An elementary proof of Fermat-Catalan conjecture.
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An basic evidence of both Catalan-Mihăilescu and Fermat-Wiles theorems and generalization to Fermat-Catalan and Beal conjectures
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Abstract

( MSC=11D04) We begin with an equation, for example: \( Y^p = X^q \pm Z^c \) and solve it.

(Keywords : Diophantine equations, Fermat-Catalan equation; Approach)

Introduction

The goal of this document is clearly to solve the Fermat-Catalan equation \( Y^p = X^q \pm Z^c \). We have some solutions, they are:

\[
\begin{align*}
 1^m + 2^3 &= 3^2 \\
 2^2 + 7^2 &= 3^4 \\
 13^2 + 7^3 &= 2^9 \\
 2^7 + 17^3 &= 71^2 \\
 3^5 + 11^4 &= 122^2 \\
 33^8 + 1549034^2 &= 15613^3 \\
 1414^3 + 2213459^2 &= 65^7 \\
 9262^3 + 15312283^2 &= 113^7 \\
 17^7 + 76271^3 &= 21063928^2 \\
 43^8 + 96222^3 &= 30042907^2
\end{align*}
\]

If we study minutely those solutions, it appears a common point, there is an exponent 2 in the formulas. It is not only the case of the Fermat-Catalan equation. Effectively, this exponent 2 appears at least in two other diophantine equations: the Fermat equation, of course, but not only, it appears also in the Catalan equation and in some Pillai equations of the form \( Y^p = X^q + a \). Our goal, here, is to show and to prove formally, with the tools of the logic and algebra, how this exponent 2 appears in the equations!

Resolution of Fermat-Catalan equation

Let Fermat-Catalan equation:

\[ Y^p = X^q + aZ^c \]

\( a = \pm 1 \)

Now, let

\[ w = \frac{\log (-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}) - \log (2Y^{p-2})}{\log (X)} \]

\( w \) exists and

\[ w \log (X) = \log (X^w) = \log (-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}) - \log (2Y^{p-2}) \]

\[ = \log \left( \frac{-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}}{2Y^{p-2}} \right) \]

Thus

\[ X^w = \frac{-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}}{2Y^{p-2}} \]
We deduce
\[ 2X^wY^{p-2} + aZ^w = \sqrt{Z^{2e} + 4Y^pX^q} \]
Or
\[ (2X^wY^{p-2} + aZ^w)^2 = Z^{2e} + 4Y^pX^q \]
\[ = 4X^{2w}Y^{2p-4} + Z^{2e} + 4aZ^X^wY^{p-2} \]
And
\[ X^{2w}Y^{2p-4} + aZ^X^wY^{p-2} = Y^pX^q \]
Or
\[ X^wY^{p-2} + aZ^w = Y^2X^{q-w} \]
Hence
\[ Y^2X^{q-w} - X^wY^{p-2} = aZ^w = Y^p - X^q = Y^2Y^{p-2} - X^wX^{q-w} \]
Thus
\[ Y^2(X^{q-w} - Y^{p-2}) + X^w(X^{q-w} - Y^{p-2}) = 0 \]
\[ = (Y^2 + X^w)(X^{q-w} - Y^{p-2}) = 0 \]
And
\[ Y^{p-2} = X^{q-w} \]
Now, let
\[ w' = \frac{\log (Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}) - \log (2X^{p-2})}{\log (Y)} \]
w' exists and
\[ w' \log (Y) = \log (Y^{w'}) = \log (Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}) - \log (2X^{p-2}) \]
\[ = \log \left( \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}}{2X^{p-2}} \right) \]
Thus
\[ Y^{w'} = \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}}{2X^{p-2}} \]
We deduce
\[ 2Y^{w'}Y^{p-2} - (Y^q - X^p) = \sqrt{(Y^q - X^p)^2 + 4Y^qX^p} \]
Or
\[ (2Y^{w'}X^{p-2} - (Y^q - X^p))^2 = (Y^q - X^p)^2 + 4Y^qX^p \]
\[ = 4Y^{2w'}X^{2p-4} + (Y^q - X^p)^2 - 4(Y^q - X^p)Y^{w'}X^{p-2} \]
And
\[ Y^{2w'}X^{2p-4} - (Y^q - X^p)Y^{w'}X^{p-2} = Y^qX^p \]
Or
\[ Y^{w'}X^{p-2} - (Y^q - X^p) = X^2Y^{q-w'} \]
Hence
\[ X^2Y^{q-w'} - Y^{w'}X^{p-2} = -Y^q + X^p = -Y^{w'}Y^{q-w'} + X^2X^{p-2} \]
Thus
\[ Y^{w'}(X^{p-2} - Y^{q-w'}) + X^2(X^{p-2} - Y^{q-w'}) = 0 \]
\[ = (Y^{w'} + X^2)(X^{p-2} - Y^{q-w'}) = 0 \]
And
\[ X^{p-2} = Y^{q-w'} \]
But
\[ X^{(p-2)^2} = Y^{(p-2)(q-w')} = X^{(q-w')(q-w')} \]
Thus
\[ (p - 2)^2 = (q - w)(q - w') \]
And
\[ X^{q-w} = Y^{p-2} = Y\sqrt{(q-w)(q-w')} \]
Thus
\[ X^{\sqrt{q-w}} = Y^{\sqrt{q-w'}} \]

But let
\[ w_1 = q - \frac{(q-w)^2}{p-2}; \quad w_2 = q - \frac{(q-w')^2}{p-2} \]

Or
\[ (q-w)^2 = (p-2)(q-w_1); \quad (q-w')^2 = (p-2)(q-w_2) \]

We have
\[ (q-w)^2(q-w')^2 = (p-2)^2(q-w_1)(q-w_2) \]

Thus
\[ (q-w)(q-w') = (q-w_1)(q-w_2) \]

And if we suppose \((p-2)(q-w)(q-w') \neq 0\)
\[ (q-w)^2 = (p-2)(q-w_1) = \sqrt{(q-w)(q-w')}(q-w_1) \]

or
\[ q - w_1 = (q-w)\sqrt{\frac{q-w'}{q-w}} \]

and
\[ (q-w')^2 = (p-2)(q-w_2) = \sqrt{(q-w)(q-w')}(q-w_2) \]
\[ q - w_2 = (q-w')\sqrt{\frac{q-w'}{q-w}} \]

But
\[ Y^{q-w_2} = Y^{(q-w')\sqrt{\frac{q-w'}{q-w}}} = X^{((p-2)\sqrt{\frac{q-w'}{q-w}})} \]
\[ = X^{(q-w)(q-w')\sqrt{\frac{q-w'}{q-w}}} = X^{q-w'} = X^{(p-2)(q-w_1)} \]

Hence
\[ Y^{q-w_2} = X^{q-w'} \]

And
\[ X^{q-w_1} = X^{(q-w)\sqrt{\frac{q-w'}{q-w}}} = X^{((p-2)\sqrt{\frac{q-w'}{q-w}})} \]
\[ = Y^{(q-w)(q-w')\frac{q-w'}{q-w}} = Y^{q-w} = Y^{(p-2)(q-w_1)} \]

Hence
\[ X^{q-w_1} = Y^{q-w}; \quad Y^{q-w_2} = X^{q-w'} \]

Then
\[ X^{\sqrt{q-w}} = Y^{\sqrt{q-w}} \]

And
\[ X^{\sqrt{q-w_1}} = Y^{\sqrt{q-w_2}} \]

But
\[ \frac{w_2 - w'}{w - w_1} = \frac{q - w' - (q-w_2)}{q - w_1 - (q-w_1)} = \frac{q - w' - (q-w')\sqrt{\frac{q-w'}{q-w}}}{(q-w)\sqrt{\frac{q-w'}{q-w}} - (q-w)} \]
\[ = \frac{(q-w)\sqrt{q-w'}}{(q-w)\sqrt{q-w}} \]

And
\[ \frac{\sqrt{q-w}(w_2 - w)}{\sqrt{q-w'}(w_2 - w_1)} = \frac{\sqrt{q-w}(w_2 - w)}{\sqrt{q-w'}(w_2 - w_1)} \]
\[ = \frac{\sqrt{q-w}(q-w') - \sqrt{q-w}(q-w_2)}{(q-w)\sqrt{q-w'} - \sqrt{q-w'}(q-w')} \]
\[ = 1 \]

Hence
\[ \frac{w_2 - w'}{w - w_1} = \left( \frac{w_2 - w}{w' - w_1} \right)^3 \]
\[ X^{q-w'} = Y^{q-w_2} \]

3
But
\[ \sqrt[4]{w_2 - w'} Y^{\frac{1}{\sqrt{w_1 - w}}} = X Y^{\frac{w - w'}{w - w_1}} \]
\[ = Y^{\frac{1}{\sqrt{w - w_1}}} Y^{\frac{w - w'}{w - w_1}} = 1 \]

But let
\[ q - w = u(q - w') \quad q - w_1 = v(q - w_2) \]

We have
\[ \frac{q - w_1}{q - w_2} = v = \left( \frac{q - w}{q - w'} \right)^2 = u^2 \]

Thus \( v = u^2 \). And
\[ (q - w_1)(q - w_2) = u^2(q - w_2)^2 = (p - 2)^2 = (q - w)(q - w') = u(q - w')^2 \]

Thus \( (q - w')^2 = u(q - w_2)^2 \)

And
\[ (q - w)^2 = u^2(q - w') = u^3(q - w_2)^2 \]

And
\[ \frac{(q - w)^2}{(q - w_1)^2} = \frac{u^2(q - w_2)^2}{u^4(q - w_2)^2} = \frac{1}{u} \]

But
\[ (q - w) \sqrt[4]{q - w'(q - w')^2} = (q - w') \sqrt[4]{q - w'(q - w)^2} \]
\[ = (q - w_1)(q - w')^2 = (q - w_2)(q - w)^2 \]

And
\[ (p - 2)^2(q - w')^2 = (p - 2)^4(q - w_2) = (q - w_1)(q - w_2)(q - w')^2 = (q - w_2)^2(q - w)^2 \]
\[ = \frac{1}{u}(q - w_1)^2(q - w_2)^2 = \frac{1}{u}(p - 2)^4 \]

And
\[ (p - 2)^2(q - w)^2 = (p - 2)^3(q - w_1) = (q - w_1)(q - w_2)(q - w)^2 = (q - w_1)^2(q - w')^2 \]
\[ = u(q - w_1)^2(q - w_2)^2 = u(p - 2)^4 \]

We deduce
\[ p - 2 = u(q - w_2) = \frac{1}{u}(q - w_1) = \sqrt{u}(q - w') = \sqrt[4]{u}(q - w) \]

But
\[ Y \sqrt[4]{w - w'} = X \sqrt[4]{w - w'} \]
\[ Y = X \sqrt[4]{\pi} \]

And
\[ X \sqrt[4]{w_2 - w_1} = Y = X \sqrt[4]{\pi} \]

Hence
\[ w_2 - w' = \sqrt[4]{w_3(w - w_1)} = (q - w') - (q - w_2) = \frac{1}{u}(q - w) - \frac{1}{u^2}(q - w_1) = \left( \frac{1}{u} - \sqrt{u} \right)(q - w) \]
\[ = \sqrt{u^3}(q - w_1 - (q - w)) = \sqrt{u^3}(\sqrt{u} - 1)(q - w) \]

Hence
\[ (u - \sqrt{u}) = \sqrt{u}(\sqrt{u} - 1) = u^2 \sqrt{u^3}(\sqrt{u} - 1) \]

Or
\[ (\sqrt{u} - 1)\sqrt{u}(u^3 - 1) = 0 \]

It means that \( u = 1 \). But as \( GCD(X, Y) = 1 \) and
\[ Y \sqrt[4]{w - w'} = Y \sqrt[4]{w - w'} = X \sqrt[4]{w - w'} \]
Then $q - w = q - w' = 0$ or $p = 2$ and $q = w = w' = w_1 = w_2$.
This calculus is available if we replace $p - 2$ by $p - 3$ and it leads as this last case does
not exclude the case $p - 2$ to $p = 2$ or $p = 3$. The two cases lead then to $p = 2$ and
$(p = 2$ or $p = 3$) which means $p = 2$. The same calculation is available to $p = 4$ and
we have $p = 2$ and ($p = 2$ or $p = 3$ or $p = 4$) and it means $p = 2$. Etc... Until infinity.
The only solution is $p = 2$.

Resolution of Catalan equation

Let Catalan equation:

$$Y^p = X^q + 1$$

Let

$$w = \frac{\log (-1 + \sqrt{1 + 4Y^p X^q}) - \log (2) - (p - 2) \log (Y)}{\log (X)}$$

existence as we see. But

$$w \log (X) = \log (X^w) = \log (-1 + \sqrt{1 + 4Y^p X^q}) - \log (2) - \log (Y^{p - 2})$$

$$= \log \left( \frac{-1 + \sqrt{1 + 4Y^p X^q}}{2Y^{p - 2}} \right)$$

Thus

$$2X^w Y^{p - 2} + 1 = \sqrt{1 + 4Y^p X^q}$$

Or

$$(2X^w Y^{p - 2} + 1)^2 = 1 + 4Y^p X^q$$

$$= 1 + 4X^{2w} Y^{2p - 4} + 4X^{w} Y^{p - 2}$$

We deduce

$$Y^p X^q - X^{2w} Y^{2p - 4} = X^w Y^{p - 2}$$

Hence

$$Y^{2q - w} - X^{w - 1} Y^{p - 2} = 1 = Y^p - X^q = Y^{p - 2} Y^2 - X^w X^{q - w}$$

Or

$$Y^{2q - w} - Y^{p - 2} + X^{w} (X^{q - w} - Y^{p - 2}) = 0$$

$$= (Y^2 + X^w)(X^{q - w} - Y^{p - 2}) = 0$$

And as $GCD(X, Y) = 1$ it leads to $p - 2 = q - w = 0$. And

$$w = q = \frac{\log (-1 + \sqrt{1 + 4Y^p X^q}) - \log (2)}{\log (X)} \in \mathbb{N}$$

This equation leads to $(X, q) = (2, 3)$. Ko Chao has already solved the case $p = 2$.

Resolution of Fermat equation

Let Fermat equation:

$$Y^n = X^n + Z^n$$

Let here too

$$w = \frac{\log (-Z^n + \sqrt{Z^{2n} + 4Y^n X^n}) - \log (2) - (n - 2) \log (Y)}{\log (X)}$$

existence as we see. But

$$w \log (X) = \log (X^w) = \log (-Z^n + \sqrt{Z^{2n} + 4Y^n X^n}) - \log (2) - \log (Y^{n - 2})$$

$$= \log \left( \frac{-Z^n + \sqrt{Z^{2n} + 4Y^n X^n}}{2Y^{n - 2}} \right)$$

Thus

$$2X^w Y^{n - 2} + Z^n = \sqrt{Z^{2n} + 4Y^n X^n}$$

Or

$$(2X^w Y^{n - 2} + Z^n)^2 = Z^{2n} + 4Y^n X^n$$

$$= Z^{2n} + 4X^{2w} Y^{2n - 4} + 4X^{w} Y^{n - 2} Z^n$$

5
We deduce
\[ Y^nX^n - X^{2w}Y^{2n-4} = X^wY^{n-2}Z^n \]

Hence
\[ Y^2X^{n-w} - X^wY^{n-2} = Z^n = Y^n - X^n = Y^{n-2}Y^2 - X^wX^{n-w} \]

Or
\[ Y^2(X^{n-w} - Y^{n-2}) + X^w(X^{n-w} - Y^{n-2}) = 0 \]
\[ = (Y^2 + X^w)(X^{n-w} - Y^{n-2}) = 0 \]

And as \( GCD(X, Y) = 1 \) it leads to \( n - 2 = n - w = 0 \). And
\[ 2 = n = \frac{\log (-1 + \sqrt{1 + 4Y^2X^2}) - \log (2)}{\log (X)} \in \mathbb{N} \]

This equation leads to the solutions of Fermat equation for \( n > 1 \) as we will see.

**Resolution of Fermat- Catalan equation**

The only solution, in all cases, is \( p = 2 \). And \( Y^2 = X^q + aZ^c \). Thus, Fermat-Catalan equation is available for
\[ q = \frac{\log (-aZ^c + \sqrt{Z^{2c} + 4Y^2X^2}) - \log (2)}{\log (X)} \in \mathbb{N} \]
\[ = \frac{\log (Y^n - X^2 + \sqrt{(Y^n - X^2)^2 + 4Y^2X^2}) - \log (2)}{\log (Y)} \in \mathbb{N} \]

If we try successively \( q = 3 \) and \( q = 4 \), etc., we will find the \( X, Y \) which satisfy the equations.

Example
\[ Z^c = 1^c \]
\[ q = \frac{\log (-1 + \sqrt{1 + 4Y^2X^2}) - \log (2)}{\log (X)} \in \mathbb{N} \]

Implies \( X = 2, Y = 3 \) and \( q = 3 \).
\[ 1^c + 2^3 = 3^2 \]
\[ aZ^c = -2^3 = -32 \]
\[ q = \frac{\log (2^5 + \sqrt{1024 + 4Y^2X^2}) - 0.69}{\log (X)} \in \mathbb{N} \]

Implies \( X = 3, q = 4 \) and \( Y = 7 \).
\[ 2^5 + 7^2 = 3^4 \]
\[ aZ^c = -7^3 = -343 \]
\[ q = \frac{\log (343 + \sqrt{117649 + 4Y^2X^2}) - 0.69}{\log (X)} \in \mathbb{N} \]

Implies \( X = 2, q = 9 \) and \( Y = 13 \).
\[ 13^2 + 7^3 = 2^9 \]
\[ aZ^c = 17^3 = 4913 \]
\[ q = \frac{\log (-4913 + \sqrt{24137569 + 4Y^2X^2}) - 0.69}{\log (X)} \in \mathbb{N} \]

Implies \( X = 2, q = 7 \) and \( Y = 71 \).
\[ 2^7 + 17^3 = 71^2 \]
\[ aZ^c = 11^4 = 14641 \]
\[ q = \frac{\log (-14641 + \sqrt{146141^2 + 4Y^2X^2}) - 0.69}{\log (X)} \in \mathbb{N} \]
Implies \( X = 3, q = 5 \) and \( Y = 122 \).

\[
3^5 + 11^4 = 122^2
\]

\[
aZ^c = -33^8
\]

\[
q = \frac{\log (33^8 + \sqrt{33^8 + 4Y^2X^7}) - 0.69}{\log (X)} \in \mathbb{N}
\]

Implies \( X = 15613, q = 3 \) and \( Y = 1549034 \).

\[
33^8 + 1549034^2 = 15613^3
\]

\[
aZ^c = -1414^3
\]

\[
q = \frac{\log (1414^3 + \sqrt{1414^3 + 4Y^2X^7}) - 0.69}{\log (X)} \in \mathbb{N}
\]

Implies \( X = 65, q = 7 \) and \( Y = 2216459 \).

\[
1414^3 + 2216459^2 = 65^7
\]

\[
aZ^c = -9262^3
\]

\[
q = \frac{\log (9262^3 + \sqrt{9262^3 + 4Y^2X^7}) - 0.69}{\log (X)} \in \mathbb{N}
\]

Implies \( X = 113, q = 7 \) and \( Y = 15312283 \).

\[
9262^3 + 15312283^2 = 113^7
\]

\[
aZ^c = 17^7
\]

\[
q = \frac{\log (-17^7 + \sqrt{17^7 + 4Y^2X^7}) - 0.69}{\log (X)} \in \mathbb{N}
\]

Implies \( X = 76271, q = 3 \) and \( Y = 21063928 \).

\[
17^7 + 76271^3 = 21063928^2
\]

\[
aZ^c = 43^8
\]

\[
q = \frac{\log (-43^8 + \sqrt{43^8 + 4Y^2X^7}) - 0.69}{\log (X)} \in \mathbb{N}
\]

Implies \( X = 96222, q = 3 \) and \( Y = 30042907 \).

\[
43^8 + 96222^3 = 30042907^2
\]

For Fermat equation, we have \( q = n = 2 = w \) and

\[
2 = \frac{\log (-aZ^2 + \sqrt{Z^4 + 4Y^2X^2}) - \log (2)}{\log (X)} \in \mathbb{N}
\]

Example:

\[
aZ^2 = 11^4
\]

\[
2 = \frac{\log (-11^4 + \sqrt{11^8 + 4Y^2X^2}) - 0.69}{\log (X)}
\]

\[
\log (2X^2) = \log (-11^4 + \sqrt{11^8 + 4Y^2X^2})
\]

\[
(2X^2 + 11^4)^2 = 11^8 + 4Y^2X^2 = 11^8 + 4X^4 + 4(11^4)X^2
\]

\[
Y^2 = X^2 + (11^4)
\]

\[
(Y - X)(Y + X) = 11^4
\]

\[
Y + X = 11^3 = 1331
\]

\[
Y - X = 11
\]

\[
2Y = 1342
\]

\[
2X = 1320
\]

\[
Y = 671
\]
\[ X = 660 \]

Or
\[ 671^2 = 660^2 + 11^4 \]
\[ = (11(61))^2 = (11(60))^2 + 11^4 \]
\[ 61^2 + 60^2 = 11^2 \]

Or
\[ aZ^2 = 13^{12} \]
\[ Y^2 = X^2 + 13^{12} \]
\[ (Y - X)(Y + X) = 13^{12} \]
\[ Y + X = 13^9 \]
\[ Y - X = 13^3 \]
\[ 2Y = 13^6 + 13^3 = 13^2(13^6 + 1) \]
\[ 2X = 13^9 - 13^3 = 13^3(13^6 - 1) \]
\[ 4Y^2 = (13^6)(13^6 + 1)^2 = (13^6)(13^6 - 1)^2 + 4(13^{12}) \]
\[ (13^6 + 1)^2 = (13^6 - 1)^2 + (2(13^3))^2 \]

Etc...

**Conclusion**

Fermat-Catalan equation \( Y^p = X^q \pm Z^c \) has solutions only for \( p = 2 \). We have shown a way to solve it.

**References**
