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Federico Della Croce, Bruno Escoffier, Vangelis Th. Paschos
Improved worst-case complexity for the MIN 3-SET COVERING problem*

Federico Della Croce1 Bruno Escoffier2 Vangelis Th. Paschos2

1 D.A.I., Politecnico di Torino, Italy, federico.dellacroce@polito.it
2 LAMSADE, CNRS UMR 7024 and Université Paris-Dauphine, France
{escoffier,paschos}@lamsade.dauphine.fr

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Abstract

We consider min set covering when the subsets are constrained to have maximum cardinality three. We propose an exact algorithm whose worst case complexity is bounded above by $O^*(1.3957^n)$. This is an improvement, based on a refined analysis, of a former result ($O^*(1.4492^n)$) by F. Della Croce and V. Th. Paschos, Computing optimal solutions for the min 3-set covering problem, Proc. ISAAC’05, LNCS 3827, pp. 685–692.

Keywords: Worst-case complexity, Exact algorithm, min set covering

In min set covering, we are given a universe $U$ of elements and a collection $S$ of (non-empty) subsets of $U$. The aim is to determine a minimum cardinality sub-collection $S' \subseteq S$ which covers $U$, i.e., $\cup_{S \in S'} S = U$ (we assume that $S$ covers $U$). The frequency $f_i$ of $u_i \in U$ is the number of subsets $S_j \in S$ in which $u_i$ is contained. The cardinality $d_j$ of $S_j \in S$ is the number of elements $u_i \in U$ that $S_j$ contains. We say that $S_j$ hits $S_k$ if both $S_j$ and $S_k$ contain an element $u_i$ and that $S_j$ double-hits $S_k$ if both $S_j$ and $S_k$ contain at least two elements $u_i, u_l$. Finally, we denote by $n$ the size (cardinality) of $S$ and by $m$ the size of $U$. In what follows, we restrict ourselves to min set covering-instances such that:

1. no element $u_i \in U$ has frequency $f_i = 1$;
2. no set $S_i \in S$ is a subset of another set $S_j \in S$.
3. no pair of elements $u_i, u_j$ exists such that every subset $S_i \in S$ containing $u_i$ contains also $u_j$.

Indeed, if item 1 is not verified, then the set containing $u_i$ belongs to any feasible cover of $U$. On the other hand, if item 2 is not verified, then $S_i$ can be replaced by $S_j$ in any solution containing $S_i$ and the resulting cover will not be worse than the one containing $S_i$. Finally, if item 3 is not verified, then element $u_j$ can be ignored as any sub-collection $S'$ covering $u_i$ will necessarily cover also $u_j$. So, for any instance of min set covering, a preprocessing of data, obviously performed in polynomial time, leads to instances where all items 1, 2 and 3 are verified.

Let $T(\cdot)$ be a super-polynomial and $p(\cdot)$ be a polynomial, both on integers. In what follows, using notations in [9], for an integer $n$, we express running-time bounds of the form $p(n).T(n)$ as $O^*(T(n))$, the asterisk meaning that we ignore polynomial factors. We denote by $T(n)$ the

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worst case time required to exactly solve the \textsc{min set covering} problem with \(n\) subsets. We recall (see, for instance, [5]) that, if it is possible to bound above \(T(n)\) by a recurrence expression of the type \(T(n) \leq \sum T(n-r_i) + O(p(n))\), we have \(T(n) = O^*(\alpha(r_1, r_2, \ldots)^n)\) where \(\alpha(r_1, r_2, \ldots)\) is the largest zero of the function \(f(x) = 1 - \sum x^{-r_i}\).

There exist to our knowledge few results on worst-case complexity of exact algorithms for \textsc{min set covering} or for cardinality-constrained versions of it. Let us note that an exhaustive algorithm computes any solution for most recent non-trivial result is the one of [6] (that has improved the result of [8]) deriving a \textsc{min set covering} \(\leq\) set (hence this set will have size 2 is the remaining instance), we consider that our benefit is size 3, then our benefit is 1. When we do not fix a set of size 3 but cover one element of this

Proof. We only prove item 1, items 2 and 3 being proved by the same kind of analysis. Assume, without loss of generality, that \(S_j\) hits \(S_k\) on \(u_i\) and \(S_j\) on \(u_p\). Suppose by contradiction that the optimal solution \(S'\) includes \(S_j\) and \(S_k\). Then, it cannot include no more \(S_i\), or else, it would not be optimal as a better cover would be obtained by excluding \(S_j\) from \(S'\). On the other hand, suppose that \(S'\) includes \(S_j\), \(S_k\) but does not include \(S_i\). Then, an equivalent optimal solution can be derived by swapping \(S_j\) with \(S_i\).

In what follows, we consider the following counting. When we fix the status of a set of size 3, then our benefit is 1. When we do not fix a set of size 3 but cover one element of this set (hence this set will have size 2 is the remaining instance), we consider that our benefit is \(\alpha \leq 1\). Obviously, when a set of size 2 is fixed, we can only consider that (in the worst case) our benefit is \(1 - \alpha\). Hence, in some cases, the benefit is increasing with \(\alpha\) while, in other cases, it is decreasing. An optimal value for \(\alpha\), following our analysis, is \(\alpha = 0.297\).

The rest of the paper is devoted to the proof of the following result.

\textbf{Theorem 1.} \textsc{min 3-set covering} can be optimally solved within time \(O^*(1.396^n)\).
The algorithm either reduces the \textit{MIN 3-SET COVERING} instance according to assumptions 1, 2 and 3 on the form of the instance (by detecting a subset \(S_j\) to be immediately included in (excluded from) \(S'\) or an element \(u_t\) to be ignored (correspondingly reducing the size of several subsets)), or applies a branching on subset \(S_j\), where the following exhaustive relevant branching cases may occur.

1. \(d_j = 2\): then no double-hitting occurs to \(S_j\) or else, due to the preprocessing step of the algorithm, \(S_j\) can be excluded from \(S'\) without branching. The following subcases occur.

(a) \(S_j\) contains elements \(u_i, u_k\) with \(f_i = f_k = 2\) where \(S_j\) hits \(S_l\) on \(u_i\) and \(S_m\) on \(u_k\).

Due to Lemma 1, if \(S_j\) is included in \(S'\), then both \(S_l\) and \(S_m\) must be excluded from \(S'\); alternatively, \(S_j\) is excluded from \(S'\) and, correspondingly, both \(S_l\) and \(S_m\) must be included in \(S'\) to cover elements \(u_i, u_k\). For the analysis, consider the two following cases.

i. \(d_l = 3\), or \(d_m = 3\), say \(d_l = 3\). Then, in both cases (including or excluding \(S_j\)) we fix \(3 - 2\alpha\) (1 for \(S_l\), (at least) \(1 - \alpha\) for \(S_j\) and \(S_m\)).

ii. \(d_l = d_m = 2\), \(S_j\) contains \(u_i\) and \(u_k\) and \(S_m\) contains \(u_i\) and \(u_m\), (with \(u_t \neq u_m\), otherwise no need to branch). By including \(S_j\) we fix \(3(1 - \alpha)\). Otherwise, \(u_i\) is contained in \(S_p\) and \(u_m\) in \(S_q\). If \(S_p \neq S_q\), then we fix at least \(3(1 - \alpha) + 2\alpha = 3 - \alpha\). Indeed, we fix \(1 - \alpha\) for any of the sets \(S_j, S_l, S_m\); by covering \(u_m\), we fix \(\alpha\) (resp., \(1 - \alpha \geq \alpha\) if \(d_p = 3\) (resp., if \(d_p = 2\), since we can exclude \(S_p\)), and the same holds for covering \(u_i\). Note that this is still valid if \(S_p = S_q\), since in this case we can exclude this set, which gives at least \(1 - \alpha \geq 2\alpha\).

In case 1(a)ii, we have \(T(n) \leq 2T(n - 3 + 2\alpha) + O(p(n))\). This results in a time-complexity of \(O^*(1.334^n)\). In case 1(a)iii, we have \(T(n) \leq T(n - 3 + 3\alpha) + T(n - 3 + \alpha) + O(p(n))\). This results in a time-complexity of \(O^*(1.336^n)\).

(b) \(S_j\) contains elements \(u_i, u_k\) with \(f_i = 2\) and \(f_k \geq 3\), where \(S_j\) hits \(S_l\) on \(u_i\) and \(S_m, S_p\) on \(u_k\). Due to Lemma 1, if \(S_j\) is included in \(S'\), then \(S_l, S_m, S_p\) must be excluded from \(S'\); alternatively, \(S_j\) is excluded from \(S'\) and, correspondingly, \(S_l\) must be included in \(S'\) to cover element \(u_i\). For the analysis, consider the two following cases.

i. \(d_l = 2\), i.e., \(S_l\) contains \(u_i, u_l\); then, \(f_l \geq 3\) (or else we are in case 1a). Then, by including \(S_j\), we fix \(4(1 - \alpha)\) \((1 - \alpha)\) for any of the sets \(S_j, S_l, S_m, S_p\); by excluding \(S_j\), we fix \(2(1 - \alpha) + 2\alpha = 2(1 - \alpha)\) for any of the sets \(S_j, S_l, S_i\), and (at least) \(\alpha\) for each set containing \(u_i\).

ii. If \(d_l \geq 3\), i.e., \(S_l\) contains at least \(u_i, u_l, u_m\), then by including \(S_j\), we fix \(3(1 - \alpha) + 1\) (since now fixing \(S_l\) gives benefit 1); by excluding \(S_j\), we fix \((1 - \alpha) + 1 + 2\alpha = 2 + \alpha\) \((\alpha\) from covering \(u_i\), \(\alpha\) from covering \(u_m\), with the same reasoning as in case 1(a)ii).

The worst case is 1(b)i where we get \(T(n) \leq T(n - 2) + T(n - 4 + 4\alpha) + O(p(n))\), resulting in a time-complexity of \(O^*(1.338^n)\).

(c) \(S_j\) contains elements \(u_i, u_k\) with \(f_i = 3\) and \(f_k \geq 3\) where \(S_j\) hits \(S_l, S_m\) on \(u_i\) and (at least) \(S_p, S_q\) on \(u_k\). Note that we can suppose that \(S_j\) hits at least one set of size 3. Due to Lemma 1, if \(S_j\) is included in \(S'\), then \(S_l, S_m, S_p, S_q\) must be excluded from \(S'\); alternatively, \(S_j\) is excluded from \(S'\). For the analysis, consider the three following cases.

i. If \(d_l = d_m = d_p = d_q = 3\), then we fix either \(5 - \alpha\), or \(1 - \alpha\).
ii. If \( d_l = 2 \) or \( d_m = 2 \), say \( d_l = 2 \), then we fix either \( 5 - 4\alpha \), or \( 1 - \alpha \). But in the case where we exclude \( S_j \) from \( S' \), then \( S_l \) has size 2 and contains \( u_i \), whose frequency is now 2. Hence, we are either in case 1a or in case 1b. In the worst case, the branching gives (with case 1(b)i) \( 5 - 4\alpha, 5(1 - \alpha) \) and \( 3 - \alpha \).

iii. Finally, if \( d_l = d_m = 3 \), then we can suppose that \( f_k \geq 4 \) (otherwise we are either in case 1(c)i or in case 1(c)ii). In this case, by including \( S_j \) we fix \( 2 + 4(1 - \alpha) \) and by excluding \( S_j \) we fix \( 1 - \alpha \).

In case 1(c)i, we get \( T(n) \leq T(n - 1 + \alpha) + T(n - 5 + 5\alpha) + O(p(n)) \), i.e., a time-complexity of \( O^*(1.3953^n) \). In case 1(c)ii, we get \( T(n) \leq T(n - 3 + \alpha) + T(n - 5 + 4\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3942^n) \). In case 1(c)iii, we get \( T(n) \leq T(n - 6 + 4\alpha) + T(n - 1 + \alpha) + O(p(n)) \), i.e., a time-complexity of \( O^*(1.389^n) \).

(d) \( S_j \) contains elements \( u_i, u_k \) with \( f_i \geq 4 \) and \( f_k \geq 4 \) where \( S_j \) hits \( S_l, S_m, S_p \) on \( u_i \) and \( S_q, S_r, S_s \) on \( u_k \). Note that we can suppose that \( S_j \) hits at least one set of size 3. Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p, S_q, S_r, S_s \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). Then, we fix either \( 7 - 6\alpha \) or \( 1 - \alpha \) getting \( T(n) \leq T(n - 1 + \alpha) + T(n - 7 + 6\alpha) + O(p(n)) \), resulting so in a time-complexity of \( O^*(1.366^n) \).

2. \( d_j = 3 \) (that is, there does not exist \( S_k \in S \) such that \( d_k = 2 \)) and there is at least one element \( u_i \) with \( f_i = 2 \). Then, \( S_j \) contains \( u_i, u_j, u_k \), and \( S_k \) contains \( u_i, u_k, u_m \) (notice that no double crossing can occur between \( S_j \) and \( S_k \) due to the preprocessing step of the algorithm). Then, either we include \( S_j \), and we fix \( 1 + 3\alpha \) new sets, or we exclude \( S_j \), and we have to include \( S_k \) fixing so \( 2 + 2\alpha \) new sets. In this case, we get \( T(n) \leq T(n - 1 - 3\alpha) + T(n - 2 - 2\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.366^n) \).

3. \( d_j = 3 \), all elements have a frequency at least 3, with \( S_j \) double-hitting one or more subsets. The following exhaustive subcases may occur.

(a) \( S_j \) double-hits at least three subsets \( S_k, S_l, S_m \). Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_k, S_l, S_m \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). This can be seen as a binary branching where either one subset \( (S_j) \) is fixed, or four subsets \( (S_j, S_k, S_l, S_m) \) are fixed and hence, \( T(n) \leq T(n - 1) + T(n - 4) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3803^n) \).

(b) \( S_j \) double-hits two subsets \( S_k, S_l \). Note that the double-hit elements must be contained by another set. Note also that (at least) one element, say \( u_i \), is in \( S_j, S_k \) and \( S_l \). Consider the two following cases.

i. If \( f_i \geq 4 \), then either we include \( S_j \) and then, by Lemma 1, we can exclude \( S_k \) and \( S_l \), or we exclude \( S_j \). Then, either we fix \( 3 + 3\alpha \) (for \( S_j, S_k, S_l \), and \( 3\alpha \) since \( u_i, u_j \) and \( u_k \) belong to at least one other set) or 1.

ii. If \( f_i = 3 \), then we must include at least one set among \( S_j, S_k, S_l \), but we can suppose that we do not include two such sets. In other words, we have a branching on the three following choices:

- taking \( S_j \) (and not \( S_k, S_l \)),
- taking \( S_k \) (and not \( S_j, S_l \)),
- taking \( S_l \) (and not \( S_j, S_k \)).

In any case, we fix \( 3 + 2\alpha \) (\( 2\alpha \) since each element has a frequency at least 3)
In the first case, \( T(n) \leq T(n - 1) + T(n - 3 - 3\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.388^n) \). In the second case, \( T(n) \leq 3T(n - 3 - 2\alpha) + O(p(n)) \), and this results in a time-complexity of \( O^*(1.358^n) \).

(c) \( S_j \) contains elements \( u_i, u_k, u_l \) and double-hits one subset \( S_k \) on elements \( u_i, u_k \). The following exhaustive subcases must be considered.

i. \( f_l = 3, f_k \geq 3, f_l \geq 3 \), with \( u_i \) contained by \( S_j, S_k, S_m, u_k \) contained at least by \( S_j, S_k, S_p \), and \( u_l \) contained at least by \( S_j, S_q, S_r \). A composite branching can be devised:
   - Suppose that \( S_j \) is included in \( S' \) and then \( S_k \) is excluded from \( S' \). In this case, we fix \( 2 + 4\alpha \) (\( \alpha \) from reduction of the sizes of \( S_m, S_p, S_q, S_r \)).
   - Suppose that \( S_j \) is excluded from \( S' \) and \( S_k \) is included in \( S' \). In this case, we fix \( 2 + 4\alpha \) (since no other double hit occurs on \( S_k \)).
   - Suppose finally that \( S_j \) and \( S_k \) are excluded from \( S' \). In this case, we have to include \( S_m \) in \( S' \). Since \( d_m = 3 \), all elements have frequency at least \( 3 \), and at most one double crossing occurs on \( S_m \); we can see that \( S_m \) hits at least three new sets. Hence, we fix \( 3 + 3\alpha \).

ii. \( f_l \geq 4, f_k \geq 4, f_l \geq 3 \), with \( u_i \) contained at least by \( S_j, S_k, S_m, S_p, u_k \) contained at least by \( S_j, S_k, S_q, S_r, u_l \) contained at least by \( S_j, S_q, S_r, S_t, S_u, S_v \). Either we include \( S_j \) in \( S' \), and then we can exclude \( S_k \) from \( S' \) and fix \( 2 + 6\alpha \), or we exclude \( S_j \) and fix 1.

In case 3(c)i, we get \( T(n) \leq 2T(n - 2 - 4\alpha) + T(n - 3 - 3\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.381^n) \). In case 3(c)ii, we get \( T(n) \leq T(n - 1) + T(n - 2 - 6\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3957^n) \).

4. \( d_j = 3 \) and no double-hitting occurs to \( S_j \) (nor to any other subset) that contains elements \( u_i, u_k, u_l \). The following subcases occur.

(a) \( f_l = 3, f_k \geq 3, f_l \geq 3 \) with \( u_i \) contained by \( S_j, S_k, S_l, u_k \) contained by \( S_j, S_m, S_p \), and \( u_l \) contained at least by \( S_j, S_q, S_r \). A composite branching can be devised:
   - if \( S_j \) is included in \( S' \), then we fix 1 + 6\alpha new sets;
   - if \( S_j \) is excluded from \( S' \) and \( S_k \) is included in \( S' \), then there exist at least five other subsets hitting \( S_k \) and hence we fix 2 + 5\alpha;
   - finally, if \( S_j, S_k \) are excluded from \( S' \), then we have to include \( S_l \) in \( S' \) (in order to cover \( u_i \)); there exist at least four other subsets hitting \( S_l \) and hence we fix \( 3 + 4\alpha \).

Thus, \( T(n) \leq T(n - 1 - 6\alpha) + T(n - 2 - 5\alpha) + T(n - 3 - 4\alpha) + O(p(n)) \), resulting in a time-complexity of \( O^*(1.378^n) \).

(b) \( f_l \geq 4, f_k \geq 4, f_l \geq 4 \), \( u_i \) is contained by \( S_j, S_k, S_l, S_m, u_k \) is contained by \( S_j, S_p, S_q, S_r \), and \( u_l \) is contained at least by \( S_j, S_t, S_u, S_v \). A composite branching on \( S_j \) can be devised:
   - if \( S_j \) is excluded from \( S' \), then we fix 1;
   - if \( S_j \) is included in \( S' \), then \( S_k, S_l, S_m \) are excluded from \( S' \); in this case we fix \( 4 + 6\alpha \);
   - finally, if \( S_j \) is included in \( S' \), then \( S_p, S_q, S_r, S_t, S_u, S_w \) are excluded from \( S' \); in this case we fix \( 7 + 3\alpha \).

Hence, \( T(n) \leq T(n - 1) + T(n - 4 - 6\alpha) + T(n - 7 - 3\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.355^n) \).
Putting things together, the global worst case complexity is \( O^*(1.3957^n) \) and the proof of the theorem is complete.

As a last word, let us note that a straightforward (improvable) analysis along the lines of Theorem 1, leads to an \( O^*(1.1679^n) \) time bound for minimum vertex covering in graphs of maximum size 3. Such a bound is the best-known dealing with search tree-based algorithms and is only dominated by the bounds in [1, 3], \((O^*(1.1252^n) \) and \( O^*(1.152^n) \), respectively) that are not based upon such algorithms. Note also, dealing with minimum dominating set in graphs of maximum size 3, analysis along the same lines reaches \( O^*(1.344^n) \), which is always the best-known search-tree complexity.

References


