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Consistency analysis of a 1D Finite Volume scheme for barotropic Euler models

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Abstract

This work is concerned with the consistency study of a 1D (staggered kinetic) finite volume scheme for barotropic Euler models. We prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

1 Introduction

The model. This work is concerned with the consistency study of a (staggered kinetic) Finite Volume (FV) scheme for barotropic Euler models

\[ \partial_t \rho + \partial_x (\rho V) = 0, \]
\[ \partial_t (\rho V) + \partial_x (\rho V^2 + p(\rho)) = 0. \]

The unknowns are the density \( \rho \) and the velocity \( V \). The pressure \( (\rho \mapsto p(\rho)) \) is assumed to be \( C^2([0,\infty)) \) with \( p(\rho) > 0, \ p'(\rho) > 0, \ p''(\rho) \geq 0, \ \forall \rho > 0. \) Thus, the sound speed \( c : \rho \mapsto \sqrt{p'(\rho)} \) is well defined and is an increasing function.

We consider the problem (1)-(2) on the bounded domain \((0,L) \times [0,T]\) with the boundary conditions \( V(0,t) = 0 = V(L,t), \ \forall t > 0 \) and the initial conditions \( \rho(x,0) = \rho_0(x), \ V(x,0) = V_0(x), \ \forall x \in (0,L) \) with \( \rho_0, \ V_0 \in L^\infty(0,L). \)

Let \( \Phi : \rho > 0 \mapsto \Phi(\rho) \) such that \( \rho \Phi'(\rho) - \Phi(\rho) = p(\rho), \ \forall \rho > 0. \) The quantity \( S = \frac{1}{2} \rho |V|^2 + \Phi(\rho) \) is an entropy of the system: entropy solutions to (1)-(2) are required to satisfy: for any \( \varphi \in C_c^\infty((0,L) \times [0,T]) \) such that \( \varphi \geq 0, \)

\[ \int_0^T \int_0^L [S \partial_t \varphi + (S + p(\rho)) V \partial_x \varphi](x,t) \, dx \, dt - \int_0^L S_0(x) \varphi(x,0) \, dx \leq 0. \]

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The meshes. We consider a set of \( J+1 \) points \( 0 = x_1 < x_2 < \ldots < x_J < x_{J+1} = L \). The \( x_j \) are the edges of the so-called primal mesh \( \mathcal{T} \). We set \( \delta x_{j+1/2} = x_{j+1} - x_j \). The centers of the primal cells, \( x_{j+1} = (x_j + x_{j+1})/2 \) for \( j \in \{1, \ldots, J\} \), realize the dual mesh \( \mathcal{T}^* \). We set \( \delta x_j = (\delta x_{j-1/2} + \delta x_{j+1/2})/2 \) for \( j \in \{2, \ldots, J-1\} \) and \( \delta x = \text{size}(\mathcal{T}) = \max_j \delta x_{j+1/2} \). The adaptive time step is \( \delta t^k \) and we set \( \delta t = \max_k \delta t^k \).

The scheme. We analyze the scheme introduced in [1]. It works on staggered grids: the densities, \( \rho_j \), \( j \in \{1, \ldots, J\} \), are evaluated at centers whereas the velocities, \( V_j \), \( j \in \{1, \ldots, J+1\} \), are evaluated at edges. We set, for \( j \in \{1, \ldots, J\} \) and \( i \in \{2, \ldots, J\} \)

\[
\rho_j^0 = \frac{1}{\delta x_{j+1/2}} \int_{x_j}^{x_{j+1}} \rho_0(x) \, dx, \quad V_i^0 = \frac{1}{\delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} V_0(x) \, dx. \tag{4}
\]

The density is first updated with a FV approximation on the primal mesh

\[
\delta x_{j+1/2}(\rho_j^{k+1} - \rho_j^k) + \delta t^k (\mathcal{F}^k_{j+1} - \mathcal{F}^k_j) = 0, \quad \forall j \in \{1, \ldots, J\}. \tag{5}
\]

Then, the velocity is updated with a FV approximation on the dual mesh:

\[
\delta x_j(\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) + \delta t^k \left( \frac{\partial^k}{\partial x_{j+1/2}} - \frac{\partial^k}{\partial x_{j-1/2}} + \frac{\varphi^k}{\partial x_{j+1/2}} - \frac{\varphi^k}{\partial x_{j-1/2}} \right) = 0, \tag{6}
\]

for \( j \in \{2, \ldots, J\} \), while \( V_1^{k+1} = V_{J+1}^{k+1} = 0 \). The density on the edges \( \rho_j^k \) is defined by

\[
2\delta x_j \rho_j^k = \delta x_{j+1/2} \rho_j^{k+1} + \delta x_{j-1/2} \rho_j^{k-1}, \quad \forall j \in \{2, \ldots, J\}. \tag{7}
\]

The definition of the fluxes relies on the kinetic framework. We refer the reader to [1] for details. Let us introduce the two following functions \( \mathcal{F}^+ \) and \( \mathcal{F}^- \)

\[
\mathcal{F}^\pm (\rho, V) = \frac{\rho}{2c(\rho)} \int_{\xi \equiv 0}^{\xi \equiv c(\rho)} \xi [\xi < V] \xi^2 \xi d\xi.
\]

We adopt the following formulas for mass fluxes: \( \mathcal{F}_1^k = \mathcal{F}_{J+1}^k = 0 \),

\[
\mathcal{F}_j^k = \mathcal{F}^+(\rho_j^{k-1/2}; V_j^k) + \mathcal{F}^-(\rho_j^{k+1/2}; V_j^k), \quad \forall j \in \{2, \ldots, J\}. \tag{8}
\]

and, for momentum fluxes:

\[
\varphi_j^{k/2} = \frac{V_j^k}{2} \mathcal{F}^-(\rho_j^{k/2}; V_j^k), \quad \varphi_{j+1/2}^k = \frac{V_j^k}{2} \mathcal{F}^+(\rho_j^{k/2}; V_j^k),
\]

\[
\varphi_{j+1/2}^k = \frac{V_j^k}{2} \mathcal{F}^+(\rho_j^{k-1/2}; V_j^k) + \mathcal{F}^+(\rho_j^{k+1/2}; V_j^k) \tag{9}
\]

\[
+ \frac{V_j^{k+1}}{2} (\mathcal{F}^-(\rho_j^{k+1/2}; V_j^k) + \mathcal{F}^-(\rho_j^{k+3/2}; V_j^k)), \quad \forall j \in \{2, \ldots, J-1\}. \tag{10}
\]

The discrete pressure gradient combines a space centered scheme and time semi implicit discretization, namely it uses

\[
\pi_{j+1/2}^{k+1} = \rho_{j+1/2}^k \Phi'(\rho_{j+1/2}^{k+1} - \Phi(\rho_{j+1/2}^k)).
\]

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Properties of the scheme. The analysis is driven by the shapes of the functions $\mathcal{F}^\pm$, see [1, Lemma 3.2]. Here, we shall use the following properties

(i) Smoothness: $(\rho, V) \in (0, \infty) \times \mathbb{R} \mapsto \mathcal{F}^\pm(\rho, V)$ are of class $C^1$,

(ii) Consistency: $\mathcal{F}^+(\rho, V) + \mathcal{F}^-(\rho, V) = \rho V, \ \forall V \in \mathbb{R}, \ \forall \rho \geq 0.$ (9)

Under CFL conditions, see [1], the scheme preserves the positivity of the discrete density and discrete kinetic and internal energies evolution equations hold.

**Lemma 1.1** Let $N \in \mathbb{N}$. Assume $\min_i (\rho_{j+1/2}^k) > 0$. For all $k \in \{0, \ldots, N-1\}$, there exists $\gamma^k > 0$, which depends only on the state $(\rho^k, V^k)$, such that if

$$
\frac{\delta t^k}{\min_j (\delta x_{j+1/2})} \gamma^k \leq 1,
$$

then, $\min_i (\rho_{j+1/2}^k) > 0, \forall k \in \{0, \ldots, N\}$ and

$$
0 \leq \sum_{k=0}^{N-1} \sum_{j=2}^J D^k_j \leq C, \quad \text{with} \quad D^k_j = \frac{1}{4} \delta x_j \rho_{j+1}^{k+1} (V_{j+1}^{k+1} - V_j^k)^2,
$$

$$
\frac{\delta x_{j+1/2}}{\delta t^k} \left[ e_{j+1/2}^{k+1} - e_j^{k+1/2} \right] + \overline{G}^k_j - \overline{G}^k_{j+1} - \overline{\rho}^k_{j+1/2} \left[ V_{j+1}^{k+1} - V_j^k \right] \leq \frac{D^k_j}{\delta t^k},
$$

$$
\frac{\delta x_j}{\delta t^k} \left[ E_{K,j}^{k+1} - E_{K,j}^k \right] + \overline{\Gamma}^k_{j+1/2} - \overline{\Gamma}^k_{j-1/2} + \left[ \overline{\rho}^k_{j+1/2} - \overline{\rho}^k_{j-1/2} \right] V_{j+1}^k + \frac{D^k_j}{\delta t^k} \leq 0,
$$

where $E_{K,j}^k = \frac{1}{2} \rho_{j+1}^k (V_j^k)^2$ and $e_{j+1/2}^k = \Phi(\rho_{j+1/2}^k)$ are the kinetic and internal energies.

The fluxes are defined by $\overline{G}^k_j = \overline{G}^k_{j+1} = 0$ and

$$
\overline{G}^k_j = \Phi(\rho_{j-1/2}^k) V_{j+1}^k - \frac{\delta x_{j-1/2}}{2 \delta t^k} \left[ \overline{\Phi}(\rho_{j-1/2}^{k+1}) - \overline{\Phi}(\rho_{j-1/2}^k) \right], \forall j \in \{2, \ldots, J\},
$$

$$
\overline{\Gamma}^k_{j+1/2} = \frac{1}{2} V_j^k V_{j+1}^k \frac{\mathcal{F}^k_j + \mathcal{F}^k_{j+1}}{2} + \frac{1}{2} (V_j^k - V_{j+1}^k)^2 \mathcal{F}^k_{j+1/2} + \mathcal{F}^k_{j+1/2}, \forall j \in \{1, \ldots, J\},
$$

$$
\overline{\rho}^k_{j+1/2} = \rho_{j-1/2}^k - \frac{2 \delta t^k}{\delta x_{j-1/2}} \left( \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k) - \mathcal{F}^-(\rho_{j-1/2}^k, V_j^k) - \rho_{j-1/2}^k (V_{j+1}^k - V_j^k) \right),
$$

and $\mathcal{F}^k_{j+1} = \mathcal{F}^k_{j-1} = 0$, $\mathcal{F}^k_{j+1} = \mathcal{F}^+(\rho_{j+1/2}^k, V_j^k) - \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k), \forall j \in \{2, \ldots, J\}.$

The function $\overline{\Phi}$ is a $C^2$ extension of the function $\Phi$ (see [1, Section 4.3]).

**Results.** As in [2], we prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

## 2 Consistency analysis

**Notation.** Assuming that $\sum_{k=0}^{N-1} \delta t^k = T$, we define the reconstructions ($i = 0, 1$)

$$
\rho_{\delta}^{(i)} = \sum_{k=0}^{N-1} \sum_{j=1}^J \rho_{j+1/2}^{k+i} \chi_{j+1/2}^{k+1/2}, \quad \pi_{\delta} = \sum_{k=0}^{N-1} \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} \chi_{j+1/2}^{k+1/2}, \quad V_{\delta} = \sum_{k=0}^{N-1} \sum_{j=1}^J V_j^k \chi_{j+1/2}^{k+1/2},
$$

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where $\chi^{k+1/2}_j = \chi[x_{j-1/2}, x_{j+1/2} \times [t^k, t^{k+1}]]$, $\chi^{k+1/2}_j = \chi[x_j, x_{j+1} \times [t^k, t^{k+1}]]$.

We also introduce the following discrete norms

$$
\|\rho_\delta\|_{\infty, T} = \max_{0 \leq k \leq N} \max_{1 \leq j \leq J} |\rho_j^{k}|, \quad \|V_\delta\|_{\infty, T^*} = \max_{0 \leq k \leq N} \max_{2 \leq j \leq J} |V_j^{k}|,
$$

$$
\|\rho_\delta\|_{1, BV, T} = \sum_{k=0}^{N} \delta t^k \sum_{j=2}^{J} |\rho_j^{k+1/2} - \rho_j^{k-1/2}|, \quad \|V_\delta\|_{1, BV, T^*} = \sum_{k=0}^{N} \delta t^k \sum_{j=1}^{J} |V_j^{k+1} - V_j^{k}|,
$$

$$
\|\rho_\delta\|_{BV, T^*} = \sum_{k=0}^{N-1} \delta x_{j+1/2} \sum_{j=1}^{J} |\rho_j^{k+1} - \rho_j^{k}|.
$$

For $\varphi \in C_0^\infty((0, L) \times [0, T])$, we set $\varphi_j^{k} = \varphi(x_{j+1/2}, t^k)$ and $\varphi_j = \varphi(x_j, t^k)$. The interpolate $\varphi_T$ of $\varphi$ on the primal mesh and its discrete derivatives are defined by

$$
\varphi_T(\cdot, 0) = \sum_{j=1}^{J} \varphi_j^{0} \chi_j^{1/2}(\cdot, 0), \quad \varphi_T(\cdot, t) = \sum_{j=1}^{J} \sum_{k=0}^{N-1} \varphi_j^{k+1/2} \chi_j^{k+1/2}(\cdot, t), \forall t > 0,
$$

$$
\partial_t \varphi_T = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \frac{\varphi_j^{k+1/2} - \varphi_j^{k-1/2}}{\delta t^k} \chi_j^{k+1/2}, \quad \partial_x \varphi_T = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \frac{\varphi_j^{k+1/2} - \varphi_j^{k-1/2}}{\delta x_j} \chi_j^{k+1/2}.
$$

Similarly, the interpolate $\varphi_{T^*}$ of $\varphi$ on $T^*$ and its discrete derivatives are given by

$$
\varphi_{T^*}(\cdot, 0) = \sum_{j=1}^{J} \varphi_j^{0} \chi_j^{1/2}(\cdot, 0), \quad \varphi_{T^*}(\cdot, t) = \sum_{j=1}^{J} \sum_{k=0}^{N-1} \varphi_j^{k+1/2} \chi_j^{k+1/2}(\cdot, t), \forall t > 0,
$$

$$
\partial_t^* \varphi_{T^*} = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \frac{\varphi_j^{k+1} - \varphi_j^{k}}{\delta t^k} \chi_j^{k+1/2}, \quad \partial_x^* \varphi_{T^*} = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \frac{\varphi_j^{k+1} - \varphi_j^{k}}{\delta x_{j+1/2}} \chi_j^{k+1/2}.
$$

**Assumptions.** Let $(T_m)_{m \geq 1}$ be a sequence of meshes s.t. size $(T_m) \to 0$ and a family of time steps $(\delta t^k_m)_{k \geq 0, m \geq 1}$ verifying $\delta t^k \to 0$ and (10). Assume that there exists $N_m \in \mathbb{N}$ s.t. $\sum_{k=0}^{N_m-1} \delta t^k_m = T$. The scheme defines $(\rho_{\delta m}^{(0)}, V_{\delta m})_{m \geq 1}$. Suppose that

$$
\|\rho_{\delta m}^{(0)}\|_{\infty, T} + \|V_{\delta m}\|_{\infty, T^*} \leq C_\infty, \quad \|\rho_{\delta m}^{(0)}\|_{1, BV, T} + \|V_{\delta m}\|_{1, BV, T^*} \leq C_{BV}
$$

holds and, in the case $(\rho \mapsto \frac{\rho'\rho}{\rho}) \not\in L_1^{\text{loc}}(0, \infty)$, $\|\rho_{\delta m}^{(0)}\|_{\infty, T} \leq C$. We assume that there exists $(\tilde{\rho}, \tilde{V}) \in L^\infty((0, T) \times (0, L))^2$ such that

$$
(\rho_{\delta m}^{(0)}, V_{\delta m}) \to (\tilde{\rho}, \tilde{V}) \text{ in } L^r((0, T) \times (0, L))^2, \quad 1 \leq r < \infty.
$$

**Main results.** The uniform bounds imply that there exists constants such that

$$
\sup_{0 \leq \rho, |V| \leq C_\infty} |\mathcal{A}(\rho, V)| \leq C_{\mathcal{A}} \text{, with } \mathcal{A} = \mathcal{F}^\pm, \partial_\rho \mathcal{F}^\pm \text{ and } \partial_V \mathcal{F}^\pm,
$$

$$
\sup_{0 \leq \rho \leq C_{\infty} + 4(C_{\infty} + C_{\mathcal{F}^\pm})} |\mathcal{B}(\rho)| \leq C_{\mathcal{B}} \text{, with } \mathcal{B} = \Phi, \Phi', \text{ and } \tilde{\Phi}'.
$$
Note also that we have \(|\Phi(\rho_{j+1/2}^k)| \leq C_{\Phi, \rho} \rho_{j+1/2}^k, \forall j, k\). Furthermore, we show that \(||\rho_{\delta_m}^{(0)}||_{\text{BV,1,T}} \leq C\) by using (5) which allows to dominate \(\delta x_{j+1/2}|\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|\) by
\[
\delta t k \left[ C_{\partial_x} \frac{\partial}{\partial x} \left( |\rho_{j+1/2}^k - \rho_{j-1/2}^k + |\rho_{j+3/2}^k - \rho_{j+1/2}^k| \right) + 2C_{\partial_x} \frac{\partial}{\partial x} |V_{j+1} - V_j| \right].
\]
Consequently, \(\rho_{\delta_m}^{(1)} \to \tilde{\rho}\) and \(\pi_{\delta_m} \to p(\tilde{\rho})\) in \(L^r((0,T) \times (0,L))\); with (4) and since \(\rho_0, V_0 \in L^\infty(0,L)\), we get \(\rho_{\delta_m}^{(0)}(., 0) \to \rho_0\) and \(V_{\delta_m}(., 0) \to V_0\) in \(L^r((0,L))\), \(1 \leq r < \infty\).

Finally, in the sequel, when a function \(\varphi \in C^\infty_c((0,L) \times [0,T))\) is given, we assume that \(\delta t_m \) and \(\delta x_m\) are sufficiently small so that \(\varphi(x, \cdot) \equiv 0, \forall x \in [0, x_{j+1/2}] \cup [x_{j+1/2}, L]\) and \(\varphi(\cdot, t) \equiv 0, \forall t \in [t^{N-1}, t^N]\). Moreover, since \(\varphi\) is smooth, \(\varphi_{t_m}, \varphi_{T_m} \to \varphi, \tilde{\partial}_t \varphi_{t_m}, \tilde{\partial}_t \varphi_{T_m} \to \partial_t \varphi, \tilde{\partial}_x \varphi_{t_m}, \tilde{\partial}_x \varphi_{T_m} \to \partial_x \varphi, \) in \(L^r((0,T) \times (0,L))\), \(1 \leq r \leq \infty\).

**Theorem 2.1** Assume (14) and (15). Then, \((\tilde{\rho}, \tilde{V})\) satisfies (1)-(2) in the distribution sense in \((C^\infty_c((0,L) \times [0,T]))')\), that is
\[
\begin{align*}
-\int_0^T \int_0^L [\tilde{\rho} \partial_t \varphi + \tilde{\rho} \tilde{V} \partial_x \varphi] (x,t) \, dx \, dt - \int_0^L \rho_0(x) \varphi(x, 0) \, dx &= 0, \quad (16) \\
-\int_0^T \int_0^L [\tilde{\rho} \tilde{V} \partial_x \varphi + (\tilde{\rho} \tilde{V}^2 + p(\tilde{\rho})) \partial_t \varphi] (x,t) \, dx \, dt - \int_0^L \rho_0(x) V_0(x) \varphi(x, 0) \, dx &= 0. \quad (17)
\end{align*}
\]
Moreover, \((\tilde{\rho}, \tilde{V})\) satisfies the entropy inequality (3).

**Proof.** Let \(\varphi \in C^\infty_c((0,L) \times [0,T))\). For the sake of simplicity, the index \(m\) is dropped.

**Mass balance.** We multiply (5) by \(\varphi_{j+1/2}^{k+1}\) and sum the results for \(0 \leq k \leq N-1\) and \(1 \leq j \leq J\) to obtain
\[
\sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2}(\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k) \varphi_{j+1/2}^{k+1} + \sum_{k=0}^{N-1} \sum_{j=1}^J \delta t k \sum_{j=1}^J (\mathcal{R}_{j+1}^k - \mathcal{R}_j^k) \varphi_{j+1/2}^{k+1} = 0.
\]
For \(T_1\), since \(\varphi_{N, j+1/2}^N = 0\), a discrete integration by part w.r.t. time yields
\[
T_1 = -\sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} \rho_{j+1/2}^k \varphi_{j+1/2}^{k+1} - \sum_{j=1}^J \delta x_{j+1/2} \rho_{j+1/2}^0 \varphi_{j+1/2}^0.
\]
Noting that
\[
\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1}} \delta t^0 \partial_t \varphi \, dx \, dt = \delta x_{j+1/2} \rho_{j+1/2}^0 \varphi_{j+1/2}^{k} - \varphi_{j+1/2}^{k+1},
\]
for \(k \in \{0, ..., N-1\}, \ j \in \{1, ..., J\}\), we get
\[
T_1 = -\int_0^T \int_0^L \rho_0^0 \partial_t \varphi \, dx \, dt - \int_0^L \rho_0^0 (x, 0) \varphi(x, 0) \, dx.
\]
For \(T_2\), by integrating by part w.r.t. space, we readily obtain
\[
T_2 = -\sum_{k=0}^{N-1} \sum_{j=2}^J \mathcal{R}_j^k \varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}.
\]
Bearing in mind that $2\delta x_j = \delta x_{j-1/2} + \delta x_{j+1/2}$, we then combine the two following expressions of mass fluxes (see (9)-(ii))

$$\mathcal{F}_j = \rho_{j-1/2}^k V_j^k + R_{j-1/2}^k \pm R_{j+1/2}^k \pm, \quad R_{j-1/2}^k = \mathcal{F}_j(\rho_{j-1/2}^k, V_j^k) = \mathcal{F}_j(\rho_{j+1/2}^k, V_j^k)$$

to write

$$\mathcal{F}_j^k = \left[ \frac{\delta x_{j-1/2}}{2\delta x_j} \rho_{j-1/2}^k \right] V_j^k + \frac{\delta x_{j+1/2}}{2\delta x_j} \rho_{j+1/2}^k V_j^k = \mathcal{F}_j(\rho_{j-1/2}^k, V_j^k) - \mathcal{F}_j(\rho_{j+1/2}^k, V_j^k).$$

This expression of the mass fluxes leads to $T_2 = -T_{2,1} - T_{2,2}$ with

$$T_{2,1} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \frac{1}{2} \left[ \delta x_{j-1/2} \rho_{j-1/2}^k + \delta x_{j+1/2} \rho_{j+1/2}^k \right] V_j^k \frac{\varphi_{j+1/2}^k - \varphi_{j-1/2}^k}{\delta x_j},$$

$$T_{2,2} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \frac{1}{2} \left[ \delta x_{j-1/2} R_{j-1/2}^k - \delta x_{j+1/2} R_{j+1/2}^k \right] V_j^k \frac{\varphi_{j+1/2}^k - \varphi_{j-1/2}^k}{\delta x_j}.$$

We now observe that, for $k \in \{0, \ldots, N - 1\}$, $j \in \{2, \ldots, J\}$,

$$\int_{t_k}^{t_{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0^k \delta \varphi \delta x \varphi_T \, dx \, dt = \delta t^k \frac{\delta x_{j-1/2}}{2} \left[ \rho_{j-1/2}^k V_j^k \frac{\varphi_{j+1/2}^k - \varphi_{j-1/2}^k}{\delta x_j} \right],$$

and

$$\int_{t_k}^{t_{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0^k \delta \varphi \delta x \varphi_T \, dx \, dt = \delta t^k \frac{\delta x_{j+1/2}}{2} \left[ \rho_{j+1/2}^k V_j^k \frac{\varphi_{j+1/2}^k - \varphi_{j-1/2}^k}{\delta x_j} \right].$$

Summing these equalities yields

$$T_{2,1} = \int_0^T \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0^k \delta \varphi \delta x \varphi_T \, dx \, dt = \int_0^T \int_0^{x_{j+1/2}} \rho_0^k \delta \varphi \delta x \varphi_T \, dx \, dt,$$

since $\delta_x \varphi_T(x, \cdot) \equiv 0$ for $x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$.

With (15), we pass to the limit in $T_1$ and $T_{2,1}$. We prove that $(\rho, V)$ satisfies the mass conservation equation (16) by showing that $T_{2,2} \to 0$ since

$$|T_{2,2}| \leq C_{\partial_x \varphi} ||\partial_x \varphi||_{L^\infty} ||V||_{L^\infty} ||\rho_0||_{1BV, \varphi_T} \delta x \lesssim \delta x.$$

**Momentum balance.** We multiply (6) by $\varphi_j^{k+1}$ and sum for $0 \leq k \leq N - 1$ and $2 \leq j \leq J$ to obtain $T_3 + T_4 + T_5 = 0$ with

$$T_3 = \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^{k} V_j^{k}) \varphi_j^{k+1},$$

$$T_4 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J (\varphi_j^{k+1} - \varphi_j^{k-1/2}) \varphi_j^{k+1}, \quad T_5 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J (\pi_j^{k+1} - \pi_j^{k-1/2}) \varphi_j^{k+1}.$$

For $T_3$, integrating by part w.r.t time yields

$$T_3 = -\sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^{k}) - \sum_{j=2}^J \delta x_j \rho_j^0 V_j^0 \varphi_j^0.$$
Next, we observe that
\[
\int_{t_k}^{t_{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} V_0 \delta_i^* \varphi \cdot \delta \tau \, dx \, dt = V_j^k \varphi_j^{k+1} - \varphi_j^k \int_{t_k}^{t_{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} \, dx \, dt
\]
\[
= \delta x \rho_\delta^k V_j^k (\varphi_j^{k+1} - \varphi_j^k).
\]
Summing these equalities for \( k \in \{0, \ldots, N - 1\}, j \in \{2, \ldots, J\} \) yields
\[
T_3 = -\int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_0 \delta_i^* \varphi \cdot \delta \tau \, dx \, dt - \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} (x, 0)V_0(x, 0)\varphi \cdot (x, 0) \, dx,
\]
\[
= -\int_0^T \int_0^L \rho_\delta^{(0)} V_0 \delta_i^* \varphi \cdot \delta \tau \, dx \, dt - \int_0^L \rho_\delta^{(0)} (x, 0)V_0(x, 0)\varphi \cdot (x, 0) \, dx.
\]
For \( T_4 \) and \( T_5 \), we first integrate by part w.r.t space and obtain
\[
T_4 = -\sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \mathcal{A}_j^{k+1/2} (\varphi_j^{k+1} - \varphi_j^k), \quad T_5 = -\sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_j^{k+1/2} (\varphi_j^{k+1} - \varphi_j^k).
\]
We then use the following expression of the momentum flux
\[
\begin{align*}
\mathcal{A}_j^{k+1/2} &= \frac{1}{2} \rho_j^{k+1/2} \left[(V_j^k)^2 + (V_{j+1}^k)^2\right] + Q_j^{k+1/2}, \\
Q_j^{k+1/2} &= -\frac{1}{2} V_j^k R_{j,+}^k + \frac{1}{2} V_{j+1}^k R_{j,-}^k - \frac{1}{2} (V_j^k - V_{j+1}^k) \left[\mathcal{F}^+(\rho_j^{k+1/2}, V_j^k) - \mathcal{F}^-(\rho_j^{k+1/2}, V_{j+1}^k)\right],
\end{align*}
\]
to write \( T_4 = -T_{4,1} - T_{4,2} \) with
\[
T_{4,1} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \frac{1}{2} \rho_j^{k+1/2} \left[(V_j^k)^2 + (V_{j+1}^k)^2\right] (\varphi_j^{k+1} - \varphi_j^k),
\]
\[
T_{4,2} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J Q_j^{k} (\varphi_j^{k+1} - \varphi_j^k).
\]
Next, we observe that
\[
\int_{t_k}^{t_{k+1}} \int_{x_j}^{x_{j+1}} \rho_\delta^{(0)} (V_0)^2 \delta_i^* \varphi \cdot \delta \tau \, dx \, dt = \delta t^k \rho_j^{k+1/2} (V_j^k)^2 + (V_{j+1}^k)^2 (\varphi_j^{k+1} - \varphi_j^k)
\]
and, consequently, summing for \( k \in \{0, \ldots, N - 1\}, j \in \{1, \ldots, J\} \),
\[
T_{4,1} = \int_0^T \int_0^L \rho_\delta^{(0)} (V_0)^2 \delta_i^* \varphi \cdot \delta \tau \, dx \, dt.
\]
Similarly, for \( T_5 \), we get
\[
T_5 = -\int_0^T \int_0^L \pi_\delta \delta_i^* \varphi \cdot \delta \tau \, dx \, dt.
\]
With (15), we pass to the limit in \( T_3, T_{4,1} \) and \( T_5 \). We obtain that \( (\tilde{\rho}, \tilde{V}) \) satisfies the momentum balance equation (17) by showing that \( T_{4,2} \to 0 \). Indeed, we have
\[
|T_{4,2}| \leq ||V_0||_{\infty, \tau^*} ||\delta_i^* \varphi||_{L^\infty} \left(C_{\partial_\delta, \partial_\varphi} ||\rho_\delta||_{1, BV, \tau} + C_{\varphi} ||V_0||_{1, BV, \tau^*}\right) \delta x \lesssim \delta x.
\]
Entropy inequality. We now assume that \( \varphi \geq 0 \).

- Kinetic energy. We multiply (13) by \( \delta t^k \varphi_j^{k+1} \) and sum for \( 0 \leq k \leq N - 1 \) and \( 2 \leq j \leq J \). We obtain to get \( T_6 + T_7 + T_8 \leq 0 \) with

\[
T_6 = \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j \left[ \frac{E_{K,j}}{E_{K,j}} - E_{K,j}^k \right] \varphi_j^{k+1}, \quad T_7 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \left[ \Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k \right] \varphi_j^{k+1},
\]

\[
T_8 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \left[ \pi_{j+1/2}^{k+1} - \pi_{j-1/2}^{k+1} \right] V_j^{k+1} \varphi_j^{k+1} + \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \varphi_j^{k+1}.
\]

Integrating by part w.r.t. space yields

\[
T_6 = -\sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j E_{K,j}^k \left[ \varphi_j^{k+1} - \varphi_j^k \right] - \sum_{j=1}^{J} \delta x_{j} E_{K,j}^0 \varphi_{j}^0
\]

\[
= -\int_0^T \int_0^L \frac{1}{2} \rho_{\delta}^{(0)} (V_\delta)^2 \partial_t^* \varphi_T \, dx \, dt - \int_0^T \int_0^L \frac{1}{2} \rho_{\delta}^{(0)} (x, 0) (V_\delta(x, 0))^2 \varphi_T (x, 0) \, dx.
\]

For \( T_7 \), we write \( \Gamma_{j+1/2}^k = \frac{1}{4} \rho_{j+1/2}^k \left[ (V_j^k)^3 + (V_{j+1}^k)^3 \right] + \frac{1}{4} \delta_{j+1/2}^k \) where

\[
S_{j+1/2}^k = V_j^k V_{j+1}^k \left[ R_{j+1/2}^k - R_{j}^k \right] + (V_j^{k+1} - V_j^k)^2 \left[ \mathcal{F}_j^{k+1} + \mathcal{F}_j^k - \rho_{j+1/2}^k (V_j^k + V_{j+1}^k) \right].
\]

Integration by part w.r.t. space leads to \( T_7 = -T_{7,1} - T_{7,2} \) with

\[
T_{7,1} = \int_0^T \int_0^L \frac{1}{2} \rho_{\delta}^{(0)} (V_\delta)^3 \partial_x^* \varphi_T \, dx \, dt,
\]

\[
T_{7,2} = \frac{1}{4} \sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta t^k \left[ \varphi_j^{k+1} - \varphi_j^k \right].
\]

Finally \( |T_{7,2}| \lesssim \delta x \) since it is dominated by

\[
\delta x |\partial_x \varphi|_{L^\infty} \| V_\delta \|_{L^\infty, T} \left[ C \frac{\partial_x \varphi}{2} \| V_\delta \|_{L^\infty, T} \| \rho_\delta \|_{L^1, BV, T} \right.
\]

\[
+ (2 C \varphi + \| V_\delta \|_{L^\infty, T} \| \rho_\delta \|_{L^1, BV, T}) \| V_\delta \|_{L^1, BV, T}\].
\]

- Internal energy. Multiply (12) by \( \delta t^k \varphi_j^{k+1} \) and sum for \( 0 \leq k \leq N - 1 \) and \( 1 \leq j \leq J \) to get \( T_9 + T_{10} + T_{11} \leq 0 \) with

\[
T_9 = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta x_j \left[ \epsilon_j^{k+1} - \epsilon_j^{k+1/2} \right] \varphi_j^{k+1}, \quad T_{10} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \left[ \mathcal{C}_j^{k+1} - \mathcal{C}_j^k \right] \varphi_j^{k+1/2},
\]

\[
T_{11} = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \pi_j^{k+1/2} (V_j^{k+1} - V_j^k) \varphi_j^{k+1} + \sum_{k=0}^{N-1} \sum_{j=1}^{J} D_j^k \varphi_j^{k+1}.
\]

Owing to integration by part w.r.t. time, we get

\[
T_9 = -\sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta x_j \Phi (\rho_{j+1/2}^k) (\varphi_j^{k+1} - \varphi_j^{k+1/2}) - \sum_{j=1}^{J} \delta x_j \Phi (\rho_{j+1/2}^0) \varphi_{j+1/2}^0
\]

\[
= -\int_0^T \int_0^L \Phi (\rho_{\delta}^{(0)}) \partial_t \varphi_T \, dx \, dt - \int_0^T \Phi (\rho_{\delta}^{(0)} (x, 0)) \varphi_T (x, 0) \, dx.
\]
For $T_{10}$, we rewrite the flux as follows

$$
\begin{align*}
G_j^k &= \frac{1}{2\delta x_j} \left[ \delta x_{j-1/2} \Phi(\rho^k_{j-1/2}) + \delta x_{j+1/2} \Phi(\rho^k_{j+1/2}) \right] V_j^k + U_{1,j}^k + U_{2,j}^k + U_{3,j}^k, \\
U_{1,j}^k &= e_{j-1/2}^k (V_{j+1}^{k+1} - V_j^k), \\
U_{2,j}^k &= -\frac{\delta x_{j+1/2}}{2\delta x_j} \left[ e_{j+1/2}^k - e_{j-1/2}^k \right] V_j^k, \\
U_{3,j}^k &= -\frac{\delta x_{j-1/2}}{2\delta x_j} \left[ \Phi(\rho_{j-1/2}^{k+1}) - \Phi(\rho_{j-1/2}^k) \right].
\end{align*}
$$

It leads to $T_{10} = -T_{10,0} - T_{10,1} - T_{10,2} - T_{10,3}$ with

$$
\begin{align*}
T_{10,0} &= \int_0^T \int_0^L \Phi(\rho^0_\delta) V_\delta \partial_x \varphi \, dx \, dt, \\
T_{10,i} &= \sum_{k=0}^{\infty} \sum_{j=2}^{J} U_{i,j}^k (\varphi_{j+1/2}^{k} - \varphi_{j-1/2}^{k}), \quad i = 1, 2, 3.
\end{align*}
$$

The term $T_{10,1}$ can be bounded as follows

$$|T_{10,1}| \leq C_{\Phi, \rho} |\partial_x \varphi| \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta t^k \sum_{j=2}^{J} \delta x_j \rho_{j-1/2}^k |V_j^{k+1} - V_j^k|. \quad (18)$$

Since $a \leq \min(a, b) + |b - a|$, we get $\rho_{j-1/2}^k \leq \rho_j^k + |\rho_{j+1/2}^k - \rho_{j-1/2}^k|$. This leads to

$$|T_{10,1}| \leq C_{\Phi, \rho} |\partial_x \varphi|_{L^\infty} \left( \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta t^k \sum_{j=2}^{J} \delta x_j \rho_{j-1/2}^k |V_j^{k+1} - V_j^k| + 2||V_\delta||_{1, \tau} ||\rho_\delta||_{1, B, \tau} \delta x \right).$$

Writing $\rho_j^k = \rho_j^{k+1} - (\rho_j^{k+1} - \rho_j^k)$ and using the Cauchy-Schwarz inequality yields

$$T^* \leq 2 (T L ||\rho_\delta||_{1, \tau})^{1/2} \left( \delta t \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \right)^{1/2} + 2 ||V_\delta||_{1, \tau} ||\rho_\delta||_{B, 1, \tau} \delta t \leq \delta t^{1/2}.$$

It finally leads to $|T_{10,1}| \leq \delta t^{1/2} + \delta x$. The term $T_{10,2}$ can be bounded as follows

$$|T_{10,2}| \leq C_{\Phi} ||V_\delta||_{1, \tau} ||\partial_x \varphi||_{L^\infty} ||\rho_\delta||_{1, B, \tau} \delta x \leq \delta x.$$

We now turn to $T_{10,3}$. We remark that

$$|\rho_{j-1/2}^{k+1} - \rho_{j-1/2}^k| \leq \frac{2\delta t^k}{\delta x_{j-1/2}} \left( C_{\rho_\delta, \varphi} \sum_{k} |\rho_{j+1/2}^k - \rho_{j-1/2}^k| + \rho_{j-1/2}^k |V_j^{k+1} - V_j^k| \right).$$

Hence, using the same bound as for $T_{10,1}$ yields

$$|T_{10,3}| \leq C_{\Phi, \rho} |\partial_x \varphi|_{L^\infty} \left( (C_{\rho_\delta, \varphi} + 2 ||V_\delta||_{1, \tau}) ||\rho_\delta||_{1, B, \tau} \delta x + T^* \right) \leq \delta t^{1/2} + \delta x.$$

- Pressure term. It remains to get the limit of $T_8 + T_{11} = -T_{12,0} - T_{12,1} - T_{12,2} - T_{12,3}$, where

$$T_{12,0} = \int_0^T \int_0^L \Phi(\rho^0_\delta) V_\delta \partial_x \varphi \, dx \, dt.$$

The term $T_{12,0}$ can be bounded as follows

$$|T_{12,0}| \leq C_{\Phi, \rho} |\partial_x \varphi|_{L^\infty} \left( \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta t^k \sum_{j=2}^{J} \delta x_j \rho_{j-1/2}^k |V_j^{k+1} - V_j^k| + 2 ||V_\delta||_{1, \tau} ||\rho_\delta||_{1, B, \tau} \delta x \right).$$

Using the Cauchy-Schwarz inequality yields

$$T^* \leq 2 (T L ||\rho_\delta||_{1, \tau})^{1/2} \left( \delta t \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \right)^{1/2} + 2 ||V_\delta||_{1, \tau} ||\rho_\delta||_{B, 1, \tau} \delta t \leq \delta t^{1/2}.$$

It finally leads to $|T_{12,0}| \leq \delta t^{1/2} + \delta x$. The term $T_{12,1}$ can be bounded as follows

$$|T_{12,1}| \leq C_{\Phi} ||V_\delta||_{1, \tau} ||\partial_x \varphi||_{L^\infty} ||\rho_\delta||_{1, B, \tau} \delta x \leq \delta x.$$
with

\[ T_{12,0} = \int_0^T \int_0^L \pi \delta V \delta x \varphi \tau \cdot \mathbf{d}x \mathbf{d}t, \quad T_{12,1} = \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k (\varphi_{j+1/2}^{k+1} - \varphi_j^{k+1}), \]

\[ T_{12,2} = \frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_j^{k+1} - V_j^k + V_{j+1}^{k+1} - V_{j+1}^k) (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}), \]

\[ T_{12,3} = -\frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_{j+1}^k) (2\varphi_{j+1/2}^{k+1} - \varphi_{j+1}^{k+1} - \varphi_j^{k+1}). \]

We bound \( T_{12,1} \) and \( T_{12,3} \) as follows

\[ T_{12,1} \leq \frac{\pi_{\infty}}{2} \left( \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \right) \delta x \leq \delta x, \quad T_{12,3} \leq \frac{C_\pi}{4} \left| \partial_{xx} \varphi \right|_{L^\infty} \| V_\delta \|_{1;BV, \tau^*} (\delta x)^2 \leq (\delta x)^2. \]

Note that \( |\pi_{j+1/2}^{k+1/2}| \leq (C_\varphi + C_{\Phi, \rho}) \rho_{j+1/2}^k \). It readily leads to

\[ |T_{12,2}| \leq (C_\varphi + C_{\Phi, \rho}) |\partial_{xx} \varphi|_{L^\infty} T^* \leq \delta t^{1/2}. \]

With (15), we pass to the limit in \( T_6, T_7, T_9, T_{10,0} \) and \( T_{12,0} \). We arrive at (3) since the other terms tend to 0.

**References**
