Π_0 by $\tilde{\Pi}_0 = \tilde{R}_{\mathrm{F}} \Pi_0 \tilde{R}_{\mathrm{F}}^{-1}$ and $H^{m+\frac{1}{2}} \oplus H^{m-\frac{1}{2}}$ by $H^m \oplus H^m$. From the expression (8.19) of $\tilde{\Pi}_0$ we see that the entries of $\tilde{s}_{-1,R}\tilde{\Pi}_0$ are of one of these three types:

$$1) \; \Psi^{-1}_{\mathrm{reg,r}}, \quad 2) \; \Psi^{-1}_{\mathrm{reg,r}}(\mathbf{1}-\pi), \quad 3) \; \Psi^{-1}_{\mathrm{reg,r}} a \circ b.$$

Terms of type 1) are simply considered as belonging to $\Psi^{-1}_{\rm as}$. To control terms of type 2) we recall that $\Psi^{-1}_{\rm reg,r} = \Psi^{-1}_{\rm as}\chi_>(h_\Sigma) + \Psi^{-1}_{\rm as}\tilde{\delta}_\Sigma$. By Lemma 8.11 (5) we know that $\Psi^{-1}_{\rm as}\chi_>(h_\Sigma)\pi \in \Psi^{-1}_{\rm as} + \langle x\rangle^{-1}\Psi^{-1}_{\rm as}B^{-\infty}$. The terms of type 3) belong to $\Psi^{-1}_{\rm as} + \langle x\rangle^{-1}\Psi^{-1}_{\rm as}B^{-\infty}$, by Lemma 8.11 (6). It follows that

$$\tilde{s}_{-1,R}\tilde{\Pi}_0 \in \Psi^{-1}_{\rm as} + \langle x\rangle^{-1}\Psi^{-1}_{\rm as}B^{-\infty}. \tag{8.27}$$

Let us now prove *i)*. From (8.27) we first deduce that $\|\tilde{s}_{-1,R}\tilde{\Pi}_0\|_{B(L^2)} \in o(R^0)$, hence we can find R_0 such that

$$\mathbf{1} + \tilde{s}_{-1,R}\tilde{\Pi}_0 :\ L^2(\Sigma; W\oplus W) \tilde{\to} L^2(\Sigma; W\oplus W).$$

Let us first assume that $m > 0$. We apply the identity

$$(\mathbf{1} - A)^{-1} = \sum_{j=0}^{n-1} A^j + A^n(\mathbf{1} - A)^{-1}$$

to $A = -\tilde{s}_{-1,R}\tilde{\Pi}_0$. By (8.27) we know that $\tilde{s}_{-1,R}\tilde{\Pi}_0 : H^m(\Sigma; W\oplus W) \to H^{m+1}(\Sigma; W\oplus W)$. We obtain taking n large enough that

$$(\mathbf{1} + \tilde{s}_{-1,R}\tilde{\Pi}_0)^{-1} : H^m(\Sigma; W\oplus W) \to H^m(\Sigma; W\oplus W),$$

which proves *i)* for $m > 0$. The same argument shows that for $m > 0$

$$\mathbf{1} + (\tilde{s}_{-1,R}\tilde{\Pi}_0)^* :\ H^m(\Sigma; W\oplus W) \tilde{\to} H^m(\Sigma; W\oplus W),$$

which by duality proves *i)* for $m < 0$.

To prove *ii)* we split $\tilde{s}_{-1,R}\tilde{\Pi}_0$ as $m_{1,R}+m_{2,R}$, where $m_{1,R} \in \Psi^{-1}_{\rm as}$ and $m_{2,R} \in \langle x\rangle^{-1}\Psi^{-1}_{\rm as}B^{-\infty}$. We can choose R_0 above large enough such that $(\mathbf{1} + m_{1,R})^{-1} \in \Psi^0$ for $R \geq R_0$. We have

$$(\mathbf{1} + \tilde{s}_{-1,R}\tilde{\Pi}_0)^{-1} = (\mathbf{1} + m_{1,R})^{-1}(\mathbf{1} - m_{2,R}(\mathbf{1} + \tilde{s}_{-1,R}\tilde{\Pi}_0)^{-1}).$$

Now $m_{2,R} :\ H^m \to \langle x\rangle^{-1}H^m$ and $(\mathbf{1} + m_{1,R})^{-1} : \langle x\rangle^{-1}H^m \to \langle x\rangle^{-1}H^m$ by pdo calculus, which implies that $(\mathbf{1} + \tilde{s}_{-1,R}\tilde{\Pi}_0)^{-1} : \langle x\rangle^{-1}H^m \to \langle x\rangle^{-1}H^m$. This completes the proof of the proposition. □

8.8. **The projection Π and the right inverse B.** We now define a projection Π and a right inverse B to K_Σ as in 3.4.3, 3.4.4.

Theorem 8.14. *Let Π_0 be given by (8.4) if $\Sigma = \mathbb{R}^d$ and (8.10) if Σ is compact. Let also $s_{-1,R}$ be the operator constructed in Prop. 8.12. Then there exists R_0 such that for all $R \geq R_0$:*

(1) *the operator*

$$\Pi := (\mathbf{1} + s_{-1,R})\Pi_0(\mathbf{1} + \Pi_0 s_{-1,R}\Pi_0)^{-1}$$

is a bounded projection on $L^2(\Sigma; W\oplus W)$.

(2) *moreover*

$$\mathbf{1} - \Pi = (\mathbf{1} - \Pi_0)(\mathbf{1} + s_{-1,R}\Pi_0)^{-1}.$$

(3) *one has*

a) $\operatorname{Ker}\Pi = \operatorname{Ran}K_\Sigma$,

b) $\lambda^\pm_{1\Sigma}$ *are positive on* $\operatorname{Ran}\Pi \cap \operatorname{Ker}K^\dagger_\Sigma$.

(4) $\Pi : \mathcal{H}(\Sigma; W) \to \mathcal{H}(\Sigma; W)$, $\Pi : \mathcal{H}'(\Sigma; W) \to \mathcal{H}'(\Sigma; W)$.

(5) *if Σ is compact then* $\Pi \in \Psi^\infty(\Sigma; W\oplus W)$.

Proof. If Π_0 is a bounded projection on a Hilbert space $\mathcal{H}$ and $\|r\| \ll 1$, then $\mathrm{Ker}\,\Pi_0$ and $(\mathbf{1}+r)\mathrm{Ran}\Pi_0$ are supplementary subspaces and it is easy to show that the projection Π with $\mathrm{Ker}\,\Pi = \mathrm{Ker}\,\Pi_0$ and $\mathrm{Ran}\Pi = (\mathbf{1}+r)\mathrm{Ran}\Pi_0$ is given by the formulas in (1) and (2). Statement (3a) follows from $\mathrm{Ker}\,\Pi = \mathrm{Ker}\,\Pi_0 = \mathrm{Ran}K_\Sigma$. Statement (3b) follows from $\mathrm{Ran}\Pi = (\mathbf{1}+s_{-1,R})\mathrm{Ran}\Pi_0 \subset (\mathbf{1}+r_{-1,R})\mathcal{H}(\Sigma; W_\Sigma \oplus W_\Sigma)$ by Prop. 8.12, and from Prop. 6.7.

Let us now prove (4). It suffices to prove the corresponding statements for $\mathbf{1}-\Pi$. Using that by Prop. 8.13 $(\mathbf{1}+s_{-1,R}\Pi_0)^{-1}$ maps $\mathcal{H}(\Sigma; W)$ and $\mathcal{H}'(\Sigma; W)$ into themselves, we can replace $\mathbf{1}-\Pi$ by $\mathbf{1}-\Pi_0$. The result follows then from the expression of Π_0 in (8.4) and statements (3), (6) of Lemma 8.11. Finally the fact that $\Pi \in \Psi^\infty$ if Σ is compact, follows from the same property of Π_0, see Lemma 8.6. This proves (5). □

Let us now define the right inverse B to K_Σ.

Proposition 8.15. *Let B_0 be given by (8.5) if $\Sigma = \mathbb{R}^d$ or by (8.12) if Σ is compact. Let*

$$B := B_0(\mathbf{1} + s_{-1,R}\Pi_0)^{-1}. \tag{8.28}$$

Then

$$K_\Sigma B = \mathbf{1} - \Pi, \quad BK_\Sigma = \mathbf{1}. \tag{8.29}$$

Moreover
(1) *if $\Sigma = \mathbb{R}^d$ then $B : \mathcal{H}(\Sigma; W) \to \langle x\rangle\mathcal{H}(\Sigma; W)$, $B : \mathcal{H}'(\Sigma; W) \to \langle x\rangle\mathcal{H}'(\Sigma; W)$.*
(2) *if Σ is compact then $B \in \Psi^\infty(\Sigma; W \oplus W, W_t \oplus W_t)$.*

Proof. The fact that $K_\Sigma B = \mathbf{1} - \Pi$ follows from the definitions of B, Π and the fact that $K_\Sigma B_0 = \mathbf{1} - \Pi_0$. The identity $BK_\Sigma = \mathbf{1}$ follows from $B_0 K_\Sigma = \mathbf{1}$ and $\Pi K_\Sigma = 0$. To prove (2) we can as in the proof of Thm. 8.14 replace B by B_0. The statement follows then for the expression (8.5) of B_0 and from (4) of Lemma 8.11. Finally, (2) follows from the fact that B_0, Π_0 belong to Ψ^∞, see Lemmas 8.6 and 8.7. □

8.9. **Proof of Thm. 1.1.** We now complete the proof of Thm. 1.1, by checking the assumptions of Thm. 3.17. We take for $c_i^\pm$ for $i = 0, 1$ the operators constructed in Prop. 6.3 for the operators $\partial_t^2 + a_i(t) = D_i$.

- $c_i^\pm$ are pseudodifferential operators, hence $c_i^\pm$ satisfy (3.15 i), ii) and $c_0^\pm$ satisfy (3.21) iii).

- $G_{i\Sigma}$ are equal to $\mathrm{i}\begin{pmatrix} J_i & 0 \\ 0 & -J_i \end{pmatrix}$, for J_i given in (4.3), hence conditions (3.9) and (3.21) i) are satisfied.

- K_Σ is a matrix of differential operators with coefficients bounded with all derivatives, by Hypothesis 1.4, hence conditions (3.14) and (3.21) ii) are satisfied.

- Π and B satisfy conditions (3.17) and (3.22), by Thm. 8.14 and Prop. 8.15.

- the positivity condition (3.23) is satisfied by Π, using Thm. 8.14 and the fact that $\mathrm{Ran}\Pi \cap \mathrm{Ker}K_\Sigma^\dagger = \Pi\mathrm{Ker}K_\Sigma^\dagger$ since $\mathrm{Ker}\Pi = \mathrm{Ran}K_\Sigma \subset \mathrm{Ker}K_\Sigma^\dagger$.

- the two-point functions $\lambda_{1\Sigma}^\pm$ are Hadamard, by Prop. 6.3. To prove that $\tilde{\lambda}_{1\Sigma}^\pm$ are also Hadamard, we need to check that $c_{1\mathrm{reg}}^\pm$ are regularizing. This delicate point is shown in Prop. 8.17 below. The proof of Thm. 1.1 is complete. □

Remark 8.16. *It is easy to deduce from (6.11) and the property* $\mathrm{Ker}\,\Pi = \mathrm{Ran}\,K_\Sigma$ *that the two-point functions $\tilde{\lambda}_{1\Sigma}^\pm$ we construct have the property that $\tilde{\lambda}_{1\Sigma}^+ + \tilde{\lambda}_{1\Sigma}^-$ is injective on* $\mathrm{Ker}K_\Sigma^\dagger$. *This issue is related to faithfulness of the state ω.*

Proposition 8.17. (1) *assume that $\Sigma = \mathbb{R}^d$. Then for any $n \in \mathbb{N}$ one has:*

i) $R_{-\infty}B : H^{-n}(\Sigma; W \oplus W) \to \langle x\rangle H^n(\Sigma; W \oplus W)$,

ii) $(\mathbf{1} - \Pi^\dagger)R_{-\infty}B : H^{-n}(\Sigma; W \oplus W) \to \langle x\rangle H^n(\Sigma; W \oplus W)$.

(2) *assume that* Σ *is compact. Then* $R_{-\infty}B$ *and* $(\mathbf{1}-\Pi^\dagger)R_{-\infty}B$ *belong to* $\Psi^{-\infty}(\Sigma; W\oplus W)$.

Proof. The proof of (2) is straightforward, since if Σ is compact we know that B, $(\mathbf{1}-\Pi^\dagger)\in\Psi^\infty$ and $R_{-\infty}\in\Psi^{-\infty}$.

We now turn to the proof of (1) which is much more delicate. The Sobolev spaces or pseudodifferential classes between the various vector bundles over Σ will be abbreviated H^m, Ψ^p, $m,p\in\mathbb{R}$.

We will work with the adapted Cauchy data. Note that because the operators R_{F} and R_{F}^{-1} are differential operators (see Lemma 4.1), the operator $R_{-\infty}$, expressed in term of adapted Cauchy data, i.e. $R_{\mathrm{F}}R_{-\infty}R_{\mathrm{F}}^{-1}$ belongs also to $\Psi^{-\infty}$, and will still be denoted by $R_{-\infty}$.

Let us first consider the operator $R_{-\infty}B_0$, which we write as a 4×4 matrix. A routine computation shows that the entries of $R_{-\infty}B_0$ are of one of the two forms

$$r_{-\infty},\ r_{-\infty}b, \tag{8.30}$$

for $r_{-\infty}\in\Psi^{-\infty}$. From Lemma 8.11 (4) we obtain that $b: H^{-m}\to\langle x\rangle H^{-m}$ for all $m\in\mathbb{N}$. Since $r_{-\infty}:\langle x\rangle H^{-n}\to\langle x\rangle H^n$ by pdo calculus, we obtain that $R_{-\infty}B_0: H^{-n}\to\langle x\rangle H^n$. By Prop. 8.13 *i)* we know that $1+s_{-1}\Pi_0: H^{-n}\to H^{-n}$. This completes the proof of *i)*.

The proof of *ii)* is more delicate. We claim that it suffices to prove that:

$$(\mathbf{1}-\Pi_0^\dagger)R_{-\infty}B_0: H^{-n}\to\langle x\rangle H^n,\ \forall n\in\mathbb{N}. \tag{8.31}$$

In fact by Thm. 8.14 we have:

$$(\mathbf{1}-\Pi^\dagger)=(\mathbf{1}+(s_{-1}\Pi_0)^\dagger)^{-1}(\mathbf{1}-\Pi_0^\dagger).$$

By Prop. 8.13 *i)* $(\mathbf{1}+s_{-1}\Pi_0)^{-1}: H^{-n}\to H^{-n}$, and by Prop. 8.13 *ii)* and duality $(\mathbf{1}+(s_{-1}\Pi_0)^\dagger)^{-1}:\langle x\rangle H^n\to\langle x\rangle H^n$. Hence *ii)* will follow from (8.31).

Let us now prove (8.31). We write $R_{-\infty}$ as a 4×2 matrix:

$$R_{-\infty}=\begin{pmatrix} r_{0t,0} & r_{0t,1}\\ r_{0\Sigma,0} & r_{0\Sigma,1}\\ r_{1t,0} & r_{1t,1}\\ r_{1\Sigma,0} & r_{1\Sigma,1}\end{pmatrix}.$$

Using that

$$\mathbf{1}-\Pi_0^\dagger=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{1} & 0\\ 0 & \mathrm{i}b^*a^* & 0 & \pi\end{pmatrix},$$

we obtain that the entries of $(\mathbf{1}-\Pi_0^\dagger)R_{-\infty}B_0$ are of the form (8.30), except for (sums) of the more singular terms

$$\begin{aligned} &(1)\ \pi r_{1\Sigma,1}, && (2)\ b^*a^*r_{0\Sigma,1},\\ &(3)\ b^*a^*r_{0\Sigma,0}b, && (4)\ \pi r_{1\Sigma,0}b,\end{aligned}$$

where as before all the $r_{i,j}$ terms belong to $\Psi^{-\infty}$. We will examine successively these 4 terms.

Term 1: by Lemma 8.11 (3) we know that $\pi: H^n\to H^n$ for all $n\in\mathbb{N}$, hence $\pi r_{1\Sigma,1}: H^{-n}\to H^n$.

Term 2: by Lemma 8.11 (6) and duality, we know that $b^*a^*: H^n\to H^n$, the same argument as before shows that $b^*a^*r_{0\Sigma,1}: H^{-n}\to H^n$.

The terms 3 and 4 will be more delicate to estimate. We will cut them into a high and low energy part. The high energy part is not affected by the infrared problem and is easy to estimate. The low energy part will be estimated by 'undoing the commutator', i.e. rewriting $R_{-\infty}$ as $c_1^+K_\Sigma-K_\Sigma c_0^+$.

Term 3: we write $r_{0\Sigma,0} = r_{0\Sigma,0}\chi_>(h_t) + r_{0\Sigma,0}\chi_<(h_t)$. We know that $\chi_>(h_t)b = \chi_>(h_t)h_t^{-1}\bar{\delta}_\Sigma \in \Psi^{-1}$, hence $r_{0\Sigma,0}\chi_>(h_t)b \in \Psi^{-\infty}$. This implies that $r_{0\Sigma,0}\chi_>(h_t)b : H^{-n} \to H^n$. Since by Lemma 8.11 (6) $b^*a^* : \langle x\rangle H^n \to \langle x\rangle H^n$ it follows that $b^*a^*r_{0\Sigma,0}\chi_>(h_t)b : H^{-n} \to \langle x\rangle H^n$.

It remains to control the term $b^*a^*r_{0\Sigma,0}\chi_<(h_t)b$. We claim that

$$b^*a^*r_{0\Sigma,0}\chi_<(h_t)b : H^{-n} \to \langle x\rangle H^n,\ \forall n \in \mathbb{N}. \tag{8.32}$$

To prove (8.32) we write $R_{-\infty}$ as $c_1^+K_\Sigma - K_\Sigma c_0^+$. Writing c_1^+ and c_0^+ in matrix form, we obtain after a routine computation that:

$$r_{0\Sigma,0} = m_1\bar{d}_\Sigma + m_2a + \bar{d}_\Sigma m_3,\ m_i \in \Psi^\infty.$$

We have hence to consider the three terms:

$$(3a)\ b^*a^*m_1\bar{d}_\Sigma\chi_<(h_t)b,\ (3b)\ b^*a^*m_2a\chi_<(h_t)b,\ (3c)\ b^*a^*\bar{d}_\Sigma m_3\chi_<(h_t)b,$$

and to show that each of them maps H^{-n} into $\langle x\rangle H^n$.

Term 3a: we have

$$b^*a^*m_1\bar{d}_\Sigma\chi_<(h_t)b = b^*a^*m_1\bar{d}_\Sigma\chi_<(h_t)h_t^{-1}\bar{\delta}_\Sigma.$$

Using Lemma 8.11 (1) and the fact that $m_1 \in \Psi^\infty$, we know that $m_1\bar{d}_\Sigma\chi_<(h_t)h_t^{-1}\bar{\delta}_\Sigma : H^{-n} \to H^n$. Next we use that by Lemma 8.11 (6) $b^*a^* : \langle x\rangle H^n \to \langle x\rangle H^n$.

Term 3b: by Lemma 8.11 (2) and the fact that $\bar{F}_t \in S_0^{-1}$ we know that $m_2a\chi_<(h_t)b : H^{-n} \to H^n$ and we can conclude the proof as for term 3a).

Term 3c: we use identity (2.18) to obtain that $b^*a^*\bar{d}_\Sigma = b^*\bar{\delta}_\Sigma a = \pi a$. Therefore:

$$b^*a^*\bar{d}_\Sigma m_3\chi_<(h_t)b = \pi a m_3\chi_<(h_t)b.$$

Since $\bar{F}_t \in S_0^{-1}$ we deduce from Lemma 8.11 (2) that $am_3\chi_<(h_t)b : H^{-n} \to H^n$. Next by Lemma 8.11 (3) we know that $\pi : H^n \to H^n$. This completes the proof of (8.32).

Term 4: we split $r_{1\Sigma,0}$ as $\chi_>(h_\Sigma)r_{1\Sigma,0} + \chi_<(h_\Sigma)r_{1\Sigma,0}$. By Lemma 8.11 (4) we know that $b : H^{-n} \to \langle x\rangle H^{-n}$. Since $r_{1\Sigma,0} \in \Psi^{-\infty}$ we know that $r_{1\Sigma,0} : \langle x\rangle H^{-n} \to \langle x\rangle H^n$. Finally by Lemma 8.11 (5) and duality $\pi\chi_>(h_\Sigma)\langle x\rangle H^n \to \langle x\rangle H^n$.

We now claim that:

$$\pi\chi_<(h_\Sigma)r_{1\Sigma,0}b : H^{-n} \to \langle x\rangle H^n. \tag{8.33}$$

Again we write $R_{-\infty}$ as $c_1^+K_\Sigma - K_\Sigma c_0^+$, obtain that

$$r_{1\Sigma,0} = m_1\bar{d}_\Sigma + m_2a + am_3,\ m_i \in \Psi^\infty,$$

and have to consider the three terms:

$$(4a)\ \pi\chi_<(h_\Sigma)m_1\bar{d}_\Sigma b,\ (4b)\ \pi\chi_<(h_\Sigma)m_2ab,\ (4c)\ \pi\chi_<(h_\Sigma)am_3b.$$

Term 4a: using that $\bar{d}_\Sigma b = \pi$, this term equals $\pi\chi_<(h_\Sigma)m_1\pi$, which maps H^{-n} into H^n by now standard arguments.

Term 4b, 4c: these two terms can be treated as term 3b), using that $\bar{F}_t \in S_0^{-1}$. □

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Appendix A. Background on pseudo-differential calculus

In this section we recall some facts about pseudo-differential calculus. We refer to [GW, Sect. 4] for more details. We need to extend slightly the situation in [GW] to include matrix-valued symbols.

A.1. **Notation.** - We denote by Σ either $\mathbb{R}^d$ or a smooth compact manifold. If Σ is compact we choose a smooth, non-vanishing density μ which allows to equip $C^\infty(\Sigma)$ with an Hilbertian scalar product. Typically μ will be the canonical density associated to some Riemannian metric on Σ. If $\Sigma = \mathbb{R}^d$ we use of course the Lebesgue density dx.

- We denote by V a finite dimensional complex vector space. For simplicity we assume that V is equipped with a Hilbertian scalar product, which allows to identify V and V^*.

- We denote by $C^\infty_{\mathrm{bd}}(\Sigma; V)$ the space of smooth functions $\Sigma \to V$ uniformly bounded with all derivatives. We equip $C^\infty_{\mathrm{bd}}(\Sigma; V)$ with its canonical Fréchet space structure.

- The Sobolev space of order m is denoted $H^m(\Sigma; V)$. Furthermore, we define the spaces

$$\mathcal{H}(\Sigma; V) := \bigcap_{m\in\mathbb{R}} H^m(\Sigma; V), \quad \mathcal{H}'(\Sigma; V) := \bigcup_{m\in\mathbb{R}} H^m(\Sigma; V),$$

equipped with their canonical topologies.

A.2. **Symbol classes.** We denote by $S^m(T^*\Sigma)$, $m \in \mathbb{R}$ the usual class of poly-homogeneous symbols of order m such that additionally

$$\partial_x^\alpha \partial_k^\beta a(x,k) \in O(\langle k\rangle^{m-|\beta|}),\ \alpha,\beta \in \mathbb{N}^d. \tag{A.1}$$

Similarly we will denote by $S^m(\mathbb{R})$ the class of poly-homogeneous functions $f : T^*\Sigma \to \mathbb{C}$.

We denote by $S^m_{\mathrm{h}}(T^*\Sigma) \subset S^m(T^*\Sigma)$ the subspace of symbols homogeneous of degree m in k away from 0.

These spaces are equipped with the Fréchet space topology given by the semi-norms:

$$\|a\|_{m,N} := \sup_{|\alpha|+|\beta|\leq N} |\langle k\rangle^{-m+|\beta|} \partial_x^\alpha \partial_k^\beta a|.$$

We set

$$S^{-\infty}(T^*\Sigma) := \bigcap_{m\in\mathbb{R}} S^m(T^*\Sigma), \quad S^\infty(T^*\Sigma) := \bigcup_{m\in\mathbb{R}} S^m(T^*\Sigma).$$

Let now V_1, V_2 be finite dimensional complex vector spaces equipped with non-degenerate hermitian sesquilinear forms. The spaces $S^m_{(\mathrm{h})}(T^*\Sigma)\otimes L(V_1, V_2)$ will be denoted by $S^m_{(\mathrm{h})}(T^*\Sigma; V_1, V_2)$ and by $S^m_{(\mathrm{h})}(T^*\Sigma; V)$ if $V_1 = V_2 = V$.

The subspace of *scalar* symbols $S^m(T^*\Sigma) \otimes \mathbf{1}_V$ will be denoted by $S^m_{\mathrm{scal}}(T^*\Sigma; V)$.

A.3. **Principal symbol and characteristic set.** For $a \in S^m(T^*\Sigma; V_1, V_2)$ we denote by $a_{\mathrm{pr}} \in S^m_{\mathrm{h}}(T^*\Sigma; V_1, V_2)$ the *principal part* of a, which is homogeneous of degree m.

The *characteristic set* of $a \in S^m(T^*\Sigma; V)$ is defined as

$$\mathrm{Char}(a) := \{(x,k) \in T^*\Sigma\backslash\{0\} :\ \det a_{\mathrm{pr}}(x,k) = 0\}, \tag{A.2}$$

which is conic in the k variable.

A symbol $a \in S^m(T^*\Sigma; V)$ is *elliptic* if $\mathrm{Char}(a) = \emptyset$.

A.4. **Pseudo-differential operators.** In this subsection we collect some well-known results about pseudo-differential calculus.

We denote by $\mathrm{Op} : a \mapsto \mathrm{Op}(a)$ a quantization procedure assigning to a symbol in $S^\infty(T^*\Sigma; V_1, V_2)$ a pseudo-differential operator on Σ. If Σ is compact, this quantization depends on the choice of a partition of unity on Σ and of associated coordinate mappings, the difference between two choices being a smoothing operator. If $\Sigma = \mathbb{R}^d$ w choose the Weyl quantization. One has

$$\mathrm{Op}(a) : \mathcal{H}(\Sigma; V_1) \to \mathcal{H}(\Sigma; V_2), \quad \mathrm{Op}(a) : \mathcal{H}'(\Sigma; V_1) \to \mathcal{H}'(\Sigma; V_2).$$

We denote by $\Psi^m_{(\mathrm{scal})}(\Sigma; V_1, V_2)$ the space $\mathrm{Op}(S^m_{(\mathrm{scal})}(\Sigma; V_1, V_2))$ and set

$$\Psi^{-\infty}(\Sigma; V_1, V_2) = \bigcap_{m\in\mathbb{R}} \Psi^m(\Sigma; V_1, V_2), \quad \Psi^\infty(\Sigma; V_1, V_2) = \bigcup_{m\in\mathbb{R}} \Psi^m(\Sigma; V_1, V_2).$$

We equip $\Psi^m(\Sigma; V_1, V_2)$ with the Fréchet space topology induced from the one of $S^m(T^*\Sigma; V_1, V_2)$.

Let $s, m \in \mathbb{R}$. Then the map

$$S^m(T^*\Sigma; V_1, V_2) \ni a \mapsto \mathrm{Op}(a) \in B(H^s(\Sigma; V_1), H^{s-m}(\Sigma; V_2)) \tag{A.3}$$

is continuous.

We denote by $\sigma : \Psi^\infty(\Sigma; V_1, V_2) \to S^\infty(T^*\Sigma; V_1, V_2)$ the inverse of Op, $\sigma(a)$ being called the (full) *symbol* of a.

If Σ is a compact manifold, different choices of Op lead of course to different maps σ, differing by a map from Ψ^m to S^{m-1}. On the other hand, the principal symbol map:

$$\sigma_{\mathrm{pr}} : \Psi^m(\Sigma; V_1, V_2) \to S^m_{\mathrm{h}}(T^*\Sigma; V_1, V_2)$$

is independent on the choice of the quantization.

An operator $\mathrm{Op}(a) \in \Psi^m(\Sigma; V)$ is *elliptic* if its principal symbol $\sigma_{\mathrm{pr}}(a)(x, k)$ is elliptic in $S^m(\Sigma; V)$. If $a \in \Psi^m$ is elliptic then there exists $b \in \Psi^{-m}$, unique modulo $\Psi^{-\infty}$ such that $ab = ba = \mathbf{1}$ modulo $\Psi^{-\infty}$. Such an operator b is called a *pseudo-inverse* or a *parametrix* of a. As a typical example $\mathbf{1} + b$ for $b \in \Psi^{-m}$, $m > 0$ is elliptic in Ψ^0.

A.5. **Functional calculus for pseudo-differential operators.** We recall without proof some well-known results about functional calculus and pseudo-differential operators.

Proposition A.1. *Let $a \in \Psi^m(\Sigma; V)$ for $m \geq 0$ be elliptic in $\Psi^m(\Sigma; V)$ and symmetric on $\mathcal{H}(\Sigma; V)$. Then:*

(1) *a is selfadjoint on $H^m(\Sigma; V)$,*
(2) *Denote by $\mathrm{res}(a)$ the resolvent set of a, with domain $H^m(\Sigma; V)$. Then for $z \in \mathrm{res}(a)$, $(z - a)^{-1} \in \Psi^{-m}(\Sigma; V)$,*
(3) *if $f \in S^p(\mathbb{R})$, $p \in \mathbb{R}$, then $f(a)$, defined by the functional calculus, belongs to $\Psi^{mp}(\Sigma; V)$.*
(4) *if f is elliptic in $S^p(\mathbb{R})$ then $\sigma_{\mathrm{pr}}(f(a)) = f_{\mathrm{pr}}(\sigma_{\mathrm{pr}}(a))$.*

A.6. **Propagators.** In this subsection we state some results about propagators, associated to elliptic operators in $\Psi^1(\Sigma; V)$. It is important to restrict oneself to operators with real and *scalar* principal symbols. The propagators in our presentation replace *Fourier integral operators* which are often used in the literature.

Let us fix a map $\epsilon(t) = \epsilon_1(t) + \epsilon_0(t)$, where $\epsilon_i(t) \in C^\infty(\mathbb{R}, \Psi^i(\Sigma; V))$ for $i = 0, 1$. We assume that

(1) $\epsilon_1(t)$ is *scalar*, i.e. belongs to $\Psi^1_{\mathrm{scal}}(\Sigma; V)$,
(2) $\epsilon_1(t)$ is elliptic in $\Psi^1(\Sigma; V)$,
(3) $\epsilon_1(t)$ is symmetric on $\mathcal{H}(\Sigma; V)$.

It follows by Prop. A.1 that $\epsilon_1(t)$ is selfadjoint with domain $H^1(\Sigma; V)$, hence $\epsilon(t)$ with domain $H^1(\Sigma; V)$ is closed, with non empty resolvent set.

We denote by $\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)$ the associated propagator defined by:

$$\begin{cases} \frac{\partial}{\partial t}\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma) = \mathrm{i}\epsilon(t)\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma), \\ \frac{\partial}{\partial s}\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma) = -\mathrm{i}\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)\epsilon(s), \\ \mathrm{Texp}(\int_s^s \mathrm{i}\epsilon(\sigma)d\sigma) = \mathbf{1}. \end{cases}$$

It is easy to see (see e.g. [GW, Subsect. 4.6]) that $\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)$is strongly continuous in (t, s) with values in $B(L^2(\Sigma; V))$.

Definition A.2. *We denote by $\Phi_\epsilon(t, s) : T^*\Sigma\backslash\{0\} \to T^*\Sigma\backslash\{0\}$ the symplectic flow associated to the time-dependent Hamiltonian $-\sigma_{\mathrm{pr}}(\epsilon)(t, x, k)$.*

Clearly $\Phi_\epsilon(t, s)$ is an homogeneous map of degree 0.

We now state a version of the *Egorov's theorem* for matrix-valued symbols.

Proposition A.3. (1) $\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)$ *is bounded on* $\mathcal{H}(\Sigma;V)$ *hence on* $\mathcal{H}'(\Sigma;V)$ *by duality.*
(2) *There exists* $m(t,s)\in C^\infty(\mathbb{R}^2;\Psi^0(\Sigma;V))$ *elliptic, invertible on* $L^2(\Sigma;V)$ *with* $m^{-1}(t,s)\in C^\infty(\mathbb{R}^2;\Psi^0(\Sigma;V))$ *such that*

$$\mathrm{Texp}(\textstyle\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma) = m(t,s)\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon_1(\sigma)d\sigma).$$

(3) *Let* $a\in\Psi^m(\Sigma;V)$. *Then*

$$a(t,s) := \mathrm{Texp}(\textstyle\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)a\mathrm{Texp}(\int_t^s \mathrm{i}\epsilon(\sigma)d\sigma)$$

belongs to $C^\infty(\mathbb{R}^2,\Psi^m(\Sigma;V))$. *Moreover*

$$\sigma_{\mathrm{pr}}(a)(t,s) = \sigma_{\mathrm{pr}}(a)\circ\Phi_\epsilon(s,t).$$

Proof. The proposition is well-known in the scalar case, i.e. if $\epsilon(t)=\epsilon_1(t)$, see eg [T, Sec. 0.9] for the proof. It is easy to extend it to our situation. Let us denote $\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)$, resp. $\mathrm{Texp}(\int_s^t \mathrm{i}\epsilon_1(\sigma)d\sigma)$ by $U(t,s)$ resp. $U_1(t,s)$. Setting

$$U(t,s) =: m(t,s)U_1(t,s),$$

we obtain that $m(t,s)$ solves the equation:

$$\begin{cases} \partial_t m(t,s) - \mathrm{i}\epsilon_0(t,s)m(t,s) = 0, \\ m(s,s) = \mathbf{1}, \end{cases}$$

for $\epsilon_0(t,s) := U_1(s,t)\epsilon_0(t)U_1(t,s)$. Note that $\epsilon_0(t,s)\in C^\infty(\mathbb{R}^2,\Psi^0(\Sigma;V))$, by Egorov's theorem for the scalar case. The solution is

$$m(t,s) = \mathrm{Texp}(\textstyle\int_s^t \mathrm{i}\epsilon_0(\sigma,s)d\sigma).$$

It is easy to see that $m(t,s)\in C^\infty(\mathbb{R}^2;\Psi^0(\Sigma;V))$, using for example Beals criterion. Moreover $m(t,s): L^2(\Sigma;V)\to L^2(\Sigma;V)$ is boundedly invertible, with inverse

$$m^{-1}(t,s) = \mathrm{Texp}(\textstyle\int_t^s \mathrm{i}\epsilon_0(\sigma,s)d\sigma).$$

The same argument shows that $m^{-1}(t,s)\in C^\infty(\mathbb{R}^2;\Psi^0(\Sigma;V))$, hence $m(t,s)$ is elliptic in $\Psi^0(\Sigma;V)$. This proves (2). (1) follows from (2) and the analogous result in the scalar case. We write then

$$a(t,s) = U_1(t,s)m(t,s)am^{-1}(t,s)U_1(s,t) = U_1(t,s)\tilde{a}(t,s)U_1(s,t),$$

where $\tilde{a}(t,s) = m(t,s)am^{-1}(t,s)\in C^\infty(\mathbb{R}^2,\Psi^m(\Sigma;V))$ has principal symbol $\sigma_{\mathrm{pr}}(a(t,s)) = \sigma_{\mathrm{pr}}(a)$. (3) follows then from Egorov's theorem for the scalar case. □

The following two results are proved in [GW, Sect. 4] for the scalar case. By the argument outlined in the proof of Prop. A.3 they immediately extend to our situation.

Proposition A.4. *For* $u\in\mathcal{H}'(\Sigma;V)$ *one has:*

$$\mathrm{WF}(\mathrm{Texp}(\textstyle\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)u) = \Phi_\epsilon(t,s)\mathrm{WF}(u),$$

hence

$$\mathrm{WF}'(\mathrm{Texp}(\textstyle\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)) = \{(x,k,x',k') :\ (x,k)=\Phi_\epsilon(t,s)(x',k')\}.$$

Lemma A.5. *Let* $\epsilon(t)\in C^\infty(\mathbb{R},\Psi^1(\Sigma;V))$ *as above,* $s_{-\infty}(t,s)\in C^\infty(\mathbb{R}^2,\Psi^{-\infty}(\Sigma;V))$. *Then*

$$\mathrm{Texp}(\textstyle\int_s^t \mathrm{i}\epsilon(\sigma)d\sigma)s_{-\infty}(t,s)\in C^\infty(\mathbb{R}^2,\Psi^{-\infty}(\Sigma;V)).$$

Appendix B. Some auxiliary results

B.1. **A Hardy inequality.**

Proposition B.1. *There exists $C > 0$ such that*

$$\bar{\delta}_\Sigma \bar{d}_\Sigma \geq C\langle x\rangle^{-2}, \text{ on } L^2(\mathbb{R}^d, |h|^{\frac{1}{2}}dx)\otimes\mathfrak{g}. \tag{B.4}$$

Proof. Let us denote by $M_j(x) \in L(\mathfrak{g})$ the operator $\mathrm{i}^{-1}\bar{A}_j(x)\wedge\cdot$ and note that $M_j(x)$ is selfadjoint on $(\mathfrak{g}, \mathfrak{k})$. Let

$$h_M := \sum_{j=1}^d (D_j + M_j(x))^2,$$

acting on $L^2(\mathbb{R}^d, dx)\otimes\mathfrak{g}$. We claim that the proposition follows from

$$h_M \geq C\langle x\rangle^{-2}. \tag{B.5}$$

In fact we have:

$$h_t = \bar{\delta}_\Sigma \bar{d}_\Sigma = |h|^{-\frac{1}{2}}(x)\sum_{j,k=1}^d (D_j + M_j(x))h^{jk}(x)|h|^{\frac{1}{2}}(x)(D_k + M_k(x)),$$

acting on $L^2(\mathbb{R}^d, |h|^{\frac{1}{2}}dx)\otimes\mathfrak{g}$. Clearly h_t is unitarily equivalent to:

$$\tilde{h}_t = |h|^{-\frac{1}{4}}(x)\sum_{j,k=1}^d (D_j + M_j(x))h^{jk}(x)|h|^{\frac{1}{2}}(x)(D_k + M_k(x))|h|^{-\frac{1}{4}}(x),$$

acting on $L^2(\mathbb{R}^d, dx)\otimes\mathfrak{g}$, by the map $U : u \mapsto |h|^{\frac{1}{4}}u$. It suffices to prove Hardy's inequality for $\tilde{h}_t$. Since $c_0 \leq |h|(x) \leq c_0^{-1}$ for some $c_0 > 0$, we can also replace $\tilde{h}_t$ by $|h|^{\frac{1}{4}}\tilde{h}_t|h|^{\frac{1}{4}}$. Finally since $|h|^{\frac{1}{4}}\tilde{h}_t|h|^{\frac{1}{4}} \geq Ch_M$ for some $C > 0$, we see that (B.5) implies (B.4).

Let us now prove (B.5). From the usual Hardy inequality we know that there exists $C > 0$ such that

$$-\Delta - C\langle x\rangle^{-2} \geq 0. \tag{B.6}$$

We use now the diamagnetic inequality:

$$\|\mathrm{e}^{-t(h_M - C\langle x\rangle^{-2})}u\| \leq \mathrm{e}^{-t(-\Delta - C\langle x\rangle^{-2})}\|u\|,\ u \in L^2(\mathbb{R}^d, dx)\otimes\mathfrak{g},\ t \geq 0, \tag{B.7}$$

where $\|u\|^2(x) = \overline{u}(x)\cdot\mathfrak{k}u(x)$. The proof of (B.7) can be done as in [CFKS, Thm. 1.3]. The key fact is that

$$D_j + \mathrm{i}M_j(x) = S_j^{-1}(x)D_jS_j(x)$$

for

$$S_j(x) = \mathrm{Texp}(-\mathrm{i}\textstyle\int_{x_j}^0 M_j(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_d)ds)$$

where $S_j(x)$ is unitary on $(\mathfrak{g}, \mathfrak{k})$. Using $a^{-1} = \int_0^{+\infty}\mathrm{e}^{-ta}dt$, we deduce from (B.7) that for $\varepsilon > 0$

$$\begin{aligned}(u|(h_M - C\langle x\rangle^{-2} + \varepsilon)^{-1}u)_{L^2\otimes\mathfrak{g}} &\leq (\|u\||(-\Delta - C\langle x\rangle^{-2} + \varepsilon)^{-1}\|u\|)_{L^2}\\ &\leq \varepsilon^{-1}(\|u\|\|u\|)_{L^2} = \varepsilon^{-1}(u|u)_{L^2\otimes\mathfrak{g}}.\end{aligned}$$

This implies that $h_M - C\langle x\rangle^2 \geq 0$ and completes the proof of the proposition. □

B.2. **Transition to the temporal gauge.** In this section we review the transition to the temporal gauge, explained in the language of connections.

We assume here that $g = -\beta(t,x)dt^2 + h_{ij}(t,x)dx^i dx^j$, i.e. that we are in the general globally hyperbolic case.

We set:

$$S(t,x) := \mathrm{Texp}(-\int_t^0 T_0(s,x)ds) \in C^\infty(M; L(W)),$$

so that

$$\begin{cases} \partial_t S(t,x) = S(t,x)T_0(t,x) \\ S(0,x) = \mathbf{1}_W. \end{cases}$$

Note that $S(t,x) = S_V(t,x) \otimes S_{\mathfrak{g}}(t,x)$, for:

$$S_V(t,x) = \mathrm{Texp}(-\int_t^0 \Gamma_0(s,x)ds),\ S_{\mathfrak{g}}(t,x) = \mathrm{Texp}(-\int_t^0 M_0(s,x)ds).$$

An easy computation using that T is metric for $g^{-1} \otimes \mathbb{k}$ shows that:

$$g^{-1}(t,x) \otimes \mathbb{k} = S^*(t,x)g^{-1}(0,x) \otimes \mathbb{k}S(t,x).$$

Again if we set

$$\tilde{T}_a := S\partial_a S^{-1} + ST_a S^{-1}, \quad \tilde{\rho} := S\rho S^{-1},$$

then setting $g_0^{-1}(t,x) := g^{-1}(0,x)$ we have:

$$\begin{aligned} &\partial_a g_0^{-1} \otimes \mathbb{k} = \tilde{T}_a^* g_0^{-1} \otimes \mathbb{k} + g_0^{-1} \otimes \mathbb{k}\tilde{T}_a, \\ &\tilde{\rho}^* g_0^{-1} \otimes \mathbb{k} = g_0^{-1} \otimes \mathbb{k}\tilde{\rho}, \\ &\tilde{T}_0 = 0. \end{aligned}$$

Setting $\widetilde{D}_1 = SD_1S^{-1}$ we have:

$$\widetilde{D}_1 = -|g|^{-\frac{1}{2}}\nabla_a^{\tilde{T}}|g|^{\frac{1}{2}}g^{ab}\nabla_b^{\tilde{T}} + \tilde{\rho}.$$

The conserved charge is:

$$\overline{\tilde{\zeta}_1}\tilde{q}\tilde{\zeta}_2 := \int_{\{t\}\times\Sigma} \overline{\mathrm{i}^{-1}\nabla_0^{\tilde{T}}\tilde{\zeta}_1} \cdot g_0^{-1} \otimes \mathbb{k}\tilde{\zeta}_2 + \overline{\tilde{\zeta}_1} \cdot g_0^{-1} \otimes \mathbb{k}\ \mathrm{i}^{-1}\nabla_0^{\tilde{T}}\tilde{\zeta}_2 |h|^{\frac{1}{2}}dx.$$

B.3. **Constraints for initial data of Yang-Mills equation.** In the main part of the text (Hypothesis 1.4, Thm. 1.1) we make several assumptions on the Cauchy data of the smooth solution $\bar{A}$ of the non-linear Yang-Mills equations, used to linearize the system. To be sure that such solution $\bar{A}$ exists, one needs to verify that the conditions on the Cauchy data are consistent with the *constraint equations*. Although there is already some literature on this subject [C-B, CC, Seg], it does not cover directly our case, we thus briefly discuss the constraint equations below.

We use the framework and the notations introduced in 4.4.1, in particular we assume that the spacetime (M,g) is ultra-static. We assume $\bar{A}$ is in the temporal gauge $\bar{A}_t \equiv 0$.

The definition $\bar{F} = \bar{d}\bar{A}$ gives

$$\bar{F}_\Sigma = \bar{d}_\Sigma \bar{A}_\Sigma, \tag{B.8}$$

$$\bar{F}_t = \partial_t \bar{A}_\Sigma. \tag{B.9}$$

The Yang-Mills equation $\bar{\delta}\bar{F} = 0$ reads

$$\bar{\delta}_\Sigma \bar{F}_t = 0, \tag{B.10}$$

$$\partial_t \bar{F}_t + \bar{\delta}_\Sigma \bar{F}_\Sigma = 0. \tag{B.11}$$

Taking the first time derivative of (B.8) and using (B.9) one gets

$$\partial_t \bar{F}_\Sigma = d_\Sigma \bar{F}_t + \bar{F}_t \wedge \bar{A}_\Sigma. \tag{B.12}$$

This allows to consider (B.9), (B.11) and (B.12) as evolution equations

$$\begin{cases} \partial_t \bar{A}_\Sigma = \bar{F}_t, \\ \partial_t \bar{F}_t = -\bar{\delta}_\Sigma \bar{F}_\Sigma, \\ \partial_t \bar{F}_\Sigma = d_\Sigma \bar{F}_t + \bar{F}_t \wedge \bar{A}_\Sigma, \end{cases} \tag{B.13}$$

subject to *constraint equations* (B.8) and (B.10):

$$\begin{cases} \bar{F}_\Sigma = \bar{d}_\Sigma \bar{A}_\Sigma, \\ \bar{\delta}_\Sigma \bar{F}_t = 0. \end{cases} \tag{B.14}$$

The first constraint (B.8) is not problematic in the sense that it does not restrict the set of allowed Cauchy data. It is also straightforward to see from (B.12) that it is preserved by the evolution (B.13).

The second constraint (B.10) does significantly restrict the set of allowed Cauchy data.

First, let us check that it is preserved by the evolution (this is a known result, cf. [CS] for the case of arbitrary globally hyperbolic spacetimes). Recall that $\bar{\delta}_\Sigma = \delta_\Sigma + \bar{A}_\Sigma \lrcorner \cdot$ (where for simplicity we assume the Cauchy data are real). Thus, using (B.9) and (B.11) one gets

$$\partial_t \bar{\delta}_\Sigma \bar{F}_t = \bar{\delta}_\Sigma \partial_t \bar{F}_t + \bar{F}_t \lrcorner \bar{F}_t = -\bar{\delta}_\Sigma \circ \bar{\delta}_\Sigma \bar{F}_\Sigma + \bar{F}_t \lrcorner \bar{F}_t. \tag{B.15}$$

Since (B.8) holds for any time slice, we have $\bar{d}_\Sigma \circ \bar{d}_\Sigma = \bar{F}_\Sigma \wedge \cdot$, and by taking the adjoint $\bar{\delta}_\Sigma \circ \bar{\delta}_\Sigma = \bar{F}_\Sigma \lrcorner \cdot$. Hence (B.15) gives in fact

$$\partial_t \bar{\delta}_\Sigma \bar{F}_t = -\bar{F}_\Sigma \lrcorner \bar{F}_\Sigma + \bar{F}_t \lrcorner \bar{F}_t.$$

Both terms identically vanish, as is easily seen by writing the expression for the interior product in an orthonormal frame. This proves that $\bar{\delta}_\Sigma \bar{F}_t = 0$ on each time slice.

B.3.1. *Existence of Cauchy data with decay at infinity.* One can construct examples of Cauchy data $\bar{F}_t$, $\bar{A}_\Sigma$ satisfying the constraint $\bar{\delta}_\Sigma \bar{F}_t = 0$ as follows.

Let us take $\bar{F}_t := \bar{\delta}_\Sigma G$, $G \in \mathcal{E}^2(\Sigma; \mathfrak{g})$. Then if we take $\bar{A}_\Sigma$ and G with disjoint supports, then $\bar{\delta}_\Sigma \bar{F}_t = 0$ is trivially satisfied. If, moreover, both the supports of $\bar{A}_\Sigma$ and G are compact, then the Cauchy data $\bar{A}_\Sigma, \bar{F}_\Sigma, \bar{F}_t$ have compact support, as requested in Thm. 1.1.

Let now S^m denote the space of $\mathfrak{g}$-valued functions (or differential forms) whose coefficients satisfy classical symbol estimates $\partial_x^\alpha f(x) \in O(\langle x \rangle^{m-|\alpha|})$. It suffices then to take $\bar{A}_\Sigma \in S^{-1}$ and $G \in S^{-1}$ with disjoint supports to ensure that $\bar{F}_\Sigma \in S^{-2}$ and $\bar{F}_t \in S^{-2}$. This provides a class of examples for Hypothesis 1.4.

B.4. **Global existence of smooth space-compact solutions for non-linear Yang-Mills equations.** In this subsection we explain how to deduce Prop. 3.19 from the arguments of Chruściel-Shatah [CS].

Proposition B.2. (1) *for each $\bar{A} \in \mathcal{E}^1_{\mathrm{sc}}(M) \otimes \mathfrak{g}$ there exists $\bar{A}' \in \mathcal{E}^1_{\mathrm{sc}}(M) \otimes \mathfrak{g}$ such that $\bar{A}'_t \equiv 0$ and $\bar{A}' \sim \bar{A}$.*

(2) *Assume that $\dim M \leq 4$. Let $\bar{A} \in \mathcal{E}^1_{\mathrm{sc}}(M) \otimes \mathfrak{g}$ be a solution of the non linear Yang-Mills equation (2.14) near a Cauchy surface Σ. Then there exists $\bar{A}' \in \mathcal{E}^1_{\mathrm{sc}}(M) \otimes \mathfrak{g}$ such that $\bar{A}' \sim \bar{A}$, $\bar{A}'_t \equiv 0$ and $\bar{A}'$ solves (2.14) globally.*

Proof. (1): recall that we assumed that G is represented as a subgroup of $L(V)$ for some finite dimensional vector space V. The gauge transformation generated by the map $M \ni x \mapsto \mathscr{G}(x) \in G$ is:

$$\bar{A}_\mu \mapsto \bar{A}'_\mu = \mathscr{G}^{-1} \bar{A}_\mu \mathscr{G} + \mathscr{G}^{-1} \partial_\mu \mathscr{G}.$$

Writing $M = \mathbb{R}_t \times \Sigma_x$, we obtain $\bar{A}'_t \equiv 0$ if $\partial_t \mathscr{G} + \bar{A}_t \mathscr{G} = 0$. This can be solved by

$$\mathscr{G}(t,x) = \mathrm{Texp}(\textstyle\int_0^t -\bar{A}_t(s,x) ds).$$

Since $\bar{A}_\mu \in C^\infty_{\rm sc}(M)\otimes\mathfrak{g}$, we obtain that $\mathscr{G}-\mathbf{1}\in C^\infty_{\rm sc}(M;G)$, hence $\bar{A}'_\mu\in C^\infty_{\rm sc}(M)\otimes\mathfrak{g}$.

(2): By (1) we can assume that $\bar{A}_t\equiv 0$, i.e. that $\bar{A}$ is in the temporal gauge. We recall the form of the Yang-Mills equations in the temporal gauge, recalled in [CS, Sect. 4]. Denoting by $\bar{F}_{\mu\nu}$ the curvature, we obtain the equations:

$$\text{(B.16)}\qquad \begin{cases} \partial_t \bar{A}_i = \bar{F}_{0i},\\ \mathscr{D}_t \bar{F}_{ij} = \mathscr{D}_j \bar{F}_{i0} - \mathscr{D}_i \bar{F}_{j0},\\ \mathscr{D}_t \bar{F}^{0i} = \mathscr{D}_j \bar{F}^{ji}, \end{cases}$$

where $\mathscr{D}_\mu = \nabla_\mu + [\bar{A}_\mu,\cdot]$, and $\mathscr{D}_t = \mathscr{D}_0$.

Another fact is that if $G_{\mu\nu} := \bar{F}_{\mu\nu} - \partial_\mu \bar{A}_\nu + \partial_\nu \bar{A}_\mu - [\bar{A}_\mu,\bar{A}_\nu]$ vanishes at $t=0$ and (B.16) holds in some region $I\times\mathscr{O}$ where I is a time interval, then $G_{\mu\nu}$ vanishes identically in $I\times\mathscr{O}$, hence $\bar{F}=\bar{d}\bar{A}$.

By [CS, Thm. 1.1] the local in time solution $(\bar{A}_i,\bar{F}_{ij},\bar{F}_{0j})$ of (B.16) extends globally as a smooth solution. Moreover since (B.16) is a symmetric hyperbolic semi-linear system of equations (see eg the proof of [CS, Prop. 4.1]), its solutions satisfy Huygens' principle, which implies that the global solution of (B.16) belongs to $\mathcal{E}^1_{\rm sc}(M)\otimes\mathfrak{g}$. Note that [CS] deals with the most difficult case $\dim M = 4$. It is easy to extend the result to lower dimensions. In fact if $\dim M = n < 4$, we consider $\tilde{M} = M\times\mathbb{R}^{4-n}_y$ with metric $g+dy^2$. A $1-$form $A = A_\mu(x)dx^\mu\in\mathcal{E}^1(M)\otimes\mathfrak{g}$ is extended to $\tilde{A} = A_\mu(x)dx^\mu\in\mathcal{E}^1(\tilde{M})\otimes\mathfrak{g}$. It is easy to see that A satisfies the Yang-Mills equation on M iff $\tilde{A}$ satisfies the YM equation on $\tilde{M}$. It follows that the Cauchy problem can be globally solved for smooth Cauchy data in M. The fact that a local space-compact solution extends as a global space-compact solution follows by the same argument based on Huygens' principle. □

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