An elementary proof of Fermat-Catalan conjecture.
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An evidence of both Catalan-Mihăilescu and Fermat-Wiles theorems and generalization to Fermat-Catalan and Beal conjectures

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Abstract

(MSC=11D04) We begin with an equation, for example: $Y^p = X^q \pm Z^c$ and solve it.

(Keywords: Diophantine equations, Fermat-Catalan equation; Approach)

Introduction

The goal of this document is clearly to solve the Fermat-Catalan equation $Y^b = X^q \pm Z^c$. We have some solutions, they are:

- $1^3 + 2^3 = 3^2$
- $2^2 + 7^2 = 3^4$
- $13^2 + 7^3 = 2^9$
- $2^7 + 17^3 = 71^2$
- $3^9 + 11^4 = 122^2$
- $33^3 + 1549034^2 = 15613^3$
- $1414^3 + 2213459^2 = 65^7$
- $9262^3 + 15312283^2 = 113^7$
- $17^7 + 76271^3 = 21063928^2$
- $43^8 + 96222^3 = 30042907^2$

If we study minutely those solutions, it appears a common point, there is an exponent 2 in the formulas. It is not only the case of the Fermat-Catalan equation.

Effectively, this exponent 2 appears at least in two other diophantine equations: the Fermat equation, of course, but not only, it appears also in the Catalan equation and in some Pillai equations of the form $Y^p = X^q + a$.

Our goal, here, is to show and to prove formally, with the tools of the logic and algebra, how this exponent 2 appears in the equations!

Resolution of Fermat-Catalan equation

Let Fermat-Catalan equation:

$$Y^p = X^q + aZ^c$$

$a = \pm 1$

Now, let

$$w = \frac{\log (-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}) - \log (2Y^{p-2})}{\log (X)}$$

exist and

$$w \log (X) = \log (X^w) = \log (-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}) - \log (2Y^{p-2})$$

$$= \log \left(\frac{-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}}{2Y^{p-2}}\right)$$

Thus

$$X^w = \frac{-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}}{2Y^{p-2}}$$
We deduce
\[ 2X^w Y^{p-2} + aZ^\nu = \sqrt{Z^{2\nu} + 4Y^p X^q} \]
Or
\[ (2X^w Y^{p-2} + aZ^\nu)^2 = Z^{2\nu} + 4Y^p X^q \]
= \[4X^{2w} Y^{2p-4} + Z^{2\nu} + 4aZ^X^w Y^{p-2} \]
And
\[ X^{2w} Y^{2p-4} + aZ^\nu X^w Y^{p-2} = Y^p X^q \]
Or
\[ X^w Y^{p-2} + aZ^\nu = Y^2 X^{q-w} \]
Hence
\[ Y^2 X^{q-w} - X^w Y^{p-2} = aZ^\nu = Y^p - X^q = Y^2 Y^{p-2} - X^w X^{q-w} \]
Thus
\[ Y^2 (X^{q-w} - Y^{p-2}) + X^w (X^{q-w} - Y^{p-2}) = 0 \]
= \[(Y^2 + X^w)(X^{q-w} - Y^{p-2}) = 0 \]
And
\[ Y^p - X^q = Y^{q-w} \]
Now, let
\[ w' = \log \left( \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p} - \log (2X^{p-2})}{\log (Y)} \right) \]
\[ w' \text{ exists and} \]
\[ w' \log (Y) = \log (Y^{w'}) = \log (Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p} - \log (2X^{p-2}) \]
= \[\log \left( \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p}}{2X^{p-2}} \right) \]
Thus
\[ Y^{w'} = \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p}}{2X^{p-2}} \]
We deduce
\[ 2Y^{w'} Y^{p-2} - (Y^q - X^p) = \sqrt{(Y^q - X^p)^2 + 4Y^q X^p} \]
Or
\[ (2Y^{w'} X^{p-2} - (Y^q - X^p))^2 = (Y^q - X^p)^2 + 4Y^q X^p \]
= \[4Y^{2w'} X^{2p-4} + (Y^q - X^p)^2 - 4(Y^q - X^p)Y^{w'} X^{p-2} \]
And
\[ Y^{2w'} X^{2p-4} - (Y^q - X^p)Y^{w'} X^{p-2} = Y^q X^p \]
Or
\[ Y^{w'} X^{p-2} - (Y^q - X^p) = X^2 Y^{q-w'} \]
Hence
\[ X^2 Y^{q-w'} - Y^{w'} X^{p-2} = -Y^q + X^p = -Y^{w'} Y^{q-w'} + X^2 X^{p-2} \]
Thus
\[ Y^{w'} (X^{p-2} - Y^{q-w'}) + X^2 (X^{p-2} - Y^{q-w'}) = 0 \]
= \[(Y^{w'} + X^2)(X^{p-2} - Y^{q-w'}) = 0 \]
And
\[ X^{p-2} = Y^{q-w'} \]
But
\[ X^{(p-2)^2} = Y^{(p-2)(q-w')} = X^{(q-w')(q-w')} \]
Thus
\[ (p - 2)^2 = (q - w')(q - w') \]
And
\[ X^{q-w} = Y^{p-2} = Y\sqrt{(q-w)(q-w')} \]
Thus
\[ X^{\sqrt{q-w}} = Y^{\sqrt{q-w'}} \]

But let
\[ w_1 = q - \frac{(q-w)^2}{p-2}; \quad w_2 = q - \frac{(q-w')^2}{p-2} \]

Or
\[ (q-w)^2 = (p-2)(q-w_1); \quad (q-w')^2 = (p-2)(q-w_2) \]

We have
\[ (q-w)^2(q-w')^2 = (p-2)^2(q-w_1)(q-w_2) \]

Thus
\[ (q-w)(q-w') = (q-w_1)(q-w_2) \]

And if we suppose \((p-2)(q-w)(q-w') \neq 0\)
\[ (q-w)^2 = (p-2)(q-w_1) = \sqrt{(q-w)(q-w')}(q-w_1) \]

or
\[ q-w_1 = (q-w)\sqrt{\frac{q-w'}{q-w}} \]

and
\[ (q-w')^2 = (p-2)(q-w_2) = \sqrt{(q-w)(q-w')}(q-w_2) \]

\[ q-w_2 = (q-w')\sqrt{\frac{q-w}{q-w'}} \]

But
\[ Y^{q-w_2} = Y^{(q-w')\sqrt{\frac{q-w'}{q-w}}} = X^{(p-2)\sqrt{\frac{q-w'}{q-w}}} \]

\[ = X^{\sqrt{(q-w)(q-w')\sqrt{\frac{q-w'}{q-w}}}} = X^{q-w'} = X^{\sqrt{(p-2)(q-w_1)}} \]

Hence
\[ Y^{q-w_2} = X^{q-w'} \]

And
\[ X^{q-w_1} = X^{(q-w)\sqrt{\frac{q-w'}{q-w}}} = Y^{(p-2)\sqrt{\frac{q-w'}{q-w}}} \]

\[ = Y^{\sqrt{(q-w)(q-w')\sqrt{\frac{q-w'}{q-w}}}} = Y^{q-w} = Y^{\sqrt{(p-2)(q-w_1)}} \]

Hence
\[ X^{q-w_1} = Y^{q-w_1}; \quad Y^{q-w_2} = X^{q-w'} \]

Then
\[ X^{\sqrt{q-w_1}} = Y^{\sqrt{q-w_2}} \]

And
\[ X^{\sqrt{p-w}} = Y^{\sqrt{q-w_2}}; \quad Y^{\sqrt{p-w}} = X^{\sqrt{q-w_1}} \]

But
\[ \frac{w_2 - w'}{w - w_1} = \frac{q - w' - (q-w_2)}{q - w_1 - (q-w)} = \frac{q - w' - (q-w')\sqrt{\frac{q-w'}{q-w}}} {(q-w)\sqrt{\frac{q-w'}{q-w}} - (q-w)} \]

\[ = \frac{(q-w')\sqrt{q-w'}}{(q-w)\sqrt{q-w}} \]

And
\[ \sqrt{\frac{q-w(w_2 - w)}{q-w(w' - w_1)}} = \sqrt{\frac{q-w(q-w') - q-w(q-w_2)}{q-w(q-w') - q-w(q-w_2)}} \]

\[ = \sqrt{\frac{q-w(q-w') - \sqrt{q-w(q-w')}}{(q-w)\sqrt{q-w'} - \sqrt{q-w(q-w')}}} = 1 \]

Hence
\[ \frac{w_2 - w'}{w - w_1} = \left( \frac{w_2 - w'}{w' - w_1} \right)^3 \]

\[ X^{q-w'} = Y^{q-w_2} \]

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But
\[ \sqrt{\frac{w_1 - w}{w_1}} Y \sqrt[3]{\frac{w_1 - w}{w_2}} = X \sqrt[3]{\frac{w_1 - w}{w_1}} Y \sqrt{\frac{w_1 - w}{w_2}} = Y \sqrt{\frac{w_1 - w}{w}} Y \sqrt[3]{\frac{w_1 - w}{w_1}} = 1 \]

But let
\[ q - u = v(q - w') ; \quad q - w_1 = v(q - w_2) \]

We have
\[ \frac{q - w_1}{q - w_2} = v = \left( \frac{q - w}{q - w'} \right)^2 = u^2 \]

Thus \( v = u^2 \). And
\[ (q - w_1)(q - w_2) = u^2(q - w_2)^2 = (p - 2)^2 = (q - w)(q - w') = u(q - w')^2 \]

Thus
\[ (q - w')^2 = u(q - w_2)^2 \]

And
\[ (q - w)^2 = u^2(q - w')^2 = u^3(q - w_2)^2 \]

And
\[ \frac{(q - w)^2}{(q - w_1)^2} = \frac{u^2(q - w_2)^2}{u^4(q - w_2)^2} = \frac{1}{u} \]

But
\[ (q - w) \sqrt{\frac{q - w}{q - w'}} (q - w')^2 = (q - w') \sqrt{\frac{q - w'}{q - w}} (q - w)^2 = (q - w_1)(q - w')^2 = (q - w_2)^2(q - w)^2 \]

And
\[ (p - 2)^2(q - w')^2 = (p - 2)^4(q - w_2) = (q - w_1)(q - w_2)^2(q - w')^2 = (q - w_2)^2(q - w)^2 \]

And
\[ \frac{1}{u}(q - w_1)^2(q - w_2)^2 = \frac{1}{u}(p - 2)^4 \]

And
\[ (p - 2)^2(q - w)^2 = (p - 2)^3(q - w_1) = (q - w_1)(q - w_2)(q - w) = (q - w_1)^2(q - w')^2 \]

We deduce
\[ p - 2 = u(q - w_2) = \frac{1}{u}(q - w_1) = \sqrt{u}(q - w') = \sqrt{u}(q - w) \]

But
\[ Y \sqrt{\frac{w - w'}{w}} = X \sqrt{\frac{w - w'}{w}} = X \sqrt{w(q - w')} \]

And
\[ Y = X \sqrt{\pi} \]

Hence
\[ \sqrt{w_2 - w'} = \sqrt{w^3(w - w_1)} = (q - w') - (q - w_2) = \frac{1}{u}(q - w) - \frac{1}{u^2}(q - w_1) = \left( \frac{1}{u} - \frac{\sqrt{u}}{w_2} \right)(q - w) \]

\[ = \sqrt{u^3}(q - w_1 - (q - w)) = \sqrt{u^3}(\sqrt{u} - 1)(q - w) \]

Hence
\[ (u - \sqrt{u}) = \sqrt{u}(\sqrt{u} - 1) = u^2 \sqrt{u^3}(\sqrt{u} - 1) \]

Or
\[ (\sqrt{u} - 1)\sqrt{u}(u^3 - 1) = 0 \]

It means that \( u = 1 \). But as \( GCD(X, Y) = 1 \) and
\[ Y \sqrt{\frac{w - w'}{w}} = Y \sqrt{\frac{w}{w}} = X \sqrt{\frac{w - w'}{w}} \]
Then \( q - w = q - w' = 0 \) or \( p = 2 \) and \( q = w = w' = w_1 = w_2 \).

This calculus is available if we replace \( p - 2 \) by \( p - 3 \) and it leads as this last case does not exclude the case \( p - 2 \) to \( p = 2 \) or \( p = 3 \). The two cases lead then to \( p = 2 \) and \( (p = 2 \) or \( p = 3 \)) which means \( p = 2 \). The same calculation is available to \( p = 4 \) and we have \( p = 2 \) and \( (p = 2 \) or \( p = 3 \) or \( p = 4 \)) and it means \( p = 2 \). Etc... Until infinity.

The only solution is \( p = 2 \).

**Resolution of Catalan equation**

Let Catalan equation:

\[ Y^p = X^q + 1 \]

Let \[ w = \frac{\log (-1 + \sqrt{1 + 4X^pY^q}) - \log (2) - (p - 2) \log (Y)}{\log (X)} \]

\( w \) exists as we see. But

\[ w \log (X) = \log (X^w) = \log (-1 + \sqrt{1 + 4X^pY^q}) - \log (2) - \log (Y^{p-2}) \]

\[ = \log \left( \frac{-1 + \sqrt{1 + 4X^pY^q}}{2Y^{p-2}} \right) \]

Thus

\[ 2X^wY^{p-2} + 1 = \sqrt{1 + 4X^pY^q} \]

Or

\[ (2X^wY^{p-2} + 1)^2 = 1 + 4X^pY^q \]

\[ = 1 + 4X^{2w}Y^{2p-4} + 4X^{2w}Y^{p-2} \]

We deduce

\[ Y^pX^q - X^{2w}Y^{2p-4} = X^wY^{p-2} \]

Hence

\[ Y^2X^{q-w} - X^{w}Y^{p-2} = 1 = Y^p - X^q = Y^{p-2}Y^2 - X^wX^{q-w} \]

Or

\[ Y^2(X^{q-w} - Y^{p-2}) + X^{w}(X^{q-w} - Y^{p-2}) = 0 \]

\[ = (Y^2 + X^{w})(X^{q-w} - Y^{p-2}) = 0 \]

And as \( GCD(X, Y) = 1 \) it leads to \( p - 2 = q - w = 0 \). And

\[ w = q = \frac{\log (-1 + \sqrt{1 + 4X^pY^q}) - \log (2)}{\log (X)} \in \mathbb{N} \]

This equation leads to \((X, q) = (2, 3). \) Ko Chao has already solved the case \( p = 2 \).

**Resolution of Fermat equation**

Let Fermat equation:

\[ Y^n = X^n + Z^n \]

Let here too

\[ w = \frac{\log (-Z^n + \sqrt{Z^{2n} + 4Y^nX^n}) - \log (2) - (n - 2) \log (Y)}{\log (X)} \]

\( w \) exists as we see. But

\[ w \log (X) = \log (X^w) = \log (-Z^n + \sqrt{Z^{2n} + 4Y^nX^n}) - \log (2) - \log (Y^{n-2}) \]

\[ = \log \left( \frac{-Z^n + \sqrt{Z^{2n} + 4Y^nX^n}}{2Y^{n-2}} \right) \]

Thus

\[ 2X^wY^{n-2} + Z^n = \sqrt{Z^{2n} + 4Y^nX^n} \]

Or

\[ (2X^wY^{n-2} + Z^n)^2 = Z^{2n} + 4Y^nX^n \]

\[ = Z^{2n} + 4X^{2w}Y^{2n-4} + 4X^{2w}Y^{n-2}Z^n \]
We deduce
\[ Y^n X^n - X^{2w} Y^{2n-4} = X^w Y^{n-2} Z^n \]

Hence
\[ Y^2 X^{n-w} - X^w Y^{n-2} = Z^n = Y^n - X^n = Y^n - 2Y^2 - X^w X^{n-w} \]

Or
\[ Y^2 (X^{n-w} - Y^{n-2}) + X^w (X^n - Y^{n-2}) = 0 \]

\[ = (Y^2 + X^w) (X^{n-w} - Y^{n-2}) = 0 \]

And as \( GCD(X,Y) = 1 \) it leads to \( n - 2 = n - w = 0 \). And
\[ 2 = n = \frac{\log (-1 + \sqrt{1 + 4Y^2X^2}) - \log (2)}{\log (X)} \in \mathbb{N} \]

This equation leads to the solutions of Fermat equation for \( n > 1 \) as we will see.

Resolution of Fermat- Catalan equation

The only solution, in all cases, is \( p = 2 \).

And \( Y^2 = X^q + aZ^c \). Thus, Fermat-Catalan equation is available for
\[ q = \frac{\log (-aZ^c + \sqrt{Z^{2c} + 4Y^2X^q}) - \log (2)}{\log (X)} \in \mathbb{N} \]

\[ = \frac{\log (Y^n - X^2 + \sqrt{(Y^n - X^2)^2 + 4Y^2X^q}) - \log (2)}{\log (Y)} \in \mathbb{N} \]

If we try successively \( q = 3 \) and \( q = 4 \), etc, we will find the \( X, Y \) which satisfy the equations.

Example
\[ Z^c = 1^c \]
\[ q = \frac{\log (-1 + \sqrt{1 + 4Y^2X^q}) - \log (2)}{\log (X)} \in \mathbb{N} \]

Implies \( X = 2, Y = 3 \) and \( q = 3 \).
\[ 1^c + 2^3 = 3^2 \]
\[ aZ^c = -2^5 = -32 \]
\[ q = \frac{\log (2^5 + \sqrt{1024 + 4Y^2X^q}) - 0.69}{\log (X)} \in \mathbb{N} \]

Implies \( X = 3, q = 4 \) and \( Y = 7 \).
\[ 2^5 + 7^2 = 3^4 \]
\[ aZ^c = -7^3 = -343 \]
\[ q = \frac{\log (343 + \sqrt{117649 + 4Y^2X^q}) - 0.69}{\log (X)} \in \mathbb{N} \]

Implies \( X = 2, q = 9 \) and \( Y = 13 \).
\[ 13^2 + 7^3 = 2^9 \]
\[ aZ^c = 17^3 = 4913 \]
\[ q = \frac{\log (-4913 + \sqrt{24137569 + 4Y^2X^q}) - 0.69}{\log (X)} \in \mathbb{N} \]

Implies \( X = 2, q = 7 \) and \( Y = 71 \).
\[ 2^7 + 17^3 = 71^2 \]
\[ aZ^c = 11^4 = 14641 \]
\[ q = \frac{\log (-14641 + \sqrt{14611^4 + 4Y^2X^q}) - 0.69}{\log (X)} \in \mathbb{N} \]
Imples \( X = 3, q = 5 \) and \( Y = 122 \).
\[
3^5 + 11^4 = 122^2
\]
\[
aZ^c = -33^8
\]
\[
q = \log \left( \frac{33^8 + \sqrt{33^8 + 4Y^2X^7}}{\log (X)} \right) - 0.69 \in \mathbb{N}
\]

Imples \( X = 15613, q = 3 \) and \( Y = 1549034 \).
\[
33^8 + 1549034^2 = 15613^3
\]
\[
aZ^c = -1414^3
\]
\[
q = \log \left( \frac{1414^3 + \sqrt{1414^3 + 4Y^2X^7}}{\log (X)} \right) - 0.69 \in \mathbb{N}
\]

Imples \( X = 65, q = 7 \) and \( Y = 2216459 \).
\[
1414^3 + 2213459^2 = 65^7
\]
\[
aZ^c = -9262^3
\]
\[
q = \log \left( \frac{9262^3 + \sqrt{9262^3 + 4Y^2X^7}}{\log (X)} \right) - 0.69 \in \mathbb{N}
\]

Imples \( X = 113, q = 7 \) and \( Y = 15312283 \).
\[
9262^3 + 15312283^2 = 113^7
\]
\[
aZ^c = 17^7
\]
\[
q = \log \left( \frac{-17^7 + \sqrt{17^7 + 4Y^2X^7}}{\log (X)} \right) - 0.69 \in \mathbb{N}
\]

Imples \( X = 76271, q = 3 \) and \( Y = 21063928 \).
\[
17^7 + 76271^3 = 21063928^2
\]
\[
aZ^c = 43^8
\]
\[
q = \log \left( \frac{-43^8 + \sqrt{43^8 + 4Y^2X^7}}{\log (X)} \right) - 0.69 \in \mathbb{N}
\]

Imples \( X = 96222, q = 3 \) and \( Y = 30042907 \).
\[
43^8 + 96222^3 = 30042907^2
\]

For Fermat equation, we have \( q = n = 2 = w \) and
\[
2 = \log \left( \frac{-aZ^2 + \sqrt{Z^4 + 4Y^2X^7}}{\log (X)} \right) - \log (2) \in \mathbb{N}
\]

Example:

\[
aZ^2 = 11^4
\]
\[
2 = \log \left( \frac{-11^4 + \sqrt{11^8 + 4Y^2X^7}}{\log (X)} \right)
\]
\[
(2X^2 + 11^4)^2 = 11^8 + 4Y^2X^2 = 11^8 + 4X^4 + 4(11^4)X^2
\]
\[
Y^2 = X^2 + (11^4)
\]
\[
(Y - X)(Y + X) = 11^4
\]
\[
Y + X = 11^3 = 1331
\]
\[
Y - X = 11
\]
\[
2Y = 1342
\]
\[
2X = 1320
\]
\[
Y = 671
\]
\[ X = 660 \]

Or
\[
671^2 = 660^2 + 11^4 \\
= (11(61))^2 = (11(60))^2 + 11^4 \\
61^2 + 60^2 = 11^2
\]

Or
\[
aZ^2 = 13^{12} \\
Y^2 = X^2 + 13^{12} \\
(Y - X)(Y + X) = 13^{12} \\
Y + X = 13^9 \\
Y - X = 13^3 \\
2Y = 13^6 + 13^3 = 13^3(13^3 + 1) \\
2X = 13^9 - 13^3 = 13^3(13^6 - 1) \\
4Y^2 = (13^6)(13^6 + 1)^2 = (13^6)(13^6 - 1)^2 + 4(13^{12}) \\
(13^6 + 1)^2 = (13^9 - 1)^2 + (2(13^3))^2
\]

Etc...

**Conclusion**

Fermat-Catalan equation \( Y^p = X^q \pm Z^r \) has solutions only for \( p = 2 \). We have shown a way to solve it.

**References**
