# Cyclic covers of affine T-varieties

Charlie Petitjean

# CYCLIC COVERS OF AFFINE $\mathbb{T}$-VARIETIES

CHARLIE PETITJEAN

Abstract. We consider normal affine $\mathbb{T}$-varieties $X$ endowed with an action of finite abelian group $G$ commuting with the action of $\mathbb{T}$. For such varieties we establish the existence of $G$-equivariant geometrico-combinatorial presentations in the sense of Altmann and Hausen. As an application, we determine explicit presentations of the Koras-Russell threefolds as bi-cyclic covers of $\mathbb{A}^3$ equipped with a hyperbolic $\mathbb{C}^*$-action.

## Introduction

Every algebraic action of the one dimensional torus $\mathbb{T} \simeq \mathbb{C}^*$ on a complex affine variety $X$ is determined by a $\mathbb{Z}$-grading $A = \oplus_{m\in\mathbb{Z}} A_m$ of its coordinate ring $A$, the spaces $A_m$ consisting of semi-invariant regular functions of weight $m$ on $X$. One possible way to construct $\mathbb{Z}$-graded algebras, which was studied by Demazure [2], is to start with a variety $Y$ and a $\mathbb{Q}$-divisor $D$ on $Y$ and to let $A = \oplus_{m\in\mathbb{Z}}\Gamma(Y, \mathcal{O}_Y(mD))$. For a well chosen pair $(Y, D)$, this algebra is finitely generated, corresponding to the ring of regular functions of an affine variety, $X = \mathbb{S}(Y, D)$ with a $\mathbb{C}^*$-action whose algebraic quotient is isomorphic to $\mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y))$. A slight variant of this construction [3] already enabled a complete description of $\mathbb{C}^*$-actions on normal surfaces $X$: namely they correspond to graded algebras of the form:

$$A = \oplus_{m<0}\Gamma(Y, \mathcal{O}_Y(mD_-)) \oplus \Gamma(Y, \mathcal{O}_Y) \oplus_{m>0} \Gamma(Y, \mathcal{O}_Y(mD_+)),$$

for suitably chosen triples $(Y, D_+, D_-)$ consisting of a smooth curve $Y$ and a pair of $\mathbb{Q}$-divisors $D_+$ and $D_-$ on it.

Demazure's construction was generalized by Altmann and Hausen [1] to give a description of all normal affine varieties $X$ equipped with an effective action of an algebraic torus $\mathbb{T} \simeq (\mathbb{C}^*)^k$, $k \geq 1$. Here the $\mathbb{Z}$-grading is replaced by a grading by the lattice $M \simeq \mathbb{Z}^k$ of characters of the torus, and the graded pieces are recovered from a datum consisting of a variety $Y$ of dimension $\dim(X) - \dim(\mathbb{T})$ and a so-called polyhedral divisor $\mathcal{D}$ on $Y$, a generalization of $\mathbb{Q}$-divisors for higher dimensional tori: $\mathcal{D}$ can be considered as a collection of $\mathbb{Q}$-divisors $\mathcal{D}(u)$ parametrized by a "weight cone " $\sigma^\vee \cap M$, for which we have $A = \oplus_{u\in\sigma^\vee\cap M}\Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))$. The $\mathbb{T}$-variety associated to a pair $(Y, \mathcal{D})$ is denoted by $\mathbb{S}(Y, \mathcal{D})$.

In this article, we consider affine $\mathbb{T}$-varieties $X$ endowed with an additional action of a finite abelian group $G$ commuting with the action of $\mathbb{T}$. The quotient $X' = X//G$ is again an affine $\mathbb{T}'$-variety for a torus $\mathbb{T}' \simeq \mathbb{T}$ obtained as a quotient of $\mathbb{T}$ by an apporpriate finite group, and our aim is to understand the relation between the presentations $X = \mathbb{S}(Y, \mathcal{D})$ of $X$ and those of $X' = \mathbb{S}(Y', \mathcal{D}')$. A pair $(Y, \mathcal{D})$ such that $X = \mathbb{S}(Y, \mathcal{D})$ is not unique but we will show that it is always possible to choose a particular pair $(Y, \mathcal{D}_G)$ consisting of a variety $Y$ endowed with a $G$-action and a $G$-invariant polyhedral divisor $\mathcal{D}_G$ such that $X$ is $G \times \mathbb{T}$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$. The $G$-invariant divisor $\mathcal{D}_G$ corresponds in turn to a certain polyhedral divisor $\mathcal{D}'$ on the quotient $Y//G$ with property that $X' = \mathbb{S}(Y//G, \mathcal{D}')$ as a $\mathbb{T}'$-variety.

More precisely, our main result reads as follows:

**Theorem.** *Let $X$ be a $\mathbb{T}$-variety and let $G$ be a finite abelian group acting on $X$ such that the two actions commute. Then the following hold:*

*1) There exist a semi-projective variety $Y$ endowed with an action of $G$ and a $G$-invariant pp-divisor $\mathcal{D}_G$ defined on $Y$ such that $X$ is $\mathbb{T} \times G$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*

*2) Moreover $X//G$ is equivariantly isomorphic to the $\mathbb{T}'$-variety $\mathbb{S}(Y//G, \mathcal{D}')$ where $\mathcal{D}'$ can be chosen such that $F_*(\mathcal{D}_G) = \varphi_G^*(\mathcal{D}')$, where $\varphi_G : Y \to Y//G$ denotes the quotient morphism and $F : M^\vee \to M'^\vee$ is a linear map induced by the inclusion between the character lattices $M'$ of $\mathbb{T}'$ and $M$ of $\mathbb{T}$.*

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We then apply this result to determine presentations of a family of exotic affine spaces of dimension 3 with hyperbolic $\mathbb{C}^*$-actions: the Koras-Russell threefolds. We exploit the fact that these threefolds arise as equivariant bi-cyclic cover of the affine space $\mathbb{A}^3$ equipped with a hyperbolic $\mathbb{C}^*$-action.

The article is organized as follows. The first section is devoted to a short recollection on Altmann-Hausen representations, with a particular focus on the methods to construct pairs $(Y, \mathcal{D})$ corresponding to a given graded algebra. The main theorem above is then established in section two. Finally, explicit Altmann-Hausen representations of the Koras-Russell threefolds are determined in section three.

## 1. Recollection on the Altmann-Hausen representation

In this section, we introduce the correspondence between normal affine $\mathbb{T}$-varieties $X$ and pairs $(Y, \mathcal{D})$ composed of a normal semi-projective variety $Y$ and a so-called polyhedral divisor $\mathcal{D}$ established by Altmann-Hausen [1]. In particular, for a given $X$, we summarize a construction of a corresponding $Y$ and explain a method to determine a possible $\mathcal{D}$.

1.1. **Normal affine $\mathbb{T}$-varieties.** Let $N \simeq \mathbb{Z}^k$ be a lattice of rank $k$ and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual. A pointed convex polyhedral cone $\sigma \subseteq N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is an intersection of finitely many closed linear half spaces in $N_{\mathbb{Q}}$ which does not contain any line. Its dual:

$$\sigma^\vee := \{v \in M_{\mathbb{Q}} \mid \forall u \in \sigma \ \langle u, v\rangle \geq 0\} \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q},$$

consists of all linear forms on $M_{\mathbb{Q}}$ that are non-negative on $\sigma$. A polytope $\Pi \subset N_{\mathbb{Q}}$ is the convex hull of finitely many points in $N_{\mathbb{Q}}$, and a convex polyhedron $\Delta \subseteq N_{\mathbb{Q}}$ is the intersection of finitely many closed affine half spaces in $N_{\mathbb{Q}}$. Every polyhedron admits a decomposition: $\Delta = \Pi_\Delta + \sigma$, where $\Pi_\Delta$ is a polytope and $\sigma$ is a pointed convex polyhedral cone, called the tail cone of $\Delta$. The set of all polyhedra which admit the same tail cone is a semigroup with Minkowski addition, which we denote by $\mathrm{Pol}^+_\sigma(N_{\mathbb{Q}})$.

**Definition 1.1.** A *$\sigma$-tailed polyhedral divisor* $\mathcal{D}$ on an algebraic variety $Y$ is a formal finite sum

$$\mathcal{D} = \sum \Delta_i \otimes D_i \in \mathrm{Pol}^+_\sigma(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{WDiv}(Y),$$

where $D_i$ are prime divisors on $Y$ and $\Delta_i$ are $\sigma$-polyhedra.

Every element $u \in \sigma^\vee \cap M$ determines a map $\mathrm{Pol}^+_\sigma(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{WDiv}(Y) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{WDiv}(Y)$ which associates to $\mathcal{D} = \sum \Delta_i \otimes D_i$ the Weil $\mathbb{Q}$-divisor $\mathcal{D}(u) = \sum \min_{v \in \Delta_i} \langle u, v\rangle D_i$ on $Y$.

Given a Weil $\mathbb{Q}$-divisor $D$ and a section $s \in \Gamma(Y, \mathcal{O}_Y(D))$, that is, an effective Weil divisor $D'$ linearly equivalent to the round-down $\lfloor D \rfloor$ of $D$, we denote by $Y_s$ the open subset $Y \setminus \mathrm{Supp}(D')$ of $Y$ .

**Definition 1.2.** ([1, Definition 2.5 and 2.7]) A *proper-polyhedral divisor*, noted pp-divisor, is a polyhedral divisor $\mathcal{D} = \sum \Delta_i \otimes D_i$ on $Y$ which satisfies the following properties:

1) Each $D_i$ is an effective divisor and $\mathcal{D}(u)$ is a $\mathbb{Q}$-Cartier divisor on $Y$ for every $u \in \sigma^\vee \cap M$ .

2) $\mathcal{D}(u)$ is semi-ample for each $u \in \sigma^\vee \cap M$, that is, for some $n \in \mathbb{Z}_{>0}$ the open subsets $Y_s$ , where $s \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(nu)))$, cover $Y$.

3) $\mathcal{D}(u)$ is big for each $u \in \mathrm{relint}(\sigma^\vee) \cap M$, that is, for some $n \in \mathbb{Z}_{>0}$ there exist a section $s \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(nu)))$ such that $Y_s$ is affine.

Recall [1, Definition 2.1] that a variety $Y$ is said to be semi-projective if $\Gamma(Y, \mathcal{O}_Y)$ is finitely generated and $Y$ is projective over $Y_0 = \mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y))$. Given a pp-divisor $\mathcal{D}$ on $Y$, the graded algebra

$$A = \bigoplus_{u \in \sigma^\vee \cap M} A_u = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))).$$

is finitely generated, and $\mathrm{Spec}(A)$ is a $\mathbb{T}$-variety for $\mathbb{T} = \mathrm{Spec}(\mathbb{C}[M]) \simeq (\mathbb{C}^*)^k$. More precisely Altmann and Hausen, showed the following:

**Theorem 1.1.** [1] *For any pp-divisor $\mathcal{D}$ on a normal semi-projective variety $Y$, the scheme*

$$\mathbb{S}(Y, \mathcal{D}) = \mathrm{Spec}\Big(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))\Big)$$

*is a normal affine* $\mathbb{T}$*-variety of dimension* $\dim(Y) + \dim(\mathbb{T})$*. Conversely any normal affine* $\mathbb{T}$*-variety is isomorphic to an* $\mathbb{S}(Y, \mathcal{D})$ *for suitable* $Y$ *and* $\mathcal{D}$*.*

1.2. **Determining the semi-projective variety.** The semi-projective variety $Y$ is not unique, however there exists a natural construction, which we will use in the remainder of the article. It can be summarized as follows ([1, section 6]).

Let $X = \mathrm{Spec}(\bigoplus_{u\in M} A_u)$ be an affine variety endowed with an effective action of the torus $\mathbb{T} = \mathrm{Spec}(\mathbb{C}[M])$. For each $u \in M$ the set of semistable points

$$X^{ss}(u) := \{x \in X \ / \ \exists n \in \mathbb{Z}_{\geq 0} \ and \ f \in A_{nu} \ such \ that \ f(x) \neq 0\}$$

is an open $\mathbb{T}$-invariant subset of $X$ which admits a good $\mathbb{T}$-quotient

$$Y_u = X^{ss}(u)//\mathbb{T} = \mathrm{Proj}_{A_0}(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_{nu}).$$

Following [1, section 6], there exists a fan $\Lambda \in M_{\mathbb{Q}}$ generated by a finite collection of cones $\lambda$ such that the following holds:

1) For any $u$ and $u'$ in the relative interior of $\lambda$, $X^{ss}(u) = X^{ss}(u')$. We denote $W_\lambda = X^{ss}(u)$ for any $u \in relint(\lambda)$

2) If $\gamma$ is a face of $\lambda$, $W_\lambda$ is an open subset of $W_\gamma$. Let $W = \bigcap_{\lambda\in\Lambda} W_\lambda = \underleftarrow{lim} W_\lambda$.

The quotient maps $q_\lambda : W_\lambda \to W_\lambda//\mathbb{T}$ form an inverse system indexed by the cones in $\Lambda$, whose inverse limit exist as a morphism $q : W \longrightarrow Z = \underleftarrow{lim} Y_\lambda$. The desired semi-projective variety $Y$ is the normalization of the closure of the image of $W$ by $q$.

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "q"] & W_\lambda \arrow[r] \arrow[d, "q_\lambda"] & W_\gamma \arrow[r] \arrow[d, "q_\gamma"] & X \arrow[dd, "q_0"] \\
Z \arrow[r] \arrow[drrr] & Y_\lambda \arrow[r] \arrow[drr] & Y_\gamma \arrow[dr] & \\
 & & & Y_0 = \mathrm{Spec}(A_0)
\end{tikzcd}$$

1.3. **Maps of proper polyhedral divisor.** Let $Y$ and $Y'$ be normal semi-projective varieties, $N$ and $N'$ be lattices and $\sigma \subset N_{\mathbb{Q}}$, $\sigma' \subset N'_{\mathbb{Q}}$ be pointed cones. Let $\mathcal{D} = \sum \Delta_i \otimes D_i$ and $\mathcal{D}' = \sum \Delta'_i \otimes D'_i$ be pp-divisors on $Y$ and $Y'$ respectively with corresponding tail cones $\sigma$ and $\sigma'$.

**Definition 1.3.** [1, Definition 8.3 ]1) For a morphism $\varphi : Y \to Y'$ such that $\varphi(Y)$ is not contained in $\mathrm{Supp}(D'_i)$ for any $i$, the polyhedral pull-back of $\mathcal{D}'$ is defined by :

$$\varphi^*(\mathcal{D}') := \sum \Delta'_i \otimes \varphi^*(D'_i)$$

Where $\varphi^*(D'_i)$ is the usual pull-back of $D'_i$. It is a polyhedral divisor on $Y$ with tail cone $\sigma'$.

2) For a linear map $F : N \to N'$ such that $F(\sigma) \subset \sigma'$, the polyhedral push forward is defined as :

$$F_*(\mathcal{D}) := \sum (F(\Delta_i) + \sigma') \otimes D_i$$

It is also a polyhedral divisor on $Y$ with tail cone $\sigma'$.

An equivariant morphism from $\mathbb{S}(Y, \mathcal{D})$ to $\mathbb{S}(Y', \mathcal{D}')$ is given by a homomorphism of algebraic groups $\psi : \mathbb{T} \to \mathbb{T}'$ and a morphism $\phi : \mathbb{S}(Y, \mathcal{D}) \to \mathbb{S}(Y', \mathcal{D}')$ satisfying $\phi(\lambda.x) = \psi(\lambda).\phi(x)$. Every such morphism is uniquely determined by a triple $(\varphi, F, f)$ defined as above consisting of a dominant morphism $\varphi : Y \to Y'$, a linear map $F : N \to N'$ as above and a *plurifunction* $f \in N' \otimes_{\mathbb{Z}} \mathbb{C}(Y)^*$ such that :

$$\varphi^*(\mathcal{D}') \leq F_*(\mathcal{D}) + \mathrm{div}(f).$$

The identity map of a pp-divisor is the triple $(\mathrm{id}, \mathrm{id}, 1)$ and the composition of two maps $(\varphi, F, f)$ and $(\varphi', F', f')$ is $(\varphi' \circ \varphi, F' \circ F, F'_*(f).\varphi^*(f'))$.

1.4. **Determining proper polyhedral divisors.** A method to determine a possible pp-divisor $\mathcal{D}$ ([1, section 11]) associated to a $\mathbb{T}$-variety $X$ with $\mathbb{T} = (\mathbb{C}^*)^k$ is to embed $X$ as a $\mathbb{T}$-stable subvariety of a toric variety. The calculation is then reduced to the toric case by considering an embedding in $\mathbb{A}^m$ with linear action for $m$ sufficiently large. In other words, $X$ is realized as a $(\mathbb{C}^*)^k$-stable subvariety of a $(\mathbb{C}^*)^m$-toric variety. The inclusion of $(\mathbb{C}^*)^k$ corresponds to an inclusion of the lattice of characters $\mathbb{Z}^k$ of $\mathbb{T}$ into $\mathbb{Z}^m$. We obtain the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^k \arrow[r, "F"'] & \mathbb{Z}^m \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^m/\mathbb{Z}^k \arrow[r] & 0 \end{tikzcd},$$

where $F$ is given by the action of $(\mathbb{C}^*)^k$ on $\mathbb{A}^m$ and $s$ is a section of $F$. The $(\mathbb{C}^*)^m$-toric variety is determined by the first integral vectors $v_i$ of the unidimensional cone generated by the i-th column vector of $P$ as rays in a $\mathbb{Z}^m$ lattice, and each $v_i$ correspond to a divisor. The support of $D_i$ is the intersection between $X$ and the divisor corresponding to $v_i$. The tail cone is $\sigma := s(\mathbb{Q}^m_{\geq 0} \cap F(\mathbb{Q}))$, and the polytopes are $\Pi_i = s(\mathbb{R}^m_{\geq 0} \cap P^{-1}(v_i))$.

## 2. Actions of finite abelian groups

Let $X = \mathrm{Spec}(A)$ be a normal affine variety with an effective action of a torus $\mathbb{T}$ and let $G$ be a finite abelian group of order $d \geq 2$ whose action on $X$ commutes with that of $\mathbb{T}$. The goal of this section is to determine the relationship between the Altmann-Hausen representations of $X$ and those of $X//G = \mathrm{Spec}(A^G)$.

Let $Y$ be a semi-projective variety equipped with an action $\psi : G \times Y \to Y$ of an algebraic group $G$. If $\mathcal{D}_G$ is a $G$-invariant pp-divisor, i.e a pp-divisor such that $\psi(g, \cdot)^*\mathcal{D}_G = \mathcal{D}_G$ for every $g \in G$, then for every $u \in \sigma^\vee \cap M$ the space $A_u = \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))$ of global sections $\mathcal{O}(\mathcal{D}_G(u))$ is endowed with a $G$-action. It follows that $\mathbb{S}(Y, \mathcal{D}_G) = \mathrm{Spec}(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))))$ admits an action of $G$ commuting with that of $\mathbb{T}$.

**Theorem 2.1.** *Let $X$ be a $\mathbb{T}$-variety and let $G$ be a finite abelian group acting on $X$ such that the two actions commute. Then the following hold:*

*1) There exist a semi-projective variety $Y$ endowed with an action of $G$ and a $G$-invariant pp-divisor $\mathcal{D}_G$ on $Y$ such that $X$ is $\mathbb{T} \times G$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*

*2) Moreover $X//G$ is equivariantly isomorphic to the $\mathbb{T}'$-variety $\mathbb{S}(Y//G, \mathcal{D}')$ where $\mathcal{D}'$ can be chosen such that $F_*(\mathcal{D}_G) = \varphi_G^*(\mathcal{D}')$, where $\varphi_G : Y \to Y//G$ denotes the quotient morphism and $F : M^\vee \to M'^\vee$ is a linear map induced by the inclusion between the character lattices $M'$ of $\mathbb{T}'$ and $M$ of $\mathbb{T}$ (see 2.3 ).*

We will divide the proof in several steps. First we will prove that the action of $G$ on $X$ induces an action of $G$ on $Y$. Secondly we will consider the case where the orbits of the $G$-action are included in the orbits of the $\mathbb{T}$-action and finally we consider the case where the action of $G \times \mathbb{T}$ is effective on $X$.

**Lemma 2.1.** *Let $Y$ a quasi-projective variety endowed with an action of a finite group $G$ and let $\widehat{Y} \to Y$ be the normalization of $Y$. Then the action of $G$ lifts to an action on $\widehat{Y}$ and the induced morphism $\widehat{Y}//G \to Y//G$ is the normalization of $Y//G$.*

*Proof.* Since $Y$ is quasi-projective and $G$ is finite, every $x \in X$ admits a $G$-invariant affine open neighborhood. The normalization being a local operation, we may assume that $Y$ is affine. Using the universal properties of the normalization and of the quotient we obtain the following commutative diagram:

$$\begin{array}{ccc} Y & \to & Y//G \\ \uparrow & & \uparrow \\ \widehat{Y} & \to & \widehat{Y//G} \\ & \searrow & \uparrow \\ & & \widehat{Y}//G \end{array}$$

Thus $\mathbb{C}[\widehat{Y//G}] \subset \mathbb{C}[\widehat{Y}]^G$. Conversely, let $f \in \mathbb{C}[\widehat{Y}]^G$. Then $g.f = f$ for all $g$ in $G$ and there exists a monic polynomial $P$ with coefficients in $\mathbb{C}[Y]$ such that $P(f) = 0$. Since $G$ is finite, $Q = \prod_{g \in G} g.P$ is a monic polynomial with $G$-invariant coefficients and $G(f) = 0$. So $f \in \mathbb{C}[\widehat{Y//G}]$ . $\square$

**Corollary 2.1.** *Let $X$ be a $\mathbb{T}$-variety and suppose that a finite abelian group $G$ acts on $X$ such the two actions commute. Then there exists a semi-projective variety $Y$ and a pp-divisor $\mathcal{D}$ on $Y$ such that $X$ is $G \times \mathbb{T}$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D})$ and the action of $G$ on $\mathbb{S}(Y, \mathcal{D})$ induces an action of $G$ on $Y$.*

*Proof.* We consider the construction of $Y$ given in section 1.2. Since the action of $G$ and $\mathbb{T}$ commute, for every $\lambda \in \Lambda$ the subset $X^{ss}(u)$ with $u \in relint(\lambda)$ is $G$-stable. Thus $W := \cap_{\lambda\in\Lambda} W_\lambda$ is also $G$-stable. Since $q' : W \to Z$ is the quotient by $\mathbb{T}$, the action of $G$ on $W$ induces one on $q'(W)$. The closure $\overline{q'(W)}$ is again $G$-stable, and since $\overline{q'(W)}$ is quasi-projective it follows from lemma 2.1 that the action of $G$ lifts to an action on $Y$. □

**Lemma 2.2.** *Let $X = \operatorname{Spec}(A)$ be a $\mathbb{T}$-variety and let $G$ a finite abelian group acting on $X$ such that the two actions commute. Then there exists a $G$-invariant pp-divisor $\mathcal{D}_G$ defined on $Y$ such that $X$ is equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*

*Proof.* By lemma 2.1 the action of $G$ on $X$ induces an action of $G$ on $Y$. By the proof of Theorem 3.4 in [1], a pp-divisor on $Y$ corresponding to $X$ is determined by the choice of a homomorphism $h$ from $M$ into the fraction field of $A$ with the property that for every $u \in M$, $h(u)$ is semi-invariant of weight $u$. Namely, if $u \in \sigma^\vee \cap M$ is any *saturated element*, that is, $u \in \sigma^\vee \cap M$ such that $\bigoplus_{n\in\mathbb{N}} A_{nu}$ is generated in degree 1, then there exist a unique Cartier divisor $\mathcal{D}(u)$ such that $A_u = h(u).\Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))$: its local equations on open subsets $Y_s$ with $s \in A_u$ are $h(u)/s$. By definition $h(u) = \frac{f}{g}$ where $f$ and $g$ are both non zero and $f \in A_{u_1}$, $g \in A_{u_2}$ such that $u_1 - u_2 = u$. Since $A_u$ is $G$-stable for all $u \in M$, we can choose $f \in A_{u_1}$, $g \in A_{u_2}$ semi-invariant for the action of $G$ with $u_1 - u_2 = u$ so that $h(u) = f/g$ is also semi-invariant for $G$. The corresponding divisor $\mathcal{D}(u)$ is then $G$-invariant. In the case of a general $u \in \sigma^\vee \cap M$, we can choose a saturated multiple $nu$ and define $\mathcal{D}(u) = \mathcal{D}(nu)/n$. □

To complete the proof of Theorem 2.1, we divide the argument into two cases. First we consider the situation where $G$ is a subgroup of $\mathbb{T}$ and secondly where the action of $G \times \mathbb{T}$ is effective.

**Lemma 2.3.** *Let $X$ be the $\mathbb{T}$-variety $\mathbb{S}(Y, \mathcal{D})$ and let $G$ be a finite abelian subgroup of $\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M])$. Then $X' = X//G$ is a $\mathbb{T}'$-variety where $\mathbb{T}' \simeq \mathbb{T}//G$ and is equivariantly isomorphic to the $\mathbb{S}(Y, F_*(\mathcal{D}))$ where $F : N = M^\vee \to N' = (M')^\vee$ is the linear map induced by the inclusion between the character lattices $M'$ of $\mathbb{T}'$ and $M$ of $\mathbb{T}$.*

*Proof.* Let $Y$ be as in 1.2. Since by hypothesis the $G$-orbits are contained in $\mathbb{T}$-orbits, the induced $G$-action on $Y$ is trivial. In this case, for each $u \in \sigma^\vee \cap M$, $A_u^G$ is either $A_u$ or $\{0\}$. Letting $M'$ be the sublattice $M$ generated by the elements $u \in \sigma^\vee \cap M$ such that $A_u^G \neq 0$,

$$X' = X//G = \operatorname{Spec}(\bigoplus_{u\in\sigma^\vee\cap M'} A_u^G)$$

is a $\mathbb{T}'$-variety where $\mathbb{T}' = \operatorname{Spec}(\mathbb{C}[M'])$ is a torus of the same dimension as $\mathbb{T}$. The inclusion $M' \hookrightarrow M$ gives rise the desired linear map $F : N = M^\vee \to N' = M'^\vee$. □

*Remark* 2.1. This case corresponds to the map of pp-divisors $(\mathrm{id}, F, 1)$ defined in as 1.3. Indeed the quotient morphism $\varphi : Y \to Y//G$ is the identity.

**Lemma 2.4.** *Let $X$ be a normal affine variety with an effective action of $G \times \mathbb{T}$ where $G$ is a finite abelian group. Then there exists a semi-projective variety $Y$ on which $G$ acts and a $G$-invariant pp-divisor $\mathcal{D}_G$ on $Y$ such that $X$ is $G \times \mathbb{T}$-equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*

*Moreover $X//G$ is $\mathbb{T}$-equivariantly isomorphic to $\mathbb{S}(Y//G, \mathcal{D}')$ where $\mathcal{D}_G = \varphi_G^*(\mathcal{D}')$.*

*Proof.* By lemmas 2.1 and 2.2, $Y$ is endowed with an action of $G$, and we can assume that $X$ is equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$. Since $\mathcal{D}_G$ is $G$-stable, for each $u \in \sigma^\vee \cap M$ , $\Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))$ is a $G$-invariant submodule of $\Gamma(X, \mathcal{O}_X)$ and moreover there exists $\mathcal{D}'$ satisfies $\varphi_G^*(\mathcal{D}') = \mathcal{D}_G$. Therefore, $\Gamma(X//G, \mathcal{O}_{X//G}) = (\bigoplus_{u\in\sigma^\vee\cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u))))^G = \bigoplus_{u\in\sigma^\vee\cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))^G$.

By assumption, $\varphi : Y \to Y//G$ is the quotient morphism, and $\mathcal{D}'$ satisfying $\varphi_G^*(\mathcal{D}') = \mathcal{D}_G$. Thus

$$\begin{aligned}
\Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))^G &= \left\{f \in \mathbb{C}(Y)^G, div(f) + \mathcal{D}_G(u) \geq 0\right\} \cup \{0\} \\
&= \{h \in \mathbb{C}(Y//G), \varphi^*(\operatorname{div}(h) + \mathcal{D}'(u)) \geq 0\} \cup \{0\} \\
&= \{h \in \mathbb{C}(Y//G), \operatorname{div}(h) + \mathcal{D}'(u) \geq 0\} \cup \{0\}.
\end{aligned}$$

We conclude that $X//G \simeq \operatorname{Spec}(\bigoplus_{u\in\sigma^\vee\cap M} \Gamma(Y//G, \mathcal{O}(\mathcal{D}'(u))))$. □

*Remark* 2.2. This lemma is the analogue of 4.1 in [2], in which Demazure established a similar result for algebras constructed from $\mathbb{Q}$-divisors. This case corresponds to the map of proper polyhedral divisors $(\varphi_G, \mathrm{id}, 1)$ defined as in 1.3.

*Proof.* (of Theorem 2.1) Consider a finite abelian group $G$ acting on $X = \mathbb{S}(Y, \mathcal{D})$ whose action commutes with that of $\mathbb{T}$. By virtue of lemmas 2.1 and 2.2, we may assume that $G$ acts on $Y$ and that $\mathcal{D}$ is $G$-invariant. Then we let $H$ be the subgroup of $G \times \mathbb{T}$ consisting of elements which act trivially on $X$. We let $G_0 \subset G$ and $\mathbb{T}_0 \subset \mathbb{T}$ be the images of $H$ by the two projections and we let $G' = G/G_0$ and $\mathbb{T}' = \mathbb{T}/\mathbb{T}_0$. Applying lemma 2.3 to $X$ equipped with the action of $G_0$, we obtain a variety $X//G_0$ endowed with an effective action of $G' \times \mathbb{T}'$ to which the lemma 2.4 can be applied. Any map $(\varphi_G, F, 1)$ is obtained by composing maps of the two types above. □

## 3. Applications in the case $\mathbb{T} = \mathbb{C}^*$

3.1. **Basic examples of $\mathbb{C}^*$-actions.** The coordinate ring of a normal affine variety $X = \operatorname{Spec}(A)$ equipped with an effective $\mathbb{C}^*$-action is $\mathbb{Z}$-graded in a natural way via $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where $A_n := \{f \in A/f(\lambda\cdot x) = \lambda^n f(x)\}$. The semi-projective variety associated to the Altmann-Hausen representation of $X$ is the irreducible component which correspond to the normalization of the closure of the image of $W$ by $q'$ (see 1.2) in the fiber product :

$$Y(X) := Y_-(X) \underset{Y_0(X)}{\times} Y_+(X)$$

where $Y_0(X) = X//\mathbb{C}^* = \operatorname{Spec}(A_0)$ , $Y_\pm(X) = \operatorname{Proj}_{A_0}(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_{\pm n})$.

A $\mathbb{C}^*$-action said to be *hyperbolic* if there is at least one $n_1 < 0$ and one $n_2 > 0$ such that $A_{n_1}$ and $A_{n_2}$ are nonzero. In this case, the tail cone $\sigma$ is equal to $\{0\}$ (see 1.4 ). If in addition $X$ is smooth, then $Y(X)$ is in fact equal to the fiber product which is itself isomorphic to the blow-up of $Y_0(X)$ with center at th closed suscheme defined by the ideal $\mathcal{I} = \langle A_d.A_{-d}\rangle$ where $d > 0$ is chosen so that $\bigoplus_{n\in\mathbb{Z}} A_{dn}$ is generated by $A_0$ and $A_{\pm d}$ ( [8] Theorem 1.9 and proposition 1.4).

In what follows, we denote by $\pi : \tilde{\mathbb{A}}^n_{(I)} \to \mathbb{A}^n$ the blow-up of the ideal $(I)$ in $\mathbb{A}^n_{(x_1,\dots,x_n)} = \operatorname{Spec}(\mathbb{C}[x_1, ..., x_n])$ .

Given an irreducible and reduced hypersurface $H = \{f(x_1, ..., x_n) = 0\} \subset \mathbb{A}^n$ containing the origin, the hypersurface $X_{n,p,f}$ of $\mathbb{A}^{n+2} = \operatorname{Spec}(\mathbb{C}[x_1, ..., x_n][y, t])$ defined by the equation

$$\frac{f(x_1 y, ..., x_n y)}{y} + t^p = 0$$

comes equipped with an effective $\mathbb{C}^*$-action induced by the linear one $\lambda\cdot(x_1, ..., x_n, y, t) = (\lambda^p x_1, ..., \lambda^p x_n, \lambda^{-p} y, \lambda t)$ on $\mathbb{A}^{n+2}$. We have $\mathbb{A}^{n+2}//\mathbb{C}^* \simeq \mathbb{A}^{n+1} = \operatorname{Spec}(\mathbb{C}[u_1, ..., u_{n+1}])$ via $u_i = x_i y$ for $i = 1, ..., n$ and $u_{n+1} = yt^p$.

**Proposition 3.1.** *The variety $X_{n,p,f}$ is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^n_{(u_1,\dots,u_n)}, \mathcal{D})$ for with $\mathcal{D} = \left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}]E$, where $E$ is the exceptional divisor of the blow up and $D$ is the strict transform of the hypersurface $H \subset \mathbb{A}^n$.*

*Proof.* We determine $Y(X_{n,p,f})$ and the pp-divisor $\mathcal{D}$ using the method described in sections 1.2 and 1.3. We consider the exact sequence :

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & \mathbb{Z}^{n+2} \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^{n+1} \arrow[r] & 0 \end{tikzcd}$$

where $F = {}^t(p, ..., p, -p, 1)$, $P = \begin{pmatrix} & & & 1 & 0 \\ & \mathrm{I}_n & & \vdots & \vdots \\ & & & 1 & 0 \\ 0 & \cdots & 0 & 1 & p \end{pmatrix}$ $\mathrm{I}_n$ being the identity matrix of rank $n \times n$ and $s = (0, ..., 0, 1)$.

The fan in $\mathbb{Z}^{n+2}$ is generated by the rays $\{v_i\}_{i=1,..,n+2}$ where $v_i$ is the first integral vector of the unidimensional cone generated by the i-th column vector of $P$. It corresponds to the blow up of the origin in $\mathbb{A}^{n+1}$, as a toric variety.

The variety $Y$ is equal to the strict transform by $\pi : \tilde{\mathbb{A}}^{n+1}_{(u_1,...,u_n)} \to \mathbb{A}^{n+1} \simeq \mathbb{A}^{n+2}//\mathbb{C}^*$ of $\{f(u_1,...,u_n) + u_{n+1} = 0\} \subset \mathbb{A}^{n+1}$, thus $Y \simeq \tilde{\mathbb{A}}^n$.

Since $\sigma := s(\mathbb{Q}^m_{\geq 0} \cap F(\mathbb{Q}))$ is $\{0\}$, applying the formula $\Pi_i = s(\mathbb{R}^m_{\geq 0} \cap P^{-1}(v_i))$, we deduce that $\mathcal{D}$ has the form $\left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}]E$, where $D$ corresponds to the restriction to $Y$ of the toric divisor given by the ray $v_{n+2}$. It is the restriction of $\{u_{n+1} = yt^p = 0\}$ to $Y$ thus $D$ is the strict transforms of the hypersurface $H \subset \mathbb{A}^n$. The divisor $E$ corresponds to the restriction to $Y$ of the toric divisor given by $v_{n+1}$, that is, the exceptional divisor. □

**Example 3.1.** Specializing the above construction we obtain examples of linear hyperbolic $\mathbb{C}^*$-actions on $\mathbb{A}^3$ which will be building blocks for further applications :

a) Choosing $n = 2$ and $f(x_1, x_2) = x_1$, we obtain that $X_{2,x_1,p}$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ with $\mathcal{D} = \left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}]E$, where $E$ is the exceptional divisor of the blow up and $D$ is the strict transform of the line $\{u = 0\} \subset \mathbb{A}^2$. Thus $X_{2,x_1,p} \subset \mathbb{A}^4$ is isomorphic to $\mathbb{A}^3$ equipped with the $\mathbb{C}^*$-action : $\lambda \cdot (x_2, y, t) = (\lambda^p x_2, \lambda^{-p} y, \lambda t)$.

b) In particular, if $p = 1$ then $X_{2,x_1,1}$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ with $\mathcal{D} = \{1\} D + [0, 1]E$. Since $\mathcal{D} = \{1\} D + [0, 1]E$ is equivalent to $\mathcal{D}' = [-1, 0]E$, we have that $X_{2,x_1,1}$ is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D}')$.

**Example 3.2.** Choosing $n = 2$ and $f(x_1, x_2) = x_1 + (x_1^d + x_2^d)^l$ yields that

$$X_{2,p,f} = \{x_1 + y^{dl-1}(x_1^d + x_2^d)^l + t^p = 0\} \subset \mathbb{A}^4$$

is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D} = \left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}]E)$ , where $E$ is the exceptional divisor of the blow up and $D$ is the strict transform of the curve $\{v + (v^d + u^d)^l = 0\} \subset \mathbb{A}^2$. Note that in contrast with the previous example, $X_{2,p,f}$ is not isomorphic to $\mathbb{A}^3$. Indeed, if it were, then by the result of Koras-Russell [7], the $\mathbb{C}^*$ action on $X_{2,p,f}$ would be linearizable. By considering the linear action induced on the tangent space of the fixed point, we find that $X_{2,p,f}$ would have to be equivariantly isomorphic to $X_{2,x_1,p}$ for some $p$. On the other hand it follows from [1, corollary 8.12] that two pp-divisors $\mathcal{D}_i$, defined on $Y_i$ respectively with the same tail cone, define equivariantly isomorphic varieties $\mathbb{S}(Y_i, \mathcal{D}_i)$ if and only if there exist projective birational morphisms $\psi_i : Y_i \to Y$ and a pp-divisor $\mathcal{D}$ on $Y$ such that $\mathcal{D}_i \simeq \psi_i^*(\mathcal{D})$ $i = 1, 2$. This would induce an automorphism $\phi$ of $\tilde{\mathbb{A}}^2$, such that $\phi^*(f) = x_1$, which is not possible, since a general fiber of $f$ is singular.

3.2. **Koras-Russell threefolds.** Smooth affine, contractible threefolds with a hyperbolic $\mathbb{C}^*$-action whose quotient is isomorphic to $\mathbb{A}^2/G$ where $G$ is a finite cyclic group have been classified by Koras and Russell [5], in the context of the linearization problem for $\mathbb{C}^*$-actions on $\mathbb{A}^3$ [7]. These threefolds, which we call Koras-Russell threefolds, provide examples of $\mathbb{T}$-varieties of complexity two. According to [6] they admit the following description:

Let $a'$,$b'$ and $c'$ be pairwise prime natural numbers with $b' \geq c'$ and let $\mu_{a'}$, the group of $a'$-th roots of unity, act on $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$ by $(u, v) \to (\lambda^{c'} u, \lambda^{b'} v)$ where $\lambda \in \mu_{a'}$. Consider a semi-invariant polynomial $f$ of weight congruent to $b'$ modulo $a'$ and with the property that $L = \{f = 0\}$ is isomorphic to a line and meets the axis $u = 0$ transversely at the origin and at $r - 1 \geq 1$ other points. With these assumptions the polynomial $s^{-c'} f(s^{c'} u, s^{b'} v)$ can be rewritten in the form $F(w, u, v)$ with $w = s^{a'}$where $F$ is semi-invariant of weight $b'$ for the $\mathbb{C}^*$-action $(w, u, v) \longmapsto (\lambda^{-a'} w, \lambda^{c'} u, \lambda^{b'} v)$. Then for any choice of pairwise prime integers $(\alpha_1, \alpha_2, \alpha_3)$ such that $\gcd(\alpha_1, a') = \gcd(\alpha_2, b') = \gcd(\alpha_2, c') = 1$, the hypersurface $X = \{(x, y, z, t) \in \mathbb{A}^4/t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x) = 0\}$ is a Koras-Russell threefold.

Here we mainly consider two families of such threefolds:

1) The first kind is defined by equations of the form:

$$\{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\},$$

where $2 \leq d$, $2 \leq \alpha_2 < \alpha_3$ with $\gcd(\alpha_2, \alpha_3) = 1$ and equipped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(\alpha_2\alpha_3, -(d-1)\alpha_2\alpha_3, \alpha_3, \alpha_2)$. These correspond to the choice of $f = u + v + v^d$.

2) The second type is defined by

$$\{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\},$$

where $2 \leq d$, $1 \leq l$ , $2 \leq \alpha_2 < \alpha_3$ with $\gcd(\alpha_2, d) = \gcd(\alpha_2, \alpha_3) = 1$ and equipped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(\alpha_2\alpha_3, -(dl-1)\alpha_2\alpha_3, d\alpha_3, \alpha_2)$. These correspond to the choice of $f = v + (u + v^d)^l$.

To obtain the Altmann-Hausen representation for these threefolds, we will exploit the fact that they arise as $\mathbb{C}^*$-equivariant bi-cyclic covers of $\mathbb{A}^3$. We will see that the polyhedral coefficients are related with the choice of $(\alpha_1, \alpha_2, \alpha_3)$ and the divisors are related with the choice of the fiber $L = \{f = 0\}$ in the construction above.

3.3. **The Russell Cubic.** We begin with the Russell cubic $X = \{x + x^2y + z^2 + t^3 = 0\}$ in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ which corresponds to the choice $a' = b' = c' = 1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and $f(u, v) = u + v + v^2$ in the construction above. By construction $X$ is equiped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(6, -6, 3, 2)$. The algebraic quotient $X/\!/\mathbb{C}^*$ is isomorphic to $\mathbb{A}^2_{(u,v)} = \mathrm{Spec}(\mathbb{C}[u, v])$ where $u = yz^2$ and $v = yx$.

**Proposition 3.2.** *(see also* [4]*) The Russell Cubic $X$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ for*

$$\mathcal{D} = \left\{\frac{1}{2}\right\} D_3 + \left\{-\frac{1}{3}\right\} D_2 + \left[0, \frac{1}{6}\right] E,$$

*where $E$ is the exceptional divisor of $\pi : \tilde{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2$, and where $D_2$ and $D_3$ are the strict transforms of the curves $\{u = 0\}$ and $\{u + v + v^2 = 0\}$ in $\mathbb{A}^2$ respectively.*

*Proof.* The two projections $\Phi_2 = \mathrm{pr}_{x,y,t} : X \to X_2 = \mathbb{A}^3$ and $\Phi_3 = \mathrm{pr}_{x,y,z} : X \to X_3 = \mathbb{A}^3$ express $X$ as cyclic Galois covers of $\mathbb{A}^3$ of degrees 2 and 3 respectively, whose Galois groups $\mu_2$ and $\mu_3$ act on $X$ by $\xi \cdot (x, y, z, t) = (x, y, \xi z, t)$ and $\zeta \cdot (x, y, z, t) = (x, y, z, \zeta t)$ respectively. Furthermore these two actions commute and the quotient $X_6 = X/\!/(\mu_2 \times \mu_3)$ is isomorphic to $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z^2])$. Letting $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be the coordinate ring of $X$ equipped with the grading corresponding to the given $\mathbb{C}^*$-action, we have in fact $X_\ell = \mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}} A_{\ell n})$, $\ell = 2, 3, 6$. This yields a $\mathbb{C}^*$-equivariant commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\Phi_2"'] \arrow[dr, "\Phi_3"] \arrow[dd, "\Phi_6"] & \\
X_2 = X/\!/\mu_2 \arrow[dr] & & X_3 = X/\!/\mu_3 \arrow[dl] \\
 & X_6 = X/\!/(\mu_2 \times \mu_3) &
\end{tikzcd}$$

where $\mathbb{C}^*$ acts linearly on $X_2$, $X_3$ and $X_6$ with weights $(3, -3, 1)$, $(2, -2, 1)$ and $(1, -1, 1)$ respectively.

Furthermore since the action of $\mu_2 \times \mu_3$ on $X$ factors through that of $\mathbb{C}^*$ we deduce from Theorem 2.1 that $\Phi_2$ corresponds to the map of proper polyhedral divisors $(\mathrm{id}, F_2, 1)$ and $\Phi_3$ corresponds to the map of proper polyhedral divisors $(\mathrm{id}, F_3, 1)$ where $F_\ell^*(\mathcal{D}) = \ell\mathcal{D}$, $\ell = 2, 3, 6$. The semi-projective varieties $Y(X)$ and $Y(X_\ell)$, $\ell = 2, 3, 6$ are all isomorphic. As observed earlier, $A_0 = \mathbb{C}[u, v]$ with $u = yz$ and $v = yx$ so that $Y_0(X_6) \simeq Y_0(X) = \mathbb{A}^2_{(u,v)}$. We further observe that $A_{-6n} = A_0 \cdot y^n \subset A$ because all semi-invariant polynomials of negative weights divisible by 6 are divisible by $y$. This implies that $Y_-(X_6) \simeq \mathrm{Proj}(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_0.y^n) \simeq Y_0(X)$. Finally, $\bigoplus_{n\in\geq 0} A_{6n} \simeq \mathrm{Sym}_{A_0} A_6$ where $A_6$ is the free $A_0$-submodule of $A$ generated by $x$ and $z$. Therefore

$$Y(X) \simeq Y(X_6) \quad = \quad Y_-(X_6) \times_{Y_0(X_6)} Y_+(X_6) \simeq Y_+(X_6)$$

is isomorphic to the blow-up $\tilde{\mathbb{A}}^2_{(u,v)}$ of $Y_0(X) = \mathbb{A}^2$ at the origin. It remains to determine the pp-divisor $\mathcal{D}$. We will construct it from those $\mathcal{D}_2$ and $\mathcal{D}_3$ corresponding to $X_2$ and $X_3$ respectively.

By Proposition 3.1, $X_2 = \mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D}_2 = \{\frac{1}{3}\}D_2 + [0, \frac{1}{3}]E)$ where $D_2$ is the strict transform of the curve $\{u = 0\}$ and $E$ is the exceptional divisor and $X_3 = \mathbb{S}(\tilde{\mathbb{A}}^2_{(u',v)}, \mathcal{D}_3 = \{\frac{1}{2}\}D_3 + [0, \frac{1}{2}]E)$ where $D_3$ is the strict transform of the curve $\{u' = 0\}$ and $E$ is the exceptional divisor. Theorem 2.1 implies in turn that $2\mathcal{D} \sim \mathcal{D}_2 = \{\frac{1}{3}\}D_2 + [0, \frac{1}{3}]E$ and $3\mathcal{D} \sim \mathcal{D}_3 = \{\frac{1}{2}\}D_3 + [0, \frac{1}{2}]E$. Thus $\mathcal{D}_2 + \mathcal{D} = \mathcal{D}_3$ and we conclude that $\mathcal{D} = \left\{\frac{1}{2}\right\} D_3 + \left\{-\frac{1}{3}\right\} D_2 + \left[0, \frac{1}{6}\right] E$ . □

*Remark.* The choice of the coefficients is not unique since $\mathcal{D}' \sim \mathcal{D} + \mathrm{div}(f)$ for any rational function $f$ on $Y$. This corresponds for example to $\mathcal{D}' \sim \mathcal{D} + D_3 + E$ and more generally for any pair $(a, b) \in \mathbb{Z}^2$ such that $3a + 2b = 1$ we have that $\mathcal{D} \sim \left\{\frac{a}{2}\right\} D_3 + \left\{\frac{b}{3}\right\} D_2 + \left[0, \frac{1}{6}\right] E$.

3.4. **Koras Russell threefolds of the first kind.** Now we will show that a similar method can be used to present all Koras-Russell threefolds of the form $X = \{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$. Namely, we consider a cyclic cover $V$ of $X$ with algebraic quotient $V//\mathbb{C}^*$ isomorphic to $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$ where $u = yz^{\alpha_2}$ and $v = yx$. A representation of $V$ is obtained by the same method as in the previous case and the representation of $X$ is deduced by applying again Theorem 2.1.

The categorical quotient $X//\mathbb{C}^*$ is isomorphic to $\mathbb{A}^2_{(u,v)}//\mu_{d-1}$ where $\mu_{d-1}$ acts by $\xi \cdot (u, v) = (\xi u, \xi v)$. So we consider $V$ a finite cyclic cover of $X$ given by the equation $X = \{x + x^d y^{d-1} + z^{\alpha_2} + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$, equipped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(\alpha_2\alpha_3, -\alpha_2\alpha_3, \alpha_3, \alpha_2)$. Furthermore $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{d-1}$ acts on $V$ by $(\zeta, \epsilon, \xi) \cdot (x, y, z, t) \to (x, \xi y, \zeta z, \epsilon t)$. Observe that the action of $\mu_{\alpha_2} \times \mu_{\alpha_3}$ factors through that of $\mathbb{C}^*$. This yields the following diagram of quotient morphisms:

$$\begin{tikzcd} & V \arrow[dl, "\Phi_{\alpha_2}"'] \arrow[d, "\Phi_{\mu_{d-1}}"'] \arrow[dr, "\Phi_{\alpha_3}"] & \\ \mathbb{A}^3 \simeq V//\mu_{\alpha_2} & X = V//\mu_{d-1} & \mathbb{A}^3 \simeq V//\mu_{\alpha_3}. \end{tikzcd}$$

By Theorem 2.1, $\Phi_{\alpha_2}$ corresponds to the map of proper polyhedral divisors $(\mathrm{id}, F_{\alpha_2}, 1)$ and $\Phi_{\alpha_3}$ corresponds to the map of proper polyhedral divisor $(\mathrm{id}, F_{\alpha_3}, 1)$ where $F^*_\ell(\mathcal{D}) = \ell\mathcal{D}$, $\ell = 2, 3, 6$. In addition we obtain that $Y(V)$ is isomorphic to the blow-up $\tilde{\mathbb{A}}^2_{(u,v)}$ of $\mathbb{A}^2$ at the origin on which $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{d-1}$ acts by $(\zeta, \epsilon, \xi) \cdot (u, v) = (\xi u, \xi v)$. This leads to the following diagram:

$$\begin{tikzcd} & Y(V) \arrow[dl, "\simeq"'] \arrow[d, "\varphi_{\mu_{d-1}}"'] \arrow[dr, "\simeq"] & \\ Y(V_{\alpha_2}) & Y(X) \simeq Y(V)//\mu_{d-1} & Y(V_{\alpha_3}). \end{tikzcd}$$

Using example 3.1 we obtain Altmann-Hausen representations of $V//\mu_{\alpha_2}$ and $V//\mu_{\alpha_3}$ in the form $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D}_{\alpha_2} = \{\frac{1}{\alpha_3}\}D_{\alpha_2} + [0, \frac{1}{\alpha_3}]E)$ where $D_{\alpha_2}$ is the strict transform of the curve $\{u = 0\}$, $E$ is the exceptional divisor and $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u',v)}, \mathcal{D}_{\alpha_3} = \{\frac{1}{\alpha_2}\}D_{\alpha_3} + [0, \frac{1}{\alpha_2}]E)$ where $D_{\alpha_3}$ is the strict transform of the curve $\{u' = 0\}$, $E$ is the exceptional divisor. This implies that $V$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ for

$$\mathcal{D} = \left\{\frac{a}{\alpha_2}\right\} D_{\alpha_3} + \left\{\frac{b}{\alpha_3}\right\} D_{\alpha_2} + \left[0, \frac{1}{\alpha_2\alpha_3}\right] E \ (*),$$

where $E$ is the exceptional divisor of $\pi : \tilde{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2$, $D_{\alpha_2}$ and $D_{\alpha_3}$ are the strict transforms of the curves $\{u = 0\}$ and $\left\{u + v + v^d = 0\right\}$ in $\mathbb{A}^2_{(u,v)}$ respectively, and $(a, b) \in \mathbb{Z}^2$ are chosen such that $a\alpha_3 + b\alpha_2 = 1$. Applying Theorem 2.1 we obtain

**Proposition 3.3.** *The Koras-Russell threefold* $X = \{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\}$ *in* $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ *is isomorphic to* $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}//\mu_{d-1}, \mathcal{D}')$ *for*

$$\mathcal{D}' = \left\{\frac{a}{\alpha_2}\right\} D'_{\alpha_3} + \left\{\frac{b}{\alpha_3}\right\} D'_{\alpha_2} + \left[0, \frac{1}{(d-1)\alpha_2\alpha_3}\right] E'$$

*where* $\mathcal{D} = \varphi^*_{\mu_{d-1}}(\mathcal{D}')$ , $\mathcal{D}$ *is defined in the relation* $(*)$ *and* $D'_{\alpha_3}$,$D'_{\alpha_2}$ *are prime divisors and* $E'$ *is the exceptional divisor of the blow-up of the singularity in* $\mathbb{A}^2//\mu_{d-1}$.

3.5. **Koras Russell threefolds of the second kind.** For Koras-Russell threefolds of the second kind $X=\{x+y(x^d+z^{\alpha_2})^l+t^{\alpha_3}=0\}$ in $\mathbb{A}^4=\mathrm{Spec}(\mathbb{C}[x,y,z,t])$ the construction will be slightly different due to the fact that the variables $z$ and $t$ do no longer play symmetric roles. We will consider again a cyclic cover $V$ of $X$, but in this case $V//\mu_{\alpha_2}$ will not be isomorphic to $\mathbb{A}^3$. Recall that by definition, $\alpha_2$ and $d$ are coprime. We consider a bi-cyclic cover $V=\{x+y^{dl-1}(x^d+z^{d\alpha_2})^l+t^{\alpha_3}=0\}$ of $X$ of order $d\times(dl-1)$, which we decompose as a cyclic cover $\phi_d: V\to V_d=\{x+y^{dl-1}(x^d+z^{\alpha_2})^l+t^{\alpha_3}=0\}$ of degree $d$, followed by a cyclic cover $\phi_{dl-1}: V_d\to X$ of degree $dl-1$. The hypersurface $V$ is equipped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(\alpha_2\alpha_3,-\alpha_2\alpha_3,\alpha_3,\alpha_2)$ and with the action of $\mu_{\alpha_2}\times\mu_{\alpha_3}\times\mu_{dl-1}\times\mu_d$ defined by $(\zeta,\epsilon,\xi,\delta)\cdot(x,y,z,t)=(x,\xi y,\zeta\delta z,\epsilon t)$. The action of $\mu_{\alpha_2}\times\mu_{\alpha_3}$ on $V$ factors through that of $\mathbb{C}^*$ and we obtain the following diagram:

$$\begin{tikzcd}
 & V \arrow[dl, "\Phi_{\alpha_2}"'] \arrow[d, "\Phi_{\mu_d}"] \arrow[dr, "\Phi_{\alpha_3}"] & \\
V_{\alpha_2}=V//\mu_{\alpha_2} & V_d=V//\mu_d \arrow[d, "\Phi_{\mu_{dl-1}}"] & \mathbb{A}^3\simeq V_{\alpha_3}=V//\mu_{\alpha_3} \\
 & X=V//(\mu_d\times\mu_{dl-1}) & 
\end{tikzcd}.$$

By Theorem 2.1, considering $\Phi_{\alpha_3}$, we obtain that $Y(V)$ is isomorphic to the blow-up $\tilde{\mathbb{A}}^2_{(u,v)}$ of $\mathbb{A}^2$ where $u=yz^{\alpha_2}$ and $v=yx$ on which $\mu_{\alpha_2}\times\mu_{\alpha_3}\times\mu_{dl-1}\times\mu_d$ acts by $(\zeta,\epsilon,\xi,\delta)\cdot(u,v)=(\xi\delta^{\alpha_2}u,\xi v)$. We obtain the following quotient diagram:

$$\begin{tikzcd}
 & Y(V) \arrow[dl, "\simeq"'] \arrow[d, "\varphi_{\mu_d}"] \arrow[dr, "\simeq"] & \\
Y(V_{\alpha_2}) & Y(V_d) \arrow[d, "\varphi_{\mu_{dl-1}}"] & Y(V_{\alpha_3}) \\
 & Y(X) & 
\end{tikzcd}.$$

Now by Proposition 3.1 $V_{\alpha_2}=\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\mathcal{D}_{\alpha_2}=\{\frac{1}{\alpha_3}\}D_{\alpha_2}+[0,\frac{1}{\alpha_3}]E)$ where $D_{\alpha_2}$ is the strict transform of the curve $\{v+(v^d+u^d)^l)=0\}$ and $E$ is the exceptional divisor, and $V_{\alpha_3}=\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\mathcal{D}_{\alpha_3}=\{\frac{1}{\alpha_2}\}D_{\alpha_3}+[0,\frac{1}{\alpha_2}]E)$ where $D_{\alpha_3}$ is the strict transform of the curve $\{u=0\}$ and $E$ is the exceptional divisor. Thus $V=\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\mathcal{D})$ for

$$\mathcal{D}=\left\{\frac{a}{\alpha_2}\right\}D_{\alpha_3}+\left\{\frac{b}{\alpha_3}\right\}D_{\alpha_2}+\left[0,\frac{1}{\alpha_2\alpha_3}\right]E,$$

where $E$ is the exceptional divisor of $\pi:\tilde{\mathbb{A}}^2_{(u,v)}\to\mathbb{A}^2$, and where $D_{\alpha_2}$ and $D_{\alpha_3}$ are the respective strict transforms of the curves $\{v+(v^d+u^d)^l)=0\}$ and $\{u=0\}$ in $\mathbb{A}^2$ and $(a,b)\in\mathbb{Z}^2$ $a\alpha_3+b\alpha_2=1$. Note that the choice of $\mathcal{D}$ up to linear equivalence does not depend of the choice on $(a,b)\in\mathbb{Z}^2$ .

Now we deduce from Theorem 2.1 that $V_d=\mathbb{S}(\tilde{\mathbb{A}}^2_{(u',v'^d)},\mathcal{D}_d)$ for

$$\mathcal{D}_d=\left\{\frac{a'}{\alpha_2}\right\}D_{d,\alpha_3}+\left\{\frac{b'}{\alpha_3}\right\}D_{d,\alpha_2}+\left[0,\frac{1}{\alpha_2\alpha_3}\right]E_d\ (**),$$

where $a'=a/d$, $b'=b$, $E_d$ is the exceptional divisor of $\pi:\tilde{\mathbb{A}}^2_{(u',v'^d)}\to\mathbb{A}^2$ due to the fact that $\tilde{\mathbb{A}}^2_{(u',v')}//\mu_d\simeq\tilde{\mathbb{A}}^2_{(u',v'^d)}$ for the action of $\mu_d$ as above, and where $D_{d,\alpha_2}$ and $D_{d,\alpha_3}$ are the strict transforms of the curves $\{v'+(u'+v'^d)^l)=0\}$ and $\{u'=0\}$ ( $u'=\varphi_d(u^d)$) in $\mathbb{A}^2=\mathrm{Spec}(\mathbb{C}[u',v'])$ respectively. Applying again Theorem 2.1 we obtain :

**Proposition 3.4.** *A Koras-Russell threefold $X=\{x+y(x^d+z^{\alpha_2})^l+t^{\alpha_3}=0\}$ in $\mathbb{A}^4=\mathrm{Spec}(\mathbb{C}[x,y,z,t])$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u',v'^d)}//\mu_{dl-1},\mathcal{D}_{d(dl-1)})$ for*

$$\mathcal{D}_{d(dl-1)} = \left\{\frac{a'}{\alpha_2}\right\} D_{d(dl-1),\alpha_3} + \left\{\frac{b'}{\alpha_3}\right\} D_{d(dl-1),\alpha_2} + \left[0, \frac{1}{(dl-1)\alpha_2\alpha_3}\right] E_{d(dl-1)},$$

*where* $\mathcal{D}_d = \varphi^*_{\mu_{dl-1}}(\mathcal{D}_{d(dl-1)})$ , $\mathcal{D}_d$ *is defined in the relation* $(**)$ *and* $E_{d(dl-1)}$ *is the exceptional divisor of the blow-up of the singularity in* $\mathbb{A}^2//\mu_{dl-1}$.

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Charlie Petitjean, Institut de MathÃ©matiques de Bourgogne, UniversitÃ© de Bourgogne, 9 Avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France

*E-mail address*: charlie.petitjean@u-bourgogne.fr