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To cite this version:
Federico Della Croce, Bruno Escoffier, Vangelis Paschos. Improved worst-case complexity for the MIN 3-SET COVERING problem. 2006. hal-00957610

HAL Id: hal-00957610
https://hal.archives-ouvertes.fr/hal-00957610
Submitted on 10 Mar 2014

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CAHIER DU LAMSADE

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Janvier 2006

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January 13, 2006

Abstract

We consider MIN SET COVERING when the subsets are constrained to have maximum cardinality three. We propose an exact algorithm whose worst case complexity is bounded above by $O^*(1.3957^n)$. This is an improvement, based on a refined analysis, of a former result ($O^*(1.4492^n)$) by F. Della Croce and V. Th. Paschos, Computing optimal solutions for the MIN 3-SET COVERING problem, Proc. ISAAC’05, LNCS 3827, pp. 685–692.

Keywords: Worst-case complexity, Exact algorithm, MIN SET COVERING

In MIN SET COVERING, we are given a universe $U$ of elements and a collection $S$ of (non-empty) subsets of $U$. The aim is to determine a minimum cardinality sub-collection $S' \subseteq S$ which covers $U$, i.e., $\cup_{S \in S'} S = U$ (we assume that $S$ covers $U$). The frequency $f_i$ of $u_i \in U$ is the number of subsets $S_j \in S$ in which $u_i$ is contained. The cardinality $d_j$ of $S_j \in S$ is the number of elements $u_i \in U$ that $S_j$ contains. We say that $S_j$ hits $S_k$ if both $S_j$ and $S_k$ contain an element $u_i$ and that $S_j$ double-hits $S_k$ if both $S_j$ and $S_k$ contain at least two elements $u_i, u_l$.

Finally, we denote by $n$ the size (cardinality) of $S$ and by $m$ the size of $U$. In what follows, we restrict ourselves to MIN SET COVERING-instances such that:

1. no element $u_i \in U$ has frequency $f_i = 1$;
2. no set $S_i \in S$ is a subset of another set $S_j \in S$.
3. no pair of elements $u_i, u_j$ exists such that every subset $S_i \in S$ containing $u_i$ contains also $u_j$.

Indeed, if item 1 is not verified, then the set containing $u_i$ belongs to any feasible cover of $U$. On the other hand, if item 2 is not verified, then $S_i$ can be replaced by $S_j$ in any solution containing $S_i$ and the resulting cover will not be worse than the one containing $S_i$. Finally, if item 3 is not verified, then element $u_j$ can be ignored as any sub-collection $S'$ covering $u_i$ will necessarily cover also $u_j$. So, for any instance of MIN SET COVERING, a preprocessing of data, obviously performed in polynomial time, leads to instances where all items 1, 2 and 3 are verified.

Let $T(\cdot)$ be a super-polynomial and $p(\cdot)$ be a polynomial, both on integers. In what follows, using notations in [9], for an integer $n$, we express running-time bounds of the form $p(n).T(n)$ as $O^*(T(n))$, the asterisk meaning that we ignore polynomial factors. We denote by $T(n)$ the

*Part of this research has been performed while the first author was in visit at the LAMSADE on a research position funded by the CNRS
Theorem 1. \textit{min 3-set covering} is the largest zero of the function $f_T$ recall (see, for instance, [5]) that, if it is possible to bound above $T(n)$ by a recurrence expression of the type $T(n) \leq \sum T(n-r_i)+O(p(n))$, we have $T(n) = O^{*}(\alpha(r_1,r_2,\ldots)^n)$ where $\alpha(r_1, r_2, \ldots)$ is the largest zero of the function $f(x) = 1 - \sum x^{-r_i}$.

There exist to our knowledge few results on worst-case complexity of exact algorithms for \textit{min set covering} or for cardinality-constrained versions of it. Let us note that an exhaustive set (hence this set will have size 2 is the remaining instance), we consider that our benefit is $\alpha \leq 1$. Obviously, when a set of size 2 is fixed, we can only consider that (in the worst case) our benefit is $1 - \alpha$. Hence, in some cases, the benefit is increasing with $\alpha$ while, in other cases, it is decreasing. An optimal value for $\alpha$, following our analysis, is $\alpha = 0.297$.

The rest of the paper is devoted to the proof of the following result.

**Theorem 1.** \textit{min 3-set covering} can be optimally solved within time $O^{*}(1.396^n)$. 

worst case time required to exactly solve the \textit{MIN SET COVERING} problem with $n$ subsets. We recall (see, for instance, [5]) that, if it is possible to bound above $T(n)$ by a recurrence expression of the type $T(n) \leq \sum T(n-r_i)+O(p(n))$, we have $T(n) = O^{*}(\alpha(r_1,r_2,\ldots)^n)$ where $\alpha(r_1, r_2, \ldots)$ is the largest zero of the function $f(x) = 1 - \sum x^{-r_i}$.

There exist to our knowledge few results on worst-case complexity of exact algorithms for \textit{min set covering} or for cardinality-constrained versions of it. Let us note that an exhaustive set (hence this set will have size 2 is the remaining instance), we consider that our benefit is $\alpha \leq 1$. Obviously, when a set of size 2 is fixed, we can only consider that (in the worst case) our benefit is $1 - \alpha$. Hence, in some cases, the benefit is increasing with $\alpha$ while, in other cases, it is decreasing. An optimal value for $\alpha$, following our analysis, is $\alpha = 0.297$.

The rest of the paper is devoted to the proof of the following result.

**Theorem 1.** \textit{min 3-set covering} can be optimally solved within time $O^{*}(1.396^n)$. 


The algorithm either reduces the min 3-set covering instance according to assumptions 1, 2 and 3 on the form of the instance (by detecting a subset \( S_j \) to be immediately included in (excluded from) \( S' \) or an element \( u_i \) to be ignored (correspondingly reducing the size of several subsets), or applies a branching on subset \( S_j \), where the following exhaustive relevant branching cases may occur.

1. \( d_j = 2 \): then no double-hitting occurs to \( S_j \) or else, due to the preprocessing step of the algorithm, \( S_j \) can be excluded from \( S' \) without branching. The following subcases occur.

   (a) \( S_j \) contains elements \( u_i, u_k \) with \( f_i = f_k = 2 \) where \( S_j \) hits \( S_l \) on \( u_i \) and \( S_m \) on \( u_k \).
   
   Due to Lemma 1, if \( S_j \) is included in \( S' \), then both \( S_l \) and \( S_m \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \) and, correspondingly, both \( S_l \) and \( S_m \) must be included in \( S' \) to cover elements \( u_i, u_k \). For the analysis, consider the two following cases.

   i. \( d_l = 3 \), or \( d_m = 3 \), say \( d_l = 3 \). Then, in both cases (including or excluding \( S_j \)) we fix \( 3 - 2\alpha \) (1 for \( S_l \), (at least) \( 1 - \alpha \) for \( S_j \) and \( S_m \)).

   ii. \( d_l = d_m = 2 \), \( S_l \) contains \( u_i \) and \( u_l \) and \( S_m \) contains \( u_k \) and \( u_m \), (with \( u_l \neq u_m \), otherwise no need to branch). By including \( S_j \) we fix \( 3(1 - \alpha) \). Otherwise, \( u_l \) is contained in \( S_p \) and \( u_m \) in \( S_q \). If \( S_p \neq S_q \), then we fix at least \( 3(1 - \alpha) + 2\alpha = 3 - \alpha \).

   Indeed, we fix \( 1 - \alpha \) for any of the sets \( S_j, S_l, S_m; \) by covering \( u_m \), we fix \( \alpha \) (resp., \( 1 - \alpha \)). If \( d_p = 3 \) (resp., if \( d_p = 2 \), since we can exclude \( S_p \)), and the same holds for covering \( u_k \). Note that this is still valid if \( S_p \neq S_q \), since in this case we can exclude this set, which gives at least \( 1 - \alpha \geq 2\alpha \).

   In case 1(a)i, we have \( T(n) \leq 2T(n - 3 + 2\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.334^n) \). In case 1(a)ii, we have \( T(n) \leq T(n - 3 + 3\alpha) + T(n - 3 + \alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.336^n) \).

   (b) \( S_j \) contains elements \( u_i, u_k \) with \( f_i = 2 \) and \( f_k \geq 3 \), where \( S_j \) hits \( S_l \) on \( u_i \) and \( S_m, S_p \) on \( u_k \). Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \) and, correspondingly, \( S_l \) must be included in \( S' \) to cover element \( u_i \). For the analysis, consider the two following cases.

   i. \( d_l = 2 \), i.e., \( S_l \) contains \( u_i, u_l \); then, \( f_i \geq 3 \) (or else we are in case 1a). Then, by including \( S_j \), we fix \( 4(1 - \alpha) \) \((1 - \alpha)\) for any of the sets \( S_j, S_l, S_m, S_p \); by excluding \( S_j \), we fix \( 2(1 - \alpha) + 2\alpha = 2 \) \((1 - \alpha)\) for any of the sets \( S_j, S_l \), and \( \) (at least) \( \alpha \) for each set containing \( u_l \).

   ii. If \( d_l \geq 3 \), i.e., \( S_l \) contains at least \( u_i, u_l, u_m \), then by including \( S_j \), we fix \( 3(1 - \alpha) + 1 \) \((1 - \alpha)\) (since now fixing \( S_l \) gives benefit 1); by excluding \( S_j \), we fix \( 1 - \alpha + 2\alpha = 2 + \alpha \) \((1 - \alpha)\) from covering \( u_l \) and \( \alpha \) from covering \( u_m \), with the same reasoning as in case 1(a)ii).

   The worst case is 1(b)ii where we get \( T(n) \leq T(n - 2) + T(n - 4 + 4\alpha) + O(p(n)) \), resulting in a time-complexity of \( O^*(1.338^n) \).

   (c) \( S_j \) contains elements \( u_i, u_k \) with \( f_i = 3 \) and \( f_k \geq 3 \) where \( S_j \) hits \( S_l, S_m \) on \( u_i \) and \( S_p, S_q \) on \( u_k \). Note that we can suppose that \( S_j \) hits at least one set of size 3.

   Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p, S_q \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). For the analysis, consider the three following cases.

   i. If \( d_l = d_m = d_p = d_q = 3 \), then we fix either \( 5 - \alpha \), or \( 1 - \alpha \).
ii. If \( d_l = 2 \) or \( d_m = 2 \), say \( d_l = 2 \), then we fix either \( 5 - 4\alpha \), or \( 1 - \alpha \). But in the case where we exclude \( S_j \) from \( S' \), then \( S_l \) has size 2 and contains \( u_i \), whose frequency is now 2. Hence, we are either in case 1a or in case 1b. In the worst case, the branching gives (with case 1(b)) \( 5 - 4\alpha \), \( 5(1 - \alpha) \) and \( 3 - \alpha \).

iii. Finally, if \( d_l = d_m = 3 \), then we can assume that \( f_k \geq 4 \) (otherwise we are either in case 1(c)i) or in case 1(c)ii). In this case, by including \( S_j \) we fix \( 2 + 4(1 - \alpha) \) and by excluding \( S_j \) we fix \( 1 - \alpha \).

In case 1(c)i, we get \( T(n) \leq T(n - 1 + \alpha) + T(n - 5 + \alpha) + O(p(n)) \), i.e., a time-complexity of \( O^*(1.3953^n) \). In case 1(c)ii, we get \( T(n) \leq T(n - 3 + \alpha) + T(n - 5 + 5\alpha) + T(n - 5 + 4\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3942^n) \). In case 1(c)iii, we get \( T(n) \leq T(n - 6 + 4\alpha) + T(n - 1 + \alpha) + O(p(n)) \), i.e., a time-complexity of \( O^*(1.389^n) \).

(d) \( S_j \) contains elements \( u_i, u_k \) with \( f_i \geq 4 \) and \( f_k \geq 4 \) where \( S_j \) hits \( S_l, S_m, S_p \) on \( u_i \) and \( S_q, S_r, S_s \) on \( u_k \). Note that we can suppose that \( S_j \) hits at least one set of size 3. Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p, S_q, S_r, S_s \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). Then, we fix either \( 7 - 6\alpha \) or \( 1 - \alpha \) getting \( T(n) \leq T(n - 1 + \alpha) + T(n - 7 + 6\alpha) + O(p(n)) \), resulting so in a time-complexity of \( O^*(1.366^n) \).

2. \( d_j = 3 \) (that is, there does not exist \( S_k \in S \) such that \( d_k = 2 \)) and there is at least one element \( u_i \) with \( f_i = 2 \). Then, \( S_j \) contains \( u_i, u_j, u_k \), and \( S_k \) contains \( u_i, u_j, u_m \) (notice that no double crossing can occur between \( S_j \) and \( S_k \) due to the preprocessing step of the algorithm). Then, either we include \( S_j \), and we fix \( 1 + 3\alpha \) new sets, or we exclude \( S_j \), and we have to include \( S_k \) fixing so \( 2 + 2\alpha \) new sets. In this case, we get \( T(n) \leq T(n - 1 - 3\alpha) + T(n - 2 - 2\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.366^n) \).

3. \( d_j = 3 \), all elements have a frequency at least 3, with \( S_j \) double-hitting one or more subsets. The following exhaustive subcases may occur.

(a) \( S_j \) double-hits at least three subsets \( S_k, S_l, S_m \). Due to Lemma 1, if \( S_j \) is included in \( S' \) then \( S_k, S_l, S_m \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). This can be seen as a binary branching where either one subset \( (S_j) \) is fixed, or four subsets \( (S_j, S_k, S_l, S_m) \) are fixed and hence, \( T(n) \leq T(n - 1) + T(n - 4) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3803^n) \).

(b) \( S_j \) double-hits two subsets \( S_k, S_l \). Note that the double-hit elements must be contained by another set. Note also that (at least) one element, say \( u_i \), is in \( S_j, S_k \) and \( S_l \).

Consider the two following cases.

i. If \( f_i \geq 4 \), then either we include \( S_j \) and then, by Lemma 1, we can exclude \( S_k \) and \( S_l \), or we exclude \( S_j \). Then, either we fix \( 3 + 3\alpha \) (3 for \( S_j, S_k, S_l \), and \( 3\alpha \) since \( u_i, u_j \) and \( u_k \) belong to at least one other set) or 1.

ii. If \( f_i = 3 \), then we must include at least one set among \( S_j, S_k, S_l \), but we can suppose that we do not include two such sets. In other words, we have a branching on the three following choices:

- taking \( S_j \) (and not \( S_k, S_l \)),
- taking \( S_k \) (and not \( S_j, S_l \)),
- taking \( S_l \) (and not \( S_j, S_k \)).

In any case, we fix \( 3 + 2\alpha \) (\( 2\alpha \) since each element has a frequency at least 3)
In the first case, \( T(n) \leq T(n - 1) + T(n - 3 - 3 \alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.388^n) \). In the second case, \( T(n) \leq 3T(n - 3 - 2 \alpha) + O(p(n)) \), and this results in a time-complexity of \( O^*(1.358^n) \).

(c) \( S_j \) contains elements \( u_i, u_k, u_l \) and double-hits one subset \( S_k \) on elements \( u_i, u_k \). The following exhaustive subcases must be considered.

i. \( f_i = 3, f_k \geq 3, f_l \geq 3 \), with \( u_i \) contained by \( S_j, S_k, S_m, u_k \) contained at least by \( S_j, S_k, S_p \) and \( u_l \) contained at least by \( S_j, S_q, S_r \). A composite branching can be devised:

- Suppose that \( S_j \) is included in \( S' \) and then \( S_k \) is excluded from \( S' \). In this case, we fix \( 2 + 4 \alpha \) (\( \alpha \) from reduction of the sizes of \( S_m, S_p, S_q, S_r \)).
- Suppose that \( S_j \) is excluded from \( S' \) and \( S_k \) is included in \( S' \). In this case, we fix \( 2 + 4 \alpha \) (since no other double hit occurs on \( S_k \)).
- Suppose finally that \( S_j \) and \( S_k \) are excluded from \( S' \). In this case, we have to include \( S_m \) in \( S' \). Since \( d_m = 3 \), all elements have frequency at least 3, and at most one double crossing occurs on \( S_m \); we can see that \( S_m \) hits at least three new sets. Hence, we fix \( 3 + 3 \alpha \).

ii. \( f_i \geq 4, f_k \geq 4, f_l \geq 3 \), with \( u_i \) contained at least by \( S_j, S_k, S_m, S_p, u_k \) contained at least by \( S_j, S_k, S_q, S_r \), and \( u_l \) contained at least by \( S_j, S_u, S_v \). Either we include \( S_j \) in \( S' \), and then we can exclude \( S_k \) from \( S' \) and fix \( 2 + 6 \alpha \), or we exclude \( S_j \) and fix 1.

In case 3(c)i, we get \( T(n) \leq 2T(n - 2 - 4 \alpha) + T(n - 3 - 3 \alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.381^n) \). In case 3(c)ii, we get \( T(n) \leq T(n - 1) + T(n - 2 - 6 \alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3957^n) \).

4. \( d_j = 3 \) and no double-hitting occurs to \( S_j \) (nor to any other subset) that contains elements \( u_i, u_k, u_l \). The following subcases occur.

(a) \( f_i = 3, f_k \geq 3, f_l \geq 3 \) with \( u_i \) contained by \( S_j, S_k, S_l, u_k \) contained by \( S_j, S_m, S_p \) and \( u_l \) contained at least by \( S_j, S_q, S_r \). A composite branching can be devised:

- if \( S_j \) is included in \( S' \), then we fix 1 + 6 \( \alpha \) new sets;
- if \( S_j \) is excluded from \( S' \) and \( S_k \) is included in \( S' \), then there exist at least five other subsets hitting \( S_k \) and hence we fix \( 2 + 5 \alpha \);
- finally, if \( S_j \) and \( S_k \) are excluded from \( S' \), then we have to include \( S_l \) in \( S' \) (in order to cover \( u_l \)); there exist at least four other subsets hitting \( S_l \) and hence we fix \( 3 + 4 \alpha \).

Thus, \( T(n) \leq T(n - 1 - 6 \alpha) + T(n - 2 - 5 \alpha) + T(n - 3 - 4 \alpha) + O(p(n)) \), resulting in a time-complexity of \( O^*(1.378^n) \).

(b) \( f_i \geq 4, f_k \geq 4, f_l \geq 4 \), \( u_i \) is contained by \( S_j, S_k, S_l, u_k \) is contained by \( S_j, S_m, S_p, S_q \), and \( u_l \) is contained at least by \( S_j, S_t, S_u, S_v \). A composite branching on \( S_j \) can be devised:

- if \( S_j \) is excluded from \( S' \), then we fix 1;
- if \( S_j \) is included in \( S' \), then \( S_k, S_l, S_m \) are excluded from \( S' \); in this case we fix \( 4 + 6 \alpha \);
- finally, if \( S_j \) is included in \( S' \), then \( S_p, S_q, S_t, S_u, S_w \) are excluded from \( S' \); in this case we fix \( 7 + 3 \alpha \).

Hence, \( T(n) \leq T(n - 1) + T(n - 4 - 6 \alpha) + T(n - 7 - 3 \alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.355^n) \).
Putting things together, the global worst case complexity is $O^*(1.3957^n)$ and the proof of the theorem is complete.

As a last word, let us note that a straightforward (improvable) analysis along the lines of Theorem 1, leads to an $O^*(1.1679^n)$ time bound for minimum vertex covering in graphs of maximum size 3. Such a bound is the best-known dealing with search tree-based algorithms and is only dominated by the bounds in [1, 3], ($O^*(1.1252^n)$ and $O^*(1.152^n)$, respectively) that are not based upon such algorithms. Note also, dealing with minimum dominating set in graphs of maximum size 3, analysis along the same lines reaches $O^*(1.344^n)$, which is always the best-known search-tree complexity.

References


