Generalized Stress Concentration Factors

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In memory of my friends
Isaac Feldman (1954–73), Amir Moses (1954–96), and Ilan Ramon (1954–2003).

Abstract: The classical stress concentration factor is regarded as the ratio between the maximal value of the stress in a body and the maximal value of the applied force for a given distribution of material properties. An optimal stress concentration factor is defined as the lowest stress concentration factor if we allow any stress field that is in equilibrium with the given load. The generalized stress concentration factor, a purely geometric property of a body, is the maximal optimal stress concentration factor for any applied force. We show that the generalized stress concentration factor is equal to the norm of an extension mapping of Sobolev functions.

Key Words: Continuum mechanics, forces, stresses, stress concentration factor, Sobolev space

1. INTRODUCTION

The present paper considers the question of how bad is the geometry of a body in terms of the magnitude of the stresses induced in it. We establish a relation between this interpretation of stress concentration and another geometric property – the norm of an extension mapping of Sobolev functions defined on the interior of the body.

The traditional stress concentration factor (see for example Peterson [1]) indicates, for a given loading condition, the ratio between the maximum of a component of the stress and the value of that component for an idealized geometry. For example, if there is a change of the cross section A along a bar subjected to tension F, it gives the ratio between the maximal stress and the nominal stress F/A. Assuming that the nominal stress is actually the stress far away from the location where the cross section changes, we may regard those nominal stresses as traction boundary conditions. Thus, the stress concentration factor may be represented by the ratio

$$K_F = \frac{\sup_{x,i,k}\{|\sigma_{ik}(x)|\}}{\sup_{i,x,y}\{|b_i(x)|\,,\,|t_i(y)|\}}, \quad x \in \mathrm{Int}B,\ y \in \partial B,$$

where b_i and t_i are the body force and surface force distributions associated with the given loading F.

In the last expression, the maximum over i in the denominator (e.g., $\max_i\{|b_i(x)|\}$) and the maximum over i, k in the numerator serve as norms on $\mathbb{R}^3$ and on the space $L(\mathbb{R}^3, \mathbb{R}^3)$ of linear mappings defined on $\mathbb{R}^3$. This particular choice, possibly reflecting the flavor of traditional stress concentration factors but not rotation invariance (e.g., $\max_i\{|b_i(x)|\}$ depends on the coordinates chosen), may be replaced by other norms on the corresponding finite-dimensional spaces. In the sequel, we use $|b(x)|$ and $|t(x)|$ to denote the norms of the values at $x \in B$ of the body force vector field b and surface force vector field t associated with the given loading F. We denote by $|\sigma(x)|$ the norm of the value of the stress at x. For example, one may use $|b(x)| = \sqrt{b_i(x)b_i(x)}$ and $|\sigma(x)| = \sqrt{\sigma_{ik}(x)\sigma_{ik}(x)}$ that are rotation invariant.

Thus, the stress concentration factor is given by

$$K_F = \frac{\sup_x \{|\sigma(x)|\}}{\sup_{x,y}\{|b(x)|, |t(y)|\}}, \qquad x \in \operatorname{Int} B, \ y \in \partial B.$$

In the following, this concept of stress concentration is developed further in a number of steps.

Traditional stress concentration factors are calculated analytically, approximated numerically or measured experimentally. Their values are given for specific distributions of material properties, usually homogeneous, isotropic, linearly elastic materials. We wish to leave the material properties of the body open and consider the following question. Assume that one is performing a process of structural optimization by varying the distribution of material properties in the body. What would be the smallest value of the stress concentration for the given load? (Clearly, one can obtain as large a stress concentration as one wishes by varying the distribution of material properties.) Specifically, we denote the collection of all stress fields in equilibrium with the loading F by Σ_F and consider

$$K^{F,\text{optimal}} = \frac{\inf_{\sigma \in \Sigma_F}\left\{\sup_x \{|\sigma(x)|\}\right\}}{\sup_{x,y}\{|b(x)|, |t(y)|\}}.$$

Next, we arrive at a purely geometric property of the body. Noting that one usually does not know the exact nature of the loading, we allow the force distribution to vary and consider the worst case, i.e.

$$K = \sup_F \left\{K^{F,\text{optimal}}\right\} = \sup_F \left\{\frac{\inf_{\sigma \in \Sigma_F}\left\{\sup_x \{|\sigma(x)|\}\right\}}{\sup_{x,y}\{|b(x)|, |t(y)|\}}\right\}.$$

We also ignore high stresses and force densities on parts of the body having zero volume (or area in case of t) so we replace the suprema above by essential suprema. Finally, the stress object $\sigma(x)$ considered here contains not only the traditional stress tensor σ_{kl} but also a self-force σ_i. The self-force, appearing in various theories of materials with microstructure, may be thought of as the reaction of a three-dimensional (elastic) foundation. Thus, $|\sigma(x)|$ is redefined appropriately and the principle of virtual work has the form (we use the summation convention and commas denote partial differentiation)

$$\int_{\mathrm{Int}B} b_i w_i \mathrm{d}V + \int_{\partial B} t_i w_i \mathrm{d}A = \int_{\mathrm{Int}B} \sigma_i w_i \mathrm{d}V + \int_{\mathrm{Int}B} \sigma_{ik} w_{i,k} \mathrm{d}V.$$

The objective of the analysis presented in this paper is to prove that K is equal to the norm of an extension mapping associated with Sobolev functions. Specifically, let $L^1_1(\mathrm{Int}\ B, \mathbb{R}^3)$ denote the Sobolev space of vector fields over the interior of the body with integrable components and integrable derivatives of the components. Then, one of the basic properties of Sobolev functions implies that each Sobolev vector field $\phi \in L^1_1(\mathrm{Int}\ B, \mathbb{R}^3)$ may be extended to B in such a way that the restriction of the extension to the boundary, its *trace*, is integrable over the boundary. Denoting this extension mapping by $\imath$, one may consider the norm

$$\|\imath\| = \sup_{\phi} \frac{\|\imath(\phi)\|}{\|\phi\|}.$$

For an appropriate norm on the extended function and using the Sobolev norm for ϕ, $\|\imath\|$ assumes the form

$$\|\imath\| = \sup_{\phi \in L^1_1(\mathrm{Int}B, \mathbb{R}^3)} \frac{\int_{\mathrm{Int}B} |\phi| \,\mathrm{d}V + \int_{\partial B} \left|\hat{\phi}\right| \mathrm{d}A}{\int_{\mathrm{Int}B} |\phi| \,\mathrm{d}V + \int_{\mathrm{Int}B} |\nabla \phi| \,\mathrm{d}V},$$

where $\hat{\phi}$ denotes the extension of ϕ to the boundary ∂B of the body. This paper proves in some detail that $K = \|\imath\|$.

In writing the paper an attempt was made to make it self-contained for a wide readership. For this reason, some standard definitions and results of analysis are incorporated. The discussion is limited to the particular application presented here. For the general definitions and theorems the reader is referred to standard texts on Sobolev spaces (e.g., [2]). Section 2 introduces the basic notation and norms used for external forces and stresses. Section 3 considers the L^1_1-Sobolev space and shows how elements of its dual space are represented in terms of essentially bounded functions that we interpret physically as stress fields. Section 4 presents the relevant results on extensions and traces of Sobolev mappings. All the preceding material is combined in Section 5 to yield the result stated above. Finally, the short discussion of Section 6 presents some of the limitations of possible application of our result.

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2. PRELIMINARIES

2.1. Analytical Preliminaries

Considering a vector space $\mathbf{W}$ with a seminorm $\|\cdot\|$, a linear functional $\psi : \mathbf{W} \to \mathbb{R}$ is continuous, or bounded, relative to the given seminorm if there is a number $C > 0$ such that

$$|\psi(w)| \leqslant C \, \|w\| \,, \quad \text{for all} \quad w \in \mathbf{W}.$$

For the given vector space and seminorm, we will denote by $\mathbf{W}^*$ its dual space, i.e. the vector space of all bounded linear functionals. The vector space $\mathbf{W}^*$ has a natural dual norm $\|\cdot\|^*$ defined as follows. For each linear functional ψ, $\|\psi\|^*$ is the smallest C that satisfies the continuity condition. Thus, formally,

$$\|\psi\|^* = \sup_{w \in \mathbf{W}} \frac{|\psi(w)|}{\|w\|} \,.$$

Assume that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathbf{W}_1$ and $\mathbf{W}_2$, with dual norms $\|\cdot\|_1^*$ and $\|\cdot\|_2^*$, respectively. If we use on the space $\mathbf{W}_1 \times \mathbf{W}_2$ the norm $\|(w_1, w_2)\| = \|w_1\|_1 + \|w_2\|_2$, then, it follows from the definition above that the corresponding dual norm on $\mathbf{W}_1^* \times \mathbf{W}_2^* = (\mathbf{W}_1 \times \mathbf{W}_2)^*$ is given by $\left\|(\psi_1, \psi_2)\right\| = \max\left\{\left\|\psi_1\right\|_1^*, \left\|\psi_2\right\|_2^*\right\}$.

In the sequel, we will use norms to measure magnitudes of elements in $\mathbb{R}^3$, its dual, the space of linear mappings $L(\mathbb{R}^3, \mathbb{R}^3)$, and its dual. For $v \in \mathbb{R}^3$, we will use $|v|$ to denote its norm. For example, we may use $|v|_1 = \sum_i |v_i|$, or $|v|_2 = \sqrt{v_i v_i}$. For the first example the dual norm of an element $f \in (\mathbb{R}^3)^*$ will be given by $|f|_1^* = |f|_\infty = \max_i |f_i|$. For the second example where the norm is induced by the inner product, $|f|_2^* = |f|_2 = \sqrt{f_i f_i}$.

Similarly, for $T \in L(\mathbb{R}^3, \mathbb{R}^3)$ (e.g., the gradient of a vector field) we will use $|T|$ to denote its norm. For example, we may use $|T|_1 = \sum_{k,l} |T_{kl}|$ or $|T|_2 = \sqrt{T_{kl} T_{kl}}$ giving the corresponding dual norms $|R|_1^* = |R|_\infty = \max_{k,l} |R_{kl}|$ and $|R|_2^* = |R|_2 = \sqrt{R_{kl} R_{kl}}$ for $R \in L(\mathbb{R}^3, \mathbb{R}^3)^*$.

We will also be concerned with $\mathbb{R}^3 \times L(\mathbb{R}^3, \mathbb{R}^3) \cong \mathbb{R}^{12}$. For an element $(v, T) \in \mathbb{R}^3 \times L(\mathbb{R}^3, \mathbb{R}^3)$ we will use the norm $|(v, T)| = |v| + |T|$ so that for $(f, R) \in \left(\mathbb{R}^3 \times L(\mathbb{R}^3, \mathbb{R}^3)\right)^*$, $|(f, R)| = \max\{|f|, |R|\}$.

2.2. Mechanical Variables

A body will be modelled mathematically by a three-dimensional compact submanifold B with smooth boundary ∂B of $\mathbb{R}^3$. The interior of the body will be denoted by Int B. The physical space will be modeled by $\mathbb{R}^3$ and a virtual velocity is a mapping $w : B \longrightarrow \mathbb{R}^3$. As customary in continuum mechanics, a force is represented by a body force b and a surface force t defined in the interior of the body and its boundary, respectively. We regard b and t as elements of $\mathbb{R}^{3*}$ acting on virtual velocity w in the forms $b_i w_i$ and $t_i w_i$ to produce power densities. Further assumptions concerning the regularity of virtual velocities, body force and surface force fields will be specified below.

We will regard a force on a body as a linear functional operating on virtual velocities to produce power in the form

$$F(w) = \int\limits_{\mathrm{Int}B} b_i w_i \mathrm{d}V + \int\limits_{\partial B} t_i w_i \mathrm{d}A.$$

Usually, stress fields $\sigma = (\sigma_{ik})$ are regarded as linear functionals on the space of tensor fields. This is in agreement with the traditional expression

$$\int\limits_{\mathrm{Int}B} \sigma_{ik} w_{i,k} \mathrm{d}V$$

for the power performed by the stress tensor. We will generalize stresses to include self-forces $\sigma_0 = (\sigma_i)$: that is, additional stress components that operate directly on the component of the virtual velocity fields. Self-forces appear in theories of continuum mechanics on manifolds and theories of materials with microstructure. Thus, together with the self-forces, the stress $\sigma = (\sigma_0, \sigma_1)$ has the components (σ_i, σ_{kl}).

We use the term *local velocity field* for an integrable field over the interior of the body valued in $\mathbb{R}^3 \times L(\mathbb{R}^3, \mathbb{R}^3)$. Thus, a local velocity field χ is represented by a collection of 12 functions $\chi = (\chi_0, \chi_1) = (\chi_i, \chi_{kl})$ on the interior of the body. For the values of a local velocity field we use the norm $|\chi(x)| = \left|\chi_0(x)\right| + \left|\chi_1(x)\right|$ as above. We regard the values of stresses (together with the self-forces) as elements of $\left(\mathbb{R}^3 \times L(\mathbb{R}^3, \mathbb{R}^3)\right)^*$. Thus, we have the action $\sigma(x)(\chi(x)) = \sigma_i(x)\chi_i(x) + \sigma_{kl}(x)\chi_{kl}(x)$ giving the density of the power of the internal forces. Hence, the norm of the value of a stress at a point will be given by

$$|\sigma(x)| = \max\{|\sigma_0(x)|, |\sigma_1(x)|\}.$$

(It is noted that traditional failure criteria for even the case of linear elastic materials are usually seminorms and not norms. Thus, any norm of the value of the stress tensor at a point may only be used to bound the "equivalent stress".)

Stress fields act on local velocity fields to produce the power of the internal forces in the form

$$\sigma(\chi) = \int\limits_{\mathrm{Int}B} \sigma_i \chi_i \mathrm{d}V + \int\limits_{\mathrm{Int}B} \sigma_{ik} \chi_{ik} \mathrm{d}V.$$

The principle of virtual work is a generalized form of the equilibrium equation and boundary conditions. We will say that the virtual velocity field w is *compatible* with a local velocity field χ if $\chi_i = w_i$ and $\chi_{ik} = w_{i,k}$. Thus, the force F is in equilibrium with the stress σ if $F(w) = \sigma(\chi)$ for all compatible pairs of virtual velocities and local virtual velocities, i.e.

$$\int\limits_{\mathrm{Int}B} b_i w_i \mathrm{d}V + \int\limits_{\partial B} t_i w_i \mathrm{d}A = \int\limits_{\mathrm{Int}B} \sigma_i \chi_i \mathrm{d}V + \int\limits_{\mathrm{Int}B} \sigma_{ik} \chi_{ik} \mathrm{d}V.$$

2.3. *The L^1 and L^∞ Norms and Their Duality*

We want to represent the maximum of the magnitude $|\sigma(x)|$ of the stress in a body as the norm of the stress field. In addition, we want to allow stresses that are unbounded on subsets of the body having zero volume. Specifically, assuming that each of the 12 components of the stress defined on Int B is essentially bounded and using the standard notation L^∞ for the class of essentially bounded mappings, the vector space of stress fields is denoted by L^∞(Int $B, \mathbb{R}^{12}$). As expected, we use the L^∞-norm

$$\|\sigma\|^{L^\infty} = \operatorname*{ess\,sup}_x\{|\sigma(x)|\} = \operatorname*{ess\,sup}_x\{|\sigma_0(x)|, |\sigma_1(x)|\}$$

on this space of stresses.

A standard result of Lebesgue integration theory (see [2, p. 41]) states that the L^∞-space of functions can be identified using integration with the dual of the space L^1 of Lebesgue integrable mappings equipped with the L^1-norm. For our situation, the space of integrable mappings χ is L^1(Int $B, \mathbb{R}^{12}$) and the L^1-norm of a field χ is

$$\|\chi\|^{L^1} = \int_{\mathrm{Int}B} |\chi|\,\mathrm{d}V = \int_{\mathrm{Int}B} |\chi_0|\,\mathrm{d}V + \int_{\mathrm{Int}B} |\chi_1|\,\mathrm{d}V.$$

The action of a bounded linear functional associated with the stress field $\sigma \in L^\infty$(Int $B, \mathbb{R}^{12}$) on fields $\chi \in L^1$(Int $B, \mathbb{R}^{12}$) is given by the familiar expression

$$\sigma(\chi) = \int_{\mathrm{Int}B} \sigma_i\chi_i\mathrm{d}V + \int_{\mathrm{Int}B} \sigma_{ik}\chi_{ik}\mathrm{d}V.$$

Moreover, the L^∞-norm we use is the natural norm induced on the dual by the L^1-norm as defined above. Thus,

$$\|\sigma\|^{L^\infty} = \|\sigma\|^{L^{1*}} = \sup_\chi \frac{|\sigma(\chi)|}{\|\chi\|^{L^1}} = \sup_{\|\chi\|^{L^1}=1} |\sigma(\chi)|.$$

2.4. *The Norm of Forces*

In analogy with the preceding paragraph we wish to regard the largest magnitude of either a body force or a surface force as the norm of a given force F. We also want to regard this norm as a dual norm to a certain norm on the space of virtual velocities. Note that we cannot use simply the L^1-norm

$$\|w\|^{L^1} = \int_B |w|\,\mathrm{d}V$$

on the space of virtual velocities because in that case a force that has a non-vanishing surface force field t will not be continuous. (A change of w on the boundary only will not be reflected in the norm because the boundary has zero volume measure. However, because of the boundary term, the virtual work is sensitive to the boundary values of the virtual displacements.)

To account for the boundary term in the work done by the external forces we require that the restriction $w|_{\partial B}$ of a virtual velocity field to the boundary be integrable over the boundary. We thus use for virtual velocities the norm $\|\cdot\|^{L^{1,\mu}}$ defined by

$$\|w\|^{L^{1,\mu}} = \|w\|^{L^1} + \|w|_{\partial B}\|^{L^1} = \int_{\mathrm{Int}B} |w|\,\mathrm{d}V + \int_{\partial B} |w|\,\mathrm{d}A,$$

i.e., the sum of the L^1-norm of the mapping w defined on the body and the L^1-norm of its restriction $w|_{\partial B}$ to the boundary relative to the area measure. Alternatively, the $L^{1,\mu}$-norm may be regarded as the L^1-norm relative to the Radon measure μ, defined on B by

$$\mu(D) = V(D \cap \mathrm{Int}B) + A(D \cap \partial B),$$

where V and A are the volume and area measures, and hence the notation. This space of μ-integrable virtual velocity fields will be denoted by $L^{1,\mu}(B, \mathbb{R}^3)$.

Forces, being elements of the dual space $L^{1,\mu}(B, \mathbb{R}^3)^*$, may be identified with elements of $L^{\infty,\mu}(B, \mathbb{R}^3)$. A force F may be identified with a continuous linear functional relative to the $L^{1,\mu}$-norm if the body force components b_i and surface force components t_i (alternatively, $|b|$ and $|t|$) are essentially bounded relative to the volume and area measures, respectively. Moreover, the dual norm of a force is the $L^{\infty,\mu}$-norm, given by

$$\|F\|^{L^{\infty,\mu}} = \|F\|^{L^{1,\mu *}} = \operatorname{ess\,sup}_{x \in \mathrm{Int}B,\, y \in \partial B} \{|b(x)|\,, |t(y)|\}$$

as anticipated.

Keeping in mind the two last sections, our objective of finding some relation between maximal stresses and maximal values of the applied force densities is translated into the mathematical question of finding a relation between the L^∞-norm of a stress and the $L^{\infty,\mu}$-norm of a force assuming that the two satisfy the principle of virtual work.

3. THE SOBOLEV SPACE $L^1_1(\mathrm{Int}\ B, \mathbb{R}^3)$ AND ITS DUAL

3.1. Basic Definitions

The Sobolev space $L^1_1(\mathrm{Int}\ B, \mathbb{R}^3)$ to be defined below, plays a central role in the forthcoming analysis although it has not been considered above. Its dual, the space $L^1_1(\mathrm{Int}\ B, \mathbb{R}^3)^*$ serves as a "mediator" between the space $L^{\infty,\mu}(B, \mathbb{R}^3)$ containing the forces, and space of stress fields $L^\infty(\mathrm{Int}\ B, \mathbb{R}^{12})$.

The space $L_1^1(\text{Int } B, \mathbb{R}^3)$ is the collection of vector fields on Int B that are integrable relative to the volume measure and whose gradients have integrable components. The norm on $L_1^1(\text{Int } B, \mathbb{R}^3)$ is given by

$$\|\phi\|^{L_1^1} = \|\phi\|^{L^1} + \|\nabla\phi\|^{L^1} = \int_{\text{Int}B} |(\phi, \nabla\phi)|\, \mathrm{d}V = \int_{\text{Int}B} (|\phi| + |\nabla\phi|)\mathrm{d}V.$$

There is a natural mapping

$$j\colon L_1^1(\text{Int}B, \mathbb{R}^3) \longrightarrow L^1(\text{Int}B, \mathbb{R}^{12})$$

taking a field ϕ to a local velocity field $\chi = j(\phi) = (\phi_i, \phi_{i,k})$. It is noted that j is a one-to-one mapping but it is not onto $L^1(\text{Int } B, \mathbb{R}^{12})$. In addition, j is norm-preserving as $\|j(\phi)\|^{L^1} = \|\phi\|^{L_1^1}$.

3.2. *The Representation of L_1^1-Forces by Stresses*

In the sequel we will refer to an element $S \in L_1^1(\text{Int } B, \mathbb{R}^3)^*$ as an L_1^1-force. The mapping j defined above enables one to represent any L_1^1-force S by stresses as follows.

Let $S\colon L_1^1(\text{Int } B, \mathbb{R}^3) \longrightarrow \mathbb{R}$ be a linear functional bounded by $\|\cdot\|^{L_1^1}$. For every $\chi \in$ Image $j \subset L^1(\text{Int } B, \mathbb{R}^{12})$ we have

$$\left|S(j^{-1}(\chi))\right| \leqslant \|S\|^{L_1^{1*}} \left\|j^{-1}(\chi)\right\|^{L_1^1} = \|S\|^{L_1^{1*}} \|\chi\|^{L^1},$$

where we used the fact that j is injective in order to apply

$$j^{-1}\colon \text{Image } j \longrightarrow L_1^1(\text{Int}B, \mathbb{R}^3),$$

and we used the fact that $\|\phi\|^{L_1^1} = \|j(\phi)\|^{L^1}$ implies

$$\left\|j^{-1}(\chi)\right\|^{L_1^1} = \|\chi\|^{L^1}.$$

It follows that

$$S \circ j^{-1}\colon \text{Image } j \subset L^1(\text{Int}B, \mathbb{R}^{12}) \longrightarrow \mathbb{R}$$

is a linear functional on the vector subspace Image $j \subset L^1(\text{Int } B, \mathbb{R}^{12})$ that is bounded relative to the $\|\cdot\|^{L^1}$-norm.

We now recall the Hahn–Banach theorem stating that if $\mathbf{U}$ is a vector subspace of the normed vector space $\mathbf{W}$ and T is a bounded linear functional on $\mathbf{U}$, then T may be extended to a bounded linear functional $\hat{T}$ on $\mathbf{W}$ such that $\left\|\hat{T}\right\|^* = \|T\|^*$. In other words, there is a bounded linear functional $\hat{T} \in \mathbf{W}^*$ such that

$$\hat{T}(v) = T(v) \quad \text{for all} \quad v \in \mathbf{U}$$

and

$$\sup_{w\in\mathbf{W}} \frac{\left|\hat{T}(w)\right|}{\|w\|} = \sup_{v\in\mathbf{U}} \frac{|T(v)|}{\|v\|}.$$

By the Hahn–Banach theorem, $S\circ j^{-1}$ may be extended to a linear functional $\hat{\sigma}: L^1(\text{Int } B, \mathbb{R}^{12}) \to \mathbb{R}$ such that

$$\hat{\sigma}(\chi) = S \circ j^{-1}(\chi) \quad \text{for all} \quad \chi \in \text{Image } j,$$

or equivalently,

$$S(\phi) = \hat{\sigma}(j(\phi)) \quad \text{for every} \quad \phi \in L_1^1(\text{Int}B, \mathbb{R}^3).$$

We recall that for a linear mapping

$$M: \mathbf{W}_1 \longrightarrow \mathbf{W}_2$$

the dual mapping

$$M^*: \mathbf{W}_2^* \longrightarrow \mathbf{W}_1^*$$

is defined by

$$M^*(\sigma)(w) = \sigma(M(w)) \quad \text{for all} \quad \sigma \in w_2^*, \quad \text{and} \quad w \in \mathbf{W}_1.$$

Thus, using the definition of the dual mapping, we conclude that every $S \in L_1^1(\text{Int } B, \mathbb{R}^3)^*$ is of the form

$$S = j^*(\hat{\sigma}) \quad \text{for some} \quad \hat{\sigma} \in L^1(\text{Int}B, \mathbb{R}^{12})^* = L^\infty(\text{Int}B, \mathbb{R}^{12}).$$

3.3. Evaluation of the Dual-L_1^1-Norm

Using the second assertion of the Hahn–Banach theorem one obtains

$$\sup_{\phi\in L_1^1(\text{Int}B,\mathbb{R}^3)} \frac{|S(\phi)|}{\|\phi\|^{L_1^1}} = \sup_{\chi'\in\text{Image } j} \frac{\left|S\circ j^{-1}(\chi')\right|}{\|\chi'\|^{L^1}} = \sup_{\chi\in L^1(\text{Int}B,\mathbb{R}^{12})} \frac{\left|\hat{\sigma}(\chi)\right|}{\|\chi\|^{L^1}},$$

hence,

$$\|S\|^{L_1^{1*}} = \left\|j^*(\hat{\sigma})\right\|^{L_1^{1*}} = \left\|\hat{\sigma}\right\|^{L^{1*}} = \left\|\hat{\sigma}\right\|^{L^\infty}.$$

Generally, for any $\sigma \in L^\infty(\text{Int } B, \mathbb{R}^{12})$

$$\| j^*(\sigma) \|^{L_1^{1*}} = \sup_\phi \frac{| j^*(\sigma)(\phi) |}{\| \phi \|^{L_1^1}} = \sup_\phi \frac{| \sigma(j(\phi)) |}{\| \phi \|^{L_1^1}} \leqslant \sup_\phi \frac{\| \sigma \|^{L^\infty} \| j(\phi) \|^{L^1}}{\| j(\phi) \|^{L^1}},$$

so $\| j^*(\sigma) \|^{L_1^{1*}} \leqslant \| \sigma \|^{L^\infty}$.

We conclude that

$$\| S \|^{L_1^{1*}} = \left\| \hat{\sigma} \right\|^{L^\infty} = \inf \| \sigma \|^{L^\infty}$$

where the infimum is take over all $\sigma \in L^\infty(\text{Int } B, \mathbb{R}^{12})$, with $S = j^*(\sigma)$.

4. THE SOBOLEV EMBEDDING THEOREM AND ITS CONSEQUENCES

Now that we presented the relation between the Sobolev forces and stresses we want to relate the Sobolev forces with the applied forces in $L^{\infty,\mu}(B, \mathbb{R}^3)$ represented by body forces and surface forces. To do that, one constructs a mapping

$$\imath\colon L_1^1(\text{Int}B, \mathbb{R}^3) \longrightarrow L^{1,\mu}(B, \mathbb{R}^3)$$

that is linear, one-to-one, and continuous–an embedding. The construction of the mapping $\imath$ uses important properties of Sobolev spaces asserted by the trace theorem. The theorem is described below for the particular application for which it is needed here. Its usual formulation is much more general (see for example [2]).

4.1. Some Properties of Sobolev Mappings

Sobolev mappings defined on open sets with smooth compact boundaries (in the sense of manifolds with boundaries as we postulated here) may be extended linearly and continuously to $\mathbb{R}^3$. That is, for each $\phi \in L_1^1(\text{Int } B, \mathbb{R}^3)$ there is a field

$$\mathcal{E}(\phi)\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \text{such that} \quad \mathcal{E}(\phi)(x) = \phi(x) \quad \text{for all} \quad x \in \text{Int}B.$$

The extension is viewed as a linear and continuous mapping

$$\mathcal{E}\colon L_1^1(\text{Int}B, \mathbb{R}^3) \longrightarrow L_1^1(\mathbb{R}^3, \mathbb{R}^3),$$

so

$$\| \mathcal{E}(\phi) \|^{L_1^1} \leqslant C_1 \| \phi \|^{L_1^1},$$

for some $C_1 > 0$. (Note that the norm of $\mathcal{E}(\phi)$ uses integration over $\mathbb{R}^3$ rather than Int B.)

Sobolev mappings may be restricted continuously and linearly to integrable mappings on a smooth surface M. That is, there is a linear mapping

$$\rho\colon L^1_1(\mathbb{R}^3, \mathbb{R}^3) \longrightarrow L^1(M, \mathbb{R}^3), \quad \text{with} \quad \|\rho(\phi)\|^{L^1} \leqslant C_2 \|\phi\|^{L^1_1},$$

for some $C_2 > 0$. (Note that the restriction of an integrable function to a surface need not be integrable on that surface. Its values on that surface may be infinity everywhere.)

Combining the two, one may first extend an L^1_1-mapping from Int B to $\mathbb{R}^3$ and then restrict it to ∂B to obtain a continuous linear mapping, the trace mapping,

$$\imath_\partial = \rho \circ \mathcal{E}\colon L^1_1(\mathrm{Int}B, \mathbb{R}^3) \longrightarrow L^1(\partial B, \mathbb{R}^3).$$

Thus, there is a $C_\partial > 0$ such that

$$\|\imath_\partial(\phi)\|^{L^1} \leqslant C_\partial \|\phi\|^{L^1_1}.$$

Clearly, as any Sobolev mapping is integrable, we have also the continuous inclusion mapping

$$\imath_0\colon L^1_1(\mathrm{Int}B, \mathbb{R}^3) \longrightarrow L^1(\mathrm{Int}B, \mathbb{R}^3),$$

satisfying

$$\|\imath_0(\phi)\|^{L^1} \leqslant \|\phi\|^{L^1_1}.$$

4.2. Implications for the Present Situation

Given a mapping $\phi \in L^1_1(\mathrm{Int}\ B, \mathbb{R}^3)$, let $\imath(\phi) : B \to \mathbb{R}^3$ denote the mapping obtained by extending ϕ to $\mathbb{R}^3$ and then restricting it to B. Clearly, $\imath(\phi)$ agrees with ϕ in Int B. We have

$$\begin{aligned} \|\imath(\phi)\|^{L^{1,\mu}} &= \|\imath_\partial(\phi)\|^{L^1} + \|\imath_0(\phi)\|^{L^1} \\ &\leqslant C_\partial \|\phi\|^{L^1_1} + \|\phi\|^{L^1_1}. \end{aligned}$$

Hence,

$$\|\imath(\phi)\|^{L^{1,\mu}} \leqslant C \|\phi\|^{L^1_1}, \quad \text{where} \quad C = 1 + C_\partial.$$

It follows that we have a mapping

$$\imath\colon L^1_1(\mathrm{Int}B, \mathbb{R}^3) \longrightarrow L^{1,\mu}(B, \mathbb{R}^3),$$

which is clearly linear, one-to-one and continuous–bounded by C.

The definition of the norm of a bounded linear mapping $M: \mathbf{W}_1 \longrightarrow \mathbf{W}_2$ is

$$\|M\| = \sup_{w \in \mathbf{W}_1} \frac{\|M(w)\|}{\|w\|},$$

i.e. it is the smallest bound C_M such that $\|M(w)\| \leqslant C_M \|w\|$ for all $w \in \mathbf{W}_1$. Returning to the mapping $\imath$, we have

$$\|\imath\| = \sup_{w} \frac{\|\imath(w)\|^{L^{1,\mu}}}{\|w\|^{L_1^1}} \leqslant C.$$

4.3. *The Relation Between the $L^{\infty,\mu}$ and Dual-L_1^1 Norms of Forces*

As $\imath$ is a linear continuous injection, the dual mapping

$$\begin{aligned} \imath^*: L^{1,\mu}(B, \mathbb{R}^3)^* &= L^{\infty,\mu}(B, \mathbb{R}^3) \longrightarrow L_1^1(\mathrm{Int} B, \mathbb{R}^3)^*, \\ \imath^*(F)(w) &= F(\imath(w)), \end{aligned}$$

for all $w \in L_1^1(\mathrm{Int}\ B, \mathbb{R}^3)$, is continuous. A basic implication of the Hahn–Banach theorem is that $\|\imath^*\| = \|\imath\|$. For the sake of completeness we present the proof in the Appendix.

Thus, for every force $F \in L^{\infty,\mu}(B, \mathbb{R}^3)$, we have

$$\sup_{F} \frac{\|\imath^*(F)\|^{L_1^{1*}}}{\|F\|^{L^{\infty,\mu}}} = \|\imath^*\| = \|\imath\|,$$

where the supremum is taken over all forces in $L^{\infty,\mu}(B, \mathbb{R}^3)$.

5. CONCLUSION

The situation so far may be described by the following diagram.

$$\begin{tikzcd}
L^{1,\mu}\left(B, \mathbb{R}^3\right) & L_1^1\left(\mathrm{Int}\ B, \mathbb{R}^3\right) \arrow[l, "\imath"'] \arrow[r, "j"] & L^1\left(\mathrm{Int}\ B, \mathbb{R}^{12}\right) \\
L^{1,\mu}\left(B, \mathbb{R}^3\right)^* \arrow[r, "\imath^*"] \arrow[d, equal] & L_1^1\left(\mathrm{Int}\ B, \mathbb{R}^3\right)^* & L^1\left(\mathrm{Int}\ B, \mathbb{R}^{12}\right)^* \arrow[l, "j^*"'] \arrow[d, equal] \\
L^{\infty,\mu}\left(B, \mathbb{R}^3\right) & & L^{\infty}\left(\mathrm{Int}\ B, \mathbb{R}^{12}\right)
\end{tikzcd}$$

We now combine the results of the two previous sections and represent $\|\imath^*(F)\|^{L_1^{1*}}$ in terms of stresses to obtain

$$\|\imath\| = \sup_{F\in L^{\infty,\mu}(B,\mathbb{R}^3)} \frac{\|\imath^*(F)\|^{L_1^{1*}}}{\|F\|^{L^{\infty,\mu}}}$$
$$= \sup_{F\in L^{\infty,\mu}(B,\mathbb{R}^3)} \frac{\inf_{\sigma,\ \imath^*(F)=j^*(\sigma)} \left\{\text{ess sup}_x \{|\sigma_0(x)|, |\sigma_1(x)|\}\right\}}{\text{ess sup}_{x,y}\{|b(x)|, |t(y)|\}}.$$

We recall that $\imath^*(F) = j^*(\sigma)$ means

$$\imath^*(F)(w) = j^*(\sigma)(w),$$

so that

$$\int_{\mathrm{Int}B} b_i w_i \mathrm{d}V + \int_{\partial B} t_i w_i \mathrm{d}A = \int_{\mathrm{Int}B} \sigma_{0i} w_i \mathrm{d}V + \int_{\mathrm{Int}B} \sigma_{ik} w_{i,k} \mathrm{d}V.$$

Thus, for each F, the infimum is taken over all stress fields σ in equilibrium with F. The right-hand side becomes K and we conclude that the generalized stress concentration factor is indeed equal to the norm of the extension mapping $\imath$ of Sobolev functions.

6. DISCUSSION

We will indicate below some limitations to the possible applicability of the foregoing result in practical stress analysis. Hopefully, some of these difficulties may be overcome by further study.

Firstly, we note that "optimal" stress fields we consider include self-forces. Thus, in order to obtain the optimal stress field for any given force, one has to provide such self-forces through some mechanical apparatus or special materials. Usually the self-forces vanish so, in such cases, the tensor part of the stress field will be larger than the optimal one. In addition the tensor part of the stress object is not symmetric and the same observation applies.

A second issue that one may consider is the existence of a material for which such an "optimal" stress field exists. For simplicity, we restrict ourselves to infinitesimal elasticity. Then, one looks for a distribution of material properties, satisfying the usual requirements of continuum mechanics, such that the resulting strain components satisfy the equations of compatibility. Assuming that there is a compliance mapping at each point, giving the strain in terms of the stress tensor, the compatibility equations will yield differential equations for the distribution of the compliance over the body. Moreover, if the application of the self-force is associated with some sort of "elastic foundations", additional compatibility, between the deformation calculated from the stress tensor and the displacement associated with the "elastic foundation", should be imposed.

Finally, we note that no boundary conditions on the displacements were imposed. Thus, unless the external load F is in equilibrium, the body will accelerate. This will change the external loading because of additional unknown inertial and possibly viscous forces. Nevertheless, since our generalized stress concentration factor applies to all forces acting on the body, it applies in particular to forces that are in equilibrium. Thus, the generalized stress concentration factor may serve as an upper bound, possibly not attainable, for statics.

APPENDIX. $\|M\| = \|M^*\|$

Consider two normed vector spaces $\mathbf{W}$ and $\mathbf{V}$ and a bounded linear mapping $M\colon \mathbf{W} \to \mathbf{V}$ with norm

$$\|M\| = \sup_{w\in\mathbf{W}} \frac{\|M(w)\|}{\|w\|}\,.$$

We prove that

$$\|M^*\| = \sup_{\eta\in\mathbf{V}^*} \frac{\|M^*(\eta)\|}{\|\eta\|} = \|M\|\,.$$

We show first that $\|M^*\| \leqslant \|M\|$. Recalling that for any $\eta \in \mathbf{V}^*$,

$$\|\eta\| = \sup_{v\in\mathbf{V}} \frac{|\eta(v)|}{\|v\|},$$

we have

$$|M^*(\eta)(w)| = |\eta(M(w))| \leqslant \|\eta\|\,\|M(w)\| \leqslant \|\eta\|\,\|M\|\,\|w\|\,.$$

Hence,

$$\|M^*(\eta)\| = \sup_{w\in\mathbf{W}} \frac{|M^*(\eta)(w)|}{\|w\|} \leqslant \|\eta\|\,\|M\|\,.$$

It follows that

$$\sup_{\eta\in\mathbf{V}^*} \frac{\|M^*(\eta)\|}{\|\eta\|} = \|M^*\| \leqslant \|M\|\,.$$

This of course implies that M^* is also bounded.

We now show that $\|M^*\| \geqslant \|M\|$. We have

$$\|M^*\| = \sup_{\eta\in\mathbf{V}^*} \frac{\|M^*(\eta)\|}{\|\eta\|} = \sup_{\eta\in\mathbf{V}^*}\sup_{w\in\mathbf{W}} \frac{|M^*(\eta)(w)|}{\|\eta\|\,\|w\|} = \sup_{\eta\in\mathbf{V}^*}\sup_{w\in\mathbf{W}} \frac{|\eta(M(w))|}{\|\eta\|\,\|w\|}.$$

Thus, it is sufficient to prove that for any $\varepsilon > 0$ there is an $\eta \in \mathbf{V}^*$ and a $w \in \mathbf{W}$ such that

$$\|M\| - \frac{|\eta(M(w))|}{\|\eta\| \, \|w\|} < \varepsilon,$$

so that $\|M\| - \|M^*\| < \varepsilon$ for all $\varepsilon > 0$, implying that $\|M\| - \|M^*\| \leqslant 0$.

From the definition of $\|M\|$ there is a $w_\varepsilon \in \mathbf{W}$ such that

$$\|M\| - \frac{\|M(w_\varepsilon)\|}{\|w_\varepsilon\|} < \varepsilon.$$

Thus, for the one-dimensional subspace $\mathbf{M}$ of $\mathbf{V}$ spanned by $M(w_\varepsilon)$, i.e. all vectors of the form $u = aM(w_\varepsilon)$, $a \in \mathbb{R}$, we define the linear η_ε by $\eta_\varepsilon(aM(w_\varepsilon)) = a\,\|M(w_\varepsilon)\|$. On $\mathbf{M}$ we have

$$\left\|\eta_\varepsilon\right\|^{\mathbf{M}} = \sup_{u \in \mathbf{M}} \frac{\left|\eta_\varepsilon(u)\right|}{\|u\|} = \sup_{a \in \mathbb{R}} \frac{\left|\eta_\varepsilon(aM(w_\varepsilon))\right|}{\|aM(w_\varepsilon)\|} = 1.$$

By the Hahn–Banach theorem η_ε may be extended to a bounded linear functional $\hat{\eta}_\varepsilon \colon \mathbf{V} \longrightarrow \mathbb{R}$ with, $\hat{\eta}_\varepsilon(M(v_\varepsilon)) = a\,\|M(v_\varepsilon)\|$, for all $a \in \mathbb{R}$, and

$$\sup_{v \in \mathbf{V}} \frac{\left|\hat{\eta}_\varepsilon(v)\right|}{\|v\|} = \left\|\hat{\eta}_\varepsilon\right\| = 1.$$

Hence,

$$\|M\| - \frac{\left|\hat{\eta}_\varepsilon(M(w_\varepsilon))\right|}{\left\|\hat{\eta}_\varepsilon\right\| \, \|w_\varepsilon\|} = \|M\| - \frac{\|M(w_\varepsilon)\|}{\|w_\varepsilon\|} < \varepsilon.$$

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