DISPERSIVE ESTIMATES FOR THE SCHRODINGER OPERATOR ON STEP 2 STRATIFIED LIE GROUPS

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HAJER BAHOURI, CLOTILDE FERMANIAN-KAMMERER, AND ISABELLE GALLAGHER

Abstract. The present paper is dedicated to the proof of dispersive estimates on stratified Lie groups of step 2, for the linear Schrödinger equation involving a sublaplacian. It turns out that the propagator behaves like a wave operator on a space of the same dimension p as the center of the group, and like a Schrödinger operator on a space of the same dimension k as the radical of the canonical skew-symmetric form, which suggests a decay rate $|t|^{-\frac{k+p-1}{2}}$. In this article, we identify a property of the canonical skew-symmetric form under which we establish optimal dispersive estimates with this rate. The relevance of this property is discussed through several examples.

1. Introduction

1.1. **Dispersive inequalities.** Dispersive inequalities for evolution equations (such as Schrödinger and wave equations) play a decisive role in the study of semilinear and quasilinear problems which appear in numerous physical applications. Dispersion phenomena amount to establishing a decay estimate for the L^∞ norm of the solutions of these equations at time t in terms of some negative power of t and the L^1 norm of the data. In many cases, the main step in the proof of this decay in time relies on the application of a stationary phase theorem on an (approximate) representation of the solution. Combined with an abstract functional analysis argument known as the TT^*-argument, dispersion phenomena yield a range of estimates involving space-time Lebesgue norms. Those inequalities, called Strichartz estimates, have proved to be powerful in the study of nonlinear equations (for instance one can consult [5] and the references therein).

In the $\mathbb{R}^d$ framework, dispersive inequalities have a long history beginning with the articles of Brenner [12], Pecher [30], Segal [32] and Strichartz [39]. They were subsequently developed by various authors, starting with the paper of Ginibre and Velo [21] (for a detailed bibliography, we refer to [23, 40] and the references therein). In [7], the authors generalize the dispersive estimates for the wave equation to the Heisenberg group $\mathbb{H}^d$ with an optimal rate of decay of order $|t|^{-1/2}$ (regardless of the dimension d) and prove that no dispersion phenomenon occurs for the Schrödinger equation. In [17], optimal results are proved for the time behavior of the Schrödinger and wave equations on H-type groups: if p is the dimension of the center of the H-type group, the author establishes sharp dispersive inequalities for the wave equation solution (with a decay rate of $|t|^{-p/2}$) as well as for the Schrödinger equation solution (with a $|t|^{-(p-1)/2}$ decay). Compared with the $\mathbb{R}^d$ framework, there is an exchange in the rates of decay between the wave and the Schrödinger equations.

Strichartz estimates in other settings have been obtained in a number of works. One can first cite various results dealing with variable coefficient operators(see for instance [24, 33]) or studies concerning domains such as [14, 22, 34]. One can also refer to the result concerning the full Laplacian on the Heisenberg group in [20], works in the framework of the real hyperbolic spaces in [1, 10, 41], or in the framework of compact and noncompact manifolds in [2, 11, 13]; finally one can mention the quasilinear framework studied in [3, 4, 25, 35], and the references therein.

In this paper our goal is to establish optimal dispersive estimates for the solutions of the Schrödinger equation on 2-step stratified Lie groups. We shall emphasize in particular the key role played by the canonical skew-symmetric form in determining the rate of decay of the solutions. It turns out that the Schrödinger propagator on G behaves like a wave operator on a space of the same dimension as the center of G, and like a Schrödinger operator on a space of the same dimension as the radical of the canonical skew-symmetric form associated with the dual of the center. This unusual behavior of the Schrödinger propagator in the case of Lie algebras whose canonical skew-symmetric form is degenerate (known as Lie algebras which are not MW, see [28], [29] for example) makes the analysis of the explicit representations of the solutions tricky and gives rise to uncommon dispersive estimates. It will also appear from our analysis that the optimal rate of decay is not always in accordance with the dimension of the center: we shall exhibit examples of 2-step stratified Lie groups with center of any dimension and for which no dispersion occurs for the Schrödinger equation. We shall actually highlight that the optimal rate of decay in the dispersive estimates for solutions to the Schrödinger equation is rather related to the properties of the canonical skew-symmetric form.

1.2. **Stratified Lie groups.** Let us recall here some basic facts about stratified Lie groups (see [16, 18, 19, 38] and the references therein for further details). A connected, simply connected nilpotent Lie group G is said stratified if its left-invariant Lie algebra $\mathfrak{g}$ (assumed real-valued and of finite dimension n) is endowed with a vector space decomposition

$$\mathfrak{g} = \oplus_{1\leq k\leq\infty}\, \mathfrak{g}_k\,,$$

where all but finitely many of the $\mathfrak{g}_k'$s are $\{0\}$, such that $[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$. Via the exponential map

$$\exp : \mathfrak{g} \to G$$

which is in that case a diffeomorphism from $\mathfrak{g}$ to G, one identifies G and $\mathfrak{g}$. It turns out that under this identification, the group law on G (which is generally not commutative) provided by the Campbell-Baker-Hausdorff formula, $(x, y) \mapsto x \cdot y$ is a polynomial map. In the following we shall denote by $\mathfrak{z}$ the center of G which is simply the last non zero $\mathfrak{g}_k$ and write

$$G = \mathfrak{v} \oplus \mathfrak{z}\,, \tag{1.1}$$

where $\mathfrak{v}$ is any subspace of G complementary to $\mathfrak{z}$.

The group G is endowed with a smooth left invariant measure $\mu(x)$, the Haar measure, induced by the Lebesgue measure on $\mathfrak{g}$ and which satisfies the fundamental translation invariance property:

$$\forall f \in L^1(G, d\mu)\,, \quad \forall x \in G\,, \quad \int_G f(y)d\mu(y) = \int_G f(x\cdot y)d\mu(y)\,.$$

Note that the convolution of two functions f and g on G is given by

$$f * g(x) := \int_G f(x\cdot y^{-1})g(y)d\mu(y) = \int_G f(y)g(y^{-1}\cdot x)d\mu(y) \tag{1.2}$$

and as in the euclidean case we define Lebesgue spaces by

$$\|f\|_{L^p(G)} := \left(\int_G |f(y)|^p\, d\mu(y)\right)^{\frac{1}{p}}\,,$$

for $p \in [1, \infty[$, with the standard modification when $p = \infty$.

Since G is stratified, there is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we can decompose X as $X = \sum X_k$ with $X_k \in \mathfrak{g}_k$, and then

$$\delta_t X := \sum t^k X_k\,.$$

This allows to define the dilation δ_t on the Lie group G via the identification by the exponential map:

$$\begin{array}{ccc} \mathfrak{g} & \stackrel{\delta_t}{\rightarrow} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ G & \underset{\exp\circ\,\delta_t\,\circ\exp^{-1}}{\rightarrow} & G \end{array}$$

To avoid heaviness, we shall still denote by δ_t the map $\exp\circ\,\delta_t\,\circ\exp^{-1}$.

Observe that the action of the left invariant vector fields X_k, for X_k belonging to $\mathfrak{g}_k$, changes the homogeneity in the following way:

$$X_k(f\circ\delta_t) = t^k X_k(f)\circ\delta_t\,,$$

where by definition $X_k(f)(y) := \frac{d}{ds} f\big(y\cdot \exp(sX_k)\big)_{|s=0}$ and the Jacobian of the dilation δ_t is t^Q where $Q := \sum_{1\leq k\leq\infty} k \dim \mathfrak{g}_k$ is called the homogeneous dimension of G:

$$\int_G f(\delta_t\, y)\, d\mu(y) = t^{-Q} \int_G f(y)\, d\mu(y)\,. \tag{1.3}$$

Let us also point out that there is a natural norm ρ on G which is homogeneous in the sense that it respects dilations: $G \ni x \mapsto \rho(x)$ satisfies

$$\forall x\in G,\ \ \rho(x^{-1}) = \rho(x)\,,\ \ \rho(\delta_t x) = t\rho(x)\,,\ \text{and}\ \ \rho(x) = 0 \iff x = 0\,.$$

We can define the Schwartz space $\mathcal{S}(G)$ as the set of smooth functions on G such that for all α in $\mathbb{N}^d$, for all p in $\mathbb{N}, x \mapsto \rho(x)^p \mathcal{X}^\alpha f(x)$ belongs to $L^\infty(G)$, where $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left invariant vector fields. The Schwartz space $\mathcal{S}(G)$ has properties very similar to those of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, particularly density in Lebesgue spaces.

1.3. **The Fourier transform.** The group G being non commutative, its Fourier transform is defined by means of irreducible unitary representations. We devote this section to the introduction of the basic concepts that will be needed in the sequel. From now on, we assume that G is a step 2 stratified Lie group and we choose $\mathfrak{v} = \mathfrak{g}_1$ in (1.1).

1.3.1. *Irreducible unitary representations.* Let us fix some notation, borrowed from [15] (see also [16] or [29]). For any $\lambda \in \mathfrak{z}^\star$ (the dual of the center $\mathfrak{z}$) we define a skew-symmetric bilinear form on $\mathfrak{v}$ by

$$\forall\, U, V \in \mathfrak{v}\,,\quad B(\lambda)(U,V) := \lambda([U,V])\,. \tag{1.4}$$

One can find a Zariski-open subset Λ of $\mathfrak{z}^\star$ such that the number of distinct eigenvalues of $B(\lambda)$ is maximum. We denote by k the dimension of the radical $\mathfrak{r}_\lambda$ of $B(\lambda)$. Since $B(\lambda)$ is skew-symmetric, the dimension of the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{v}$ is an even number which we shall denote by $2d$. Therefore, there exists an orthonormal basis

$$\big(P_1(\lambda),\ldots,P_d(\lambda),Q_1(\lambda),\ldots,Q_d(\lambda),R_1(\lambda),\ldots,R_k(\lambda)\big)$$

such that the matrix of $B(\lambda)$ takes the following form

$$\begin{pmatrix}
0 & \ldots & 0 & \eta_1(\lambda) & \ldots & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \ldots & 0 & 0 & \ldots & \eta_d(\lambda) & 0 & \cdots & 0\\
-\eta_1(\lambda) & \ldots & 0 & 0 & \ldots & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \ldots & -\eta_d(\lambda) & 0 & \ldots & 0 & 0 & \cdots & 0\\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \cdots & 0
\end{pmatrix},$$

where each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree one in $\lambda = (\lambda_1,\ldots,\lambda_p)$ and the basis vectors are chosen to depend smoothly on λ in Λ. Decomposing $\mathfrak{v}$ as

$$\mathfrak{v} = \mathfrak{p}_\lambda + \mathfrak{q}_\lambda + \mathfrak{r}_\lambda$$

with

$$\mathfrak{p}_\lambda := \mathrm{Span}\,\big(P_1(\lambda),\ldots,P_d(\lambda)\big)\,,\quad \mathfrak{q}_\lambda := \mathrm{Span}\,\big(Q_1(\lambda),\ldots,Q_d(\lambda)\,,\quad \mathfrak{r}_\lambda := \quad \mathrm{Span}\,\big(R_1(\lambda),\ldots,R_k(\lambda)\big)$$

any element $V\in\mathfrak{v}$ will be written in the following as $P+Q+R$ with $P\in\mathfrak{p}_\lambda$, $Q\in\mathfrak{q}_\lambda$ and $R\in\mathfrak{r}_\lambda$. We then introduce irreducible unitary representations of G on $L^2(\mathfrak{p}_\lambda)$:

$$u_X^{\lambda,\nu}\phi(\xi) := \mathrm{e}^{-i\nu(R)-i\lambda(Z+[\xi+\frac12 P,Q])}\phi(P+\xi)\,,\ \lambda\in\mathfrak{z}^*,\ \nu\in\mathfrak{r}_\lambda^*\,, \tag{1.5}$$

for any $x = \exp(X) \in G$ with $X = X(\lambda,x) := \big(P(\lambda,x),Q(\lambda,x),R(\lambda,x),Z(x)\big)$ and $\phi\in L^2(\mathfrak{p}_\lambda)$. In order to shorten notation, we shall omit the dependence on (λ,x) whenever there is no risk of confusion.

1.3.2. *The Fourier transform.* In contrast with the euclidean case, the Fourier transform is defined on the manifold $\mathfrak{r}(\Lambda)$ whose fibre above $\lambda\in\Lambda$ is $\mathfrak{r}^*_\lambda\sim\mathbb{R}^k$ and is valued in the space of bounded operators on $L^2(\mathfrak{p}_\lambda)$. More precisely, the Fourier transform of a function f in $L^1(G)$ is defined as follows: for any $(\lambda,\nu)\in\mathfrak{r}(\Lambda)$

$$\mathcal{F}(f)(\lambda,\nu):=\int_G f(x)u^{\lambda,\nu}_{X(\lambda,x)}\,d\mu(x)\,.$$

Note that for any (λ,ν), the map $u^{\lambda,\nu}_{X(\lambda,x)}$ is a group homomorphism from G into the group $U(L^2(\mathfrak{p}_\lambda))$ of unitary operators of $L^2(\mathfrak{p}_\lambda)$, so functions f of $L^1(G)$ have a Fourier transform $(\mathcal{F}(f)(\lambda,\nu))_{\lambda,\nu}$ which is a bounded family of bounded operators on $L^2(\mathfrak{p}_\lambda)$. One may check that the Fourier transform exchanges convolution, whose definition is recalled in (1.2), and composition:

$$\mathcal{F}(f\star g)(\lambda,\nu)=\mathcal{F}(f)(\lambda,\nu)\circ\mathcal{F}(g)(\lambda,\nu)\,.\tag{1.6}$$

Besides, the Fourier transform can be extended to an isometry from $L^2(G)$ onto the Hilbert space of two-parameter families $A=\{A(\lambda,\nu)\}_{(\lambda,\nu)\in\mathfrak{r}(\Lambda)}$ of operators on $L^2(\mathfrak{p}_\lambda)$ which are Hilbert-Schmidt for almost every $(\lambda,\nu)\in\mathfrak{r}(\Lambda)$, with $\|A(\lambda,\nu)\|_{HS(L^2(\mathfrak{p}_\lambda))}$ measurable and with norm

$$\|A\|:=\left(\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}^*_\lambda}\|A(\lambda,\nu)\|^2_{HS(L^2(\mathfrak{p}_\lambda))}|\mathrm{Pf}(\lambda)|d\nu\,d\lambda\right)^{\frac12}<\infty\,,$$

where $|\mathrm{Pf}(\lambda)|:=\prod_{j=1}^d\eta_j(\lambda)$ is the Pfaffian of $B(\lambda)$. We have the following Fourier-Plancherel formula: there exists a constant $\kappa>0$ such that

$$\int_G|f(x)|^2\,dx=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}^*_\lambda}\|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS(L^2(\mathfrak{p}_\lambda))}|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,.\tag{1.7}$$

Finally, we have an inversion formula as stated in the following proposition which is proved in the Appendix page 21.

Proposition 1.1. *There exists $\kappa>0$ such that for $f\in\mathcal{S}(G)$ and for almost all $x\in G$ the following inversion formula holds:*

$$f(x)=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}^*_\lambda}\mathrm{tr}\left((u^{\lambda,\nu}_{X(\lambda,x)})^\star\mathcal{F}f(\lambda,\nu)\right)|Pf(\lambda)|\,d\nu\,d\lambda\,.\tag{1.8}$$

1.3.3. *The sublaplacian.* Let $(V_1,\dots,V_m)$ be an orthonormal basis of $\mathfrak{g}_1$, then the sublaplacian on G is defined by

$$\Delta_G:=\sum_{j=1}^m V_j^2.\tag{1.9}$$

It is a self-adjoint operator which is independent of the orthonormal basis $(V_1,\dots,V_m)$, and homogeneous of degree 2 with respect to the dilations in the sense that :

$$\delta_t^{-1}\Delta_G\,\delta_t=t^2\Delta_G\,.$$

To write its expression in Fourier space, we consider the basis of Hermite functions $(h_n)_{n\in\mathbf{N}}$, normalized in $L^2(\mathbb{R})$ and satisfying for all real numbers ξ:

$$h_n''(\xi)-\xi^2h_n(\xi)=-(2n+1)h_n(\xi)\,.$$

Then, for any multi-index $\alpha\in\mathbb{N}^d$, we define the functions $h_{\alpha,\eta(\lambda)}$ by

$$\begin{aligned}&\forall\,\Xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d\,,\quad h_{\alpha,\eta(\lambda)}(\Xi):=\prod_{j=1}^d h_{\alpha_j,\eta_j(\lambda)}(\xi_j)\quad\text{and}\\&\forall(n,\beta)\in\mathbb{N}\times\mathbb{R}^+\,,\forall\xi\in\mathbb{R}\,,\quad h_{n,\beta}(\xi):=\beta^{\frac14}h_n\big(\beta^{\frac12}\xi\big)\,.\end{aligned}\tag{1.10}$$

The sublaplacian Δ_G defined in (1.9) satisfies

$$\mathcal{F}(-\Delta_Gf)(\lambda,\nu)=\mathcal{F}(f)(\lambda,\nu)\left(H(\lambda)+|\nu|^2\right)\,,\tag{1.11}$$

where $|\nu|$ denotes the euclidean norm of the vector ν in $\mathbb{R}^k$ and $H(\lambda)$ is the diagonal operator defined on $L^2(\mathbb{R}^d)$ by

$$H(\lambda)h_{\alpha,\eta(\lambda)} = \sum_{j=1}^{d}(2\alpha_j+1)\eta_j(\lambda)\, h_{\alpha,\eta(\lambda)}\,.$$

In the following we shall denote the "frequencies" associated with $P_j^2(\lambda)+Q_j^2(\lambda)$ by

$$\zeta_j(\alpha,\lambda) := (2\alpha_j+1)\eta_j(\lambda)\,,\quad (\alpha,\lambda)\in\mathbb{N}^d\times\Lambda\,, \tag{1.12}$$

and those associated with $H(\lambda)$ by

$$\zeta(\alpha,\lambda) := \sum_{j=1}^{d}\zeta_j(\alpha,\lambda)\,,\quad (\alpha,\lambda)\in\mathbb{N}^d\times\Lambda\,. \tag{1.13}$$

Note that Δ_G is directly related to the harmonic oscillator via $H(\lambda)$ since eigenfunctions associated with the eigenvalues $\zeta(\alpha,\lambda)$ are the products of 1-dimensional Hermite functions. Also observe that $\zeta(\alpha,\lambda)$ is smooth and homogeneous of degree one in $\lambda=(\lambda_1,\dots,\lambda_p)$. Moreover, $\zeta(\alpha,\lambda)=0$ if and only if $B(\lambda)=0$, or equivalently by (1.4), $\lambda=0$.

Notice also that there is a difference in homogeneity in the variables λ and ν. Namely, in the variable ν, the sublaplacian acts as in the euclidean case (homogeneity 2) while in λ, it has the homogeneity 1 of a wave operator.

Finally, for any smooth function Φ, we define the operator $\Phi\left(-\Delta_G\right)$ by the formula

$$\mathcal{F}\big(\Phi(-\Delta_G)f\big)(\lambda,\nu) := \Phi(H(\lambda)+|\nu|^2)\mathcal{F}(f)(\lambda,\nu)\,, \tag{1.14}$$

which also reads

$$\mathcal{F}\big(\Phi(-\Delta_G)f\big)(\lambda,\nu)h_{\alpha,\eta(\lambda)} := \Phi\left(|\nu|^2+\zeta(\alpha,\lambda)\right)\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\,,$$

for all $(\lambda,\nu)\in\mathfrak{r}(\Lambda)$ and $\alpha\in\mathbb{N}^d$.

1.3.4. *Strict spectral localization.* Let us introduce the following notion of spectral localization, which we shall call strict spectral localization and which will be very useful in the following.

Definition 1.2. *A function f belonging to $L^1(G)$ is said to be strictly spectrally localized in a set $\mathcal{C}\subset\mathbb{R}$ if there exists a smooth function θ, compactly supported in $\mathcal{C}$, such that for all $1\le j\le d$,*

$$\mathcal{F}(f)(\lambda,\nu) = \mathcal{F}(f)(\lambda,\nu)\,\theta\big((P_j^2+Q_j^2)(\lambda)\big)\,,\ \ \forall(\lambda,\nu)\in\mathfrak{r}(\Lambda)\,. \tag{1.15}$$

Remark 1.3. One could expect the notion of spectral localization to relate to the Laplacian instead of each individual vector field $P_j^2+Q_j^2$, assuming rather the less restrictive condition

$$\mathcal{F}(f)(\lambda,\nu) = \mathcal{F}(f)(\lambda,\nu)\,\theta\big(H(\lambda)\big)\,,\ \ \forall(\lambda,\nu)\in\mathfrak{r}(\Lambda)\,.$$

The choice we make here is more restrictive due to the anisotropic context (namely the fact that $\eta_j(\lambda)$ depends on j): in the case of the Heisenberg group or more generally H-type groups, the notion of "strict spectral localization" in a ring $\mathcal{C}$ of $\mathbb{R}^p$ actually coincides with the more usual definition of "spectral localization" since as recalled in the next paragraph $\eta_j(\lambda)=4|\lambda|$ (for a complete presentation and more details on spectrally localized functions, we refer the reader to [6, 8, 9]). Assumption (1.15) guarantees a lower bound, which roughly states that for $\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\lambda}$ to be non zero, then

$$\forall j\in\{1,\dots,d\},\ \ (2\alpha_j+1)\eta_j(\lambda)\ge c>0\,, \tag{1.16}$$

hence each η_j must be bounded away from zero, rather than the sum over j. These lower bounds are important ingredients of the proof (see Section 3.3).

1.4. **Examples.** Let us give a few examples of well-known stratified Lie groups with a step 2 stratification. Note that nilpotent Lie groups which are connected, simply connected and whose Lie algebra admits a step 2 stratification are called Carnot groups.

1.4.1. *The Heisenberg group.* The Heisenberg group $\mathbb{H}^d$ is defined as the space $\mathbb{R}^{2d+1}$ whose elements can be written $w = (x, y, s)$ with $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, endowed with the following product law:

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' - 2(x \mid y') + 2(y \mid x')),$$

where $(\cdot \mid \cdot)$ denotes the euclidean scalar product on $\mathbb{R}^d$. In that case the center consists in the points of the form $(0, 0, s)$ and is of dimension 1. The Lie algebra of left invariant vector fields is generated by

$$X_j := \partial_{x_j} + 2y_j\partial_s\,,\ \ Y_j := \partial_{y_j} - 2x_j\partial_s \ \text{ with } 1 \leq j \leq d\,,\ \text{ and } S := \partial_s = \frac{1}{4}[Y_j, X_j]\,.$$

The canonical skew-symmetric form associated with the frequencies $\lambda \in \mathbb{R}^*$ writes

$$B(\lambda)(U, V) = \lambda\,([U, V])\,,\ \ \forall\, U, V \in \mathrm{Vect}\,(X_j, Y_j, 1 \leq j \leq d)$$

and its radical reduces to $\{0\}$ with $\Lambda = \mathbb{R}^*$ and $|\eta_j(\lambda)| = 4\,|\lambda|$ for all $j \in \{1, \dots, d\}$. Note in particular that strict spectral localization and spectral localization are equivalent.

1.4.2. *H-type groups.* These groups are canonically isomorphic to $\mathbb{R}^{m+p}$, and are a multidimensional version of the Heisenberg group. The group law is of the form

$$(x^{(1)}, x^{(2)}) \cdot (y^{(1)}, y^{(2)}) := \begin{pmatrix} x_j^{(1)} + y_j^{(1)},\ \ j = 1, ..., m \\ x_j^{(2)} + y_j^{(2)} + \frac{1}{2}\langle x^{(1)}, U^{(j)}y^{(1)}\rangle,\ \ j = 1, ..., p \end{pmatrix}$$

where $U^{(j)}$ are $m \times m$ linearly independent orthogonal skew-symmetric matrices satisfying the property

$$U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0$$

for every $r, s \in \{1, ..., p\}$ with $r \neq s$. In that case the center is of dimension p and may be identified with $\mathbb{R}^p$ and the radical of the canonical skew-symmetric form associated with the frequencies λ is again $\{0\}$. For example the Iwasawa subgroup of semi-simple Lie groups of split rank one (see [26]) is of this type. On H-type groups, m is an even number which we denote by 2ℓ and the Lie algebra of left invariant vector fields is spanned by the following vector fields, where we have written $z = (x, y)$ in $\mathbb{R}^\ell \times \mathbb{R}^\ell$: for j running from 1 to ℓ and k from 1 to p,

$$X_j := \partial_{x_j} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{2\ell} z_l\, U_{l,j}^{(k)}\partial_{s_k}\,,\ \ Y_j := \partial_{y_j} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{2\ell} z_l\, U_{l,j+\ell}^{(k)}\partial_{s_k} \quad \text{and} \quad \partial_{s_k}.$$

In that case, we have $\Lambda = \mathbb{R}^p \setminus \{0\}$ with $\eta_j(\lambda) = \sqrt{\lambda_1^2 + \cdots + \lambda_p^2}$ for all $j \in \{1, \dots, \ell\}$ (here again, strict spectral localization and spectral localization are equivalent).

1.4.3. *Diamond groups.* These groups which occur in crystal theory (for more details, consult [27, 31]), are of the type $\Sigma \ltimes \mathbb{H}^d$ where Σ is a connected Lie group acting smoothly on $\mathbb{H}^d$. One can find examples for which the radical of the canonical skew-symmetric is of any dimension k, $0 \leq k \leq d$. For example, one can take for Σ the k-dimensional torus, acting on $\mathbb{H}^d$ by

$$\theta(w) := (\theta \cdot z, s) := \left(\mathrm{e}^{i\theta_1} z_1, \dots, \mathrm{e}^{i\theta_k} z_k, z_{k+1}, \dots, z_d, s\right),\ \ w = (z, s)$$

where the element $\theta = (\theta_1, \dots, \theta_k)$ corresponds to the element $\left(\mathrm{e}^{i\theta_1}, \dots, \mathrm{e}^{i\theta_k}\right)$ of $\mathbb{T}^k$. Then, the product law on $G = \mathbb{T}^k \ltimes \mathbb{H}^d$ is given by

$$(\theta, w) \cdot (\theta', w') = \left(\theta + \theta', w.(\theta(w'))\right),$$

where $w.(\theta(w'))$ denotes the Heisenberg product of w by $\theta(w')$. As a consequence, the center of G is of dimension 1 since it consists of the points of the form $(0, 0, s)$ for $s \in \mathbb{R}$. Let us choose for simplicity $k = d = 1$, the algebra of left-invariant vector fields is generated by the vector fields ∂_θ, ∂_s, $\Gamma_{\theta,x}$ and $\Gamma_{\theta,y}$ where

$$\begin{aligned} \Gamma_{\theta,x} &= \cos\theta\partial_x + \sin\theta\partial_y + 2(y\cos\theta - x\sin\theta)\partial_s, \\ \Gamma_{\theta,y} &= -\sin\theta\partial_x + \cos\theta\partial_y - 2(y\sin\theta + x\cos\theta)\partial_s. \end{aligned}$$

It is not difficult to check that the the radical of $B(\lambda)$ is of dimension 1. In the general case, where $k \leq d$, the algebra of left-invariant vector fields is generated by the vector fields ∂_s, the $d-k-1$ vectors

$$X_\ell = \partial_{x_\ell} + 2y_\ell \partial_s, \;\; Y_\ell = \partial_{y_\ell} - 2x_\ell \partial_s,$$

and the k vectors defined for $1 \leq j \leq k$ by : ∂_{θ_j},, Γ_{θ_j,x_j} and Γ_{θ_j,y_j} where

$$\begin{aligned} \Gamma_{\theta_j,x_j} &= \cos\theta_j \partial_{x_j} + \sin\theta_j \partial_{y_j} + 2(y_j \cos\theta_j - x_j \sin\theta_j)\partial_s, \\ \Gamma_{\theta_j,y_j} &= -\sin\theta_j \partial_{x_j} + \cos\theta_j \partial_{y_j} - 2(y_j \sin\theta_j + x_j \cos\theta_j)\partial_s. \end{aligned}$$

and this provides an example with a radical of dimension k.

1.4.4. *The tensor product of Heisenberg groups.* Consider $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ the set of elements (w_1, w_2) in $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$, which can be written $(w_1, w_2) = (x_1, y_1, s_1, x_2, y_2, s_2)$ in $\mathbb{R}^{2d_1+1} \times \mathbb{R}^{2d_2+1}$, equipped with the law of product:

$$(w_1, w_2) \cdot (w_1', w_2') = (w_1 \cdot w_1', w_2 \cdot w_2'),$$

where $w_1 \cdot w_1'$ and $w_2 \cdot w_2'$ denote respectively the product in $\mathbb{H}^{d_1}$ and $\mathbb{H}^{d_2}$. Clearly $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ is a step 2 stratified Lie group with center of dimension 2 and radical index null. Moreover, for $\lambda = (\lambda_1, \lambda_2)$ in the dual of the center, the canonical skew bilinear form $B(\lambda)$ has radical $\{0\}$ with $\Lambda = \mathbb{R}^* \times \mathbb{R}^*$, and one has $\eta_1(\lambda) = 4\,|\lambda_1|$ and $\eta_2(\lambda) = 4\,|\lambda_2|$. In that case, strict spectral localization is a more restrictive condition than spectral localization. Indeed, if f is spectrally localized, one has $\lambda_1 \neq 0$ **or** $\lambda_2 \neq 0$ on the support of $\mathcal{F}(f)(\lambda)$, while one has $\lambda_1 \neq 0$ **and** $\lambda_2 \neq 0$ on the support of $\mathcal{F}(f)(\lambda)$ if f is strictly spectrally localized.

1.4.5. *Tensor product of H-type groups.* The group $\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2}$ is easily verified to be a step 2 stratified Lie group with center of dimension p_1+p_2, a radical index null and a skew bilinear form $B(\lambda)$ defined on $\mathbb{R}^{m_1+m_2}$ with $m_1 = 2\ell_1$ and $m_2 = 2\ell_2$. The Zariski open set associated with B is given by $\Lambda = (\mathbb{R}^{p_1} \setminus \{0\}) \times (\mathbb{R}^{p_2} \setminus \{0\})$ and for $\lambda = (\lambda_1, \cdots, \lambda_{p_1+p_2})$, we have

$$\begin{aligned} &\eta_j(\lambda) = \sqrt{\lambda_1^2 + \cdots + \lambda_{p_1}^2}, \quad \text{for all} \quad j \in \{1, \dots, \ell_1\} \quad \text{and} \\ &\eta_j(\lambda) = \sqrt{\lambda_{p_1+1}^2 + \cdots + \lambda_{p_1+p_2}^2} \quad \text{for all} \quad j \in \{\ell_1+1, \dots, \ell_1+\ell_2\}. \end{aligned} \tag{1.17}$$

1.5. **Main results.** The purpose of this paper is to establish optimal dispersive inequalities for the linear Schrödinger equation on step 2 stratified Lie groups associated with the sublaplacian. In view of (1.11) and the fact that the "frequencies" $\zeta(\alpha, \lambda)$ associated with $H(\lambda)$ given by (1.13) are homogeneous of degree one in λ, the Schrödinger operator on G behaves like a wave operator on a space of the same dimension p as the center of G, and like a Schrödinger operator on a space of the same dimension k as the radical of the canonical skew-symmetric form. By comparison with the classical dispersive estimates, the expected result would be a dispersion phenomenon with an optimal rate of decay of order $|t|^{-\frac{k+p-1}{2}}$. However, as will be seen through various examples, this anticipated rate is not always achieved. To reach this maximum rate of dispersion, we require a condition on $\zeta(\alpha, \lambda)$.

Assumption 1.4. *For each multi-index α in $\mathbb{N}^d$, the Hessian matrix of the map $\lambda \mapsto \zeta(\alpha, \lambda)$ satisfies*

$$\operatorname{rank} D_\lambda^2 \zeta(\alpha, \lambda) = p - 1$$

where p is the dimension of the center of G.

Remark 1.5. As was observed in Paragraph 1.3.3, $\zeta(\alpha, \lambda)$ is a smooth function, homogeneous of degree one on Λ. By homogeneity arguments, one therefore has $D_\lambda^2 \zeta(\alpha, \lambda)\lambda = 0$. It follows that there always holds

$$\operatorname{rank} D_\lambda^2 \zeta(\alpha, \lambda) \leq p - 1\,,$$

hence Assumption 1.4 may be understood as a maximal rank property.

Let us now present the dispersive inequality for the Schrödinger equation. Recall that the linear Schrödinger equation writes as follows on G:

$$\left\{ \begin{array}{l} (i\partial_t - \Delta_G)\, f = 0 \\ f_{|t=0} = f_0\,, \end{array} \right. \tag{1.18}$$

where the function f with complex values depends on $(t,x) \in \mathbb{R} \times G$.

Theorem 1. *Let G be a step 2 stratified Lie group with center of dimension p with $1 \leq p < n$ and radical index k. Assume that Assumption 1.4 holds. A constant C exists such that if f_0 belongs to $L^1(G)$ and is strictly spectrally localized in a ring of $\mathbb{R}$ in the sense of Definition 1.2, then the associate solution f to the Schrödinger equation (1.18) satisfies*

$$\|f(t,\cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{\frac{k}{2}}(1+|t|^{\frac{p-1}{2}})} \|f_0\|_{L^1(G)}\,, \tag{1.19}$$

for all $t \neq 0$ and the result is sharp in time.

The fact that a spectral localization is required in order to obtain the dispersive estimates is not surprising. Indeed, recall that in the $\mathbb{R}^d$ case for instance, the dispersive estimate for the Schrödinger equation derives immediately (without any spectral localization assumption) from the fact that the solution $u(t)$ to the free Schrödinger equation on $\mathbb{R}^d$ with Cauchy data u_0 writes for $t \neq 0$

$$u(t,\cdot) = u_0 * \frac{1}{(-2i\pi t)^{\frac{d}{2}}} \mathrm{e}^{-i\frac{|\cdot|^2}{4t}}\,,$$

where $*$ denotes the convolution product in $\mathbb{R}^d$ (for a detailed proof of this fact, see for instance Proposition 8.3 in [5]). However proving dispersive estimates for the wave equation in $\mathbb{R}^d$ requires more elaborate techniques (including oscillating integrals) which involve an assumption of spectral localization in a ring. In the case of a step 2 stratified Lie group G, the main difficulty arises from the complexity of the expression of Schrödinger propagator that mixes a wave operator behavior with that of a Schrödinger operator. This explains on the one hand the decay rate in Estimate (1.19) and on the other hand the hypothesis of strict spectral localization.

Let us now discuss Assumption 1.4. As mentioned above, there is no dispersion phenomenon for the Schrödinger equation on the Heisenberg group $\mathbb{H}^d$ (see [7]). Actually the same holds for the tensor product of Heisenberg groups $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ whose center is of dimension $p = 2$ and radical index null, and more generally to the case of 2 step stratified Lie groups, decomposable on non trivial 2 step stratified Lie groups : we derive indeed from Theorem 1 the following corollary.

Corollary 1.6. *Let $G = \otimes_{1\leq m\leq r} G_m$ be a decomposable, 2 step stratified Lie group where the groups G_m are non trivial 2-step stratified Lie groups satisfying Assumption 1.4, of radical index k_m and with centers of dimension p_m. Then the dispersive estimates holds with rate $|t|^{-q}$,*

$$q := \frac{1}{2}\sum_{1\leq m\leq r}(k_m + p_m - 1) = \frac{1}{2}(k+p-r)\,,$$

where p is the dimension of the center of G and k its radical index. Besides, this rate is optimal.

Corollary 1.6 is a direct consequence of Theorem 1 and the simple observation that $\Delta_G = \otimes_{1\leq m\leq r}\Delta_{G_m}$. This result applies for example to the tensor product of Heisenberg groups, for which there is no dispersion, and to the tensor product of H-type groups $\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2}$ for which the dispersion rate is $t^{-(p_1+p_2-2)/2}$ (see [17]). Corollary 1.6 therefore shows that it can happen that the "best" rate of decay $|t|^{-(k+p-1)/2}$ is not reached, in particular for decomposable Lie groups. This suggests that Assumption 1.4 could be related with decomposability.

More generally, a large class of groups which do not satisfy the Assumption 1.4 is given by step 2 stratified Lie group G for which $\zeta(0,\lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Of course, the Heisenberg group and any tensor product of Heisenberg group is of that type. We then have the following result which illustrates that there exists examples of groups without any dispersion and which do not satisfy Assumption 1.4.

Proposition 1.7. *Consider a step 2 stratified Lie group G whose radical index is null and for which $\zeta(0,\lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Then, there exists $f_0 \in \mathcal{S}(G)$, $x \in G$ and $c_0 > 0$ such that*

$$\forall t \in \mathbb{R}^+,\ \ |\mathrm{e}^{-it\Delta_G} f_0(x)| \geq c_0.$$

Finally we point out that the dispersive estimate given in Theorem 1 can be regarded as a first step towards space-time estimates of the Strichartz type. However due to the strict spectral localization assumption, the Besov spaces which should appear in the study (after summation over frequency bands) are naturally anisotropic; thus proving such estimates is likely to be very technical, and is postponed to future works.

1.6. **Strategy of the proof of Theorem 1.** In the statement of Theorem 1, there are two different results: the dispersive estimate in itself on the one hand, and its optimality on the other. Our strategy or proof is closely related to the method developed in [7] and [17] with additional non negligible technicalities.

In the situation of [7] where the Heisenberg group $\mathbb{H}^d$ is considered, the authors prove that there is no dispersion by exhibiting explicitly a Cauchy data f_0 for which the solution $f(t,\cdot)$ to the Schrödinger equation (1.18) satisfies

$$\forall\, q\in[1,\infty]\,,\quad \|f(t,\cdot)\|_{L^q(\mathbb{H}^d)}=\|f_0\|_{L^q(\mathbb{H}^d)}\,. \tag{1.20}$$

More precisely, they take advantage of the fact that the Laplacian-Kohn operator $\Delta_{\mathbb{H}^d}$ can be recast under the form

$$\Delta_{\mathbb{H}^d}=4\sum_{j=1}^{d}(Z_j\overline{Z}_j+i\partial_s)\,, \tag{1.21}$$

where $\left\{Z_1,\overline{Z}_1,\ldots,\ Z_d,\overline{Z}_d,\partial_s\right\}$ is the canonical basis of Lie algebra of left invariant vector fields on $\mathbb{H}^d$ (see [8] and the references therein for more details). This implies that for a non zero function f_0 belonging to Ker $\left(\sum_{j=1}^d Z_j\overline{Z}_j\right)$, the solution of the Schrödinger equation on the Heisenberg group $f(t)=\mathrm{e}^{-it\Delta_{\mathbb{H}^d}}f_0$ actually solves a transport equation:

$$f(z,s,t)=\mathrm{e}^{4dt\partial_s}f_0(z,s)=f_0(z,s+4dt)$$

and hence satisfies (1.20). The arguments used in [17] for general H-type groups are similar to the ones developed in [7]: the dispersive estimate is obtained using an explicit formula for the solution, coming from Fourier analysis, combined with a stationary phase theorem. The Cauchy data used to prove the optimality is again in the kernel of an adequate operator, by a decomposition similar to (1.21).

As in [7] and [17], the first step of the proof of Theorem 1 consists in writing an explicit formula for the solution of the equation by use of the Fourier transform. Let us point out that in the setting of [7] and [17], irreducible representations are isotropic with respect to the dual of the center of the group; this isotropy allows to reduce to a one-dimensional framework and deduce the dispersive effect from a careful use of a stationary phase argument of [37]. As we have already seen in Paragraph 1.3.1, the irreducible representations are no longer isotropic in the general case of stratified Lie groups, and thus we adopt a more technical approach making use of Schrödinger representation and taking advantage of some properties of Hermite functions appearing in the explicit representation of the solutions derived by Fourier analysis (see Section 3.3). The optimality of the inequality is obtained as in [7] and [17], by an adequate choice of the initial data.

1.7. **Organization of the paper.** The article is organized as follows. In Section 2, we write an explicit formulation of the solutions of the Schrödinger equation. Then, Section 3 is devoted to the proof of Theorem 1 and in Section 4, we discuss the optimality of the result and prove Proposition 1.7.

Finally, we mention that the letter C will be used to denote a universal constant which may vary from line to line. We also use $A\lesssim B$ to denote an estimate of the form $A\leq CB$ for some constant C.

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2. Explicit representation of the solutions

2.1. **The convolution kernel.** Let f_0 belong to $\mathcal{S}(G)$ and let us consider $f(t,\cdot)$ the solution to the free Schrödinger equation (1.18). In view of (1.11), we have

$$\mathcal{F}(f(t,\cdot))(\lambda,\nu)=\mathcal{F}(f_0)(\lambda,\nu)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\,,$$

which implies easily (arguing as in the Appendix) that $f(t,\cdot)$ belongs to $\mathcal{S}(G)$. Assuming that f_0 is strictly spectrally localized in the sense of Definition 1.2, there exists a smooth function θ compactly supported in a ring $\mathcal{C}$ of $\mathbb{R}$ such that, defining

$$\Theta(\lambda):=\prod_{j=1}^{d}\theta\big((P_j^2+Q_j^2)(\lambda)\big)\,,$$

then

$$\mathcal{F}(f(t,\cdot))(\lambda,\nu)=\mathcal{F}(f_0)(\lambda,\nu)\,\Theta(\lambda)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\,.$$

Therefore by the inverse Fourier transform (1.8), we deduce that the function $f(t,\cdot)$ may be decomposed in the following way:

$$f(t,x)=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\mathrm{tr}\,\Big((u^{\lambda,x}_{X(\lambda,\nu)})^\star\,\mathcal{F}(f_0)(\lambda,\nu)\,\Theta(\lambda)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\Big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,. \tag{2.1}$$

We set for $X\in\mathbb{R}^n$,

$$k_t(X):=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\mathrm{tr}\,\Big((u^{\lambda,\nu}_{X})^\star\,\Theta(\lambda)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\Big)\,|\mathrm{Pf}(\lambda)|\,d\nu d\lambda\,. \tag{2.2}$$

The function k_t plays the role of a convolution kernel in the variables of the Lie algebra and we have the following result.

Proposition 2.1. *If the function k_t defined in (2.2) satisfies*

$$\forall t\in\mathbb{R}\,,\quad \|k_t\|_{L^\infty(\mathbb{R}^n)}\leq\frac{C}{|t|^{\frac{k}{2}}(1+|t|^{\frac{p-1}{2}})}\,, \tag{2.3}$$

then Theorem 1 holds.

Proof. We write, according to (2.1),

$$\begin{aligned}f(t,x)&=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{y\in G}\mathrm{tr}\,\Big((u^{\lambda,\nu}_{X(\lambda,x)})^*u^{\lambda,\nu}_{X(\lambda,y)}\,\Theta(\lambda)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\Big)f_0(y)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\mu(y)\\&=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{y\in G}\mathrm{tr}\,\Big(u^{\lambda,\nu}_{X(\lambda,y)}\,\Theta(\lambda)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\Big)f_0(x\cdot y)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\mu(y)\,.\end{aligned}$$

We use the exponential law $y\mapsto Y=(P(\lambda,y),Q(\lambda,y),Z,R(\lambda,y))$ and the fact that $d\mu(y)=dY$ the Lebesgue measure, then we perform a linear orthonormal change of variables

$$(P(\lambda,y),Q(\lambda,y),R(\lambda,y))\mapsto(\tilde P,\tilde Q,\tilde R)$$

so that $d\mu(y)=dY=d\tilde P\,d\tilde Q\,dZ\,d\tilde R$ and we write

$$\begin{aligned}f(t,x)=\kappa\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{(\tilde P,\tilde Q,Z,\tilde R)\in\mathbb{R}^n}&\mathrm{tr}\,\Big(u^{\lambda,\nu}_{(\tilde P,\tilde Q,Z,\tilde R)}\,\Theta(\lambda)\,\mathrm{e}^{it|\nu|^2+itH(\lambda)}\Big)\\&\times f_0(x\cdot\exp(\tilde P,\tilde Q,Z,\tilde R))|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\tilde P\,d\tilde Q\,dZ\,d\tilde R\,.\end{aligned}$$

Thanks to the Fubini Theorem and Young inequalities, we can write (dropping the ˜ on the variables),

$$\begin{aligned}|f(t,x)|&=\left|\int_{(P,Q,Z,R)\in\mathbb{R}^n}k_t(P,Q,Z,R)f_0(x\cdot\exp(P,Q,Z,R))dP\,dQ\,dR\,dZ\right|\\&\leq\|k_t\|_{L^\infty(G)}\left|\int_{(P,Q,Z,R)\in\mathbb{R}^n}f_0(x\cdot\exp(P,Q,Z,R))dP\,dQ\,dR\,dZ\right|\\&\leq\|k_t\|_{L^\infty(G)}\|f_0\|_{L^1(G)}.\end{aligned}$$

Proposition 2.1 is proved. $\square$

In the next subsections, we make preliminary work by transforming the expression of k_t and reducing the proof to statements equivalent to (2.3).

2.2. **Transformation of k_t: expression in terms of Hermite functions.** Decomposing the operator $H(\lambda)$ in the basis of Hermite functions, and recalling notation (1.12) replaces (2.2) with

$$k_t(X) = \kappa \sum_{\alpha\in\mathbb{N}^d} \int_\Lambda \int_{\nu\in\mathfrak{r}_\lambda^*} e^{it|\nu|^2+it\zeta(\alpha,\lambda)} \prod_{j=1}^d \theta\big(\zeta_j(\alpha,\lambda)\big)\big(u_X^{\lambda,\nu} h_{\alpha,\eta(\lambda)}|h_{\alpha,\eta(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,,\quad X\in\mathbb{R}^n\ .$$

Using the explicit form of $u_X^{\lambda,\nu}$ recalled in (1.5) we find the following result.

Lemma 2.2. *There is a constant $\widetilde{\kappa}$ and a smooth function F such that with the above notation, we have for $t\neq 0$*

$$k_t(P,Q,tZ,R) = \frac{\widetilde{\kappa}\,\mathrm{e}^{-i\frac{|R|^2}{4t}}}{t^{\frac{k}{2}}} \sum_{\alpha\in\mathbb{N}^d}\int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)} G_\alpha\big(P,Q,\eta(\lambda)\big)\,|\mathrm{Pf}(\lambda)|\,F(\lambda)\,d\lambda\,,$$

where the phase Φ_α is given by

$$\Phi_\alpha(Z,\lambda) := \zeta(\alpha,\lambda)-\lambda(Z)\,,$$

with Notation (1.13), *and the function G_α is given by the following formula, for all $(P,Q,\eta)\in\mathbb{R}^{3d}$:*

$$G_\alpha(P,Q,\eta) := \prod_{j=1}^d \theta\big(\zeta_j(\alpha,\lambda)\big)\,g_{\alpha_j}\Big(\sqrt{\eta_j}\,P_j,\sqrt{\eta_j}\,Q_j\Big) \tag{2.4}$$

while for each (ξ_1,ξ_2,n) in $\mathbb{R}^2\times\mathbb{N}$, using Notation (1.10),

$$g_n(\xi_1,\xi_2) := e^{-i\frac{\xi_1\xi_2}{2}}\int_{\mathbb{R}} e^{-i\xi_2\xi}h_n(\xi_1+\xi)h_n(\xi)\,d\xi\,. \tag{2.5}$$

Notice that $(g_n)_{n\in\mathbf{N}}$ is uniformly bounded in $\mathbb{R}^2$ thanks to the Cauchy-Schwarz inequality and the fact that $\|h_n\|_{L^2(\mathbb{R})}=1$, and hence the same holds for $(G_\alpha)_{\alpha\in\mathbf{N}^d}$ (in $\mathbb{R}^{3d}$).

Proof. We begin by observing that for $X=(P,Q,R,Z)$,

$$\begin{aligned} I &:= \Big(u_X^{\lambda,\nu}h_{\alpha,\eta(\lambda)}\,|\,h_{\alpha,\eta(\lambda)}\Big)\\ &= \mathrm{e}^{-i\nu(R)-i\lambda(Z)}\int_{\mathbb{R}^d}\mathrm{e}^{-i\lambda([\xi+\frac12 P,Q])}h_{\alpha,\eta(\lambda)}(P+\xi)h_{\alpha,\eta(\lambda)}(\xi)d\xi\,,\end{aligned}$$

with in view of (1.4)

$$\lambda\big([\xi+\frac12P,Q]\big) = B(\lambda)\big(\xi+\frac12P,Q\big) = \sum_{1\le j\le d}\eta_j(\lambda)Q_j\big(\xi_j+\frac12P_j\big)\,.$$

As a consequence,

$$I = \mathrm{e}^{-i\nu(R)-i\lambda(Z)}\prod_{1\le j\le d}\int_{\mathbb{R}}\mathrm{e}^{-i\eta_j(\lambda)(\xi_j+\frac12P_j)Q_j}h_{\alpha_j,\eta_j(\lambda)}(P_j+\xi_j)h_{\alpha_j,\eta_j(\lambda)}(\xi_j)d\xi_j\,.$$

The change of variables $\widetilde{\xi_j}=\sqrt{\eta_j(\lambda)}\,\xi_j$ gives, dropping the $\widetilde{}$ for simplicity,

$$I = \mathrm{e}^{-i\nu(R)-i\lambda(Z)}\prod_{1\le j\le d}\int_{\mathbb{R}}\mathrm{e}^{-i\sqrt{\eta_j(\lambda)}\,Q_j\big(\xi_j+\frac12\sqrt{\eta_j(\lambda)}P_j\big)}h_{\alpha_j}\big(\xi_j+\sqrt{\eta_j(\lambda)}P_j\big)h_{\alpha_j}(\xi_j)d\xi_j\,,$$

which implies that

$$k_t(P,Q,tZ,R) = \kappa\sum_{\alpha\in\mathbb{N}^d}\int_{\mathfrak{r}(\Lambda)}\mathrm{e}^{-it\lambda(Z)-i\nu(R)}\mathrm{e}^{it\zeta(\alpha,\lambda)+it|\nu|^2}G_\alpha\big(P,Q,\eta(\lambda)\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,.$$

It is well known (see for instance Proposition 1.28 in [5]) that for $t\neq0$

$$\int_{\mathbb{R}^k}\mathrm{e}^{-i(\nu|R)+it|\nu|^2}\,d\nu = \Big(\frac{i\pi}{t}\Big)^{\frac{k}{2}}\mathrm{e}^{-i\frac{|R|^2}{4t}}\,, \tag{2.6}$$

where $(\cdot \mid \cdot)$ denotes the euclidean scalar product on $\mathbb{R}^k$. This implies that, for $t \neq 0$

$$|k_t(P,Q,tZ,R)| \lesssim \frac{1}{|t|^{\frac{k}{2}}} \Big| \sum_{\alpha \in \mathbb{N}^d} \int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)} G_\alpha\big(P,Q,\eta(\lambda)\big) \, |\mathrm{Pf}(\lambda)| \, F(\lambda)\, d\lambda \Big| \, ,$$

with F the Jacobian of the change of variables $f : \mathfrak{r}_\lambda^* \longrightarrow \mathbb{R}^k$, which is a smooth function. Lemma 2.2 is proved. $\square$

2.3. **Transformation of the kernel k_t: change of variable.** We are then reduced to establishing that the kernel $\widetilde{k}_t(P,Q,tZ)$ defined by

$$\widetilde{k}_t(P,Q,tZ) := \sum_{\alpha \in \mathbb{N}^d} \int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)} G_\alpha\big(P,Q,\eta(\lambda)\big) \, |\mathrm{Pf}(\lambda)| \, F(\lambda)\, d\lambda$$

satisfies

$$\forall t \in \mathbb{R}\,, \quad \|\widetilde{k}_t\|_{L^\infty(G)} \leq \frac{C}{1+|t|^{\frac{p-1}{2}}} \,\cdot \tag{2.7}$$

To this end, let us define $m := |\alpha| = \sum_{j=1}^d \alpha_j$, and in the case when $m \neq 0$, let us set $\gamma := m\lambda \in \mathbb{R}^p$. By construction of $\eta(\lambda)$ (which is homogeneous of degree one), we have

$$\forall m \neq 0\,, \quad \eta(\lambda) = \widetilde{\eta}_m(\gamma) := \frac{1}{m}\eta(\gamma)\,. \tag{2.8}$$

Let us check that if λ lies in the support of $\theta\big(\zeta_j(\alpha,\cdot)\big)$, then γ lies in a fixed ring $\mathcal{C}$ of $\mathbb{R}^p$, independent of α. On the one hand we note that there is a constant $C > 0$ such that on the support of $\theta\big(\zeta_j(\alpha,\lambda)\big)$, the variable γ must satisfy

$$\forall m \neq 0\,, \quad (2\alpha_j+1)\eta_j(\gamma) \leq Cm\,, \tag{2.9}$$

for all $\alpha \in \mathbf{N}^d$ such that $|\alpha| = m$. Since for each j we know that $\eta_j(\gamma)$ is positive and homogeneous of degree one, we infer that the function $\eta_j(\gamma)$ goes to infinity with $|\gamma|$ so (2.9) implies that γ must remain bounded on the support of $\theta\big(\zeta_j(\alpha,\lambda)\big)$. Moreover thanks to (2.9) again, it is clear that the bound may be made uniform in m.

Now let us prove that γ may be bounded from below uniformly. We know that there is a positive constant c such that for λ on the support of $\theta\big(\zeta_j(\alpha,\lambda)\big)$, we have

$$\forall m \neq 0\,, \quad \zeta_j(\alpha,\gamma) \geq cm\,. \tag{2.10}$$

Writing $\gamma = |\gamma|\hat\gamma$ with $\hat\gamma$ on the unit sphere of $\mathbb{R}^p$, we find

$$|\gamma| \geq \frac{cm}{\zeta_j(\alpha,\hat\gamma)}\,\cdot$$

Defining

$$C_j := \max_{|\hat\gamma|=1} \eta_j(\hat\gamma) < \infty\,,$$

it is easy to deduce that if (2.10) is satisfied, then

$$|\gamma| \geq \frac{cm}{(2m+d)\max\limits_{1\leq j\leq d} C_j}\,,$$

hence γ lies in a fixed ring of $\mathbb{R}^p$, independent of $\alpha \neq 0$. This fact will turn out to be important to perform the stationary phase argument.

Then we can rewrite the expression of $\widetilde{k}_t(P,Q,tZ)$ in terms of the variable γ, which in view of the homogeneity of the Pfaffian produces the following formula:

$$\begin{aligned}\widetilde{k}_t(P,Q,tZ) = &\int_\Lambda \mathrm{e}^{it\Phi_0(Z,\lambda)} G_0(P,Q,\eta(\lambda)) |\mathrm{Pf}(\lambda)|\, F(\lambda)\, d\lambda \\ &+ \sum_{m\in\mathbb{N}^*} \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|=m}} m^{-p-d} \int \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})} G_\alpha\big(P,Q,\widetilde{\eta}_m(\gamma)\big)\, |\mathrm{Pf}(\gamma)|\, F\big(\frac{\gamma}{m}\big)\, d\gamma\,.\end{aligned}$$

Since the functions G_α are uniformly bounded with respect to $\alpha\in\mathbb{N}^d$ and F is smooth, there is a positive constant C such that

$$\forall t\in\mathbb{R}\,,\quad \|\widetilde{k}_t\|_{L^\infty(G)} \le C\,.$$

In order to establish the dispersive estimate, it suffices then to prove that

$$\forall t\neq 0\,,\quad \|\widetilde{k}_t\|_{L^\infty(G)} \le \frac{C}{|t|^{\frac{p-1}{2}}}\,. \tag{2.11}$$

3. End of the proof of the dispersive estimate

In order to prove (2.11), we decompose $\widetilde{k}_t$ into two parts, writing

$$\widetilde{k}_t(P,Q,tZ) = k_t^1(P,Q,tZ) + k_t^2(P,Q,tZ)\,,$$

with, for a constant c_0 to be fixed later on independently of m,

$$\begin{aligned} k_t^1(P,Q,tZ) := &\int_{|\nabla_\lambda\Phi_0(Z,\lambda)|\le c_0} \mathrm{e}^{it\Phi_0(Z,\lambda)} G_0\big(P,Q,\eta(\lambda)\big) |\mathrm{Pf}(\lambda)|\, F(\lambda)\, d\lambda \\ &+ \sum_{m\in\mathbb{N}^*} \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|=m}} m^{-p-d} \int_{|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|\le c_0} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}\, G_\alpha\big(P,Q,\widetilde{\eta}_m(\gamma)\big) F\big(\frac{\gamma}{m}\big) |\mathrm{Pf}(\gamma)|\, d\gamma\,.\end{aligned} \tag{3.1}$$

In the following subsections, we successively show (2.11) for k_1^t and k_2^t.

3.1. **Stationary phase argument for k_1^t.** To establish Estimate (2.11), let us first concentrate on k_t^1. To this end we shall, as usual in this type of problem, define for each integral of the series defining k_t^1 a vector field that commutes with the phase, prove an estimate for each term and finally check the convergence of the series. More precisely, in the case when $\alpha\neq 0$ and $t>0$ (the case $t<0$ is dealt with exactly in the same manner), we consider the following first order operator:

$$\mathcal{L}_\alpha^1 := \frac{\mathrm{Id} - i\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\cdot\nabla_\gamma}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2}\,.$$

Clearly we have

$$\mathcal{L}_\alpha^1\, \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})} = \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}\,.$$

Let us admit the next lemma for the time being.

Lemma 3.1. *For any integer N, there is a smooth function θ_N compactly supported on a ring of $\mathbb{R}^p$ and a positive constant C_N such that defining*

$$\psi_\alpha(\gamma) := G_\alpha\big(P,Q,\widetilde{\eta}_m(\gamma)\big) F\big(\frac{\gamma}{m}\big)\, |\mathrm{Pf}(\gamma)|\,, \tag{3.2}$$

recalling notation (2.8)*, we have*

$$|({}^t\mathcal{L}_\alpha^1)^N \psi_\alpha(\gamma)\,| \le C_N\, m^N\, \theta_N(\gamma)\,\big(1+\big|t^{\frac12}\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\big|^2\big)^{-N}\,.$$

Returning to k_t^1, let us define (recalling that γ belongs to a fixed ring $\mathcal{C}$)

$$\mathcal{C}_\alpha(Z) := \big\{\gamma\in\mathcal{C};\, |\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|\le c_0\big\}$$

and let us write for any integer N and $\alpha \neq 0$ (which we assume to be the case for the rest of the computations)

$$
\begin{aligned}
I_\alpha(Z) &:= \int_{\mathcal{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}\psi_\alpha(\gamma)\,d\gamma \\
&= \int_{\mathcal{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}({}^t\mathcal{L}^1_\alpha)^N\psi_\alpha(\gamma)\,d\gamma\,,
\end{aligned}
\tag{3.3}
$$

where $\psi_\alpha(\gamma)$ has been defined in (3.2). Then by Lemma 3.1 we find that for each integer N there is a constant C_N such that

$$
|I_\alpha(Z)| \leq C_N\, m^N \int_{\mathcal{C}_\alpha(Z)} \theta_N(\gamma)\big(1+t\big|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\big|^2\big)^{-N}\,d\gamma\,. \tag{3.4}
$$

Then the end of the proof relies on three steps:

(1) a careful analysis of the properties of the support of the integral,
(2) a change of variables which leads to the estimate in $t^{-(p-1)/2}$,
(3) a control in m in order to prove the convergence of the sum over m.

Before entering into details for each step, let us observe that by definition, we have

$$
\Phi_\alpha(Z,\frac{\gamma}{m}) = \frac{1}{m}\big(\zeta(\alpha,\gamma)-\gamma(Z)\big)\,,
$$

with $\gamma(Z) = (A\gamma|Z) = (\gamma|{}^tAZ)$ for some invertible matrix A. Performing a change of variables in γ if necessary, we can assume without loss of generality that $A = \mathrm{Id}$. Thus we write

$$
\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m})) = \frac{1}{m}\big(\nabla_\gamma\zeta(\alpha,\gamma)-Z\big)\,. \tag{3.5}
$$

3.1.1. *Analysis of the support of the integral defining $I_\alpha(Z)$.* Let us prove the following result.

Proposition 3.2. *One can choose the constant c_0 in* (3.1) *small enough so that if γ belongs to $\mathcal{C}_\alpha(Z)$, then $\gamma\cdot Z\neq 0$.*

Proof. Let us first prove that we can choose c_0 small enough so that if $\mathcal{C}_\alpha(Z)$ is not empty, then one has $Z\neq 0$. Equivalently, let us prove that we can choose c_0 small enough such that $\mathcal{C}_\alpha(0)=\emptyset$. We first notice that thanks to (3.5),

$$
\nabla_\gamma\zeta(\alpha,\gamma) = \nabla_\gamma\Phi_\alpha(0,\frac{\gamma}{m})\,.
$$

Now since the function $\zeta(\alpha,\cdot)$ is homogeneous of degree 1, then $\nabla_\gamma\zeta(\alpha,\cdot)$ is homogeneous of degree 0. As a consequence, if $\nabla\zeta(\alpha,\gamma_0)=0$ with $\gamma_0\in\mathcal{C}$, then the function $\mathbb{R}\ni t\mapsto\zeta(\alpha,t\gamma_0)$ is constant and identically equal to 0. This is in contradiction with the localization on a ring by the function

$$
\widetilde{\theta}(\gamma) := \prod_{j=1}^{d}\theta(\zeta_j(\alpha,m^{-1}\gamma))\,,
$$

which implies the existence of a positive constant c_1 such that on the support of $\widetilde{\theta}$, $|\nabla_\gamma\zeta(\alpha,\gamma)|>2mc_1$. This ensures the result if we assume that the constant c_0 in the definition of k^1_t is smaller than c_1.

Let us now prove that if the constant c_0 is chosen small enough, then for all $\gamma\in\mathcal{C}_\alpha(Z)$ we have $\gamma\cdot Z\neq 0$. Indeed writing

$$
\gamma\cdot Z = \gamma\cdot\nabla_\gamma\zeta(\alpha,\gamma)+\gamma\cdot(Z-\nabla_\gamma\zeta(\alpha,\gamma))\,,
$$

and observing that thanks to homogeneity arguments $\gamma\cdot\nabla_\gamma\zeta(\alpha,\gamma)=\zeta(\alpha,\gamma)$, we deduce that for any $\gamma\in\mathcal{C}_\alpha(Z)$

$$
|\gamma\cdot Z| \geq |\zeta(\alpha,\gamma)| - |\gamma|\,|Z-\nabla_\gamma\zeta(\alpha,\gamma)|\,.
$$

Since as argued above, γ belongs to a fixed ring and $\zeta(\alpha,\lambda)=0$ if and only if $\lambda=0$ (as noticed in Section 1.3.3), there is a positive constant c such that for any $\gamma\in\mathcal{C}_\alpha(Z)$

$$
|\zeta(\alpha,\gamma)| \geq mc\,,
$$

which implies in view of the definition of $\mathcal{C}_\alpha(Z)$ that there is a positive constant $\widetilde{c}$ depending only on the ring $\mathcal{C}$ such that

$$
|\gamma\cdot Z| \geq mc - mc_0\,\widetilde{c}\,.
$$

This ensures the desired result, by choosing the constant c_0 in the definition of k_t^1 both smaller than $c/\widetilde{c}$ and the constant c_1 defined above. Proposition 3.2 is proved. □

3.1.2. *A change of variables: the diffeomorphism $\mathcal{H}$.* We can assume without loss of generality (if not the integral is zero) that $\mathcal{C}_\alpha(Z)$ is not empty, and in view of Proposition 3.2, we can write for any $\gamma \in \mathcal{C}_\alpha(Z)$ the following orthogonal decomposition (since $Z \neq 0$):

$$\frac{1}{m}\nabla_\gamma\zeta(\alpha,\gamma) = \widetilde{\Gamma}_1\,\hat{Z}_1 + \widetilde{\Gamma}'\,, \quad \text{with} \quad \widetilde{\Gamma}_1 := \Big(\frac{1}{m}\nabla_\gamma\zeta(\alpha,\gamma)\Big|\hat{Z}\Big) \text{ and } \hat{Z}_1 := \frac{Z}{|Z|}\,\cdot \tag{3.6}$$

Since $\widetilde{\Gamma}'$ is orthogonal to the vector Z, we infer that

$$\Big|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\Big| = \frac{1}{m}\,|Z - \nabla_\gamma\zeta(\alpha,\gamma)| \geq |\widetilde{\Gamma}'|\,. \tag{3.7}$$

Let us consider in $\mathbb{R}^p$ an orthonormal basis $(\hat{Z}_1,\ldots,\hat{Z}_p)$. Thanks to Proposition 3.2, we have $\gamma\cdot\hat{Z}_1 \neq 0$ on the support of the integral defining $I_\alpha(Z)$. Obviously, the vector $\widetilde{\Gamma}'$ defined by (3.6) belongs to the vector space generated by $(\hat{Z}_2,\ldots,\hat{Z}_p)$. To investigate the integral $I_\alpha(Z)$ defined in (3.3), let us consider the map $\mathcal{H} : \gamma \mapsto \widetilde{\gamma}'$ defined by

$$\mathcal{C}_\alpha(Z) \ni \gamma \longmapsto \mathcal{H}(\gamma) := (\gamma\cdot\hat{Z}_1)\,\hat{Z}_1 + \sum_{k=2}^{p}(\widetilde{\Gamma}'\cdot\hat{Z}_k)\,\hat{Z}_k =: \sum_{k=1}^{p}\widetilde{\gamma}'_k\,\hat{Z}_k\,. \tag{3.8}$$

Proposition 3.3. *The map $\mathcal{H}$ realizes a diffeomorphism from $\mathcal{C}_\alpha(Z)$ into a fixed compact set of $\mathbb{R}^p$.*

Proof. It is clear that the smooth function $\mathcal{H}$ maps $\mathcal{C}_\alpha(Z)$ into a fixed compact set $\mathcal{K}$ of $\mathbb{R}^p$ and that

$$\widetilde{\gamma}'_1 = \gamma\cdot\hat{Z}_1\,, \quad \text{and for} \quad 2\leq k\leq p\,,\ \ \widetilde{\gamma}'_k = \frac{1}{m}\nabla_\gamma\zeta(\alpha,\gamma)\cdot\hat{Z}_k\,.$$

Now let us prove that thanks to Assumption 1.4, the map $\mathcal{H}$ constitutes a diffeomorphism. Indeed, by straightforward computations we find that $D\mathcal{H}$, the differential of $\mathcal{H}$ satisfies:

$$\begin{aligned}
&\langle D\mathcal{H}(\gamma)\hat{Z}_1,\hat{Z}_1\rangle = 1\\
&\langle D\mathcal{H}(\gamma)\hat{Z}_1,\hat{Z}_k\rangle = \Big\langle\frac{1}{m}D^2_\gamma\zeta(\alpha,\gamma)\hat{Z}_1,\hat{Z}_k\Big\rangle \quad \text{for } 2\leq k\leq p\,,\\
&\langle D\mathcal{H}(\gamma)\hat{Z}_j,\hat{Z}_k\rangle = \Big\langle\frac{1}{m}D^2_\gamma\zeta(\alpha,\gamma)\hat{Z}_j,\hat{Z}_k\Big\rangle \quad \text{for } 2\leq j,k\leq p \quad \text{and}\\
&\langle D\mathcal{H}(\gamma)\hat{Z}_j,\hat{Z}_1\rangle = 0 \quad \text{for } 2\leq j\leq p\,.
\end{aligned}$$

Proving that $\mathcal{H}$ is a diffeomorphism amounts to showing that for any $\gamma\in\mathcal{C}_\alpha(Z)$, the kernel of $D\mathcal{H}(\gamma)$ reduces to $\{0\}$. In view of the above formulas, if $V = \displaystyle\sum_{j=1}^{p} V_j\hat{Z}_j$ belongs to the kernel of $D\mathcal{H}(\gamma)$ then $V_1 = V\cdot\hat{Z}_1 = 0$ and $D^2_\gamma\zeta(\alpha,\gamma)V\cdot\hat{Z}_k = 0$ for $2\leq k\leq p$. Thus we can write $D^2_\gamma\zeta(\alpha,\gamma)V = \tau\hat{Z}_1$ for some $\tau\in\mathbb{R}$. Let us point out that since the function $\zeta(\alpha,\cdot)$ is homogeneous of degree 1, then $D^2_\gamma\zeta(\alpha,\gamma)\gamma = 0$. We deduce that

$$0 = D^2_\gamma\zeta(\alpha,\gamma)\gamma\cdot V = \gamma\cdot D^2_\gamma\zeta(\alpha,\gamma)V = \tau\,\gamma\cdot\hat{Z}_1\,.$$

Since for all $\gamma\in\mathcal{C}_\alpha(Z)$, $\gamma\cdot\hat{Z}_1\neq 0$, we find that $\tau = 0$ and therefore $D^2_\gamma\zeta(\alpha,\gamma)V = 0$. But Assumption 1.4 states that the Hessian $D^2_\gamma\zeta(\alpha,\gamma)$ is of rank $p-1$, so we conclude that V is colinear to γ. But we have seen that $V\cdot\hat{Z}_1 = 0$, which contradicts the fact that $\gamma\cdot\hat{Z}_1\neq 0$. This entails that V is null and ends the proof of the proposition. □

We can therefore perform the change of variables defined by (3.8) in the right-hand side of (3.4), to obtain

$$|I_\alpha(Z)| \leq C_N\,m^N\int_{\mathcal{K}}\frac{1}{(1+t|\widetilde{\gamma}'|^2)^N}\,d\widetilde{\gamma}'\,d\widetilde{\gamma}_1\,.$$

3.1.3. *End of the proof: convergence of the series.* Choosing $N = p-1$ implies by the change of variables $\gamma^\sharp = t^{\frac12}\widetilde{\gamma}'$ that there is a constant C such that

$$|I_\alpha(Z)| \leq C|t|^{-\frac{p-1}{2}}\, m^{p-1}\,,$$

which gives rise to

$$\Big|\int_{\mathcal{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}\psi_\alpha(\gamma)d\gamma\Big| \leq C|t|^{-\frac{p-1}{2}}m^{p-1}.$$

We get exactly in the same way that

$$\Big|\int_{|\nabla_\lambda\Phi_0(Z,\lambda)|\leq c_0} \mathrm{e}^{it\Phi_0(Z,\lambda)}G_0\big(P,Q,\eta(\lambda)\big)|\mathrm{Pf}(\lambda)|\,F(\lambda)\,d\lambda\Big| \leq C|t|^{-\frac{p-1}{2}}.$$

Finally returning to the kernel k_t^1 defined in (3.1), we get

$$\begin{aligned}|k_t^1(P,Q,tZ)| &\leq C|t|^{-\frac{p-1}{2}} + C\,|t|^{-\frac{p-1}{2}}\sum_{m\in\mathbb{N}^*} m^{d-1}m^{-d-p}\,m^{p-1}\\ &\leq C|t|^{-\frac{p-1}{2}}\,,\end{aligned}$$

since the series over m is convergent. The dispersive estimate is thus proved for k_t^1.

3.2. **Stationary phase argument for k_2^t.** We now prove (2.11) for k_t^2, which is easier since the gradient of the phase is bounded from below. We claim that there is a constant C such that

$$|k_t^2(P,Q,tZ)| \leq \frac{C}{t^{\frac{p-1}{2}}}\,\cdot \tag{3.9}$$

This can be achieved as above by means of adequate integrations by parts. Indeed, in the case when $\alpha\neq 0$, consider the following first order operator:

$$\mathcal{L}_\alpha^2 := -i\frac{\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\cdot\nabla_\gamma}{|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2}\,\cdot$$

Note that when $\alpha = 0$, the arguments are the same without performing the change of variable $\lambda = \gamma/m$. The operator $\mathcal{L}_\alpha^2$ obviously satisfies

$$\mathcal{L}_\alpha^2\,\mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})} = t\,\mathrm{e}^{it\,\Phi_\alpha(Z,\frac{\gamma}{m})}\,,$$

hence by repeated integrations by parts, we get

$$\begin{aligned}J_\alpha(P,Q,tZ) &:= \int_{|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|\geq c_0}\mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}\psi_\alpha(\gamma)\,d\gamma\\ &= \frac{1}{t^N}\int_{|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|\geq c_0}\mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m}))}({}^t\mathcal{L}_\alpha^2)^N\psi_\alpha(\gamma)\,d\lambda\,.\end{aligned}$$

Let us admit the following lemma for a while.

Lemma 3.4. *For any integer N, there is a smooth function θ_N compactly supported on a compact set of $\mathbb{R}^p$ such that*

$$\big|({}^t\mathcal{L}_\alpha^2)^N\psi_\alpha(\gamma)\big| \leq \frac{\theta_N(\gamma)\,m^N}{|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^N}\,\cdot$$

One then observes that if γ is in the support of the integral defining k_t^2, the lemma implies

$$\big|({}^t\mathcal{L}_\alpha^2)^N\psi_\alpha(\gamma)\big| \leq \frac{\theta_N(\gamma)}{c_0^N}\,m^N.$$

This estimate ensures the result as in Section 3.1 by taking $N = p-1$.

3.3. **Proofs of Lemma 3.1 and Lemma 3.4.** Lemma 3.1 is an obvious consequence of the following Lemma 3.5, taking $(a,b)\equiv(0,0)$. We omit the proof of Lemma 3.4 which consists in a straightforward modification of the arguments developed below.

Lemma 3.5. *For any integer N, one can write*

$$({}^t\mathcal{L}^1_\alpha)^N\psi_\alpha(\gamma)=f_{N,m}\big(\gamma,t^{\frac12}\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\big)\,,$$

with $|\alpha|=m$, and where $f_{N,m}$ is a smooth function supported on $\mathcal{C}\times\mathbb{R}^p$ with $\mathcal{C}$ a fixed ring of $\mathbb{R}^p$, such that for any couple $(a,b)\in\mathbb{N}^p\times\mathbb{N}^p$, there is a constant C (independent of m) such that

$$|\nabla^a_\gamma\nabla^b_\Theta f_{N,m}(\gamma,\Theta)|\leq C\,m^{N+|a|}(1+|\Theta|^2)^{-N-\frac{|b|}{2}}\,.$$

Proof of Lemma 3.5. Let us prove the result by induction over N. We start with the case when N is equal to zero. Notice that in that case the function $f_{0,m}(\gamma,\Theta)=\psi_\alpha(\gamma)$ does not depend on the quantity $\Theta=t^{\frac12}\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))$, so we need to check that for any $a\in\mathbb{N}^p$, there is a constant C such that

$$|\nabla^a_\gamma\psi_\alpha(\gamma)|\leq C\,m^{|a|}\,, \tag{3.10}$$

when $|\alpha|=m$. The case when $a=0$ is obvious thanks to the uniform bound on G_α. To deal with the case $|a|\geq1$, we state the following technical result which will be proved at the end of this paragraph.

Lemma 3.6. *For any integer k, there is a constant C such that the following bound holds for the function g_n defined in* (2.5)*, $n\in\mathbb{N}$:*

$$\forall(\xi_1,\xi_2)\in\mathbb{R}^2\,,\quad\big|(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})^k g_n(\xi_1,\xi_2)\big|\leq Cn^k\,.$$

Let us now compute $\nabla^a_\gamma\psi_\alpha(\gamma)$. Recall that according to (3.2),

$$\begin{aligned}\psi_\alpha(\gamma)&=G_\alpha\big(P,Q,\widetilde\eta_m(\gamma)\big)F\Big(\frac{\gamma}{m}\Big)\,|\mathrm{Pf}(\gamma)|\\&=F\Big(\frac{\gamma}{m}\Big)\prod_{j=1}^d\psi_{\alpha,j}(\gamma)\,,\end{aligned}$$

where

$$\psi_{\alpha,j}(\gamma):=\eta_j(\gamma)\widetilde\theta\big((2\alpha_j+1)\widetilde\eta_{j,m}(\gamma)\big)\,g_{\alpha_j}\big(\sqrt{\widetilde\eta_{j,m}(\gamma)}P_j,\sqrt{\widetilde\eta_{j,m}(\gamma)}Q_j\big)\,,\quad\widetilde\eta_{j,m}(\gamma):=\frac1m\eta_j(\gamma)\,.$$

We compute

$$\nabla^a_\gamma\psi_{\alpha,j}(\gamma)=\sum_{\substack{b\in\mathbb{N}^p\\0\leq|b|\leq|a|}}\binom{b}{a}\nabla^b_\gamma\big(\theta\big((2\alpha_j+1)\widetilde\eta_{j,m}(\gamma)\big)\big)\nabla^{a-b}_\gamma\big(\eta_j(\gamma)g_{\alpha_j}\big(\sqrt{\widetilde\eta_{j,m}(\gamma)}P_j,\sqrt{\widetilde\eta_{j,m}(\gamma)}Q_j\big)\big)\,.$$

Let us assume first that $|a-b|=1$. Then we write, for some $1\leq\ell\leq p$,

$$\begin{aligned}\partial_{\gamma_\ell}\Big(\eta_j(\gamma)g_{\alpha_j}\big(\sqrt{\widetilde\eta_{j,m}(\gamma)}P_j,\sqrt{\widetilde\eta_{j,m}(\gamma)}Q_j\big)\Big)&=\partial_{\gamma_\ell}\eta_j(\gamma)g_{\alpha_j}\big(\sqrt{\widetilde\eta_{j,m}(\gamma)}P_j,\sqrt{\widetilde\eta_{j,m}(\gamma)}Q_j\big)\\&\quad+\eta_j(\gamma)\frac{\partial_{\gamma_\ell}\widetilde\eta_{j,m}(\gamma)}{2\widetilde\eta_{j,m}(\gamma)}\times\big((\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})g_{\alpha_j}\big)\Big(\sqrt{\widetilde\eta_{j,m}(\gamma)}P_j,\sqrt{\widetilde\eta_{j,m}(\gamma)}Q_j\Big)\,.\end{aligned}$$

Next we use the fact that there is a constant C such that on the support of $\theta\big((2\alpha_j+1)\widetilde\eta_{j,m}(\gamma)\big)$,

$$\widetilde\eta_{j,m}(\gamma)\geq\frac{1}{Cm}\quad\text{and}\quad|\partial_{\gamma_\ell}\widetilde\eta_{j,m}(\gamma)|\leq\frac{C}{m}\,,$$

so applying Lemma 3.6 gives

$$|\nabla_\gamma\psi_{\alpha,j}(\gamma)|\lesssim\alpha_j\,.$$

Recalling that $\alpha_j\leq m$ and that, for all $i\in\{1,\cdots,d\}$, $\psi_{\alpha,j}$ is uniformly bounded, this easily achieves the proof of Estimate (3.10) in the case $|a|=1$ by taking the product over j. Once we have noticed that

$$\alpha_1^{a_1}\cdots\alpha_d^{a_d}\lesssim(\alpha_1+\cdots+\alpha_d)^{a_1+\cdots+a_d}\,,$$

the general case (when $|a|>1$) is dealt with identically, we omit the details.

Finally let us proceed with the induction: assume that for some integer N one can write

$$({}^t\mathcal{L}^1_\alpha)^{N-1}\psi_\alpha(\gamma) = f_{N-1,m}\big(\gamma, t^{\frac12}\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\big)$$

where $f_{N-1,m}$ is a smooth function supported on $\mathcal{C}\times\mathbb{R}^p$, such that for any couple $(a,b)\in\mathbb{N}^p\times\mathbb{N}^p$, there is a constant C (independent of m) such that

$$|\nabla^a_\gamma\nabla^b_\Theta f_{N-1,m}(\gamma,\Theta)| \leq C\, m^{N-1+|a|}(1+|\Theta|^2)^{-(N-1)-\frac{|b|}{2}}\,. \tag{3.11}$$

We compute for any function $\Psi(\gamma)$,

$$\begin{aligned}
{}^t\mathcal{L}^1_\alpha\Psi(\gamma) &= i\frac{\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\cdot\nabla_\gamma\Psi(\gamma)}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2} + \frac{1+i\Delta(\Phi_\alpha(Z,\frac{\gamma}{m}))}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2}\Psi(\gamma)\\
&- 2it\sum_{1\leq j,k\leq p}\frac{\partial_{\gamma_j}\partial_{\gamma_k}(\Phi_\alpha(Z,\frac{\gamma}{m}))\partial_{\gamma_j}(\Phi_\alpha(Z,\frac{\gamma}{m}))\partial_{\gamma_k}(\Phi_\alpha(Z,\frac{\gamma}{m}))}{(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2)^2}\Psi(\gamma)\,.
\end{aligned}$$

We apply that formula to $\Psi := f_{N-1}\big(\gamma, t^{\frac12}\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\big)$ and estimating each of the three terms separately we find (using the fact that $m\geq 1$),

$$\begin{aligned}
\Big|{}^t\mathcal{L}^1_\alpha\Big(f_{N-1}\big(\gamma,t^{\frac12}\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\big)\Big)\Big| &\leq C\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2\big)^{-1}\\
&\quad\times m^{N-1+1}(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2)^{-(N-1)}\\
&+ C\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2\big)^{-1}\\
&\quad\times m^{N-1}(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2)^{-(N-1)}\\
&+ Ct\,|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2\big)^{-2}\\
&\quad\times m^{N-1}(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2)^{-(N-1)}
\end{aligned}$$

thanks to the induction assumption (3.11) along with (3.10) and the fact that on $\mathcal{C}_\alpha(Z)$, all the derivatives of the function $\nabla_\gamma\big(\Phi_\alpha\big(Z,\frac{\gamma}{m}\big)\big)$ are uniformly bounded with respect to α and Z. A similar argument allows to control derivatives in γ and Θ, so Lemma 3.5 is proved. □

Proof of Lemma 3.6. One important ingredient in the proof is the fact that for all integers k, there is a constant C such that

$$\big\|\,|\cdot|^k h_n\big\|_{L^2}\leq Cn^{\frac{k}{2}}\,. \tag{3.12}$$

This comes from the well-known relations

$$\begin{aligned}
h_n'(\xi)+\xi h_n(\xi) &= \sqrt{2n}\,h_{n-1}\\
h_n'(\xi)-\xi h_n(\xi) &= -\sqrt{2(n+1)}\,h_{n+1}\,,
\end{aligned}\tag{3.13}$$

which imply $\|\xi h_n(\xi)\|_{L^2_\xi}\leq\sqrt{2(n+1)}$. Using $h_n''(\xi)=-(2n+1)h_n(\xi)+\xi^2h_n(\xi)$, an integration by parts gives

$$\begin{aligned}
\int\xi^{2(k+1)}h_n^2(\xi)d\xi &= \int\xi^{2k}\big(h_n''(\xi)+(2n+1)h_n(\xi)\big)h_n(\xi)d\xi\\
&= -\int h_n'(\xi)\left(2k\xi^{2k-1}h_n(\xi)+\xi^{2k}h_n'(\xi)\right)d\xi + (2n+1)\|\,|\cdot|^kh_n\|^2_{L^2}\,.
\end{aligned}$$

Hence

$$\int\xi^{2(k+1)}h_n^2(\xi)d\xi\leq -2k\int\xi^{2k-1}h_n(\xi)h_n'(\xi)d\xi+(2n+1)\|\,|\cdot|^kh_n\|^2_{L^2}\,.$$

Finally using (3.13) we find

$$\begin{aligned}
\|\,|\cdot|^{k+1}h_n\|^2_{L^2} &\leq -2k\int\xi^{2k-1}h_n(\xi)\big(-(2n+2)^{\frac12}h_{n+1}(\xi)+\xi h_n(\xi)\big)d\xi+(2n+1)\|\,|\cdot|^kh_n\|^2_{L^2}\\
&\leq 2k(2n+2)^{\frac12}\|\,|\cdot|^kh_n\|_{L^2}\||\cdot|^{k-1}h_{n+1}\|_{L^2}+(2n+1)\|\,|\cdot|^kh_n\|^2_{L^2}
\end{aligned}$$

and we obtain (3.12) by a recursive argument.
Then, we consider the case when $k=1$ and we write, using the formula defining g_n in (2.5) and an integration by parts,

$$\begin{aligned}(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})g_n(\xi_1,\xi_2) &= \mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\int \mathrm{e}^{-i\xi_2\xi}\xi_1 h_n'(\xi_1+\xi)h_n(\xi)d\xi \\ &\qquad - i\xi_2\mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\int \mathrm{e}^{-i\xi_2\xi}(\xi_1+\xi)h_n(\xi_1+\xi)h_n(\xi)d\xi \\ &= \mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\int \mathrm{e}^{-i\xi_2\xi}\xi_1 h_n'(\xi_1+\xi)h_n(\xi)d\xi \\ &\qquad - \mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\int \mathrm{e}^{-i\xi_2\xi}\partial_\xi\left[(\xi_1+\xi)h_n(\xi_1+\xi)h_n(\xi)\right]d\xi \\ &= -g_n(\xi_1,\xi_2)-\mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\int \mathrm{e}^{-i\xi_2\xi}\left(\xi h_n'(\xi_1+\xi)h_n(\xi)+(\xi_1+\xi)h_n(\xi_1+\xi)h_n'(\xi)\right)d\xi\,.\end{aligned}$$

Therefore

$$|(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})g_n(\xi_1,\xi_2)|\leq |g_n(\xi_1,\xi_2)|+2\|\xi h_n(\xi)\|_{L^2_\xi}\|h_n'(\xi)\|_{L^2_\xi}\leq C\,n.$$

This concludes the proof in the case when $k=1$ thanks to the Cauchy-Schwarz inequality.

Let us now consider $k\geq 1$ and $(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})^k g_n(\xi_1,\xi_2)$. By induction on k, one can prove that $(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})^k g_n(\xi_1,\xi_2)$ is a linear combination of terms of the form

$$\xi_1^{k_1}\xi_2^{k_2}\partial_{\xi_1}^{k_3}\partial_{\xi_2}^{k_4}g_n(\xi_1,\xi_2)\ \text{ with }\ k_1+k_2+k_3+k_4\leq k$$

and the coefficients of the combinations are controled in terms of k or, equivalently, a linear combination of terms of the form

$$\mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\int \mathrm{e}^{-i\xi_2\xi}(\xi_1+\xi)^{k_1}\xi^{k_2}h_n^{(k_3)}(\xi_1+\xi)h_n^{(k_4)}(\xi)d\xi$$

with coefficients controled in terms of k. Similarly, by induction on j and thanks to (3.13), one gets

$$h_n^{(j)}(\xi)=\sum_{p=0}^{j}a_{j,p,n}\xi^{j-p}n^{\frac{p}{2}}h_{p+n}(\xi)\,,$$

where the $a_{j,p,n}$ are bounded by a constant independent of n. We are left with a linear combination of terms of the form

$$I_{k_1,k_2,k_3,k_4,p,p'}:=n^{\frac{p+p'}{2}}\int \mathrm{e}^{-i\frac{\xi_1\xi_2}{2}}\mathrm{e}^{-i\xi_2\xi}(\xi_1+\xi)^{k_1+k_3-p}\xi^{k_2+k_4-p'}h_{n+p}(\xi_1+\xi)h_{n+p'}(\xi)d\xi$$

with coefficients depending on k and controled uniformly with respect to n. By (3.12), we obtain by the Cauchy-Schwartz inequality

$$|I_{k_1,k_2,k_3,k_4,p,p'}|\leq Cn^{\frac{p+p'}{2}+\frac{k_1+k_3-p}{2}+\frac{k_2+k_4-p'}{2}}\leq Cn^k\,.$$

The proposition is proved. □

4. Optimality of the dispersive estimates

In this section, we first end the proof of Theorem 1 by proving the optimality of the dispersive estimates for groups satisfying Assumption 1.4. Then, we prove Proposition 1.7.

4.1. **Optimality for groups satisfying Assumption 1.4.** Let us now end the proof of Theorem 1 by establishing the optimality of the dispersive estimate (1.19). We use the fact that there always exists $\lambda^*\in\Lambda$ such that

$$\nabla_\lambda\zeta(0,\lambda^*)\neq 0, \tag{4.1}$$

where the function ζ is defined in (1.12). Indeed, if not, the map $\lambda\mapsto\zeta(0,\lambda)$ would be constant which is in contradiction with the fact that ζ is homogeneous of degree 1. We prove the following proposition, which yields the optimality of the dispersive estimate of Theorem 1.

Proposition 4.1. *Let $\lambda^* \in \Lambda$ satisfying (4.1). There is a function $g \in \mathcal{D}(\mathbb{R}^p)$ compactly supported in a connected open neighborhood of λ^* in Λ, such that for the initial data f_0 defined by*

$$\forall (\lambda,\nu) \in \mathfrak{r}(\Lambda),\ \ \mathcal{F}(f_0)(\lambda,\nu) h_{\alpha,\eta(\lambda)} = 0 \text{ for } \alpha \neq 0 \text{ and } \mathcal{F}(f_0)(\lambda,\nu) h_{0,\eta(\lambda)} = g(\lambda) h_{0,\eta(\lambda)}\,, \tag{4.2}$$

there exists $c_0 > 0$ and $x \in G$ such that

$$|\mathrm{e}^{-it\Delta_G} f_0(x)| \geqslant c_0\, |t|^{-\frac{k+p-1}{2}}\,.$$

Proof. Let g be any smooth compactly supported function over $\mathbb{R}^p$, and define f_0 by (4.2). For any point $x = e^X \in G$ under the form $X = (P = 0, Q = 0, Z, R)$, the inversion formula gives

$$\mathrm{e}^{-it\Delta_G} f_0(x) = \kappa \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}^*_\lambda} \mathrm{e}^{it|\nu|^2 + it\zeta(0,\lambda) - i\lambda(Z) - i\nu(R)} g(\lambda) |\mathrm{Pf}(\lambda)| d\nu d\lambda\,.$$

To simplify notations, we set $\zeta_0(\lambda) := \zeta(0,\lambda)$. Setting $Z = tZ^*$ with $Z^* := \nabla_\lambda \zeta(0,\lambda^*) \neq 0$, we get as in (2.6)

$$\left|\mathrm{e}^{-it\Delta_G} f_0(x)\right| = c_1 |t|^{-\frac{k}{2}} \left| \int_{\lambda \in \mathbb{R}^p} \mathrm{e}^{it(\lambda\cdot Z^* - \zeta_0(\lambda))} g(\lambda) |\mathrm{Pf}(\lambda)| d\lambda \right|$$

for some constant $c_1 > 0$. Without loss of generality, we can assume

$$\lambda^* = (1, 0, \dots, 0)$$

(if not, we perform a change of variables $\lambda \mapsto \Omega\lambda$ where Ω is a fixed orthogonal matrix), and we now shall perform a stationary phase in the variable λ', where we have written $\lambda = (\lambda_1, \lambda')$. For any fixed λ_1, the phase

$$\Phi_{\lambda_1}(\lambda', Z) := Z \cdot \lambda - \zeta_0(\lambda)$$

has a stationary point λ' if and only if $Z' = \nabla_{\lambda'} \zeta_0(\lambda)$ (with the same notation $Z = (Z_1, Z')$). We observe that the homogeneity of the function ζ_0 and the definition of Z^* imply that

$$Z^* = \nabla_\lambda \zeta_0(1, 0, \dots, 0) = \nabla_\lambda \zeta_0(\lambda_1, 0, \dots, 0),\ \ \forall \lambda_1 \in \mathbb{R}\,,$$

hence if $\lambda' = 0$, then the phase $\Phi_{\lambda_1}(0, Z^*)$ has a stationary point.

From now on we choose g supported near those stationary points $(\lambda_1, 0)$, and vanishing in the neighborhood of any other stationary point.

Let us now study the Hessian of Φ_{λ_1} in $\lambda' = 0$. Again because of the homogeneity of the function ζ_0, we have

$$[\mathrm{Hess}\, \zeta_0(\lambda)]\, \lambda = 0,\ \ \forall \lambda \in \mathbb{R}^p\,.$$

In particular, for all $\lambda_1 \neq 0$, $\mathrm{Hess}\, \zeta_0(\lambda_1, 0, \cdots, 0)\, (\lambda_1, 0, \cdots, 0) = 0$ and the matrix $\mathrm{Hess}\, \zeta_0(\lambda_1, 0, \cdots, 0)$ in the canonical basis is of the form

$$\mathrm{Hess}\, \zeta_0(\lambda_1, 0, \cdots, 0) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Hess}_{\lambda',\lambda'}\, \zeta_0(\lambda_1, 0, \cdots, 0) \end{pmatrix}.$$

Using that $\mathrm{Hess}\, \zeta_0(\lambda_1, 0, \cdots, 0)$ is of rank $p-1$, we deduce that $\mathrm{Hess}_{\lambda',\lambda'}\, \zeta_0(\lambda_1, 0, \cdots, 0)$ is also of rank $p-1$ and we conclude by the stationary phase theorem ([36], Chap. VIII.2), choosing g so that the remaining integral in λ_1 does not vanish. □

4.2. **Proof of Proposition 1.7.** Assume that G is a step 2 stratified Lie group whose radical index is null and for which $\zeta(0,\lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Let g be a smooth nonnegative function supported in one of the connected components of Λ and define f_0 by

$$\mathcal{F}(f_0)(\lambda) h_{\alpha,\eta(\lambda)} = 0 \text{ for } \alpha \neq 0 \text{ and } \mathcal{F}(f_0)(\lambda) h_{0,\eta(\lambda)} = g(\lambda) h_{0,\eta(\lambda)}\,.$$

By the inverse Fourier formula, if $x = e^X \in G$ is such that $X = (P = 0, Q = 0, tZ)$, then we have

$$\mathrm{e}^{-it\Delta_G}(x) = \kappa \int \mathrm{e}^{-it\,\lambda(Z)} \mathrm{e}^{it\zeta(0,\lambda)} g(\lambda)\, |\mathrm{Pf}(\lambda)|\, d\lambda\,.$$

Since $\zeta(0,\lambda)$ is a linear form on each connected component of Λ, there exists Z_0 in $\mathfrak{z}$ such that for

$$\forall \lambda \in \mathfrak{z}^* \cap \mathrm{supp}\, g,\ \ -\lambda(Z_0) + \zeta(0,\lambda) = 0\,.$$

As a consequence, choosing $Z = Z_0$, we obtain

$$e^{-it\Delta_G}(x) = \kappa \int g(\lambda)\, |\mathrm{Pf}(\lambda)|\, d\lambda \neq 0,$$

which ends the proof of the result.

Appendix A. On the inversion formula in Schwartz space

This section is dedicated to the proof of the inversion formula in the Schwartz space $\mathcal{S}(G)$ (Proposition 1.1 page 4).

Proof. We first observe that to establish (1.8), it suffices to prove that

$$f(0) = \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \mathrm{tr}\,\Big(\mathcal{F}(f)(\lambda,\nu)\Big)\, |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda\,. \tag{A.1}$$

Indeed, introducing the auxiliary function g defined by $g(x') := f(x\cdot x')$ which obviously belongs to $\mathcal{S}(G)$ and satisfies $\mathcal{F}(g)(\lambda,\nu) = u^{\lambda,\nu}_{X(\lambda,x^{-1})} \circ \mathcal{F}(f)(\lambda,\nu)$, and assuming (A.1) holds, we get

$$\begin{aligned} f(x) = g(0) &= \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \mathrm{tr}\,\Big(\mathcal{F}(g)(\lambda,\nu)\Big)\, |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda \\ &= \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \mathrm{tr}\,\Big(u^{\lambda,\nu}_{X(\lambda,x^{-1})}\mathcal{F}(f)(\lambda,\nu)\Big)\, |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda\,, \end{aligned}$$

which is the desired result.

Let us now focus on (A.1). In order to compute the right-hand side of Identity (A.1), we introduce

$$\begin{aligned} A &:= \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \mathrm{tr}\,\Big(\mathcal{F}(f)(\lambda,\nu)\Big)\, |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda \\ &= \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{x\in G} \sum_{\alpha\in\mathbb{N}^d} \big(u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} | h_{\alpha,\eta(\lambda)}\big)\, |\mathrm{Pf}(\lambda)|\, f(x)\, d\mu(x)\, d\nu\, d\lambda\,, \end{aligned}$$

with the notation of Section 1.3. In order to carry on the calculations, we need to resort to a Fubini argument which comes from the following identity:

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \|\mathcal{F}(f)(\lambda,\nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda \,<\infty\,. \tag{A.2}$$

We postpone the proof of (A.2) to the end of this section. Thanks to (A.2), the order of integration does not matter and we can transform the expression of A: we use the fact that for any $\alpha\in\mathbb{N}^d$

$$\big(u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} | h_{\alpha,\eta(\lambda)}\big) = e^{-i\nu(R)-i\lambda(Z)} \int_{\mathbb{R}^d} e^{-i\sum_j \eta_j(\lambda)\,(\xi_j+\frac12 P_j)\,Q_j}\, h_{\alpha,\eta(\lambda)}(P+\xi)\, h_{\alpha,\eta(\lambda)}(\xi) d\xi\,,$$

where we have identified $\mathfrak{p}_\lambda$ with $\mathbb{R}^d$, and this gives rise to

$$\begin{aligned} A = \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{x\in G} \int_{\xi\in\mathbb{R}^d} \sum_{\alpha\in\mathbb{N}^d} & e^{-i\nu(R)-i\lambda(Z)} e^{-i\sum_j \eta_j(\lambda)\,(\xi_j+\frac12 P_j)\,Q_j}\, h_{\alpha,\eta(\lambda)}(P+\xi) \\ & \times\, h_{\alpha,\eta(\lambda)}(\xi)\, |\mathrm{Pf}(\lambda)|\, f(x)\, d\mu(x)\, d\xi\, d\nu\, d\lambda\,, \end{aligned}$$

where we recall that

$$h_{\alpha,\eta(\lambda)}(\xi) = \prod_{j=1}^d h_{\alpha_j,\eta_j(\lambda)}(\xi_j) \quad \text{with} \quad h_{\alpha_j,\eta_j(\lambda)}(\xi_j) = \eta_j(\lambda)^{\frac14}\, h_{\alpha_j}\Big(\sqrt{\eta_j(\lambda)}\,\xi_j\Big)\,.$$

Performing the change of variables

$$\begin{cases} \widetilde{\xi}_j = \sqrt{\eta_j(\lambda)}\,\xi_j \\ \widetilde{P}_j = \sqrt{\eta_j(\lambda)}\,P_j \\ \widetilde{Q}_j = \sqrt{\eta_j(\lambda)}\,Q_j \end{cases}$$

for $j \in \{1, \dots, d\}$, we obtain, dropping the $\widetilde{}$ on the variables,

$$A = \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}_\lambda^*} \int_{(P,Q,R,Z) \in \mathbb{R}^n} \int_{\xi \in \mathbb{R}^d} \sum_{\alpha \in \mathbb{N}^d} \mathrm{e}^{-i\nu(R) - i\lambda(Z)} \mathrm{e}^{-i \sum_\ell (\xi_\ell + \frac{1}{2} P_\ell) \cdot Q_\ell} \prod_{j=1}^d h_{\alpha_j}(P_j + \xi_j)\, h_{\alpha_j}(\xi_j)$$
$$\times\, f\big(\eta^{-\frac{1}{2}}(\lambda)\, P, \eta^{-\frac{1}{2}}(\lambda)\, Q, R, Z\big)\, dP\, dQ\, dR\, dZ\, d\xi\, d\nu\, d\lambda\,,$$

with $\eta^{-\frac{1}{2}}(\lambda)\, P := (\eta_1^{-\frac{1}{2}}(\lambda)\, P_1, \dots, \eta_d^{-\frac{1}{2}}(\lambda)\, P_d)$ and similarly for Q.

Then using the change of variables $\xi'_j = \xi_j + P_j$, for $j \in \{1, \dots, d\}$, gives

$$A = \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}_\lambda^*} \int_{(\xi',Q,R,Z) \in \mathbb{R}^n} \int_{\xi \in \mathbb{R}^d} \sum_{\alpha \in \mathbb{N}^d} \mathrm{e}^{-i\nu(R) - i\lambda(Z)} \mathrm{e}^{-\frac{i}{2} \sum_\ell (\xi_\ell + \xi'_\ell) \cdot Q_\ell} \prod_{j=1}^d h_{\alpha_j}(\xi'_j)\, h_{\alpha_j}(\xi_j)$$
$$\times\, f\big(\eta^{-\frac{1}{2}}(\lambda)\, (\xi' - \xi), \eta^{-\frac{1}{2}}(\lambda)\, Q, R, Z\big)\, d\xi'\, dQ\, dR\, dZ\, d\xi\, d\nu\, d\lambda\,.$$

Because $(h_\alpha)_{\alpha \in \mathbf{N}^d}$ is a Hilbert basis of $L^2(\mathbb{R}^d)$, we have for all $\varphi \in L^2(\mathbb{R}^d)$

$$\varphi(\xi) = \sum_{\alpha \in \mathbb{N}^d} \int_{\xi' \in \mathbb{R}^d} \varphi(\xi')\, h_\alpha(\xi')\, d\xi'\, h_\alpha(\xi)\,,$$

which leads to

$$A = \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}_\lambda^*} \int_{(Q,R,Z) \in \mathbb{R}^{d+k+p}} \int_{\xi \in \mathbb{R}^d} \mathrm{e}^{-i\nu(R) - i\lambda(Z)} \mathrm{e}^{-i\xi \cdot Q}\, f\big(0, \eta^{-\frac{1}{2}}(\lambda)\, Q, R, Z\big)\, dQ\, dR\, dZ\, d\xi\, d\nu\, d\lambda\,.$$

Applying the Fourier inversion formula successively on $\mathbb{R}^d$, $\mathbb{R}^k$ and on $\mathbb{R}^p$ (and identifying $\mathfrak{r}(\Lambda)$ with $\mathbb{R}^p \times \mathbb{R}^k$), we conclude that there exists a constant $\kappa > 0$ such that

$$A = \kappa\, f(0)\,,$$

which ends the proof of (A.1).

Let us conclude the proof by showing (A.2). We choose M a nonnegative integer. According to the obvious fact that the function $(\mathrm{Id} - \Delta_G)^M f$ also belongs to $\mathcal{S}(G)$ (hence to $L^1(G)$), we get in view of Identity (1.11)

$$\mathcal{F}(f)(\lambda, \nu) h_{\alpha, \eta(\lambda)} = \Big(1 + |\nu|^2 + \zeta(\alpha, \lambda)\Big)^{-M} \mathcal{F}\left((\mathrm{Id} - \Delta_G)^M f\right)(\lambda, \nu) h_{\alpha, \eta(\lambda)}\,.$$

In view of the definition of the Fourier transform on the group G, we thus have

$$\|\mathcal{F}(f)(\lambda, \nu) h_{\alpha, \eta(\lambda)}\|^2_{L^2(\mathfrak{p}_\lambda)} = \Big(1 + |\nu|^2 + \zeta(\alpha, \lambda)\Big)^{-2M}$$
$$\times \int_{\mathfrak{p}_\lambda} \bigg(\int_G \Big((\mathrm{Id} - \Delta_G)^M f(x)\Big) u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)}(\xi)\, d\mu(x) \overline{\int_G \Big((\mathrm{Id} - \Delta_G)^M f(x')\Big) u^{\lambda,\nu}_{X(\lambda,x')} h_{\alpha,\eta(\lambda)}(\xi)\, d\mu(x')}\bigg) d\xi\,.$$

Now, by Fubini's theorem, we get

$$\|\mathcal{F}(f)(\lambda, \nu) h_{\alpha, \eta(\lambda)}\|^2_{L^2(\mathfrak{p}_\lambda)} = \Big(1 + |\nu|^2 + \zeta(\alpha, \lambda)\Big)^{-2M}$$
$$\times \int_G \int_G (\mathrm{Id} - \Delta_G)^M f(x) \overline{(\mathrm{Id} - \Delta_G)^M f(x')} (u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} \,|\, u^{\lambda,\nu}_{X(\lambda,x')} h_{\alpha,\eta(\lambda)})_{L^2(\mathfrak{p}_\lambda)}\, d\mu(x)\, d\mu(x')\,.$$

Since the operators $u^{\lambda,\nu}_{X(\lambda,x)}$ and $u^{\lambda,\nu}_{X(\lambda,x')}$ are unitary on $\mathfrak{p}_\lambda$ and the family $(h_{\alpha,\eta(\lambda)})_{\alpha \in \mathbb{N}^d}$ is a Hilbert basis of $\mathfrak{p}_\lambda$, we deduce that

$$\|\mathcal{F}(f)(\lambda, \nu) h_{\alpha, \eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} \leq \Big(1 + |\nu|^2 + \zeta(\alpha, \lambda)\Big)^{-M} \|(\mathrm{Id} - \Delta_G)^M f\|_{L^1(G)}\,.$$

Because

$$\mathrm{Card}\Big(\Big\{\alpha \in \mathbb{N}^d \,/\, |\alpha| = m\Big\}\Big) = \binom{m + d - 1}{m} \leq C(m+1)^{d-1}\,,$$

this ensures that

$$\sum_{\alpha\in\mathbb{N}^d}\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)}|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\lesssim\|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)}$$
$$\times\sum_m (m+1)^{d-1}\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\Big(1+|\nu|^2+\zeta(\alpha,\lambda)\Big)^{-M}|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,.$$

Hence taking $M=M_1+M_2$, with $M_2>\dfrac{k}{2}$ implies that

$$\sum_{\alpha\in\mathbb{N}^d}\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)}|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\lesssim\|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)}$$
$$\times\sum_m (m+1)^{d-1}\int_{\lambda\in\Lambda}\Big(1+\zeta(\alpha,\lambda)\Big)^{-M_1}|\mathrm{Pf}(\lambda)|\,d\lambda\,.$$

Noticing that $\zeta(\alpha,\lambda)=0$ if and only if $\lambda=0$ and using the homogeneity of degree 1 of ζ, yields that there exists $c>0$ such that $\zeta(\alpha,\lambda)\geq c\,m\,|\lambda|$. Therefore, we can end the proof of (A.2) by choosing M_1 large enough and performing the change of variable $\mu=m\,\lambda$ in each term of the above series.

Proposition 1.1 is proved. $\square$

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