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Abstract

This work is concerned with the consistency study of a 1D (staggered kinetic) finite volume scheme for barotropic Euler models. We prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

1 Introduction

The model. This work is concerned with the consistency study of a (staggered kinetic) Finite Volume (FV) scheme for barotropic Euler models

$$\partial_t \rho + \partial_x(\rho V) = 0, \tag{1}$$

$$\partial_t(\rho V) + \partial_x(\rho V^2 + p(\rho)) = 0. \tag{2}$$

The unknowns are the density ρ and the velocity V. The pressure $\big(\rho \mapsto p(\rho)\big)$ is assumed to be $\mathcal{C}^2([0,\infty))$ with $p(\rho) > 0,\ p'(\rho) > 0,\ p''(\rho) \geqslant 0,\ \forall \rho > 0$. Thus, the sound speed $c : \rho \mapsto \sqrt{p'(\rho)}$ is well defined and is an increasing function.

We consider the problem (1)-(2) on the bounded domain $(0, L) \times [0, T]$ with the boundary conditions $V(0,t) = 0 = V(L,t),\ \forall t > 0$ and the initial conditions $\rho(x,0) = \rho_0(x),\ V(x,0) = V_0(x),\ \forall x \in (0,L)$ with $\rho_0,\ V_0 \in L^\infty(0,L)$.

Let $\Phi : \rho > 0 \mapsto \Phi(\rho)$ such that $\rho\Phi'(\rho) - \Phi(\rho) = p(\rho),\ \forall \rho > 0$. The quantity $\mathcal{S} = \frac{1}{2}\rho|V|^2 + \Phi(\rho)$ is an entropy of the system: entropy solutions to (1)-(2) are required to satisfy: for any $\varphi \in \mathcal{C}_c^\infty((0,L) \times [0,T))$ such that $\varphi \geqslant 0$,

$$-\int_0^T\int_0^L \Big[\mathcal{S}\partial_t\varphi + (\mathcal{S} + p(\rho))V\partial_x\varphi\Big](x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^L \mathcal{S}_0(x)\varphi(x,0)\,\mathrm{d}x \leqslant 0. \tag{3}$$

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The meshes. We consider a set of $J+1$ points $0 = x_1 < x_2 < ... < x_J < x_{J+1} = L$. The x_j are the edges of the so-called primal mesh $\mathcal{T}$. We set $\delta x_{j+1/2}{=}x_{j+1}{-}x_j$. The centers of the primal cells, $x_{j+1/2} = (x_j + x_{j+1})/2$ for $j \in \{1, .., J\}$, realize the dual mesh $\mathcal{T}^\star$. We set $\delta x_j = (\delta x_{j-1/2} + \delta x_{j+1/2})/2$ for $j \in \{2, .., J-1\}$ and $\delta x{=}\text{size}(\mathcal{T}){=}\max_j \delta x_{j+1/2}$. The adaptive time step is δt^k and we set $\delta t{=}\max_k \delta t^k$.

The scheme. We analyze the scheme introduced in [1]. It works on staggered grids: the densities, $\rho_{j+1/2}$, $j \in \{1, .., J\}$, are evaluated at centers whereas the velocities, V_j, $j \in \{1, .., J+1\}$, are evaluated at edges. We set, for $j \in \{1, .., J\}$ and $i \in \{2, .., J\}$

$$\rho^0_{j+1/2} = \frac{1}{\delta x_{j+1/2}} \int_{x_j}^{x_{j+1}} \rho_0(x)\,\mathrm{d}x, \qquad V^0_i = \frac{1}{\delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} V_0(x)\,\mathrm{d}x. \tag{4}$$

The density is first updated with a FV approximation on the primal mesh

$$\delta x_{j+1/2}(\rho^{k+1}_{j+1/2} - \rho^k_{j+1/2}) + \delta t^k(\mathscr{F}^k_{j+1} - \mathscr{F}^k_j) = 0, \quad \forall j \in \{1, .., J\}. \tag{5}$$

Then, the velocity is updated with a FV approximation on the dual mesh:

$$\delta x_j(\rho^{k+1}_j V^{k+1}_j - \rho^k_j V^k_j) + \delta t^k \Big(\mathscr{G}^k_{j+1/2} - \mathscr{G}^k_{j-1/2} + \pi^{k+1/2}_{j+1/2} - \pi^{k+1/2}_{j-1/2}\Big) = 0, \tag{6}$$

for $j \in \{2, .., J\}$, while $V^{k+1}_1 = V^{k+1}_{J+1} = 0$. The density on the edges ρ^k_j is defined by

$$2\delta x_j \rho^k_j = \delta x_{j+1/2}\rho^k_{j+1/2} + \delta x_{j-1/2}\rho^k_{j-1/2}, \quad \forall j \in \{2, .., J\}.$$

The definition of the fluxes relies on the kinetic framework. We refer the reader to [1] for details. Let us introduce the two following functions $\mathscr{F}^+$ and $\mathscr{F}^-$

$$\mathscr{F}^\pm(\rho, V) = \frac{\rho}{2c(\rho)} \int_{\xi \gtrless 0} \xi \, \mathbb{1}_{|\xi - V| \leqslant c(\rho)} \mathrm{d}\, \xi.$$

We adopt the following formulas for mass fluxes: $\mathscr{F}^k_1 = \mathscr{F}^k_{J+1} = 0$,

$$\mathscr{F}^k_j = \mathscr{F}^+(\rho^k_{j-1/2}, V^k_j) + \mathscr{F}^-(\rho^k_{j+1/2}, V^k_j), \quad \forall j \in \{2, .., J\}, \tag{7}$$

and, for momentum fluxes: $\mathscr{G}^k_{3/2} = \dfrac{V^k_2}{2}\mathscr{F}^-(\rho^k_{5/2}, V^k_2)$, $\mathscr{G}^k_{J+1/2} = \dfrac{V^k_J}{2}\mathscr{F}^+(\rho^k_{J-1/2}, V^k_J)$,

$$\begin{aligned} \mathscr{G}^k_{j+1/2} = &\frac{V^k_j}{2}\big(\mathscr{F}^+(\rho^k_{j-1/2}, V^k_j) + \mathscr{F}^+(\rho^k_{j+1/2}, V^k_{j+1})\big) \\ &+ \frac{V^k_{j+1}}{2}\big(\mathscr{F}^-(\rho^k_{j+1/2}, V^k_j) + \mathscr{F}^-(\rho^k_{j+3/2}, V^k_{j+1})\big), \ \forall j \in \{2, .., J-1\}. \end{aligned} \tag{8}$$

The discrete pressure gradient combines a space centered scheme and time semi implicit discretization, namely it uses

$$\pi^{k+1/2}_{j+1/2} = \rho^k_{j+1/2}\Phi'(\rho^{k+1}_{j+1/2}) - \Phi(\rho^k_{j+1/2}).$$

Properties of the scheme. The analysis is driven by the shapes of the functions $\mathscr{F}^{\pm}$, see [1, Lemma 3.2]. Here, we shall use the following properties

$$\begin{aligned} &\text{(i) Smoothness: } (\rho, V) \in (0,\infty)\times\mathbb{R} \mapsto \mathscr{F}^{\pm}(\rho, V) \text{ are of class } C^1,\\ &\text{(ii) Consistency: } \mathscr{F}^{+}(\rho, V) + \mathscr{F}^{-}(\rho, V) = \rho V, \quad \forall V \in \mathbb{R},\ \forall \rho \geqslant 0. \end{aligned} \tag{9}$$

Under CFL conditions, see [1], the scheme preserves the positivity of the discrete density and discrete kinetic and internal energies evolution equations hold.

Lemma 1.1 *Let $N \in \mathbb{N}$. Assume $\min_i (\rho^0_{i+1/2}) > 0$. For all $k \in \{0, .., N-1\}$, there exists $\mathcal{V}^k > 0$, which depends only on the state (ρ^k, V^k), such that if*

$$\frac{\delta t^k}{\min_j (\delta x_{j+1/2})} \mathcal{V}^k \leqslant 1, \tag{10}$$

then, $\min_i (\rho^k_{i+1/2}) > 0, \forall k \in \{0, .., N\}$ and

$$0 \ \leqslant\ \sum_{k=0}^{N-1}\sum_{j=2}^{J} D_j^k \ \leqslant\ C, \quad \textit{with} \quad D_j^k = \frac{1}{4}\delta x_j \rho_j^{k+1}(V_j^{k+1} - V_j^k)^2, \tag{11}$$

$$\frac{\delta x_{j+1/2}}{\delta t^k}\Big[e^{k+1}_{j+1/2} - e^k_{j+1/2}\Big] + \overline{G}^k_{j+1} - \overline{G}^k_j + \pi^{k+1/2}_{j+1/2}\Big[V^{k+1}_{j+1} - V^{k+1}_j\Big] \leqslant \frac{D^k_j}{\delta t^k}, \tag{12}$$

$$\frac{\delta x_j}{\delta t^k}\Big[E^{k+1}_{K,j} - E^k_{K,j}\Big] + \Gamma^k_{j+1/2} - \Gamma^k_{j-1/2} + \Big[\pi^{k+1/2}_{j+1/2} - \pi^{k+1/2}_{j-1/2}\Big]V^{k+1}_j + \frac{D^k_j}{\delta t^k} \leqslant 0, \tag{13}$$

where $E^k_{K,j} = \frac{1}{2}\rho^k_j (V^k_j)^2$ and $e^k_{j+1/2} = \Phi(\rho^k_{j+1/2})$ are the kinetic and internal energies. The fluxes are defined by $\overline{G}^k_1 = \overline{G}^k_{J+1} = 0$ and

$$\overline{G}^k_j = \Phi(\rho^k_{j-1/2})V^{k+1}_j - \frac{\delta x_{j-1/2}}{2\delta t^k}\Big[\bar{\Phi}\Big(\overline{\overline{\rho^{k+1}_{j-1/2}}}\Big) - \bar{\Phi}(\rho^k_{j-1/2})\Big],\ \forall j \in \{2, .., J\},$$

$$\Gamma^k_{j+1/2} = \frac{1}{2}V^k_j V^k_{j+1}\frac{\mathscr{F}^k_j + \mathscr{F}^k_{j+1}}{2} + \frac{1}{2}(V^k_j - V^k_{j+1})^2\frac{\mathscr{F}^{k,|\cdot|}_j + \mathscr{F}^{k,|\cdot|}_{j+1}}{2},\ \forall j \in \{1, .., J\},$$

$$\overline{\overline{\rho^{k+1}_{j-1/2}}} = \rho^k_{j-1/2} - \frac{2\delta t^k}{\delta x_{j-1/2}}\Big(\mathscr{F}^-(\rho^k_{j+1/2}, V^k_j) - \mathscr{F}^-(\rho^k_{j-1/2}, V^k_j) - \rho^k_{j-1/2}(V^{k+1}_j - V^k_j)\Big),$$

and $\mathscr{F}^{k,|\cdot|}_1 = \mathscr{F}^{k,|\cdot|}_{J+1} = 0$, $\mathscr{F}^{k,|\cdot|}_j = \mathscr{F}^+(\rho^k_{j-1/2}, V^k_j) - \mathscr{F}^-(\rho^k_{j+1/2}, V^k_j)$, $\forall j \in \{2, .., J\}$. The function $\bar{\Phi}$ is a $\mathcal{C}^2$ extension of the function Φ (see [1, Section 4.3]).

Results. As in [2], we prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

2 Consistency analysis

Notation. Assuming that $\sum_{k=0}^{N-1} \delta t^k = T$, we define the reconstructions $(i = 0, 1)$

$$\rho^{(i)}_\delta = \sum_{k=0}^{N-1}\sum_{j=1}^{J} \rho^{k+i}_{j+1/2}\chi^{k+1/2}_{j+1/2}, \quad \pi_\delta = \sum_{k=0}^{N-1}\sum_{j=1}^{J} \pi^{k+1/2}_{j+1/2}\chi^{k+1/2}_{j+1/2}, \quad V_\delta = \sum_{k=0}^{N-1}\sum_{j=2}^{J} V^k_j \chi^{k+1/2}_j,$$

where $\chi_j^{k+1/2} = \chi_{[x_{j-1/2},\ x_{j+1/2}[\times[t^k,\ t^{k+1}[}, \quad \chi_{j+1/2}^{k+1/2} = \chi_{[x_j,\ x_{j+1}[\times[t^k,\ t^{k+1}[}$.

We also introduce the following discrete norms

$$||\rho_\delta||_{\infty,\mathcal{T}} = \max_{0\leqslant k\leqslant N} \max_{1\leqslant j\leqslant J} |\rho_{j+1/2}^k|, \qquad ||V_\delta||_{\infty,\mathcal{T}^\star} = \max_{0\leqslant k\leqslant N} \max_{2\leqslant j\leqslant J} |V_j^k|,$$
$$||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}} = \sum_{k=0}^{N} \delta t^k \sum_{j=2}^{J} |\rho_{j+1/2}^k - \rho_{j-1/2}^k|, \quad ||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star} = \sum_{k=0}^{N} \delta t^k \sum_{j=1}^{J} |V_{j+1}^k - V_j^k|,$$
$$||\rho_\delta||_{\mathrm{BV};1,\mathcal{T}} = \sum_{j=1}^{J} \delta x_{j+1/2} \sum_{k=0}^{N-1} |\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|.$$

For $\varphi \in \mathcal{C}_c^\infty((0,L)\times[0,T))$, we set $\varphi_{j+1/2}^k = \varphi(x_{j+1/2}, t^k)$ and $\varphi_j^k = \varphi(x_j, t^k)$. The interpolate $\varphi_\mathcal{T}$ of φ on the primal mesh and its discrete derivatives are defined by

$$\varphi_\mathcal{T}(\cdot,0) = \sum_{j=1}^{J} \varphi_{j+1/2}^0 \chi_{j+1/2}^{1/2}(\cdot,0), \qquad \varphi_\mathcal{T}(\cdot,t) = \sum_{k=0}^{N-1}\sum_{j=1}^{J} \varphi_{i+1/2}^{k+1} \chi_{j+1/2}^{k+1/2}(\cdot,t),\ \forall t>0,$$
$$\eth_t \varphi_\mathcal{T} = \sum_{k=0}^{N-1}\sum_{j=1}^{J} \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k}{\delta t^k} \chi_{j+1/2}^{k+1/2}, \quad \eth_x \varphi_\mathcal{T} = \sum_{k=0}^{N-1}\sum_{j=2}^{J} \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \chi_j^{k+1/2}.$$

Similarly, the interpolate $\varphi_{\mathcal{T}^\star}$ of φ on $\mathcal{T}^\star$ and its discrete derivatives are given by

$$\varphi_{\mathcal{T}^\star}(\cdot,0) = \sum_{j=2}^{J} \varphi_j^0 \chi_j^{1/2}(\cdot,0), \qquad \varphi_{\mathcal{T}^\star}(\cdot,t) = \sum_{k=0}^{N-1}\sum_{j=2}^{J} \varphi_j^{k+1} \chi_j^{k+1/2}(\cdot,t),\ \forall t>0,$$
$$\eth_t^\star \varphi_{\mathcal{T}^\star} = \sum_{k=0}^{N-1}\sum_{j=2}^{J} \frac{\varphi_j^{k+1} - \varphi_j^k}{\delta t^k} \chi_j^{k+1/2}, \qquad \eth_x^\star \varphi_{\mathcal{T}^\star} = \sum_{k=0}^{N-1}\sum_{j=1}^{J} \frac{\varphi_{j+1}^{k+1} - \varphi_j^{k+1}}{\delta x_{j+1/2}} \chi_{j+1/2}^{k+1/2}.$$

Assumptions. Let $(\mathcal{T}_m)_{m\geqslant 1}$ be a sequence of meshes s. t. size$(\mathcal{T}_m) \to 0$ and a familly of time steps $(\delta t_m^k)_{k\geqslant 0,\ m\geqslant 1}$ verifying $\delta t_m \to 0$ and (10). Assume that there exists $N_m \in \mathbb{N}$ s. t. $\sum_{k=0}^{N_m-1} \delta t_m^k = T$. The scheme defines $(\rho_{\delta_m}^{(0)}, V_{\delta_m})_{m\geqslant 1}$. Suppose that

$$||\rho_{\delta_m}^{(0)}||_{\infty,\mathcal{T}} + ||V_{\delta_m}||_{\infty,\mathcal{T}^\star} \leqslant C_\infty, \qquad ||\rho_{\delta_m}^{(0)}||_{1;\mathrm{BV},\mathcal{T}} + ||V_{\delta_m}||_{1;\mathrm{BV},\mathcal{T}^\star} \leqslant C_{\mathrm{BV}} \tag{14}$$

holds and, in the case $(\rho \mapsto \frac{p'(\rho)}{\rho}) \notin L^1_{\mathrm{loc}}(0,\infty)$, $||(\rho_{\delta_m}^{(0)})^{-1}||_{\infty,\mathcal{T}} \leqslant C$. We assume that there exists $(\bar\rho, \bar V) \in L^\infty((0,T)\times(0,L))^2$ such that

$$(\rho_{\delta_m}^{(0)}, V_{\delta_m}) \longrightarrow (\bar\rho, \bar V) \text{ in } L^r((0,T)\times(0,L))^2,\ 1\leqslant r<\infty. \tag{15}$$

Main results. The uniform bounds imply that there exists constants such that

$$\begin{aligned} \sup_{0\leqslant\rho,|V|\leqslant C_\infty} |\mathscr{A}(\rho,V)| &\leqslant C_\mathscr{A}, \quad \text{with } \mathscr{A} = \mathscr{F}^\pm,\ \partial_\rho\mathscr{F}^\pm \text{ and } \partial_V\mathscr{F}^\pm, \\ \sup_{0\leqslant\rho\leqslant C_\infty+4(C_\infty^2+C_{\mathscr{F}^\pm})} |\mathscr{B}(\rho)| &\leqslant C_\mathscr{B}, \quad \text{with } \mathscr{B} = \Phi,\ \Phi', \text{ and } \bar\Phi'. \end{aligned}$$

Note also that we have $|\Phi(\rho^k_{j+1/2})| \leqslant C_{\Phi,\rho}\rho^k_{j+1/2},\ \forall j,k$. Furthermore, we show that $||\rho^{(0)}_{\delta_m}||_{\mathrm{BV};1,\mathcal{T}} \leqslant C$ by using (5) which allows to dominate $\delta x_{j+1/2}|\rho^{k+1}_{j+1/2}-\rho^k_{j+1/2}|$ by

$$\delta t^k\Big[C_{\partial_\rho \mathscr{F}^\pm}(|\rho^k_{j+1/2}-\rho^k_{j-1/2}|+|\rho^k_{j+3/2}-\rho^k_{j+1/2}|)+2C_{\partial_V\mathscr{F}^\pm}|V^k_{j+1}-V^k_j|\Big].$$

Consequently, $\rho^{(1)}_{\delta_m} \to \bar\rho$ and $\pi_{\delta_m} \to p(\bar\rho)$ in $L^r((0,T)\times(0,L))$; with (4) and since $\rho_0, V_0 \in L^\infty(0,L)$, we get $\rho^{(0)}_{\delta_m}(\cdot,0)\to\rho_0$ and $V_{\delta_m}(\cdot,0)\to V_0$ in $L^r((0,L)),\ 1\leqslant r<\infty$.

Finally, in the sequel, when a function $\varphi\in\mathcal{C}^\infty_c\big((0,L)\times[0,T)\big)$ is given, we assume that δt_m and δx_m are sufficiently small so that $\varphi(x,\cdot)\equiv 0,\ \forall x\in[0,x_{3/2}]\cup[x_{J+1/2},L]$ and $\varphi(\cdot,t)\equiv 0,\ \forall t\in[t^{N-1},t^N]$. Moreover, since φ is smooth, $\varphi_{\mathcal{T}_m},\ \varphi_{\mathcal{T}^\star_m}\to\varphi$, $\eth_t\varphi_{\mathcal{T}_m},\ \eth^\star_t\varphi_{\mathcal{T}^\star_m}\to\partial_t\varphi$, and $\eth_x\varphi_{\mathcal{T}_m},\ \eth^\star_x\varphi_{\mathcal{T}^\star_m}\to\partial_x\varphi$, in $L^r((0,T)\times(0,L)),\ 1\leqslant r\leqslant\infty$.

Theorem 2.1 *Assume* (14) *and* (15). *Then,* $(\bar\rho,\bar V)$ *satisfies* (1)-(2) *in the distribution sense in* $\big(\mathcal{C}^\infty_c((0,L)\times[0,T))\big)'$, *that is*

$$-\int_0^T\int_0^L\Big[\bar\rho\partial_t\varphi+\bar\rho\bar V\partial_x\varphi\Big](x,t)\,\mathrm{d}x\,\mathrm{d}t-\int_0^L\rho_0(x)\varphi(x,0)\,\mathrm{d}x=0, \quad (16)$$

$$-\int_0^T\int_0^L\Big[\bar\rho\bar V\partial_t\varphi+(\bar\rho\bar V^2+p(\bar\rho))\partial_x\varphi\Big](x,t)\,\mathrm{d}x\,\mathrm{d}t-\int_0^L\rho_0(x)V_0(x)\varphi(x,0)\,\mathrm{d}x=0. \quad (17)$$

Moreover, $(\bar\rho,\bar V)$ *satisfies the entropy inequality* (3).

Proof. Let $\varphi\in\mathcal{C}^\infty_c((0,L)\times[0,T))$. For the sake of simplicity, the index m is dropped.

Mass balance. We multiply (5) by $\varphi^{k+1}_{j+1/2}$ and sum the results for $0\leqslant k\leqslant N-1$ and $1\leqslant j\leqslant J$ to obtain

$$\underbrace{\sum_{k=0}^{N-1}\sum_{j=1}^{J}\delta x_{j+1/2}(\rho^{k+1}_{j+1/2}-\rho^k_{j+1/2})\varphi^{k+1}_{j+1/2}}_{:=T_1}+\underbrace{\sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}(\mathscr{F}^k_{j+1}-\mathscr{F}^k_j)\varphi^{k+1}_{j+1/2}}_{:=T_2}=0.$$

For T_1, since $\varphi^N_{j+1/2}=0$, a discrete integration by part w.r.t. time yields

$$T_1=-\sum_{k=0}^{N-1}\sum_{j=1}^{J}\delta x_{j+1/2}\rho^k_{j+1/2}(\varphi^{k+1}_{j+1/2}-\varphi^k_{j+1/2})-\sum_{j=1}^{J}\delta x_{j+1/2}\rho^0_{j+1/2}\varphi^0_{j+1/2}.$$

Noting that

$$\int_{t^k}^{t^{k+1}}\int_{x_j}^{x_{j+1}}\rho^{(0)}_\delta\eth_t\varphi_{\mathcal{T}}\,\mathrm{d}x\,\mathrm{d}t=\delta x_{j+1/2}\rho^k_{j+1/2}(\varphi^{k+1}_{j+1/2}-\varphi^k_{j+1/2})$$

for $k\in\{0,..,N-1\}$, $j\in\{1,..,J\}$, we get

$$T_1=-\int_0^T\int_0^L\rho^{(0)}_\delta\eth_t\varphi_{\mathcal{T}}\,\mathrm{d}x\,\mathrm{d}t-\int_0^L\rho^{(0)}_\delta(x,0)\varphi_{\mathcal{T}}(x,0)\,\mathrm{d}x.$$

For T_2, by integrating by part w.r.t. space, we readily obtain

$$T_2=-\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^{J}\mathscr{F}^k_j(\varphi^{k+1}_{j+1/2}-\varphi^{k+1}_{j-1/2}).$$

Bearing in mind that $2\delta x_j = \delta x_{j-1/2} + \delta x_{j+1/2}$, we then combine the two following expressions of mass fluxes (see (9)-(ii))

$$\mathscr{F}_j^k = \rho_{j\pm 1/2}^k V_j^k \mp R_j^{k,\pm} \text{ with } R_j^{k,\pm} = \mathscr{F}^\pm(\rho_{j+1/2}^k, V_j^k) - \mathscr{F}^\pm(\rho_{j-1/2}^k, V_j^k)$$

to write

$$\mathscr{F}_j^k = \Big[\frac{\delta x_{j-1/2}}{2\delta x_j}\rho_{j-1/2}^k V_j^k + \frac{\delta x_{j+1/2}}{2\delta x_j}\rho_{j+1/2}^k V_j^k\Big] + \Big[\frac{\delta x_{j-1/2}}{2\delta x_j} R_j^{k,-} - \frac{\delta x_{j+1/2}}{2\delta x_j} R_j^{k,+}\Big].$$

This expression of the mass fluxes leads to $T_2 = -T_{2,1} - T_{2,2}$ with

$$T_{2,1} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \frac{1}{2}\Big[\delta x_{j-1/2}\rho_{j-1/2}^k + \delta x_{j+1/2}\rho_{j+1/2}^k\Big] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j},$$

$$T_{2,2} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \frac{1}{2}\Big[\delta x_{j-1/2} R_j^{k,-} - \delta x_{j+1/2} R_j^{k,+}\Big] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j}.$$

We now observe that, for $k \in \{0,..,N-1\}$, $j \in \{2,..,J\}$,

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_j} \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \,\mathrm{d}x\,\mathrm{d}t = \delta t^k \frac{\delta x_{j-1/2}}{2}\Big[\rho_{j-1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j}\Big],$$

and

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \,\mathrm{d}x\,\mathrm{d}t = \delta t^k \frac{\delta x_{j+1/2}}{2}\Big[\rho_{j+1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j}\Big].$$

Summing these equalities yields

$$T_{2,1} = \int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_0^L \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \,\mathrm{d}x\,\mathrm{d}t,$$

since $\eth_x \varphi_{\mathcal{T}}(x,\cdot) \equiv 0$ for $x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$.

With (15), we pass to the limit in T_1 and $T_{2,1}$. We prove that $(\bar\rho, \bar V)$ satisfies the mass conservation equation (16) by showing that $T_{2,2} \to 0$ since

$$|T_{2,2}| \leqslant C_{\partial_\rho \mathscr{F}^\pm}\ |\partial_x \varphi|_{L^\infty}\ ||V_\delta||_{\infty,\mathcal{T}^\star}\ ||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\ \delta x\ \lesssim\ \delta x.$$

Momentum balance. We multiply (6) by φ_j^{k+1} and sum for $0 \leqslant k \leqslant N-1$ and $2 \leqslant j \leqslant J$ to obtain $T_3 + T_4 + T_5 = 0$ with

$$T_3 = \sum_{k=0}^{N-1}\sum_{j=2}^{J} \delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k)\varphi_j^{k+1},$$

$$T_4 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} (\mathscr{G}_{j+1/2}^k - \mathscr{G}_{j-1/2}^k)\varphi_j^{k+1}, \quad T_5 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} (\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2})\varphi_j^{k+1}.$$

For T_3, integrating by part w.r.t time yields

$$T_3 = -\sum_{k=0}^{N-1}\sum_{j=2}^{J} \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^k) - \sum_{j=2}^{J} \delta x_j \rho_j^0 V_j^0 \varphi_j^0.$$

Next, we observe that

$$\int_{t^k}^{t^{k+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \eth_t^\star \varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t = V_j^k \frac{\varphi_j^{k+1}-\varphi_j^k}{\delta t^k}\int_{t^k}^{t^{k+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \delta x_j \rho_j^k V_j^k(\varphi_j^{k+1}-\varphi_j^k).$$

Summing these equalities for $k\in\{0,..,N-1\}$, $j\in\{2,..,J\}$ yields

$$T_3 = -\int_0^T\int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \eth_t^\star \varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t - \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)}(x,0)V_\delta(x,0)\varphi_{\mathcal{T}^\star}(x,0)\,\mathrm{d}x,$$
$$= -\int_0^T\int_0^L \rho_\delta^{(0)} V_\delta \eth_t^\star \varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t - \int_0^L \rho_\delta^{(0)}(x,0)V_\delta(x,0)\varphi_{\mathcal{T}^\star}(x,0)\,\mathrm{d}x.$$

For T_4 and T_5, we first integrate by part w.r.t space and obtain

$$T_4 = -\sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}\mathscr{G}_{j+1/2}^k(\varphi_{j+1}^{k+1}-\varphi_j^{k+1}),\quad T_5 = -\sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}\pi_{j+1/2}^{k+1/2}(\varphi_{j+1}^{k+1}-\varphi_j^{k+1}).$$

We then use the following expression of the momentum flux

$$\begin{cases} \mathscr{G}_{j+1/2}^k &= \dfrac{1}{2}\rho_{j+1/2}^k\Big[\big(V_j^k\big)^2+\big(V_{j+1}^k\big)^2\Big]+Q_{j+1/2}^k,\\ Q_{j+1/2}^k &= -\dfrac{1}{2}V_j^k R_j^{k,+}+\dfrac{1}{2}V_{j+1}^k R_{j+1}^{k,-}\\ & \qquad -\dfrac{1}{2}\big(V_{j+1}^k-V_j^k\big)\Big[\mathscr{F}^+(\rho_{j+1/2}^k,V_{j+1}^k)-\mathscr{F}^-(\rho_{j+1/2}^k,V_j^k)\Big]. \end{cases}$$

to write $T_4 = -T_{4,1}-T_{4,2}$ with

$$T_{4,1} = \sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}\frac{1}{2}\rho_{j+1/2}^k\Big[\big(V_j^k\big)^2+\big(V_{j+1}^k\big)^2\Big](\varphi_{j+1}^{k+1}-\varphi_j^{k+1})$$
$$T_{4,2} = \sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}Q_{j+1/2}^k(\varphi_{j+1}^{k+1}-\varphi_j^{k+1}).$$

Next, we observe that

$$\int_{t^k}^{t^{k+1}}\int_{x_j}^{x_{j+1}} \rho_\delta^{(0)}\big(V_\delta\big)^2\eth_x^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t = \delta t^k\rho_{j+1/2}^k\frac{\big(V_j^k\big)^2+\big(V_{j+1}^k\big)^2}{2}(\varphi_{j+1}^{k+1}-\varphi_j^{k+1})$$

and, consequently, summing for $k\in\{0,..,N-1\}$, $j\in\{1,..,J\}$,

$$T_{4,1} = \int_0^T\int_0^L \rho_\delta^{(0)}\big(V_\delta\big)^2\eth_x^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t.$$

Similarly, for T_5, we get

$$T_5 = -\int_0^T\int_0^L \pi_\delta\eth_x^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t.$$

With (15), we pass to the limit in T_3, $T_{4,1}$ and T_5. We obtain that $(\bar\rho,\bar V)$ satisfies the momentum balance equation (17) by showing that $T_{4,2}\to 0$. Indeed, we have

$$|T_{4,2}| \leqslant ||V_\delta||_{\infty,\mathcal{T}^\star}|\partial_x\varphi|_{L^\infty}\Big(C_{\partial_\rho\mathscr{F}^\pm}||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}+C_{\mathscr{F}^\pm}||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star}\Big)\delta x \;\lesssim\; \delta x.$$

Entropy inequality. We now assume that $\varphi \geqslant 0$.

• Kinetic energy. We multiply (13) by $\delta t^k \varphi_j^{k+1}$and sum for $0 \leqslant k \leqslant N-1$ and $2 \leqslant j \leqslant J$. We obtain to get $T_6 + T_7 + T_8 \leqslant 0$ with

$$T_6 = \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j \Big[E_{K,j}^{k+1} - E_{K,j}^k\Big] \varphi_j^{k+1}, \quad T_7 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \Big[\Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k\Big] \varphi_j^{k+1},$$

$$T_8 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \Big[\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}\Big] V_j^{k+1} \varphi_j^{k+1} + \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \varphi_j^{k+1}.$$

Integrating by part w.r.t. time yields

$$\begin{aligned} T_6 &= -\sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j E_{K,j}^k \Big[\varphi_j^{k+1} - \varphi_j^k\Big] - \sum_{j=1}^{J} \delta x_j E_{K,j}^0 \varphi_j^0 \\ &= -\int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} \big(V_\delta\big)^2 \eth_t^\star \varphi_{\mathcal{T}^\star} \,\mathrm{d}x \,\mathrm{d}t - \int_0^L \frac{1}{2} \rho_\delta^{(0)}(x,0) \big(V_\delta(x,0)\big)^2 \varphi_{\mathcal{T}^\star}(x,0) \,\mathrm{d}x. \end{aligned}$$

For T_7, we write $\Gamma_{j+1/2}^k = \frac{1}{4} \rho_{j+1/2}^k \big[\big(V_j^k\big)^3 + \big(V_{j+1}^k\big)^3\big] + \frac{1}{4} S_{j+1/2}^k$ where

$$S_{j+1/2}^k = V_j^k V_{j+1}^k \Big[R_{j+1}^{k,-} - R_j^{k,+}\Big] + (V_{j+1}^k - V_j^k)^2 \Big[\mathscr{F}_j^{k,|\cdot|} + \mathscr{F}_{j+1}^{k,|\cdot|} - \rho_{j+1/2}^k (V_j^k + V_{j+1}^k)\Big].$$

Integration by part w.r.t space leads to $T_7 = -T_{7,1} - T_{7,2}$ with

$$T_{7,1} = \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} \big(V_\delta\big)^3 \eth_x^\star \varphi_{\mathcal{T}^\star} \,\mathrm{d}x \,\mathrm{d}t, \quad T_{7,2} = \frac{1}{4} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} S_{j+1/2}^k \Big[\varphi_{j+1}^{k+1} - \varphi_j^{k+1}\Big].$$

Finally $|T_{7,2}| \lesssim \delta x$ since it is dominated by

$$\begin{aligned} \delta x |\partial_x \varphi|_{L^\infty} ||V_\delta||_{\infty,\mathcal{T}^\star} \Bigg[& \frac{C_{\partial_\rho \mathscr{F}^\pm}}{2} ||V_\delta||_{\infty,\mathcal{T}^\star} ||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}} \\ & + \big(2 C_{\mathscr{F}^\pm} + ||V_\delta||_{\infty,\mathcal{T}^\star} ||\rho_\delta||_{\infty,\mathcal{T}}\big) ||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star} \Bigg]. \end{aligned}$$

• Internal energy. Multiply (12) by $\delta t^k \varphi_{j+1/2}^{k+1}$ and sum for $0 \leqslant k \leqslant N-1$ and $1 \leqslant j \leqslant J$ to get $T_9 + T_{10} + T_{11} \leqslant 0$ with

$$T_9 = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta x_{j+1/2} \Big[e_{j+1/2}^{k+1} - e_{j+1/2}^k\Big] \varphi_{j+1/2}^{k+1}, \; T_{10} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \Big[\overline{G}_{j+1}^k - \overline{G}_j^k\Big] \varphi_{j+1/2}^{k+1},$$

$$T_{11} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1}) \varphi_{j+1/2}^{k+1} - \sum_{k=0}^{N-1} \sum_{j=1}^{J} D_j^k \varphi_{j+1/2}^{k+1}.$$

Owing to integration by part w.r.t. time, we get

$$\begin{aligned} T_9 &= -\sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta x_{j+1/2} \Phi(\rho_{j+1/2}^k) (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k) - \sum_{j=1}^{J} \delta x_{j+1/2} \Phi(\rho_{j+1/2}^0) \varphi_{j+1/2}^0, \\ &= -\int_0^T \int_0^L \Phi(\rho_\delta^{(0)}) \eth_t \varphi_{\mathcal{T}} \,\mathrm{d}x \,\mathrm{d}t - \int_0^L \Phi(\rho_\delta^{(0)}(x,0)) \varphi_{\mathcal{T}}(x,0) \,\mathrm{d}x. \end{aligned}$$

For T_{10}, we rewrite the flux as follows

$$\begin{cases} \overline{G}_j^k = \dfrac{1}{2\delta x_j}\Big[\delta x_{j-1/2}\Phi(\rho_{j-1/2}^k) + \delta x_{j+1/2}\Phi(\rho_{j+1/2}^k)\Big]V_j^k + U_{1,j}^k + U_{2,j}^k + U_{3,j}^k, \\ U_{1,j}^k = e_{j-1/2}^k(V_j^{k+1} - V_j^k), \qquad U_{2,j}^k = -\dfrac{\delta x_{j+1/2}}{2\delta x_j}\Big[e_{j+1/2}^k - e_{j-1/2}^k\Big]V_j^k, \\ U_{3,j}^k = -\dfrac{\delta x_{j-1/2}}{2\delta t^k}\Big[\bar{\Phi}(\overline{\overline{\rho_{j-1/2}^{k+1}}}) - \bar{\Phi}(\rho_{j-1/2}^k)\Big]. \end{cases}$$

It leads to $T_{10} = -T_{10,0} - T_{10,1} - T_{10,2} - T_{10,3}$ with

$$\begin{cases} T_{10,0} &= \displaystyle\int_0^T\int_0^L \Phi(\rho_\delta^{(0)})V_\delta \eth_x\varphi_{\mathcal{T}}\,\mathrm{d}x\,\mathrm{d}t, \\ T_{10,i} &= \displaystyle\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^{J}U_{i,j}^k(\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}), \quad i = 1,2,3. \end{cases}$$

The term $T_{10,1}$ can be bounded as follows

$$|T_{10,1}| \leqslant C_{\Phi,\rho}|\partial_x\varphi|\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^{J}\delta x_j\rho_{j-1/2}^k|V_j^{k+1} - V_j^k|. \tag{18}$$

Since $a \leqslant \min(a,b) + |b-a|$, we get $\rho_{j-1/2}^k \leqslant \rho_j^k + |\rho_{j+1/2}^k - \rho_{j-1/2}^k|$. This leads to

$$|T_{10,1}| \leqslant C_{\Phi,\rho}|\partial_x\varphi|_{L^\infty}\Big(\underbrace{\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^{J}\delta x_j\rho_j^k|V_j^{k+1} - V_j^k|}_{:=T^\star} + 2||V_\delta||_{\infty,\mathcal{T}^\star}||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\delta x\Big).$$

Writing $\rho_j^k = \rho_j^{k+1} - (\rho_j^{k+1} - \rho_j^k)$ and using the Cauchy-Schwarz inequality yields

$$T^\star \leqslant 2\Big(TL||\rho_\delta||_{\infty,\mathcal{T}}\Big)^{1/2}\Big(\delta t\sum_{k=0}^{N-1}\sum_{j=2}^{J}D_j^k\Big)^{1/2} + 2||V_\delta||_{\infty,\mathcal{T}^\star}||\rho_\delta||_{\mathrm{BV};1,\mathcal{T}}\delta t \;\lesssim\; \delta t^{1/2}.$$

It finally leads to $|T_{10,1}| \lesssim \delta t^{\frac{1}{2}} + \delta x$. The term $T_{10,2}$ can be bounded as follows

$$|T_{10,2}| \leqslant C_{\Phi'}||V_\delta||_{\infty,\mathcal{T}^\star}|\partial_x\varphi|_{L^\infty}||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\;\delta x \;\lesssim\; \delta x.$$

We now turn to $T_{10,3}$. We remark that

$$\Big|\overline{\overline{\rho_{j-1/2}^{k+1}}} - \rho_{j-1/2}^k\Big| \leqslant \frac{2\delta t^k}{\delta x_{j-1/2}}\Big(C_{\partial_\rho\mathscr{F}^\pm}|\rho_{j+1/2}^k - \rho_{j-1/2}^k| + \rho_{j-1/2}^k|V_j^{k+1} - V_j^k|\Big).$$

Hence, using the same bound as for $T_{10,1}$ yields

$$|T_{10,3}| \leqslant C_{\bar{\Phi}'}|\partial_x\varphi|_{L^\infty}\Big(\Big(C_{\partial_\rho\mathscr{F}^\pm} + 2||V_\delta||_{\infty,\mathcal{T}^\star}\Big)||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\delta x + T^\star\Big) \;\lesssim\; \delta t^{1/2} + \delta x.$$

- Pressure terms. It remains to get the limit of $T_8 + T_{11} = -T_{12,0} - T_{12,1} - T_{12,2} - T_{12,3}$

with

$$T_{12,0} = \int_0^T \int_0^L \pi_\delta V_\delta \eth_x^\star \varphi_{\mathcal{T}^\star} \,\mathrm{d}x\,\mathrm{d}t, \qquad T_{12,1} = \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \big(\varphi_{j+1/2}^{k+1} - \varphi_j^{k+1}\big),$$

$$T_{12,2} = \frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} \big(V_j^{k+1} - V_j^k + V_{j+1}^{k+1} - V_{j+1}^k\big)\big(\varphi_{j+1}^{k+1} - \varphi_j^{k+1}\big),$$

$$T_{12,3} = -\frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} \big(V_{j+1}^{k+1} - V_j^{k+1}\big)\big(2\varphi_{j+1/2}^{k+1} - \varphi_{j+1}^{k+1} - \varphi_j^{k+1}\big).$$

We bound $T_{12,1}$ and $T_{12,3}$ as follows

$$T_{12,1} \leqslant \frac{|\partial_x \varphi|_{L^\infty}}{2} \Big(\sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \Big) \delta x \lesssim \delta x, \quad T_{12,3} \leqslant \frac{C_\pi}{4} |\partial_{xx}\varphi|_{L^\infty} ||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star} (\delta x)^2 \lesssim (\delta x)^2.$$

Note that $|\pi_{j+1/2}^{k+1/2}| \leqslant (C_{\Phi'} + C_{\Phi,\rho})\rho_{j+1/2}^k$. It readily leads to

$$|T_{12,2}| \leqslant (C_{\Phi'} + C_{\Phi,\rho})|\partial_x \varphi|_{L^\infty} T^\star \ \lesssim \ \delta t^{\frac{1}{2}}.$$

With (15), we pass to the limit in T_6, $T_{7,1}$, T_9, $T_{10,0}$ and $T_{12,0}$. We arrive at (3) since the other terms tend to 0. ■

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