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Marcel Morales, Dung Nguyen Thi

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Veronese transform, and Castelnuovo-Mumford regularity of modules

MARCEL MORALES
Université de Grenoble I, Institut Fourier, UMR 5582, B.P.74,
38402 Saint-Martin D’Hères Cedex,
and IUFM de Lyon, 5 rue Anselme,
69317 Lyon Cedex (FRANCE)

NGUYEN THI DUNG
Thai Nguyen University of Agriculture and Forestry,
Thai Nguyen,
Vietnam,

Abstract

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. In this paper we present an original, elementary way to compute the Hilbert-Poincaré series of these rings, as a consequence we compute their Castelnuovo-Mumford regularity, and also the leading term of the $h$–vector. Moreover, we can compute the Castelnuovo-Mumford regularity of the $n$–Veronese Module of any finitely generated Cohen-Macaulay graded module.

1 Introduction

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. It is well known that these rings are arithmetically Cohen-Macaulay, hence their Hilbert-Poincaré series can be written $P_R(t) = \frac{Q_R(t)}{(1-t)^{\dim R}}$, where $Q_R(t)$ is a polynomial on $t$ with $Q_R(1) \neq 0$ having positive integer coefficients, the sequence of the coefficients of $Q_R(t)$ is also called the $h$–vector of $R$. The degree of $Q_R(t)$ is the Castelnuovo-Mumford regularity (c.f.[7][Chapter 4]), and the leading term of $Q_R(t)$ is the highest graded Betti number of $R$. By using very original and elementary methods we are able to compute the leading term of $Q_R(t)$. Our results allows to compute the Castelnuovo-Mumford regularity of the $n$–Veronese Module of any finitely generated Cohen-Macaulay graded module, and the rings called of Veronese type.
Note that this result can be proved easily by using local cohomology, but our purpose is to give a very elementary proof.

Our main results improves partially [1] and [5].

**Theorem.** Let consider the Segre-Veronese ring \( R_{b,n} \), \( \dim R_{b,n} = b_1 + \ldots + b_m + 1 \). Let \( P_{R_{b,n}}(t) = \frac{Q_{R_{b,n}}}{(1-t)^{r_{b,n}}} \) be the Hilbert-Poincaré series of \( R_{b,n} \), with \( Q_{R_{b,n}} = h_0 + h_1 t + \ldots + h_{r_{b,n}} t^{r_{b,n}} \), where \( r_{b,n} \) is the Castelnuovo-Mumford regularity of \( R_{b,n} \). We set \( \alpha_{b,n} = \dim R_{b,n} - r_{b,n} \).

After a permutation of \( b_1, \ldots, b_m \), we can assume that, for all \( i = 1, \ldots, m \), \( \lceil \frac{b_1 + 1}{n_1} \rceil > b_i \). Then

\[
\alpha_{b,n} = \lceil \frac{b_1 + 1}{n_1} \rceil,
\]

and the highest Betti number of \( R_{b,n} \) is

\[
\beta_{r_{b,n}} = h_{r_{b,n}} = \left( n_1 \alpha_{b,n} - 1 \right) \ldots \left( n_m \alpha_{b,n} - 1 \right)
\]

**Theorem.** Fix integers \( d, n \in \mathbb{N}^* \), \( \tau \in \mathbb{Z} \). Let \( (a_l)_{l \in \mathbb{Z}} \) be a sequence of complex numbers, such that \( a_l = 0 \) for \( l << 0 \), set :

\[
f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nt+\tau} t^l.
\]

If \( f(t) = \frac{h(t)}{(1-t)^\tau} \) with \( h(t) \in \mathbb{C}[t, t^{-1}] \) then \( f^{<n,\tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^\tau} \) with \( h^{<n>}(t) \in \mathbb{C}[t, t^{-1}] \) such that:

- \( \deg h^{<n,\tau>}(t) \leq d - \left\lfloor \frac{d - \deg h(t) + \tau}{n} \right\rfloor \),
- If all the coefficients of \( h(t) \) are positive real numbers then \( \deg h^{<n,\tau>}(t) = d - \left\lfloor \frac{d - \deg h(t) + \tau}{n} \right\rfloor \),
- If \( \deg h(t) = d \) then \( \deg h^{<n>}(t) = d \).

2 Preliminaries on toric rings and Hilbert-Poincaré series

Let \( S = K[x_0, \ldots, x_s, x_0^{-1}, \ldots, x_s^{-1}] \) be a Laurent polynomial ring over a field \( K \) on a finite set of variables. For any finite set \( \mathcal{M} \) of monomials in \( S \), let \( K[\mathcal{M}] \subset S \) be the subring of \( S \) generated by the set
It is the toric ring defined by the semigroup generated by \( \mathcal{M} \). In what follows we consider the special case where \( S = K[x_0, \ldots, x_s] \) is a polynomial ring over the field \( K \) and all the monomials in \( \mathcal{M} \) are of the same degree.

**Example 2.1.** Let \( S = K[x_0, \ldots, x_b] \oplus l \in \mathbb{N} S_l \), and \( \mathcal{M} = \{ x_0^{\alpha_0} \ldots x_b^{\alpha_b} \mid \alpha_0 + \ldots + \alpha_b = n \} \). So that

\[
R_{b,n} = K[\mathcal{M}] = \bigoplus_{l \in \mathbb{N}} S_{nl}.
\]

This toric ring is known as the \( n \)-Veronese embedding of \( S \).

**Example 2.2.** More generally, let \( X_1, \ldots, X_m \), \( m \) sets of independent disjoint variables, with \( \text{Card}(X_i) = b_i + 1 \). Let \( S_i = K[X_i] \) for \( i = 1, \ldots, m \), \( S = K[X_1 \cup X_2 \cup X_m] \), and \( \mathcal{M} = \{ x_1x_2 \ldots x_m \mid x_i \in X_i \} \). So that

\[
R_{b_1,\ldots,b_m} = K[\mathcal{M}] = \bigoplus_{l \in \mathbb{N}} (S_1)_l \otimes \ldots \otimes (S_m)_l.
\]

This toric ring is known as the Segre embedding of the \( m \) polynomial rings \( S_1, \ldots, S_m \).

**Example 2.3.** Let \( X_1, \ldots, X_m \), \( m \) sets of independent disjoint variables such that \( X_i = \{ x_{i,0}, \ldots, x_{i,b_i} \} \), \( S_i = K[X_i] \) for \( i = 1, \ldots, m \), and \( n_1, \ldots, n_m \in \mathbb{N} \), Let \( S = K[X_1 \cup X_2 \cup X_m] \), and

\[
\mathcal{M} = \{ \mathbb{Z}_1^{\alpha_1} \ldots \mathbb{Z}_m^{\alpha_m} \mid \alpha_i = n_i \},
\]

where \( \alpha_i = (\alpha_{i,0}, \ldots, \alpha_{i,b_i}) \), \( x_i^{\alpha_i} = x_{i,0}^{\alpha_{i,0}} \ldots x_{i,b_i}^{\alpha_{i,b_i}} \), and \( |\alpha_i| = \alpha_{i,0} + \ldots + \alpha_{i,b_i} \). The Segre-Veronese embedding:

\[
R_{\underline{b},\underline{n}} = K[\mathcal{M}] = \bigoplus_{l \in \mathbb{N}} (S_1)_{n_1l} \otimes \ldots \otimes (S_m)_{n_ml},
\]

where \( \underline{b} = (b_1, \ldots, b_m) \), \( \underline{n} = (n_1, \ldots, n_m) \).

Let \( S = K[x_0, \ldots, x_s] \) be a polynomial ring over the field \( K \), graded by the standard graduation, that is \( \deg x_i = 1 \), for all \( i \). Let \( R := S/I \), where \( I \subset S \) is a graded ideal, let \( M = \bigoplus_{l \in \mathbb{Z}} M_l \) be a finitely generated graded \( R \)-module, hence \( M \) is also a \( S \)-module.

The Hilbert-function of \( M \) is defined by \( H_M(l) = \dim_K M_l \), for all \( l \in \mathbb{Z} \), and the Hilbert-Poincaré series of \( M \):

\[
P_M(t) = \sum_{l \in \mathbb{Z}} H_M(l)t^l.
\]
It is well known that

\[ P_M(t) = \frac{Q_M(t)}{(1-t)^{\dim M}} , \]

where \( Q_M(t) \) is a Laurent polynomial on \( t, t^{-1} \) with \( Q_M(1) \neq 0 \). Moreover if \( M \) is a Cohen-Macaulay \( S \)-module, all the coefficients of \( Q_M(t) \) are natural integers, and the Castelnuovo-Mumford regularity of \( M \) is the degree of \( Q_M(t) \). For more details on Hilbert-Poincare series see [10], [4][Chapter 4], [7][Chapter 4].

**Theorem 2.4.** (Hilbert’s Theorem) Let \( M = \oplus_{l \in \mathbb{Z}} M_l \) be a finitely generated graded \( S \)-module. There exists a polynomial with integer coefficients \( \Phi_{H_M}(l) \) such that \( H_M(l) = \Phi_{H_M}(l) \), for \( l \) large enough. Moreover the leading term of \( \Phi_{H_M}(l) \) can be written as : \( \frac{\deg(M)}{d!} l^d \), where \( d + 1 \) is the dimension of \( M \) and \( \deg(M) \) the degree or multiplicity of \( M \).

**Remark 2.5.** The postulation number of the Hilbert function is the biggest integer \( l \) such that \( H_M(l) \neq \Phi_{H_M}(l) \). It is well known , ([10], [4][Chapter 4]), that the postulation number equals the degree of the rational fraction defining the Poincare series.

**Remark 2.6.** We recall that binomial coefficients can be defined in a more general setting than natural numbers, indeed for \( k \in \mathbb{N} \), binomial coefficients are polynomial functions in the variable \( n \). More precisely:

1. If \( k = 0 \) then let \( \binom{n}{0} = 1 \), for all \( n \in \mathbb{C} \).
2. If \( k > 0 \) then let \( \binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} \), for all \( n \in \mathbb{C} \).

Note that for all \( n \in \mathbb{C} \), \( \binom{n}{k} = (-1)^k \binom{k-n-1}{k} \) and if \( n \in \mathbb{N}, n < k \), then \( \binom{n}{k} = 0 \).

**Example 2.7.** Let \( S = K[x_0, ..., x_b] \), be a polynomial ring. Then

\[ H_S(l) = \begin{cases} \binom{l+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases} , \quad P_S(t) = \frac{1}{(1-t)^{b+1}}. \]

Note that in fact \( \forall l \geq -b, H_S(l) = \binom{l+b}{b} \) and \( 0 = H_S(-b-1) \neq \binom{-b-1+b}{b} = (-1)^b \), so the postulation number of \( S \) is \(-(b+1)\).
Example 2.8. Let \( S = K[x_0, \ldots, x_b], \mathcal{M} = \{x_0^{\alpha_0} \cdots x_b^{\alpha_b} \mid \alpha_0 + \cdots + \alpha_b = n\}, \) and \( R_{b,n} = K[\mathcal{M}] \) the \( n \)–Veronese embedding. Then
\[
H_{R_{b,n}}(l) = H_S(nl) = \begin{cases} \binom{nl+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}.
\]

Note that \( \binom{nl+b}{b} = \binom{nl+1}{b} \cdots \binom{nl+b}{b} b! \) is a polynomial on \( l \) with leading term \( \frac{1}{b!} \), so that \( \deg(R_{b,n}) = \binom{b+1}{n} \), \( \dim R_{b,n} = b + 1 \). Note also that \( \forall l \geq -\lceil \frac{b+1}{n} \rceil \), \( H_{R_{b,n}}(l) = \binom{nl+b}{b} \) and \( 0 = H_{R_{b,n}}(-\lceil \frac{b+1}{n} \rceil) \neq (-\lceil \frac{b+1}{n} \rceil)^{n+b} = (-1)^b (\lceil \frac{b+1}{n} \rceil - 1)^{n-1} \), so the postulation number of \( R_{b,n} \) is \( -\lceil \frac{b+1}{n} \rceil \). More generally the postulation number of \( R_{b,n}[\tau] \) is \( \lceil \frac{b+1+\tau}{n} \rceil \).

3 Veronese of generating series

In a recent paper [2], Brenti and Walker prove that taking the \( n \)–Veronese transform of the \( h \) polynomial is a linear function, in this section we improve this result giving an elementary proof of the fact that taking the shifted \( n \)–Veronese transform of the \( h \) polynomial is a linear function on \( h \).

Let recall the following fact:

**Theorem 3.1.** Let \((a_l)_{l \in \mathbb{Z}}\) be a sequence of complex numbers, such that \( a_l = 0 \) for \( l << 0 \), set : \( f(t) = \sum_{l \in \mathbb{Z}} a_lt^l \), TFAE:

- There exists \( h(t) \in \mathbb{C}[t, t^{-1}] \) and a natural integer \( d \) such that \( f(t) = h(t) \frac{(1-t)^d}{(1-\tau)^d} \).
- There exists \( \Phi(t) \in \mathbb{C}[t, t^{-1}] \) of degree \( d-1 \) with leading coefficient \( e_0/(d-1)! \), such that \( \Phi(l) = a_l \) for \( l \) large enough.

Moreover \( h(1) = e_0 \).

**Definition 3.2.** Fix integers \( d, n \in \mathbb{N}^*, \tau \in \mathbb{Z} \). Let \((a_l)_{l \in \mathbb{Z}}\) be a sequence of complex numbers, such that \( a_l = 0 \) for \( l << 0 \), set :
\[
f(t) = \sum_{l \in \mathbb{Z}} a_lt^l, \quad f^{<n,\tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau}t^l.
\]

By the Theorem 3.1 if \( f(t) = \frac{h(t)}{(1-t)^d} \) with \( h(t) \in \mathbb{C}[t, t^{-1}] \) then \( f^{<n,\tau>}(t) = \frac{h^{<n,\tau>}(t)}{(1-t)^d} \) with \( h^{<n,\tau>}(t) \in \mathbb{C}[t, t^{-1}] \).
Let us introduce some notations. To any non zero polynomial \( h(t) = h_\sigma t^\sigma + \ldots + h_0 + h_1 t + \ldots + h_s t^s \in \mathbb{C}[t, t^{-1}] \) we associate the \( h \)-vector \( \overrightarrow{h} = (\ldots, 0, h_\sigma, \ldots, h_s, 0, \ldots) \), and we set \( \deg \overrightarrow{h} = \deg h(t) \). For \( j \in \mathbb{Z} \), let \( \overrightarrow{e}^j \) be the \( h \)-vector of the polynomial \( t^j \). Let denote by \([t^k]h(t)\) the coefficient of \( t^k \) in the polynomial \( h(t) \). For any \( i, j \in \mathbb{Z} \) define \( D_{i,j} \) by

\[
D_{i,j} = [t^{in-j}](\frac{(1-t^n)^d}{(1-t)^d}) = [t^{in-j}](1 + t + \ldots t^{n-1})^d.
\]

Note that

\[
D_{i,j} = \text{Card}\{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid \forall l, x_l \leq n - 1; x_1 + \ldots + x_d = in - j\}.
\]

Finally let \( D[\sigma, \tau] \) be the infinite square matrix \( D[\sigma, \tau] = (D_{i+\sigma,j+\tau}) \).

For \( \sigma = \tau = 0 \) we write \( D \) instead \( D[0,0] \). We can give some properties of the numbers \( D_{i,j} \).

**Lemma 3.3.** Let \( i, j, k \in \mathbb{Z} \), we have:

- \( D_{i,j} = 0 \) if either \( in - j < 0 \) or \( in - j > d(n - 1) \).
- For any \( i, j \), \( D_{i,j} = D_{d-i,d-j} \). That is \( D \) is symmetrical around the point \((d/2, d/2)\).
- For \( 0 \leq k \leq n - 1 \), \( D_{d,d+k} = (k+d-1) \).
- \( D_{1,0} = (n+d-1) - d \), and for \( 1 \leq k \leq n \), \( D_{1,k} = (n-k+d-1) \).
- For any integers \( q, k \), \( D_{d+q,nq+k} = D_{d,k} \).
- For any \( i \), let \( d - i = nq - k \) with \( q = \lceil \frac{d-i}{n} \rceil \), \( 0 \leq k < n \), then

\[
D_{d-[\frac{d-i}{n}],i} = \binom{k+d-1}{d-1} = \binom{n-[\frac{d-i}{n}] + i - 1}{d - 1}.
\]

**Proof.** The first claim is trivial. In order to prove the other claims, let remark that the map \((x_1, \ldots, x_d) \mapsto (y_1, \ldots, y_d)\), where \( y_l = (n - 1) - x_l \) for \( l = 1, \ldots, d \), establishes a bijection between

\[
\{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_l \leq n - 1 \text{ for } l = 1, \ldots, d; \ x_1 + \ldots + x_d = in - j\}
\]

and

\[
\{(y_1, \ldots, y_d) \in \mathbb{N}^d \mid y_l \leq n - 1 \text{ for } l = 1, \ldots, d; \ y_1 + \ldots + y_d = (d-i)n - (d-j)\}.
\]
The third claim follows from the second claim, because if $0 \leq k \leq n - 1$, then the sets
\[
\{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, \ldots, d; \ x_1+\ldots+x_d = dn-d-k\}
\]
and
\[
\{(y_1, \ldots, y_d) \in \mathbb{N}^d \mid y_1 + \ldots + y_d = k\}
\]
are in bijection.

The fourth claim follows trivially from the previous items.

The fifth claim follows from the equality: $(d+q)n - (nq + k) = dn - k$.

Finally the sixth claim follows from the third claim, since, if $d - i = nq - k$ with $0 \leq k < n$, then $(d - q)n - i = dn - (d + k)$, hence $D_{d-q,i} = D_{d,d+k}$, and $n\left\lfloor \frac{d-i}{n} \right\rfloor + i - 1 = k + d - 1$.

\begin{remark}
With the notations introduced in 3.2, it is clear that $f^{<n, kn+\tau>}(t) = t^{-k} f^{<n, \tau>}(t)$, which implies $h^{<n, kn+\tau>}(t) = t^{-k} h^{<n, \tau>}(t)$ for any integer numbers $k, \tau$.
\end{remark}

The following Theorem improves and gives a simpler proof of [2, Theorem 1.1]:

\begin{theorem}
Fix integers $d, n \in \mathbb{N}^*$, $\tau \in \mathbb{Z}$. Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l << 0$, set :
\[
f(t) = \sum_{l \in \mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d},
\]
\[
f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l = \frac{h^{<n, \tau>}(t)}{(1-t)^d},
\]
where $h(t), h^{<n, \tau>}(t) \in \mathbb{C}[t, t^{-1}]$. Then
\[
\text{for any } \tau \in \mathbb{Z}, h^{<n, kn+\tau>} = D[-k, -\tau] h.
\]
\end{theorem}

\begin{proof}
Because the Remark 3.4 we have to compute $h^{<n, \tau>}(t)$ only for $0 \leq \tau \leq n - 1$. The following formula is clear:
\[
f^{<n,0>}(t^n) + tf^{<n, 1>}(t^n) + \ldots + t^{n-1} f^{<n, n-1>}(t^n) = f(t),
\]
hence
\[
\frac{h^{<n,0>}(t^n) + th^{<n, 1>}(t^n) + \ldots + t^{n-1} h^{<n, n-1>}(t^n)}{(1-t^n)^d} = \frac{h(t)}{(1-t)^d}.
\]

\end{proof}
and

\[ h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + \ldots + t^{n-1}h^{<n,n-1>}(t^n) = h(t)\frac{(1 - t^n)^d}{(1 - t)^d}, \]

\[ t^\tau h^{<n,\tau>}(t^n) \] equals the sum of all the terms \( A_\beta t^\beta \) of \( h(t)\frac{(1 - t^n)^d}{(1 - t)^d} \) with \( \beta \equiv \tau \mod n \). In particular \( h^{<n,\tau>}(t) \) is a linear function of \( h(t) \).

So it is enough to compute \( h^{<n,\tau>}(t) \) for the canonical basis \( \{ \varepsilon_j := t^j, j \in \mathbb{Z} \} \) of \( \mathbb{C}[t, t^{-1}] \). We have

\[ [t^i](h^{<n,\tau>}(t)) = [t^{ni+\tau}](h(t)\frac{(1 - t^n)^d}{(1 - t)^d}), \]

hence

\[ \forall j \in \mathbb{Z}; [t^i](\varepsilon_j^{<n,\tau>}(t)) = [t^{ni+\tau}](t^j)\frac{(1 - t^n)^d}{(1 - t)^d} = [t^{ni+\tau-j}](\frac{(1 - t^n)^d}{(1 - t)^d}) \]

which proves our statement.

\[ \square \]

**Corollary 3.6.** Fix an integer \( d \in \mathbb{N}^* \). For \( j \in \mathbb{Z} \), let \( \varepsilon_j \) be the \( h \)-vector of the polynomial \( t^j \). Then for any \( n \in \mathbb{N}^* \), we have \( \deg \varepsilon_j^{<n>} = d - \left\lfloor \frac{d - j}{n} \right\rfloor \). Moreover the leading coefficient of \( \varepsilon_j^{<n>} \) is \( \binom{n\left\lfloor \frac{d - j}{n} \right\rfloor + j - 1}{d - 1} \).

**Proof.** Let remark that the set of \( t^j, j \in \mathbb{Z} \) is the canonical basis of \( \mathbb{C}[t, t^{-1}] \). We have by Theorem 3.5 that \( D\varepsilon_j = \varepsilon_j^{<n>} \), hence \( \varepsilon_j^{<n>} \) is the \( j \) column vector of \( D \). By the Example 2.8, we have that \( \deg \varepsilon_j^{<n>} = d - \left\lfloor \frac{d - j}{n} \right\rfloor \).

The last claim follows from the Lemma 3.3. Indeed for any \( j \in \mathbb{Z} \), we have \( D_{d - \left\lfloor \frac{d - j}{n} \right\rfloor,j} = \binom{n\left\lfloor \frac{d - j}{n} \right\rfloor + j - 1}{d - 1} \). This proves that the leading coefficient of \( \varepsilon_j^{<n>} \) is \( \binom{n\left\lfloor \frac{d - j}{n} \right\rfloor + j - 1}{d - 1} \). \[ \square \]

**Example 3.7.** Let \( d = 2 \) and \( n \in \mathbb{N}^* \), we can describe the matrix \( D \)

| \( ij \) | \(- (n+1) \) | \(-1 \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( \ldots \) | \( n \) | \( n + 1 \) | \( n + 2 \) | \( \ldots \) | \( 2n \) |
|-----|-------------|-----|-----|-----|-----|-----|-------------|-----|-----|-----|-------------|
| \(-1\) | 2 | \ldots | 0 | 0 | 0 | 0 | \ldots | 0 | 0 | 0 | \ldots | 0 |
| 0 | \( n-2 \) | \ldots | 2 | 1 | 0 | 0 | \ldots | 0 | 0 | 0 | \ldots | 0 |
| 1 | 0 | \ldots | \( n-2 \) | \( n-1 \) | \( n \) | \( n-1 \) | \( n-2 \) | \ldots | 1 | 0 | 0 | \ldots | 0 |
| 2 | 0 | \ldots | 0 | 0 | 0 | 1 | \ldots | \( n-1 \) | \( n \) | \( n-1 \) | \ldots | 1 |
| 3 | 0 | \ldots | 0 | 0 | 0 | 0 | \ldots | 0 | 0 | 1 | \ldots | \( n-1 \) |
Theorem 3.8. Fix integers \( d, n \in \mathbb{N}^*, \tau \in \mathbb{Z} \). Let \((a_l)_{l \in \mathbb{Z}}\) be a sequence of complex numbers, such that \( a_l = 0 \) for \( l << 0 \), set:

\[
f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l.
\]

If \( f(t) = \frac{h(t)}{(1-t)^\tau} \) with \( h(t) \in \mathbb{C}[t, t^{-1}] \), then \( f^{<n,\tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^\tau} \) with \( h^{<n>}(t) \in \mathbb{C}[t, t^{-1}] \) such that:

- \( \deg h^{<n,\tau>}(t) \leq d - \left\lfloor \frac{d-\deg h(t)+\tau}{n} \right\rfloor \),
- if all the coefficients of \( h(t) \) are positive real numbers then \( \deg h^{<n,\tau>}(t) = d - \left\lfloor \frac{d-\deg h(t)+\tau}{n} \right\rfloor \),
- if \( \deg h(t) = d \) then \( \deg h^{<n>}(t) = d \).

Proof. Let \( f(t) = \sum_{l \in \mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^\tau} \), where \( h(t) \in \mathbb{C}[t, t^{-1}] \).\( h(t) = \gamma_0 t^\sigma + \ldots + \gamma_s t^s \) with \( \deg h(t) = s, \gamma_s \neq 0 \). It follows that \( h\overrightarrow{\tau} = \sum_{l=\sigma}^s \gamma_l t^{l-\tau} \). We multiply this relation on the left by \( D[-\tau] \), so Theorem 3.5 implies \( h^{<n,\tau>} = \sum_{l=\sigma}^s \gamma_l t^{s-l} \). Since \( \deg <n,\tau> \leq \deg<\sigma,\tau+1> \ldots \leq \deg<\sigma-\tau,t> \), we have, \( \deg h^{<n,\tau>}, \gamma_l t^{s-l} \). It is clear that if all the coefficients of \( h(t) \) are positive real numbers then \( \deg h^{<n,\tau>} = \deg<\sigma,t> \).

In the special case \( s = d \), we have seen that for \( 0 \leq l \leq d-1 \) and any \( n \in \mathbb{N}^* \), \( \deg<\sigma,t> = d - \left\lfloor \frac{d-l}{n} \right\rfloor \leq d-1 \), and \( \deg<\sigma,t> = d \), which implies \( \deg h^{<n,t>} = d \). \( \square \)

Theorem 3.9. Let \( n \in \mathbb{N}^* \), \( S \) be a standard graded polynomial ring, \( M = \bigoplus_{l \in \mathbb{Z}} M_l \) be a finitely generated Cohen-Macaulay graded \( S \)-module of dimension \( d \geq 1 \), and \( M^{<n>} = \bigoplus_{l \in \mathbb{Z}} M_{nl} \). Let \( Q(t) = \frac{Q(t)}{(1-t)^d} \) be the Hilbert-Poincaré series of \( M \), where \( Q(t) = \gamma_0 t^\sigma + \ldots + \gamma_s t^s \in \mathbb{C}[t, t^{-1}] \) is the \( h \)-polynomial of \( M \), with \( \text{reg}(M) = \deg Q(t) = s \). Then

- \( \text{reg} M^{<n>} = d - \left\lfloor \frac{d-\text{reg} M}{n} \right\rfloor \). Moreover by taking the sum over all index \( l \) such that \( \left\lfloor \frac{d-l}{n} \right\rfloor = \left\lfloor \frac{d-\text{reg} M}{n} \right\rfloor \), we will get the leading coefficient of \( Q^{<n>}(t) \):

\[
\sum_{l \mid \left\lfloor \frac{d-l}{n} \right\rfloor = \left\lfloor \frac{d-\text{reg} M}{n} \right\rfloor} \gamma_l \left(n\left\lfloor \frac{d-l}{n} \right\rfloor + l - 1\right).
\]
• If \( \text{reg} M \leq d - 1 \) and \( n \geq d \) then \( \text{reg} M^{<n>} = d - 1 \), and the leading coefficient of \( Q^{<n>}(t) \) is

\[
\sum_{l=0}^{d-1} \gamma_l \left( n \left\lfloor \frac{d-l}{n} \right\rfloor + l - 1 \right).
\]

• If \( n > \text{reg} M \geq d \) then \( \text{reg} M^{<n>} = d \), and the leading coefficient of \( Q^{<n>}(t) \) is

\[
\sum_{l=d}^{\text{reg} M} \gamma_l \left( l - 1 \right).
\]

Proof. We have \( \overrightarrow{Q} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{e_l} \). We multiply this relation on the left by \( D \), so Theorem 3.5 implies that for any \( n \in \mathbb{N}^* \), \( \overrightarrow{Q}^{<n>} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{e_l}^{<n>} \). Since \( \gamma_l \geq 0 \) for all \( l \), \( \gamma_s > 0 \), and \( \deg \overrightarrow{e_s}^{<n>} \leq \deg \overrightarrow{e_{s+1}}^{<n>} \leq \ldots \leq \deg \overrightarrow{e_{s'}}^{<n>} = \deg \overrightarrow{e_s}^{<n>} = d - \left\lceil \frac{d-\text{reg} M}{n} \right\rceil \), this number is \( \text{reg}(M^{<n>}) \) since \( M^{<n>} \) is a Cohen-Macaulay \( S \)-module. The computation of the leading coefficient of \( Q^{<n>}(t) \) is immediate from Lemma 3.6. \( \square \)

4 \textit{h}-vector of the Segre-Veronese embedding.

The next Theorem improves partially [1] and [5].

**Theorem 4.1.** Let consider the Segre-Veronese ring \( R_{b_\mathbb{N}} \), \( \dim R_{b_\mathbb{N}} = b_1 + \ldots + b_m + 1 \). Let \( \Pr_{b_\mathbb{N}}(t) = \frac{Q_{R_{b_\mathbb{N}}}(t)}{(1-t)^{\dim R_{b_\mathbb{N}}}} \) be the Hilbert-Poincaré series of \( R_{b_\mathbb{N}} \), with \( Q_{R_{b_\mathbb{N}}}(t) = h_0 + h_1 t + \ldots + h_{r_{b_\mathbb{N}}} t^{r_{b_\mathbb{N}}} \), where \( r_{b_\mathbb{N}} = \deg Q_{R_{b_\mathbb{N}}}(t) \) is the Castelnuovo-Mumford regularity of \( R_{b_\mathbb{N}} \). We set \( \alpha_{b_\mathbb{N}} = \dim R_{b_\mathbb{N}} - r_{b_\mathbb{N}} \). After a permutation of \( b_1, \ldots, b_m \), we can assume that \( \left\lceil \frac{b_i+1}{n_i} \right\rceil > \frac{b_i}{n_i} \) \( \forall i \), then

\[
\alpha_{b_\mathbb{N}} = \left\lceil \frac{b_1+1}{n_1} \right\rceil \quad r_{b_\mathbb{N}} = (b_1 + \ldots + b_m + 1) - \left\lfloor \frac{b_1+1}{n_1} \right\rfloor,
\]

and the highest Betti number of \( R_{b_\mathbb{N}} \) is

\[
\beta_{r_{b_\mathbb{N}}} = h_{r_{b_\mathbb{N}}} = \left( \frac{n_1 \alpha_{b_\mathbb{N}} - 1}{b_1} \right) \ldots \left( \frac{n_m \alpha_{b_\mathbb{N}} - 1}{b_m} \right).
\]
Proof. The proof is by double induction on $m$ and $b_m$. The case $m = 1$ is given by the Example 2.8 and Corollary 3.6, so we can assume $m \geq 2$. We have that $\left\lceil \frac{b_1+1}{n_1} \right\rceil > \frac{b_m}{n_m} > \frac{b_m-1}{n_m}$, so by induction hypothesis the theorem is true for $R_{\underline{b} - \epsilon_m, \underline{n}}$, where $\underline{b} - \epsilon_m = (b_1, \ldots, b_{m-1}, b_m - 1)$. On the other hand the Hilbert function of $R_{\underline{b}, \underline{n}}$ is $H_{R_{\underline{b}, \underline{n}}} (l) = (n_1l+b_1) \cdots (n_ml+b_m)$, so

$$H_{R_{\underline{b}, \underline{n}}} (l) = (1 + \frac{n_m}{b_m} l) H_{R_{\underline{b} - \epsilon_m, \underline{n}}} (l).$$

(1)

Let $P_{R_{\underline{b} - \epsilon_m, \underline{n}}}(t) = \frac{Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}(t)}{(1-t)^{r_{b_1} + \cdots + r_{b_m}}}$ be the Hilbert-Poincaré series of $R_{\underline{b} - \epsilon_m, \underline{n}}$, where $Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}(t) = h_0 + h_1 t + \cdots + h_{r_{b_1} - \epsilon_m} t^{r_{b_1} - \epsilon_m}$, with $h_{r_{b_1} - \epsilon_m} \neq 0$. In order to avoid any confusion we also set: $P_{R_{\underline{b}, \underline{n}}}(t) = \frac{Q_{R_{\underline{b}, \underline{n}}}(t)}{(1-t)^{r_{b_1} + \cdots + r_{b_m} + 1}}$ be the Hilbert-Poincaré series of $R_{\underline{b}, \underline{n}}$, where

$$Q_{R_{\underline{b}, \underline{n}}}(t) = \hat{h}_0 + \cdots + \hat{h}_{r_{b_1} - \epsilon_m} t^{r_{b_1} - \epsilon_m},$$

with $\hat{h}_{r_{b_1} - \epsilon_m} \neq 0$.

Let $\beta = \frac{b_m}{n_m}$, by simple calculations from (1) we get:

$$P_{R_{\underline{b}, \underline{n}}}(t) = P_{R_{\underline{b} - \epsilon_m, \underline{n}}}(t) + \beta t P_{R_{\underline{b} - \epsilon_m, \underline{n}}}'(t).$$

(2)

Hence $\dim R_{\underline{b}, \underline{n}} = \dim R_{\underline{b} - \epsilon_m, \underline{n}} + 1$, and

$$Q_{R_{\underline{b}, \underline{n}}}(t) = Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}(t) + t[Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}(t)(\beta R_{\underline{b} - \epsilon_m, \underline{n}} - 1) + \beta Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}'(t) - \beta t Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}'(t)],$$

note that $Q_{R_{\underline{b}, \underline{n}}}(1) = \beta \dim R_{\underline{b} - \epsilon_m, \underline{n}} Q_{R_{\underline{b} - \epsilon_m, \underline{n}}}(1) \neq 0$.

In particular we have $r_{b_1 - \epsilon_m} \leq r_{b_1 - \epsilon_m} + 1$ and for all $k = 0, \ldots, r_{b_1 - \epsilon_m} + 1$ we have

$$\hat{h}_k = h_{k-1}(\frac{n_m}{b_m} \dim R_{\underline{b} - \epsilon_m, \underline{n}} - (k-1) \frac{n_m}{b_m}) + h_k(k \frac{n_m}{b_m} + 1).$$

(3)

By induction hypothesis we have $\alpha_{\underline{b} - \epsilon_m, \underline{n}} = \left\lfloor \frac{b_1+1}{n_1} \right\rfloor \neq \frac{b_m}{n_m}$, so we put $k = r_{b_1 - \epsilon_m, \underline{n}} + 1$ in equality (3), and we get:

$$\hat{h}_{r_{b_1 - \epsilon_m, \underline{n}} + 1} = h_{r_{b_1 - \epsilon_m, \underline{n}}}(\frac{n_m \alpha_{\underline{b} - \epsilon_m, \underline{n}} - b_m}{b_m}) \neq 0.$$ 

Hence $\hat{h}_{r_{b_1 - \epsilon_m, \underline{n}} + 1}$ is the leading coefficient of $Q_{R_{\underline{b}, \underline{n}}}$ and $r_{b_1, \underline{n}} = r_{b_1 - \epsilon_m, \underline{n}} + 1$ and $\alpha_{b_1, \underline{n}} = \alpha_{b_1 - \epsilon_m, \underline{n}} = \left\lfloor \frac{b_1+1}{n_1} \right\rfloor$. By induction hypothesis

$$h_{r_{b_1 - \epsilon_m, \underline{n}}} = \left(\frac{n_1 \alpha_{b_1, \underline{n}} - 1}{b_1}\right) \cdots \left(\frac{n_m \alpha_{b_1, \underline{n}} - 1}{b_m - 1}\right) \left(\frac{n_m \alpha_{b_1, \underline{n}} - 1}{b_m}\right).$$
so that
\[
\hat{h}_{\alpha_{b,n}} = \left(\frac{n_1\alpha_{b,n} - 1}{b_1}\right) \ldots \left(\frac{n_{m-1}\alpha_{b,n} - 1}{b_{m-1}}\right) \left(\frac{n_m\alpha_{b,n} - 1}{b_m}\right) \frac{n_m\alpha_{b,n} - b_m}{b_m}.
\]

5 Rings of Veronese type

Let \(b, n \in \mathbb{N}^*, \mathbf{a} = (a_0, \ldots, a_b) \in \mathbb{N}^{b+1}\) such that \(1 \leq a_i \leq n, a_0 + \ldots + a_b > n\), and \(\mathcal{M}_{b,n,a}\) be the set of monomials of the polynomial ring \(K[x_0, \ldots, x_b]\):

\[
\mathcal{M}_{b,n,a} = \{x_0^{\alpha_0} \ldots x_b^{\alpha_b} \mid \alpha_0 + \ldots + \alpha_b = n, \alpha_i \leq a_i, \forall i = 0, \ldots, b\}.
\]

Let denote by \(R_{b,n,a}\) the toric subring of \(K[x_0, \ldots, x_b]\) generated by \(\mathcal{M}_{b,n,a}\). It is well known that \(R_{b,n,a}\) is a Cohen-Macaulay ring. Let \(S\) be the collection of subsets of \(\{0, \ldots, b\}\) such that : \(S \in S\) if and only if \(S \subset \{0, \ldots, b\}\), and \(\Sigma S := \sum_{i \in S} a_i < n\).

**Theorem 5.1.** ([9]) With the above notations the Hilbert-function of \(R_{b,n,a}\) is

\[
\forall l \geq 0; \quad H_{b,n,a}(l) = \sum_{S \in S} (-1)^{|S|} \binom{l(n - \Sigma S) - |S| + b}{b}
\]

We have \(\dim(R_{b,n,a}) = b + 1\), and its degree or multiplicity is

\[
\deg(R_{b,n,a}) = \sum_{S \in S} (-1)^{|S|} (n - \Sigma S)^b.
\]

Our aim is to study the Hilbert-Poincaré series of \(R_{b,n,a}\):

\[
P_{R_{b,n,a}} = \sum_{S \in S} (-1)^{|S|} \sum_{l \geq 0} \binom{l(n - \Sigma S) - |S| + b}{b} t^l.
\]

The following Corollary follows immediately from 3.6.

**Corollary 5.2.** For any \(S \in S\) and \(k \in \mathbb{N}^*\), we have:

\[
\sum_{l \geq 0} \binom{kl - |S| + b}{b} t^l = \frac{Q_{S,k}(t)}{(1 - t)^{b+1}},
\]

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where $Q_{S,k}(t)$ is a polynomial with $Q_{S,k}(1) \neq 0$, with leading term

$$\left(\frac{k\alpha_{S,k} + |S| - 1}{b}\right)t^{b+1 - \alpha_{S,k}},$$

with $\alpha_{S,k} = \left\lceil \frac{b+1-|S|}{n-\Sigma S} \right\rceil$.

The following theorem is immediate from 5.1 and Corollary 5.2. It improves the description of the Hilbert Poincaré series given in [9].

**Theorem 5.3.** With the above notations, let $S$ be the collection of subsets of $\{0, ..., b\}$ such that $S \in S$ if and only if $S \subset \{0, ..., b\}$, and $\Sigma S := \sum_{i \in S} a_i < n$. Then we can write the Hilbert-Poincaré series of $R_{b,n,a}$:

$$P_{R_{b,n,a}} = Q_{b,n,a}(t)(1-t)^{b+1},$$

with $Q_{b,n,a}(t) = \sum_{S \in S} (-1)^{|S|}Q_{S,n}(t)$, where $Q_{S,n}(t)$ is a polynomial with $Q_{S,n}(1) \neq 0$, with leading term

$$\left(\frac{(n-\Sigma S)\alpha_{S,n-\Sigma S} + |S| - 1}{b}\right)t^{b+1 - \alpha_{S,n-\Sigma S}},$$

where $\alpha_{S,n-\Sigma S} = \left\lceil \frac{b+1-|S|}{n-\Sigma S} \right\rceil$.

Part one of the following corollary improves [9][Cor. 2.12].

**Corollary 5.4.** With the above notations:

1. $\text{reg}(R_{b,n,a}) \leq b + 1 - \left\lceil \frac{b+1}{n} \right\rceil$, and the equality is true if and only if
   $$\sum_{S \in S, \alpha_{S,n-\Sigma S} = \left\lceil \frac{b+1}{n-\Sigma S} \right\rceil} (-1)^{|S|}\left(\frac{(n-\Sigma S)\alpha_{S,n-\Sigma S} + |S| - 1}{b}\right) \neq 0$$

2. If $b + 1 > n^2$ then $\text{reg}(R_{b,n,a}) = b + 1 - \left\lceil \frac{b+1}{n} \right\rceil$. Moreover the leading term of $Q_{b,n,a}(t)$ is $\left(\frac{n-\Sigma S}{n}\right)t^{b+1 - \left\lceil \frac{b+1}{n} \right\rceil}$.

**Proof.**

1. It is enough to prove that $\min_{S \in S} \left\lceil \frac{b+1-|S|}{n-\Sigma S} \right\rceil = \left\lceil \frac{b+1}{n} \right\rceil$.

   We consider two cases,
   - if $b + 1 < n$ then $\left\lceil \frac{b+1}{n} \right\rceil = 1 \leq \left\lceil \frac{b+1-|S|}{n-\Sigma S} \right\rceil, \forall S \in S$.  
   - if $b + 1 > n^2$ then $\left\lceil \frac{b+1}{n} \right\rceil = b + 1 - \left\lceil \frac{b+1}{n} \right\rceil$. Moreover the leading term of $Q_{b,n,a}(t)$ is $\left(\frac{n-\Sigma S}{n}\right)t^{b+1 - \left\lceil \frac{b+1}{n} \right\rceil}$. 

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• If $b + 1 \geq n$, then
\[
\frac{b + 1}{n} \leq \frac{b + 1 - |S|}{n - \Sigma S} \iff (b+1)(n-\Sigma S) \leq n(b+1- |S|) \iff (b+1)\Sigma S \geq n \quad |S|,
\]
this is true since by hypothesis $\frac{b+1}{n} \geq 1 \geq \frac{|S|}{\Sigma S}$.

2. Let $b + 1 > n^2$ and $S \neq \emptyset$. By definition $\lceil \frac{b+1}{n} \rceil$ is the integer $q$ such that $b + 1 = qn - r$, with $0 \leq r < n$ and $q \geq n + 1$. We have
\[
b + 1 - |S| = qn - r - |S| = q(n - \Sigma S) - r - |S| + q\Sigma S,
\]
and $q\Sigma S - |S| \geq (n+1)\Sigma S - |S| \geq n\Sigma S > r$, so that $q\Sigma S - |S| - r > 0$, hence $\lceil \frac{b+1-|S|}{n-\Sigma S} \rceil > q = \lceil \frac{b+1}{n} \rceil$.

In general leading terms of the alternating sum can cancel, as we can see in the next example.

**Example 5.5.** Let consider the ring $R_{4,3,(1,1,1,1,1)}$, the sets $S$ can have 0, 1 or 2 elements, and we have: If $S = \emptyset$ then $\alpha_{\emptyset,3} = \lceil \frac{5}{3} \rceil = 2$, if $|S| = 1$ then $\alpha_{S,3} = \lceil \frac{4}{2} \rceil = 2$, and finally if $|S| = 2$ then $\alpha_{S,2} = \lceil \frac{3}{1} \rceil = 3$. By using Theorem 5.3 we can write
\[
PR_{R_{4,3,(1,1,1,1,1)}} = \frac{Q_0(t) - 5Q_1(t) + 10Q_2(t)}{(1-t)^{5}},
\]
with $Q_0(t) = 5t^3 + ...; Q_1(t) = t^3 + ...; Q_2(t) = t^2 + ...$. Note that in this case $Q_0(t) - 5Q_1(t) + 10Q_2(t) = h_0 + h_1 + h_2 t^2$, where $h_0 = 1, h_1 = 5$ and since $h_0 + h_1 + h_2 = \deg(R_{4,3,(1,1,1,1,1)}) = 11$, we get $h_2 = 5$, so that
\[
PR_{R_{4,3,(1,1,1,1,1)}} = \frac{1 + 5t + 5t^2}{(1-t)^{5}}.
\]

**References**


