Collision of shock waves in Einstein-Maxwell theory with a cosmological constant: A special solution

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Abstract

Post–collision space–times of the Cartesian product form $M' \times M''$, where M' and M'' are 2–dimensional manifolds, are known with M' and M'' having constant curvatures of equal and opposite sign (for the collision of electromagnetic shock waves) or of the same sign (for the collision of gravitational shock waves). We construct here a new explicit post–collision solution of the Einstein–Maxwell vacuum field equations with a cosmological constant for which M' has constant (non–zero) curvature and M'' has zero curvature.

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1 Introduction

The space–time following the head–on collision of two homogeneous, plane, electromagnetic shock waves was found by Bell and Szekeres [1] and is a solution of the vacuum Einstein–Maxwell field equations. The metric tensor is that of a Cartesian product of two 2–dimensional manifolds of equal but opposite sign constant curvatures and is the Bertotti–Robinson ([2], [3]) space–time. Recently we have shown ([5], [6]) that the Nariai–Bertotti ([2], [7]) space–time, with metric that of a Cartesian product of two 2–dimensional manifolds of equal constant curvatures, coincides with the space–time following the head–on collision of two homogeneous, plane, gravitational shock waves and is a solution of Einstein's vacuum field equations with a cosmological constant. We construct here a metric for a space–time which is a Cartesian product of two 2–dimensional manifolds, one having *non–zero constant curvature* and one having *zero curvature*, and show that the metric is: (I) that of the post collision region of space–time following the head–on collision of two plane light–like signals each consisting of combined gravitational and electromagnetic shock waves, one signal specified by a real parameter a and the second signal specified by a real parameter b and (II) is a solution of the vacuum Einstein–Maxwell field equations with a cosmological constant $\Lambda = 2ab$. The appearance of a cosmological constant term on the left hand side of the Einstein field equations is equivalent to the appearance of an energy–momentum–stress tensor for a perfect fluid for which the sum of the matter proper density and the isotropic pressure vanishes. Thus our space–time consists of an anti–collision region which is vacuum and a post–collision region which is non–vacuum in this sense. Vacuum and non–vacuum regions of space–time are familiar from solving the field equations for so–called interior and exterior solutions.

2 Cartesian Product Space–Time

We consider a pseudo–Riemannian space–time M of the form $M = M' \times M''$ where M' is a 2–dimensional manifold of non–zero constant curvature and M'' is a 2–dimensional flat manifold. So that the 4–dimensional manifold M has the correct Lorentzian signature we consider the two cases in which (i) M' is pseudo–Riemannian and M'' is Riemannian and (ii) M' is Riemannian and M'' is pseudo–Riemannian. In either case take ξ, x as local coordinates on M' and η, y as local coordinates on M''. With a, b real constants we take $ab < 0$ for case (i) and write the line element of M as

$$ds^2 = d\xi^2 - \cos^2(2\sqrt{-ab}\xi)\, dx^2 - d\eta^2 - dy^2 \ . \tag{2.1}$$

In terms of the basis 1–forms $\vartheta^1 = d\xi$ and $\vartheta^2 = \cos(2\sqrt{-ab}\xi)\, dx$ the single non–vanishing Riemann curvature tensor component on the dyad defined by this basis, for the manifold M', is

$$R_{1212} = 4ab \ , \tag{2.2}$$

indicating that the pseudo–Riemannian manifold M' has non–zero constant Riemannian curvature (see, for example, [4]) $-4ab > 0$. Clearly the manifold M'' is Riemannian and flat. For case (ii) we take $ab > 0$ and write the line element of M as

$$ds^2 = -d\xi^2 - \cos^2(2\sqrt{ab}\xi)\, dx^2 + d\eta^2 - dy^2 \ . \tag{2.3}$$

Now M' is Riemannian. In terms of the basis 1–forms $\vartheta^1 = d\xi$ and $\vartheta^2 = \cos(2\sqrt{ab}\xi)\, dx$ the non–vanishing component of the Riemann curvature tensor for M', on the dyad defined by the basis 1–forms, is

$$R_{1212} = -4ab \ , \tag{2.4}$$

indicating that the Riemannian manifold M' has non–zero Gaussian curvature $K = -R_{1212} = 4ab > 0$. In this case the manifold M'' is pseudo–Riemannian and flat. Now for case (i) make the transformation

$$\xi = \frac{au - bv}{\sqrt{-2ab}} \ , \quad \eta = \frac{au + bv}{\sqrt{-2ab}} \ , \tag{2.5}$$

while for case (ii) make the transformation

$$\xi = \frac{au - bv}{\sqrt{2ab}} \ , \quad \eta = \frac{au + bv}{\sqrt{2ab}} \ . \tag{2.6}$$

In both cases the line elements (2.1) and (2.3) become

$$ds^2 = -\cos^2\{\sqrt{2}(au - bv)\}dx^2 - dy^2 + 2\, du\, dv \ . \tag{2.7}$$

We can write this line element in the form

$$ds^2 = -(\vartheta^1)^2 - (\vartheta^2)^2 + 2\,\vartheta^3\vartheta^4 = g_{ab}\vartheta^a\vartheta^b \ , \tag{2.8}$$

with the basis 1–forms given, for example, by $\vartheta^1 = \cos\{\sqrt{2}(au - bv)\}dx$, $\vartheta^2 = dy$, $\vartheta^3 = dv$, $\vartheta^4 = du$. Thus the constants g_{ab} are the components of the metric tensor on the half–null tetrad defined via the basis 1–forms. The components R_{ab} of the Ricci tensor on this tetrad vanish except for

$$R_{11} = -4ab \ , \quad R_{33} = -2b^2 \ , \quad R_{34} = 2ab \ , \quad R_{44} = -2a^2 \ . \tag{2.9}$$

With

$$F=\frac{1}{2}F_{ab}\vartheta^a\wedge\vartheta^b=a\,\vartheta^1\wedge\vartheta^4+b\,\vartheta^3\wedge\vartheta^1\ ,\tag{2.10}$$

and $\Lambda=2ab$ we have here a solution of the Einstein–Maxwell vacuum field equations with a cosmological constant:

$$R_{ab}=\Lambda\,g_{ab}+2E_{ab}\ ,\tag{2.11}$$

and

$$dF=0=d^*F\ ,\tag{2.12}$$

where d denotes the exterior derivative, ${}^*F=a\,\vartheta^2\wedge\vartheta^4+b\,\vartheta^2\wedge\vartheta^3$ is the Hodge dual of the Maxwell 2–form (2.10) with components F_{ab} on the tetrad given by (2.10) and $E_{ab}=F_{ac}F_b{}^c-\frac{1}{4}g_{ab}\,F_{cd}F^{cd}$ is the electromagnetic energy–momentum tensor. Tetrad indices are raised with g^{ab} where $g^{ab}g_{bc}=\delta^a_c$. In Newman–Penrose [8] notation, the Weyl tensor has components

$$\Psi_0=b^2\ ,\quad \Psi_1=0\ ,\quad \Psi_2=\frac{1}{3}ab\ ,\quad \Psi_3=0\ ,\quad \Psi_4=a^2\ ,\tag{2.13}$$

which is type D in the Petrov classification and the Maxwell tensor, given by (2.10), has components

$$\Phi_0=b\ ,\quad \Phi_1=0\ ,\quad \Phi_2=a\ .\tag{2.14}$$

3 Collision of Light–Like Signals

To demonstrate that the space–time with line element (2.7) and the Maxwell field (2.10) describe the gravitational and electromagnetic fields following the head–on collision of two homogeneous, plane, light–like signals, each composed of an electromagnetic shock wave accompanied by a gravitational shock wave, we replace u,v in the argument of the cosine in (2.7) by $u_+=u\vartheta(u), v_+=v\vartheta(v)$ where $\vartheta(u)$ is the Heaviside step function which is equal to unity for $u>0$ and is zero for $u<0$ (and similarly for $\vartheta(v)$) so that the line element we now consider reads

$$ds^2=-\cos^2\{\sqrt{2}(au_+-bv_+)\}dx^2-dy^2+2\,du\,dv\ .\tag{3.1}$$

Writing this line element in the form (2.8) with basis 1–forms now given by $\vartheta^1=\cos\{\sqrt{2}(au_+-bv_+)\}dx,\ \vartheta^2=dy,\ \vartheta^3=dv,\ \vartheta^4=du$ we find that the components R_{ab} of the Ricci tensor on the tetrad defined by this basis of 1–forms vanish except for

$$\begin{aligned}R_{11}&=-4ab\,\vartheta(u)\vartheta(v)\ ,\quad R_{33}=b\sqrt{2}\,\delta(v)\tan(\sqrt{2}\,au_+)-2b^2\vartheta(v)\ ,\\ R_{34}&=2ab\,\vartheta(u)\vartheta(v)\ ,\quad R_{44}=a\sqrt{2}\,\delta(u)\tan(\sqrt{2}\,bv_+)-2a^2\vartheta(u)\ .\end{aligned}\tag{3.2}$$

This Ricci tensor can be written in the form

$$R_{ab} = \Lambda\, g_{ab} + 2E_{ab} + S_{ab}\ , \tag{3.3}$$

with $\Lambda = 2\,ab\vartheta(u)\vartheta(v)$, E_{ab} the tetrad components of the electromagnetic energy–momentum tensor calculated with the Maxwell field given by the 2–form

$$F = b\,\vartheta(v)\vartheta^3\wedge\vartheta^1 + a\,\vartheta(u)\vartheta^1\wedge\vartheta^4\ , \tag{3.4}$$

and S_{ab} are the components of the surface stress–energy tensor [11] concentrated on the portions of the null hypersurfaces $u=0, v>0$ and $v=0, u>0$ and given by

$$S_{ab} = b\sqrt{2}\,\delta(v)\,\tan(\sqrt{2}\,au_+)\delta^3_a\delta^3_b + a\sqrt{2}\,\delta(u)\tan(\sqrt{2}\,bv_+)\delta^4_a\delta^4_b\ . \tag{3.5}$$

We emphasize that in the post collision domain $(u>0, v>0)$ the field equations (3.3) can be written in the form

$$R_{ab} - \frac{1}{2}\,g_{ab}\,R = T_{ab} + 2\,E_{ab}\quad\text{with}\quad T_{ab} = -2\,a\,b\,g_{ab}\ , \tag{3.6}$$

where R denotes the Ricci scalar.While the term T_{ab} on the right hand side here has the form of a cosmological constant term it is equivalent to the energy–momentum–stress tensor for a perfect fluid for which the sum of the matter proper density and the isotropic pressure vanishes.

The Newman–Penrose components of the Maxwell field (3.4) are thus

$$\Phi_0 = b\,\vartheta(v)\ ,\quad \Phi_1 = 0\ ,\quad \Phi_2 = a\,\vartheta(u)\ , \tag{3.7}$$

while the Newman–Penrose components of the Weyl tensor are

$$\begin{aligned}
\Psi_0 &= -\frac{1}{\sqrt{2}}b\,\delta(v)\tan(\sqrt{2}\,au_+) + b^2\vartheta(v)\ ,\quad \Psi_1 = 0\ ,\\
\Psi_2 &= \frac{1}{3}ab\,\vartheta(u)\vartheta(v)\ ,\\
\Psi_3 &= 0\ ,\quad \Psi_4 = -\frac{1}{\sqrt{2}}a\,\delta(u)\,\tan(\sqrt{2}\,bv_+) + a^2\vartheta(u)\ .
\end{aligned} \tag{3.8}$$

On account of the appearance of the trigonometric functions in (3.5) and (3.8) we see that the coordinate u has the range $0\le u<\pi/2\sqrt{2}\,a$ on $v=0$ and the coordinate v has the range $0\le v<\pi/2\sqrt{2}\,b$ on $u=0$. Such restrictions are also exhibited in the Bell–Szekeres [1] solution and are discussed in [9].

We are now in a position to interpret physically what these equations are describing. First we consider the region of space–time corresponding to $v < 0$. Now $R_{ab} = 2E_{ab}$ with E_{ab} constructed from the Maxwell field $a\,\vartheta(u)\vartheta^1 \wedge \vartheta^4$. All Newman–Penrose components of the Weyl tensor vanish except $\Psi_4 = a^2\vartheta(u)$. We have here a solution of the vacuum Einstein–Maxwell field equations for $u > 0$ corresponding to an electromagnetic shock wave accompanied by a gravitational shock wave, each having propagation direction $\partial/\partial v$ in the space–time with line element

$$ds^2 = -\cos^2\{\sqrt{2}au_+\}dx^2 - dy^2 + 2\,du\,dv\ . \tag{3.9}$$

The wave amplitudes are simply related via the parameter a, which could be thought of as a form of "fine tuning". We note that the space–time is flat and the fields vanish if, in addition to $v < 0$, we have $u < 0$. A similar situation arises in the region of space–time corresponding to $u < 0$ with the gravitational shock wave described by $\Psi_0 = b^2\,\vartheta(v)$ and the electromagnetic shock wave described by $b\,\vartheta(v)\vartheta^3 \wedge \vartheta^1$, each having now propagation direction $\partial/\partial u$ in the space–time with line element

$$ds^2 = -\cos^2\{\sqrt{2}bv_+\}dx^2 - dy^2 + 2\,du\,dv\ . \tag{3.10}$$

The wave amplitudes are again "fine tuned" via the parameter b. The electromagnetic and gravitational fields are non–vanishing in the region $v > 0$ and vanish in the flat region $v < 0$. After these two light–like signals collide at $u = v = 0$ we obtain the post–collision region of space–time $u \geq 0, v \geq 0$. Clearly the subset $u > 0, v > 0$ is given by the Cartesian product space–time described in Section 2. This space–time includes a cosmological constant which has been considered in some works [10] as a possible candidate for dark energy and appears here as a consequence of the collision. On the boundary $u = 0, v > 0$ of this region we see from (3.5) that there is a light–like shell of matter with this boundary as history in space–time (a 2–plane of matter traveling with the speed of light, for example [11]) and from the last equation in (3.8) there is an impulsive gravitational wave with this boundary as history in space–time. Similarly the boundary $v = 0, u > 0$ is the history in space–time of a light–like shell of matter following from (3.5) and of an impulsive gravitational wave following from the first equation in (3.8). These products of the collision, the light–like shells, the impulsive gravitational waves, the cosmological constant, can be thought of as a complicated redistribution of the energy in the incoming light–like signals. Such phenomena occur in most collisions involving thin shells, impulsive waves and shock waves, and are a consequence of the interactions between matter and the electromagnetic and gravitational fields [11]. Additionally one can have black hole production from the collision of two ultra–relativistic particles [12], the mass inflation phenomenon inside a black hole [13], [14] and the production of radiation from the collision of shock waves [15], [16].

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