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Improved worst-case complexity for the MIN 3-SET COVERING problem

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Abstract

We consider MIN SET COVERING when the subsets are constrained to have maximum cardinality three. We propose an exact algorithm whose worst case complexity is bounded above by $O^*(1.3957^n)$. This is an improvement, based on a refined analysis, of a former result ($O^*(1.4492^n)$) by F. Della Croce and V. Th. Paschos, Computing optimal solutions for the MIN 3-SET COVERING problem, Proc. ISAAC’05, LNCS 3827, pp. 685–692.

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In MIN SET COVERING, we are given a universe $U$ of elements and a collection $S$ of (non-empty) subsets of $U$. The aim is to determine a minimum cardinality sub-collection $S' \subseteq S$ which covers $U$, i.e., $\cup_{S \in S'} S = U$ (we assume that $S$ covers $U$). The frequency $f_i$ of $u_i \in U$ is the number of subsets $S_j \in S$ in which $u_i$ is contained. The cardinality $d_j$ of $S_j \in S$ is the number of elements $u_i \in U$ that $S_j$ contains. We say that $S_j$ hits $S_k$ if both $S_j$ and $S_k$ contain an element $u_i$ and that $S_j$ double-hits $S_k$ if both $S_j$ and $S_k$ contain at least two elements $u_i, u_l$. Finally, we denote by $n$ the size (cardinality) of $S$ and by $m$ the size of $U$. In what follows, we restrict ourselves to MIN SET COVERING-instances such that:

1. no element $u_i \in U$ has frequency $f_i = 1$;
2. no set $S_i \in S$ is a subset of another set $S_j \in S$.
3. no pair of elements $u_i, u_j$ exists such that every subset $S_i \in S$ containing $u_i$ contains also $u_j$.

Indeed, if item 1 is not verified, then the set containing $u_i$ belongs to any feasible cover of $U$. On the other hand, if item 2 is not verified, then $S_i$ can be replaced by $S_j$ in any solution containing $S_i$ and the resulting cover will not be worse than the one containing $S_i$. Finally, if item 3 is not verified, then element $u_j$ can be ignored as any sub-collection $S'$ covering $u_i$ will necessarily cover also $u_j$. So, for any instance of MIN SET COVERING, a preprocessing of data, obviously performed in polynomial time, leads to instances where all items 1, 2 and 3 are verified.

Let $T(\cdot)$ be a super-polynomial and $p(\cdot)$ be a polynomial, both on integers. In what follows, using notations in [9], for an integer $n$, we express running-time bounds of the form $p(n).T(n)$ as $O^*(T(n))$, the asterisk meaning that we ignore polynomial factors. We denote by $T(n)$ the

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worst case time required to exactly solve the MIN SET COVERING problem with $n$ subsets. We recall (see, for instance, [5]) that, if it is possible to bound above $T(n)$ by a recurrence expression of the type $T(n) \leq \sum T(n-r_i) + O(p(n))$, we have $T(n) = O^*(\alpha(r_1, r_2, \ldots)^n)$ where $\alpha(r_1, r_2, \ldots)$ is the largest zero of the function $f(x) = 1 - \sum x^{-r_i}$.

There exist to our knowledge few results on worst-case complexity of exact algorithms for MIN SET COVERING or for cardinality-constrained versions of it. Let us note that an exhaustive set (hence this set will have size 2 is the remaining instance), we consider that our benefit is $\leq 1$. When we do not fix a set of size 3 but cover one element of this set has cardinality at most 2) is polynomially solvable by matching techniques ([2, 7]).

The rest of the paper is devoted to the proof of the following result.

**Theorem 1.** MIN 3-SET COVERING can be optimally solved within time $O^*(1.396^n)$. 

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** Lemma 1.** There exists at least one optimal solution of MIN SET COVERING where:

1. for any subset $S_j$ with $d_j = 2$ containing elements $u_i, u_p$, if $S_j$ is included in $S'$, then all subsets $S_k$ hitting $S_j$ are excluded from $S'$;

2. for any subset $S_j$ with $d_j = 3$ containing elements $u_i, u_p, u_q$, where $S_j$ double-hits another subset $S_k$ with $d_k = 3$ on $u_i$ and $u_p$, if $S_j$ is included in $S'$ then $S_k$ must be excluded from $S'$ and viceversa;

3. for any subset $S_j$ with $d_j = 3$ containing elements $u_i, u_p, u_q$, if $S_j$ is included in $S'$, then either all subsets $S_k$ hitting $S_j$ on element $u_i$ are excluded from $S'$, or all subsets $S_k$ hitting $S_j$ on elements $u_p$ and $u_q$ are excluded from $S'$.

** Proof.** We only prove item 1, items 2 and 3 being proved by the same kind of analysis. Assume, without loss of generality, that $S_j$ hits $S_k$ on $u_i$ and $S_l$ on $u_p$. Suppose by contradiction that the optimal solution $S'$ includes $S_j$ and $S_k$. Then, it cannot include no more $S_l$, or else, it would not be optimal as a better cover would be obtained by excluding $S_j$ from $S'$. On the other hand, suppose that $S'$ includes $S_j, S_k$ but does not include $S_l$. Then, an equivalent optimal solution can be derived by swapping $S_j$ with $S_l$.

In what follows, we consider the following counting. When we fix the status of a set of size 3, then our benefit is 1. When we do not fix a set of size 3 but cover one element of this set (hence this set will have size 2 is the remaining instance), we consider that our benefit is $\alpha \leq 1$. Obviously, when a set of size 2 is fixed, we can only consider that (in the worst case) our benefit is $1 - \alpha$. Hence, in some cases, the benefit is increasing with $\alpha$ while, in other cases, it is decreasing. An optimal value for $\alpha$, following our analysis, is $\alpha = 0.297$.

The rest of the paper is devoted to the proof of the following result.
The algorithm either reduces the min 3-set covering instance according to assumptions 1, 2 and 3 on the form of the instance (by detecting a subset $S_j$ to be immediately included in (excluded from) $S'$ or an element $u$ to be ignored (correspondingly reducing the size of several subsets)), or applies a branching on subset $S_j$, where the following exhaustive relevant branching cases may occur.

1. $d_j = 2$: then no double-hitting occurs to $S_j$ or else, due to the preprocessing step of the algorithm, $S_j$ can be excluded from $S'$ without branching. The following subcases occur.

   (a) $S_j$ contains elements $u$, $u_k$ with $f_i = f_k = 2$ where $S_j$ hits $S_t$ on $u$ and $S_t$ on $u_k$.

   Due to Lemma 1, if $S_j$ is included in $S'$, then both $S_t$ and $S_m$ must be excluded from $S'$; alternatively, $S_j$ is excluded from $S'$ and, correspondingly, both $S_t$ and $S_m$ must be included in $S'$ to cover elements $u$, $u_k$. For the analysis, consider the two following cases.

   i. $d_t = 3$, or $d_m = 3$, say $d_t = 3$. Then, in both cases (including or excluding $S_j$) we fix $3 - 2\alpha$ (1 for $S_t$, (at least) $1 - \alpha$ for $S_j$ and $S_m$).

   ii. $d_t = d_m = 2$, $S_t$ contains $u$ and $u$ and $S_m$ contains $u_k$ and $u_m$, (with $u \neq u_m$), otherwise no need to branch). By including $S_j$ we fix $3(1 - \alpha)$. Otherwise, $u$ is contained in $S_p$ and $u_m$ in $S_q$. If $S_p \neq S_q$, then we fix at least $3(1 - \alpha) + 2\alpha = 3 - \alpha$. Indeed, we fix $1 - \alpha$ for any of the sets $S_j$, $S_t$, $S_m$; by covering $u_m$, we fix $\alpha$ (resp., $1 - \alpha \geq \alpha$) if $d_p = 3$ (resp., if $d_p = 2$, since we can exclude $S_p$), and the same holds for covering $u$. Note that this is still valid if $S_p = S_q$, since in this case we can exclude this set, which gives at least $1 - \alpha \geq 2\alpha$.

   In case 1(a)i, we have $T(n) \leq 2T(n - 3 + 2\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.334^n)$. In case 1(a)ii, we have $T(n) \leq T(n - 3 + 3\alpha) + T(n - 3 + \alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.336^n)$.

   (b) $S_j$ contains elements $u$, $u_k$ with $f_i = 2$ and $f_k \geq 3$, where $S_j$ hits $S_t$ on $u$ and $S_m$, $S_p$ on $u_k$.

   Due to Lemma 1, if $S_j$ is included in $S'$, then $S_t$, $S_m$, $S_p$ must be excluded from $S'$; alternatively, $S_j$ is excluded from $S'$ and, correspondingly, $S_t$ must be included in $S'$ to cover element $u$. For the analysis, consider the two following cases.

   i. $d_t = 2$, i.e., $S_t$ contains $u$, $u_t$; then, $f_t \geq 3$ (or else we are in case 1a). Then, by including $S_j$, we fix $4(1 - \alpha)$ (1 - $\alpha$) for any of the sets $S_j$, $S_t$, $S_m$, $S_p$); by excluding $S_j$, we fix $2(1 - \alpha) + 2\alpha = 2((1 - \alpha)$ for any of the sets $S_j$, $S_t$, and (at least) $\alpha$ for each set containing $u_t$).

   ii. If $d_t \geq 3$, i.e., $S_t$ contains at least $u$, $u_t$, $u_m$, then by including $S_j$, we fix $3(1 - \alpha) + 1$ (since now fixing $S_t$ gives benefit 1); by excluding $S_j$, we fix $(1 - \alpha) + 1 + 2\alpha = 2 + \alpha$ ($\alpha$ from covering $u$, and $\alpha$ from covering $u_m$, with the same reasoning as in case 1(a)ii). The worst case is 1(b)i where we get $T(n) \leq T(n - 2) + T(n - 4 + 4\alpha) + O(p(n))$, resulting in a time-complexity of $O^*(1.338^n)$.

   (c) $S_j$ contains elements $u$, $u_k$ with $f_i = 3$ and $f_k \geq 3$ where $S_j$ hits $S_t$, $S_m$ on $u_i$ and (at least) $S_p$, $S_q$ on $u_k$. Note that we can suppose that $S_j$ hits at least one set of size 3. Due to Lemma 1, if $S_j$ is included in $S'$, then $S_t$, $S_m$, $S_p$, $S_q$ must be excluded from $S'$; alternatively, $S_j$ is excluded from $S'$. For the analysis, consider the three following cases.

   i. If $d_t = d_m = d_p = d_q = 3$, then we fix either $5 - \alpha$, or $1 - \alpha$.
ii. If $d_l = 2$ or $d_m = 2$, say $d_l = 2$, then we fix either $5 - 4\alpha$, or $1 - \alpha$. But in the case where we exclude $S_j$ from $S'$, then $S_l$ has size 2 and contains $u_i$, whose frequency is now 2. Hence, we are either in case 1a or in case 1b. In the worst case, the branching gives (with case 1(b)i) $5 - 4\alpha$, $5(1 - \alpha)$ and $3 - \alpha$.

iii. Finally, if $d_l = d_m = 3$, then we can suppose that $f_k \geq 4$ (otherwise we are either in case 1(c)i or in case 1(c)ii). In this case, by including $S_j$ we fix $2 + 4(1 - \alpha)$ and by excluding $S_j$ we fix $1 - \alpha$.

In case 1(c)i, we get $T(n) \leq T(n - 1 + \alpha) + T(n - 3 + \alpha) + O(p(n))$, i.e., a time-complexity of $O^*(1.3953^n)$. In case 1(c)ii, we get $T(n) \leq T(n - 3 + \alpha) + T(n - 5 + 5\alpha) + T(n - 5 + 4\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.3942^n)$. In case 1(c)iii, we get $T(n) \leq T(n - 6 + 4\alpha) + T(n - 1 + \alpha) + O(p(n))$, i.e., a time-complexity of $O^*(1.389^n)$.

(d) $S_j$ contains elements $u_i, u_k$ with $f_i \geq 4$ and $f_k \geq 4$ where $S_j$ hits $S_j, S_m, S_p$ on $u_i$ and $S_q, S_r, S_s$ on $u_k$. Note that we can suppose that $S_j$ hits at least one set of size 3. Due to Lemma 1, if $S_j$ is included in $S'$, then $S_l, S_m, S_p, S_q, S_r, S_s$ must be excluded from $S'$; alternatively, $S_j$ is excluded from $S'$. Then, we fix either $7 - 6\alpha$ or $1 - \alpha$ getting $T(n) \leq T(n - 1 + \alpha) + T(n - 7 + 6\alpha) + O(p(n))$, resulting so in a time-complexity of $O^*(1.366^n)$.

2. $d_j = 3$ (that is, there does not exist $S_k \in S$ such that $d_k = 2$) and there is at least one element $u_i$ with $f_i = 2$. Then, $S_j$ contains $u_i, u_j, u_k$, and $S_k$ contains $u_i, u_j, u_k$ (notice that no double crossing can occur between $S_j$ and $S_k$ due to the preprocessing step of the algorithm). Then, either we include $S_k$, and we fix 1 + $3\alpha$ new sets, or we exclude $S_j$, and we have to include $S_k$ fixing so 2 + $2\alpha$ new sets. In this case, we get $T(n) \leq T(n - 1 - 3\alpha) + T(n - 2 - 2\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.366^n)$.

3. $d_j = 3$, all elements have a frequency at least 3, with $S_j$ double-hitting one or more subsets. The following exhaustive subcases may occur.

(a) $S_j$ double-hits at least three subsets $S_k, S_l, S_m$. Due to Lemma 1, if $S_j$ is included in $S'$ then $S_k, S_l, S_m$ must be excluded from $S'$; alternatively, $S_j$ is excluded from $S'$. This can be seen as a binary branching where either one subset ($S_j$) is fixed, or four subsets ($S_j, S_k, S_l, S_m$) are fixed and hence, $T(n) \leq T(n - 1) + T(n - 4) + O(p(n))$. This results in a time-complexity of $O^*(1.3803^n)$.

(b) $S_j$ double-hits two subsets $S_k, S_l$. Note that the double-hit elements must be contained by another set. Note also that (at least) one element, say $u_i$, is in $S_j, S_k$ and $S_l$. Consider the two following cases.

i. If $f_i \geq 4$, then either we include $S_j$ and then, by Lemma 1, we can exclude $S_k$ and $S_l$, or we exclude $S_j$. Then, either we fix 3 + $3\alpha$ (3 for $S_j, S_k, S_l$, and $3\alpha$ since $u_i, u_j$ and $u_k$ belong to at least one other set) or 1.

ii. If $f_i = 3$, then we must include at least one set among $S_j, S_k, S_l$, but we can suppose that we do not include two such sets. In other words, we have a branching on the three following choices:

- taking $S_j$ (and not $S_k, S_l$),
- taking $S_k$ (and not $S_j, S_l$),
- taking $S_l$ (and not $S_j, S_k$).

In any case, we fix $3 + 2\alpha$ (2$\alpha$ since each element has a frequency at least 3)
In the first case, \( T(n) \leq T(n - 1) + T(n - 3 - 3\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.388^n) \). In the second case, \( T(n) \leq 3T(n - 3 - 2\alpha) + O(p(n)) \), and this results in a time-complexity of \( O^*(1.358^n) \).

(c) \( S_j \) contains elements \( u_i, u_k, u_l \) and double-hits one subset \( S_k \) on elements \( u_i, u_k \). The following exhaustive subcases must be considered.

i. \( f_i = 3, f_k \geq 3, f_l \geq 3 \), with \( u_i \) contained by \( S_j, S_k, S_m, u_k \) contained at least by \( S_j, S_k, S_p \) and \( u_l \) contained at least by \( S_j, S_q, S_r \). A composite branching can be devised:

- Suppose that \( S_j \) is included in \( S' \) and then \( S_k \) is excluded from \( S' \). In this case, we fix \( 2+4\alpha \) (\( \alpha \) from reduction of the sizes of \( S_m, S_p, S_q, S_r \)).
- Suppose that \( S_j \) is excluded from \( S' \) and \( S_k \) is included in \( S' \). In this case, we fix \( 2+4\alpha \) (since no other double hit occurs on \( S_k \)).
- Suppose finally that \( S_j \) and \( S_k \) are excluded from \( S' \). In this case, we have to include \( S_m \) in \( S' \). Since \( d_m = 3 \), all elements have frequency at least 3, and at most one double crossing occurs on \( S_m \); we can see that \( S_m \) hits at least three new sets. Hence, we fix \( 3+3\alpha \).

ii. \( f_i \geq 4, f_k \geq 4, f_l \geq 3 \), with \( u_i \) contained at least by \( S_j, S_k, S_m, S_p, u_k \) contained at least by \( S_j, S_k, S_q, S_r \) and \( u_l \) contained at least by \( S_j, S_q, S_r \). Either we include \( S_j \) in \( S' \), and then we can exclude \( S_k \) from \( S' \) and fix \( 2+6\alpha \), or we exclude \( S_j \) and fix 1.

In case 3(c)i, we get \( T(n) \leq 2T(n - 2 - 4\alpha) + T(n - 3 - 3\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.381^n) \). In case 3(c)ii, we get \( T(n) \leq T(n - 1) + T(n - 2 - 6\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3957^n) \).

4. \( d_j = 3 \) and no double-hitting occurs to \( S_j \) (nor to any other subset) that contains elements \( u_i, u_k, u_l \). The following subcases occur.

(a) \( f_i = 3, f_k \geq 3, f_l \geq 3 \) with \( u_i \) contained by \( S_j, S_k, S_l, u_k \) contained by \( S_j, S_m, S_p \) and \( u_l \) contained at least by \( S_j, S_q, S_r \). A composite branching can be devised:

- if \( S_j \) is included in \( S' \), then we fix \( 1+6\alpha \) new sets;
- if \( S_j \) is excluded from \( S' \) and \( S_k \) is included in \( S' \), then there exist at least five other subsets hitting \( S_k \) and hence we fix \( 2+5\alpha \);
- finally, if \( S_j, S_k \) are excluded from \( S' \), then we have to include \( S_l \) in \( S' \) (in order to cover \( u_i \)); there exist at least four other subsets hitting \( S_l \) and hence we fix \( 3+4\alpha \).

Thus, \( T(n) \leq T(n - 1 - 6\alpha) + T(n - 2 - 5\alpha) + T(n - 3 - 4\alpha) + O(p(n)) \), resulting in a time-complexity of \( O^*(1.378^n) \).

(b) \( f_i \geq 4, f_k \geq 4, f_l \geq 4 \), with \( u_i \) contained by \( S_j, S_k, S_l, u_k \) contained by \( S_j, S_p, S_q, S_r \) and \( u_l \) contained at least by \( S_j, S_t, S_u, S_v \). A composite branching on \( S_j \) can be devised:

- if \( S_j \) is excluded from \( S' \), then we fix 1;
- if \( S_j \) is included in \( S' \), then \( S_k, S_l, S_m \) are excluded from \( S' \); in this case we fix \( 4+6\alpha \);
- finally, if \( S_j \) is included in \( S' \), then \( S_p, S_q, S_r, S_t, S_u, S_v \) are excluded from \( S' \); in this case we fix \( 7+3\alpha \).

Hence, \( T(n) \leq T(n - 1) + T(n - 4 - 6\alpha) + T(n - 7 - 3\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.355^n) \).
Putting things together, the global worst case complexity is $O^*(1.3957^n)$ and the proof of the theorem is complete.

As a last word, let us note that a straightforward (improvable) analysis along the lines of Theorem 1, leads to an $O^*(1.1679^n)$ time bound for minimum vertex covering in graphs of maximum size 3. Such a bound is the best-known dealing with search tree-based algorithms and is only dominated by the bounds in [1, 3], ($O^*(1.1252^n)$ and $O^*(1.152^n)$, respectively) that are not based upon such algorithms. Note also, dealing with minimum dominating set in graphs of maximum size 3, analysis along the same lines reaches $O^*(1.344^n)$, which is always the best-known search-tree complexity.

References


