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CAUCHY'S THEOREM ON MANIFOLDS

R. SEGEV AND G. RODNAY

ABSTRACT. A generalization of the Cauchy theory of forces and stresses to the geometry of differentiable manifolds is presented using the language of differential forms. Body forces and surface forces are defined in terms of the power densities they produce when acting on generalized velocity fields. The normal to the boundary is replaced by the tangent space equipped with the outer orientation induced by outward pointing vectors. Assuming that the dimension of the material manifold is m , stresses are modeled as $m - 1$ covector valued forms. Cauchy's formula is replaced by the restriction of the stress form to the tangent space of the boundary while the outer orientation of the tangent space is taken into account. The special cases of volume manifolds and Riemannian manifolds are discussed.

Keywords. Continuum mechanics, forces, stresses, manifolds, differential forms, balance laws, Cauchy's theorem.

1. INTRODUCTION

Cauchy's postulates and theorem, the fundamental notions of stress theory in continuum mechanics, are formulated traditionally using the affine and metric properties of space. Using the affine structure one is able to integrate vector fields when formulating the balance of forces. The unit normal is used in the construction of Cauchy's theorem on the existence of stress and in Cauchy's formula to which it leads. As a result, the Cauchy stress is determined by both the force system and a particular placement of the body in space. In order to separate the information regarding the force system contained in the Cauchy stress tensor from the kinematics, the Piola Kirchhoff stresses have been introduced. However, the existence of the Piola Kirchhoff stresses depends on the existence theory for stresses in space, and their representations as tensors depend on the reference configuration chosen. It is noted that Noll in his works [11], [12], and [13] defines bodies as manifolds that do not have a particular reference configuration.

Therefore, the construction of a generalized Cauchy theory of stresses on general differentiable manifolds, where the affine and metric structures are not available, is of theoretical significance. Apart from the fact that less geometrical structure will be used in the formulation, the resulting mathematical object describing the stress will unify the Cauchy and Piola Kirchhoff stresses. In fact, these stress tensors will be just representatives of the stress object for some particular affine and metric structures. The situation will be analogous to a single linear mapping

represented by various matrices for a number of bases for the corresponding vector spaces.

Additional reason to avoid the use of affine and metric structures in the formulation of stress theory originates from the various applications of the continuum mechanics methods to generalized continua. For such continua (e.g. [10], [2], [3], the state of the body is described by internal state variables, or order parameters, for which we might not have affine or metric properties.

Over the years, research work contributed to the subject by weakening the postulates needed to prove Cauchy's theorem, by making the proof more rigorous and establishing the existence of stresses using alternative methods and approaches (e.g., [7, 8], [9], [5], [15, 17]). This paper presents the generalization of Cauchy's theory of forces and stresses, to general differentiable manifolds. The terminology of differential forms is used to formulate Cauchy's postulates and Cauchy's theorem.

In a previous paper, [16], the corresponding theory was introduced for balance laws of real valued properties (such as mass). Here, the theory is extended to forces valued in the dual of a vector bundle W over an m -dimensional material manifold $\mathcal{S}$. Thus, forces are presented as entities that produce power densities when acting on generalized velocity fields—sections of W . The usual assumption that forces consist of body forces and surface forces is retained. Thus, following [4], a body force $\beta_{\mathcal{R}}$ on a body $\mathcal{R}$ is a section of $L(W, \wedge^m(T\mathcal{R}))$ and a surface force $\tau_{\mathcal{R}}$ is a section of $L(W, \wedge^{m-1}(T\partial\mathcal{R}))$. Using integration of differential forms, the power densities of the body force and surface force are combined to generate the force functional $F_{\mathcal{R}}$ as

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \beta_{\mathcal{R}}(w) + \int_{\partial\mathcal{R}} \tau_{\mathcal{R}}(w).$$

These and other preliminary topics are considered in Section 2.

As a unit normal is not available in the case of general manifolds, the local dependence of the surface force on the normal is replaced by dependence on the tangent space to the boundary—an element of the Grassmann manifold of hyperplanes in the tangent space to $\mathcal{S}$. In addition, the outwards direction of the normal is replaced by the outwards *inclination* or *outer orientation* of the tangent space to the boundary.

In the setting presented here, stresses are $(m - 1)$ -forms, valued in W^* , defined on the material manifold. The analog of the operation $S^T(\mathbf{n})$ that appears in Cauchy's theorem, is the *inclined restriction* of forms. Instead of “taking the normal component” of the stress tensor, the stress form is restricted to the tangent space of the boundary. However, while a simple restriction is not sensitive to the outwards inclination (outer orientation) of the tangent space at the point on the boundary under consideration, the inclined restriction is an odd function of the inclination. This is done using a sign rule that is induced by an orientation of the material manifold. While the orientation is assumed to be given, odd forms can be used to render the presentation independent of the arbitrary choice of orientation

(see [18]). Since the orientation is needed only locally, there is no need to require of orientability of the material manifold.

One additional aspect of the theory that needs attention is the continuous dependence of the surface force on both the point on the boundary and the hyperplane. Since, in the case of manifolds, it is not possible to compare hyperplanes at distinct points, the mapping that assigns the surface force to the inclined tangent space is presented as a continuous section—the *Cauchy section*—of a fiber bundle, the *bundle of interaction densities*, over the bundle of inclined hyperplanes. Following [18], inclined hyperplanes, Cauchy sections and inclined restrictions are considered in Section 3.

Section 4 presents the generalized Cauchy postulates and theorem. The only aspect of the theory not mentioned above is the balance equation. The balance equation is formulated in terms of the total expanded power in the form

$$|F_{\mathcal{R}}(w)| = \left| \int_{\mathcal{R}} \beta(w) + \int_{\partial \mathcal{R}} \tau_{\mathcal{R}}(w) \right| \leq \int_{\mathcal{R}} \varsigma(w),$$

where ς is some m -form that we assume to exist and that satisfies this inequality for any body $\mathcal{R}$. This form of the balance equation emphasizes the fact that we regard the balance law as a regularity assumption on the force fields for the various bodies, or alternatively, as a boundedness assumption on the collection of force functionals for the various bodies (c.f. [17]).

Section 5 considers the situation where a volume element is given on the material manifold $\mathcal{S}$ (e.g., a mass density). In such a case the stress form can be represented by a tensor—a vector bundle morphism $W \rightarrow T\mathcal{S}$. It also considers a somewhat generalized version of the normal projection of the stress. Finally, Section 6 assumes that a Riemannian metric is given on $\mathcal{S}$ and the classical Cauchy formula is recovered. As a final remark it is noted that in the current framework, the Cauchy stress and the first Piola-Kirchhoff stresses are simply two representations of the stress form under two different Riemannian metrics.

2. PRELIMINARY DEFINITIONS AND ASSUMPTIONS

2.1. Kinematics. It is assumed that there is a *material manifold* $\mathcal{S}$ of dimension m containing all bodies as subsets. A *body* is an m -dimensional compact submanifold with corners of $\mathcal{S}$.

The *physical space* is modeled as a manifold without boundary $\mathcal{M}$, and a *configuration* of a body $\mathcal{R}$ in space is a smooth mapping $\kappa: \mathcal{R} \rightarrow \mathcal{M}$. One usually requires that the configuration is an embedding although this will not have any affect on the following discussion.

A generalized *velocity* field at a given configuration κ is a mapping $w: \mathcal{R} \rightarrow T\mathcal{M}$ such that $\tau_{\mathcal{M}} \circ w = \kappa$. Here $T\mathcal{M}$ is the tangent bundle of the space manifold and $\tau_{\mathcal{M}}: T\mathcal{M} \rightarrow \mathcal{M}$ is the tangent bundle projection.

For a given configuration κ , consider the pull-back

$$\kappa^*(\tau_{\mathcal{M}}): \kappa^*(T\mathcal{M}) \rightarrow \mathcal{R},$$

of $T\mathcal{M}$ by κ . A velocity field at the configuration κ can be regarded also as the section $w': \mathcal{R} \rightarrow \kappa^*(T\mathcal{M})$, where $w'(x)$ is identified with the element $w(x) \in T_{\kappa(x)}\mathcal{M}$. Since the configuration κ will be fixed during the following discussion, the second point of view is preferable as we can regard the velocity field as a section of a fixed vector bundle. Thus, in the sequel, we will consider a fixed vector bundle $W \rightarrow \mathcal{S}$ over the material manifold. We can restrict this vector bundle naturally to any body $\mathcal{R} \subset \mathcal{S}$ and we will abuse the notation by writing $W \rightarrow \mathcal{R}$ for the restricted bundle. Thus, a velocity field over a body $\mathcal{R}$ will be a section $w: \mathcal{R} \rightarrow W$.

2.2. Forces. For a general manifold geometry, forces may be defined invariantly as covectors or linear functionals. For example, a concentrated force f acting on the material point $x \in \mathcal{S}$ at the configuration κ is an element of the cotangent space $T_{\kappa(x)}^*\mathcal{M}$. Such a force may be evaluated for a velocity $w(x)$ and the value $f(w(x))$ is interpreted as the power that the force f expends while the material point x travels at the velocity $w(x)$. Clearly, using the pullback of $T\mathcal{M}$ by κ the velocity becomes a member of W and the force will be a member of W^* .

Force densities operate on the velocity fields to produce power densities that may be integrated. It is assumed henceforth that force densities (and the resulting power densities) are continuous. Thus, a *force density* on an n -dimensional submanifold P of $\mathcal{S}$ is a continuous section of $L(W, \wedge^n(TP))$. Here $\wedge^n(TP)$ is the bundle of n -forms on P , and for two vector bundles U and V over $\mathcal{S}$, $L(U, V)$ is the vector bundle whose fiber at x is the vector space of linear mappings $U_x \rightarrow V_x$. The power the force density γ produces for the generalized velocity w is

$$\int_P \gamma(w) = \int_P \gamma(x)(w(x)).$$

In the sequel we will often abuse notation by using the same symbol for both a differential form and its value at a point.

It is usually assumed in continuum mechanics that the force on a body consists of a body force distributed over the volume of the body and a surface force distributed over the boundary. Hence, the force F on a body $\mathcal{R}$ is represented by two force densities, the *body force* β , a section of $L(W, \wedge^m(TS))$, and a *boundary* or *surface force* τ , a section of $L(W, \wedge^{m-1}(T\partial\mathcal{R}))$. Thus, for $x \in \mathcal{R}$

$$\beta(x): W_x \rightarrow \bigwedge^m T_x \mathcal{S}$$

is a linear mapping, and so is

$$\tau(y): W_y \rightarrow \bigwedge^{m-1} T_y \partial\mathcal{R}.$$

for some $y \in \partial\mathcal{R}$.

We conclude that the power expended by a combination of a body force and a surface force is of the form

$$\int_{\mathcal{R}} \beta(w) + \int_{\partial\mathcal{R}} \tau(w).$$

It is note that the last expression may be conceived as the evaluation of a linear functional F on the space of velocity fields on a particular field w to produce the global power.

Thus, we will refer to such a linear functional F as a *force acting on the body* $\mathcal{R}$ represented by $\{\beta, \tau\}$, the body force and the surface force. The power that the force F produces for the generalized velocity field w is represented in the form

$$F(w) = \int_{\mathcal{R}} \beta(w) + \int_{\partial\mathcal{R}} \tau(w).$$

So far we were concerned with one particular body $\mathcal{R}$. When the family of all bodies within the material manifold and the force operating on each is considered, we use the notation $F_{\mathcal{R}}$ to denote the force acting on the body $\mathcal{R}$. Similarly, $\beta_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ will denote the corresponding body force and boundary force. Thus, we have

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \beta_{\mathcal{R}}(w) + \int_{\partial\mathcal{R}} \tau_{\mathcal{R}}(w).$$

We will refer to the collection $\{(\beta_{\mathcal{R}}, \tau_{\mathcal{R}})\}$ for the various bodies, as a *force system* on $\mathcal{S}$. A force system may be regarded as a set function assigning to each body $\mathcal{R}$ the corresponding two force fields $(\beta_{\mathcal{R}}, \tau_{\mathcal{R}})$. Usually, it is assumed that the value of the body force at a point does not depend on the particular body containing that point. Thus, in fact, only the surface force field $\tau_{\mathcal{R}}$ depends on $\mathcal{R}$. It is the stress that allows one to specify the fields $\tau_{\mathcal{R}}$ for the various bodies with a single field on $\mathcal{S}$.

2.3. Vector Valued Forms. Mathematically, the body force β and the surface force τ are similar—at each $x \in \mathcal{R}$ they are linear mappings from W_x into a space of alternating multi-linear mappings. We will identify such objects with (co-)vector valued forms as follows.

Let $\gamma_x \in L(W_x, \wedge^k(T_x P))$, be a linear mapping, where P is an arbitrary submanifold of $\mathcal{S}$ and $k \leq \dim(P)$. For a fixed collection of k vectors $v_1, \dots, v_k$,

$$\gamma_x(u)(v_1, \dots, v_k) \in \mathbb{R}$$

is linear in u . Hence, there is an element $\psi \in W_x^*$ such that

$$\psi(u) = \gamma_x(u)(v_1, \dots, v_k).$$

Since ψ depends on the collection $v_1, \dots, v_k$ multi-linearly and in alternating way, there is an alternating mapping, a W_x^* -valued form

$$\tilde{\gamma}_x: (T_x P)^n \rightarrow W_x^*,$$

such that

$$\tilde{\gamma}_x(v_1, \dots, v_k)(u) = \gamma_x(u)(v_1, \dots, v_k).$$

It is noted that the same relation can be used to define a unique element of $L(W_x, \wedge^k(T_x P))$ once a W_x^* vector valued k -form is given. (The isomorphism is of course a particular case of isomorphisms of the type $L(E, L(E, F)) \rightarrow L^2(E, F)$ for two vector spaces E and F .)

Clearly, this can be globalized and in the sequel we will use

$$\bigwedge^k(TP, W^*)$$

for the bundle of W^* -valued k -forms. Because of the isomorphisms

$$L(W, \bigwedge^m(T\mathcal{R})) \rightarrow \bigwedge^m(T\mathcal{R}, W^*)$$

we will often view body forces as sections of either bundle and the meaning will be clear from the context. A similar convention will apply to surface forces as a result of the isomorphism

$$L(W, \bigwedge^{m-1}(T\partial\mathcal{R})) \rightarrow \bigwedge^{m-1}(T\partial\mathcal{R}, W^*).$$

3. HYPERPLANES, ORIENTATION AND INCLINATION

One expects that a theory of stresses will enable the formulation and proof of an analogue of Newton's third law of action and reaction. In order to do that, a notion of a "sense" for a tangent hyperplane should be defined so we can distinguish between the interactions on two bodies that are tangent at a point but are situated at opposite sides of the common tangent hyperplane.

In the following, a *hyperplane* H at a point x means an $(m-1)$ -dimensional subspace of $T_x\mathcal{S}$. The collection of hyperplanes at x , a Grassmann manifold, will be denoted by $G_{m-1}(T_x\mathcal{S})$. Clearly, a hyperplane H is uniquely defined by an arbitrary form ϕ in the one-dimensional subspace $H^+ \subset T_x^*\mathcal{S}$ of annihilators of H .

The union

$$G_{m-1}(T\mathcal{S}) = \bigcup_{x \in \mathcal{S}} G_{m-1}(T_x\mathcal{S})$$

is the Grassmann bundle of hyperplanes.

3.1. Inclinations of Hyperplanes. Let H be a hyperplane at x . For two vectors $v_1, v_2 \in T_x\mathcal{S}$ that are transversal to H , one says that v_1 and v_2 are *on the same side of H* if for some $\phi \in H^+$, $\phi(v_1)$ and $\phi(v_2)$ in $\mathbb{R}$ have the same sign. Otherwise, one says that the two vectors are *on opposite sides of H* . We note that if $\phi(v_1)$ and $\phi(v_2)$ have the same sign, then, $\phi'(v_1)$ and $\phi'(v_2)$ will have the same sign for any other $\phi' \in H^+$. Clearly, $-v$ and v are on opposite sides. In addition, the relation " v_1 and v_2 are on the same side of H " is an equivalence relation. The quotient space of $T_x\mathcal{S} - H$ by this equivalence relation contains two elements—the two *sides* of H . An *inclined hyperplane* is a hyperplane H together with a side—the *positive inclination* of H .

In [14] the term *outer orientation* is used for what is referred to here as "inclination". We chose to use the latter for the sake of brevity.

For an inclined hyperplane h , there exists an inclined hyperplane $-h$ of the opposite inclination. A vector $v \in T_x\mathcal{S}$ is *positively inclined* if it belongs to the equivalence class corresponding to the positive inclination. The collection of inclined hyperplanes at $x \in \mathcal{S}$ will be denoted by $G_{m-1}^\perp(T_x\mathcal{S})$.

Given an inclined hyperplane, there is a subset $H^\perp$ of H^+ containing the forms ϕ such that $\phi(v) > 0$ for any positively inclined vector v . Consider the equivalence relation by which $\phi_1 \sim \phi_2$ if $\phi_2 = a\phi_1$ for some $a > 0$. Clearly, any two elements of $H^\perp$ are so related. Thus, $H^\perp$ is an element of the quotient space $T_x^* \mathcal{S} / \sim$. Since any covector $\phi \in T_x^* \mathcal{S}$ determines a unique inclined hyperplane that it annihilates, we can identify $G_{m-1}^\perp(T_x \mathcal{S})$ with this quotient space. Clearly, the operation $h \mapsto -h$, is a natural involution on $G_{m-1}^\perp(T_x \mathcal{S})$.

In addition, we have a natural mapping

$$p_x: G_{m-1}^\perp(T_x \mathcal{S}) \rightarrow G_{m-1}(T_x \mathcal{S})$$

that ignores the inclination of the hyperplanes. For a given inclined hyperplane h , we will often use H to denote its non-inclined counterpart $p(h)$.

Using a metric to identify $H^\perp$ with a unit normal vector to h , $G_{m-1}^\perp(T_x \mathcal{S})$ can be given the differentiable structure of the $(m-1)$ -sphere. Thus, the construction can be globalized to the entire $\mathcal{S}$ manifold to generate the fiber bundle

$$G_{m-1}^\perp(T \mathcal{S}) = \bigcup_{x \in \mathcal{S}} G_{m-1}^\perp(T_x \mathcal{S}) \rightarrow \mathcal{S}$$

of inclined hyperplanes, and a fiber bundle morphism

$$p: G_{m-1}^\perp(T \mathcal{S}) \rightarrow G_{m-1}(T \mathcal{S}).$$

It is noted that if $x \in \mathcal{S}$ is on an $(m-1)$ -dimensional face on the boundary of an m -dimensional submanifold with corners $\mathcal{R}$ of $\mathcal{S}$, then, outward pointing vectors are defined invariantly as tangents to curves in $\mathcal{S}$ through x that lie outside $\mathcal{R}$ for $0 < t < \varepsilon$, and some $\varepsilon > 0$. This allows one to assign a specific inclination to $T_x \partial \mathcal{R}$.

3.2. Interaction Densities and Cauchy Sections. The basic Cauchy postulate for force systems, states that the surface force at point $x \in \partial \mathcal{R}$ depends on the outwards pointing normal to the boundary at that point. In the current framework, the dependence on the normal is replaced by dependence on the inclined hyperplane h , $T_x \partial \mathcal{R}$ with the outwards inclination.

For an inclined hyperplane h associated with $T_x(\partial \mathcal{R})$, $\tau_{\mathcal{R}}(x)$ belongs to the vector space $\bigwedge^{m-1}(H, W_x^*)$, where, $H = p(h)$. Consider the vector bundle

$$\bigwedge^{m-1} (G_{m-1}(T \mathcal{S}), W^*) \rightarrow G_{m-1}(T \mathcal{S})$$

whose fiber at $H \in G_{m-1}(T \mathcal{S})$, with $H \subset T_x \mathcal{S}$, is $\bigwedge^{m-1}(H, W_x^*)$. One can pull back this vector bundle using $p: G_{m-1}^\perp(T \mathcal{S}) \rightarrow G_{m-1}(T \mathcal{S})$ to obtain

$$D = p^* \left(\bigwedge^{m-1} (G_{m-1}(T \mathcal{S}), W^*) \right) \rightarrow G_{m-1}^\perp(T \mathcal{S}),$$

to which we will refer as the *interaction density bundle*.

Thus, mathematically, the analog of Cauchy's postulate is the assumption that there is a section, the *Cauchy section*

$$\tau: G_{m-1}^\perp(T \mathcal{S}) \rightarrow D,$$

such that

$$\tau_{\mathcal{R}}(x) = \tau(h),$$

where h is the outwards inclined tangent space to $\partial\mathcal{R}$ at x .

3.3. Inclined Restrictions of Forms. In order to formulate the generalized version of Cauchy's formula $\mathbf{t} = S^T(\mathbf{n})$ we will need the notion of inclined restriction.

Consider an oriented hyperplane h at x whose inclination is specified by the transversal vector v . We will use $\mathcal{I}: H = p(h) \rightarrow T_x\mathcal{S}$ to denote the inclusion mapping so

$$\mathcal{I}^*: \bigwedge^k (T_x\mathcal{S}, W^*) \rightarrow \bigwedge^k (H, W^*),$$

given by

$$\mathcal{I}^*(\gamma)(v_1, \dots, v_k) = \gamma(\mathcal{I}(v_1), \dots, \mathcal{I}(v_k)),$$

is the restriction of forms.

For the definition, it is assumed that an orientation of $\mathcal{S}$ is given in a neighborhood of x . (See [18] for the required generalization in case an orientation is not specified.) For an $(m-1)$, W^* -valued form $\sigma \in \bigwedge^{m-1}(T_x\mathcal{S}, W_x^*)$, the *inclined restriction* $\iota_h^*(\sigma) \in \bigwedge^{m-1}(p(h))$ is defined by the requirement that

$$\iota_h^*(\sigma)(v_1, \dots, v_{m-1}) = \mathcal{I}^*(\sigma)(v_1, \dots, v_{m-1})$$

if $(v, v_1, \dots, v_{m-1})$ are positively oriented and

$$\iota_h^*(\sigma)(v_1, \dots, v_{m-1}) = -\mathcal{I}^*(\sigma)(v_1, \dots, v_{m-1})$$

if $(v, v_1, \dots, v_{m-1})$ are negatively oriented relative to the given orientation. It follows from the definition that

$$\iota_h^*(\sigma) = -\iota_{-h}^*(\sigma).$$

Let σ be a W^* -valued, $(m-1)$ -form. Then, the mapping

$$h \mapsto \iota_h^*(\sigma) \in \bigwedge^{m-1}(p(h), W^*)$$

is clearly a Cauchy section. We will refer to the form σ as a *stress form*. Indeed, Cauchy's postulates guarantee that the surface forces are generated by a unique stress form.

4. THE GENERALIZED CAUCHY'S POSTULATES AND THEOREM

The following are the generalized Cauchy postulates for a force system $\{(\beta_{\mathcal{R}}, \tau_{\mathcal{R}}); \mathcal{R} \subset \mathcal{S}\}$.

- (i) For every $x \in \mathcal{S}$ and every body $\mathcal{R}$, $\beta_{\mathcal{R}}(x) = \beta(x)$, that is, the value of the body force at a point is independent of the body containing it. Accordingly, we will omit the subscript $\mathcal{R}$.

(ii) There is a Cauchy section

$$\tau: G_{m-1}^\perp(T\mathcal{S}) \rightarrow D,$$

such that for any body $\mathcal{R}$,

$$\tau_{\mathcal{R}}(x) = \tau(h),$$

where h is the tangent hyperplane to $\mathcal{R}$ at x with the outwards pointing inclination.

(iii) The Cauchy section τ is continuous.

(iv) There is a W^* -valued m -form ς on $\mathcal{S}$ such that

$$|F_{\mathcal{R}}(w)| = \left| \int_{\mathcal{R}} \beta(w) + \int_{\partial\mathcal{R}} \tau_{\mathcal{R}}(w) \right| \leq \int_{\mathcal{R}} \varsigma(w)$$

for every body $\mathcal{R}$.

Remark 4.1. It is noted that the last Cauchy postulate implies that $F_{\mathcal{R}}$ is a linear functional on the space of vector fields over $\mathcal{R}$ which is continuous with respect to the C^0 -topology. In fact, it is a statement of absolute continuity relative to ς .

Remark 4.2. If we consider the balance of a real valued property, or any other situation where the vector bundle W is trivial, one can consider the power for constant velocity fields. In the case of a balance of a real valued property, it is natural to define the total (resultant) force $\bar{F}_{\mathcal{R}} = F_{\mathcal{R}}(1)$. In this case, the fourth postulate implies that

$$|\bar{F}_{\mathcal{R}}| = \left| \int_{\mathcal{R}} \beta + \int_{\partial\mathcal{R}} \tau_{\mathcal{R}} \right| \leq \int_{\mathcal{R}} \varsigma.$$

This is the relevant Cauchy postulate in the case of balance laws for real valued properties considered in [16].

Remark 4.3. Let $\{F_{\mathcal{R}} = (\beta_{\mathcal{R}}, \tau_{\mathcal{R}})\}$ be a force system that satisfies Cauchy's postulates and let w be a vector field on $\mathcal{S}$. Then,

$$\{F_{\mathcal{R}}(w) = (\beta_{\mathcal{R}}(w), \tau_{\mathcal{R}}(w))\},$$

is a force system for a balance of a real valued property on $\mathcal{S}$ for which the corresponding Cauchy postulates are satisfied.

Proposition 4.4 (The generalized Cauchy theorem). Assume that an orientation is given on $\mathcal{S}$ and let $\{F_{\mathcal{R}} = (\beta_{\mathcal{R}}, \tau_{\mathcal{R}})\}$ be a force system that satisfies the generalized Cauchy postulates. Then, there exists a unique differential stress form σ , a section of $\wedge^{m-1}(T\mathcal{S}, W^*)$, such that for any body $\mathcal{R}$ and $x \in \partial\mathcal{R}$,

$$\tau_{\mathcal{R}}(x) = \iota_h^*(\sigma(x)),$$

where, h is the tangent space to $\partial\mathcal{R}$ at x equipped with the outwards pointing inclination.

Proof. The sketch of proof described below is based on the corresponding detailed proof for the real valued case in [16]. Another simple method for the construction of the proof is implied by the discussion pertaining to Riemannian manifolds (see Remark 6.2).

Since the expression in the assertion of the proposition is invariant and local, it is enough to prove that it holds in a vector bundle chart on W . As any vector field w can be written locally as a linear combination $w = w^i \mathbf{e}_i$, of the $\mu = \dim W_x$ base vector fields $\mathbf{e}_i$ corresponding to the chart, we may consider only the scalar valued surface forces $\tau_{\mathcal{R}i} = \tau_{\mathcal{R}}(\mathbf{e}_i)$. However, by Cauchy's theorem for scalar valued balance laws (see [16]), for each $i = 1, \dots, m$ there is a unique $(m-1)$ -form σ_i , such that

$$\tau_{\mathcal{R}i}(x) = l_h^*(\sigma_i(x)).$$

Clearly, the forms σ_i transform as components of an element of W^* —the required $(m-1)$, W^* -valued form σ . □

Remark 4.5. In the case where an orientation is not given on $\mathcal{S}$, the Proposition still holds if one replaces the form σ by an odd form (see [18] for the details of the required generalization). Orientation is assumed here merely to make the presentation clearer.

5. THE CASE OF VOLUME MANIFOLDS

5.1. The Stress Tensor. In this Section we restrict ourselves to the situation where a volume element θ is given on the material manifold $\mathcal{S}$. Evidently, such a volume element induces an orientation on $\mathcal{S}$. We will first make a straightforward extension of the contraction of a form by a vector.

Definition 5.1. Let $\phi \in \wedge^k(T_x \mathcal{S})$ be a form and let $S \in L(W_x, T_x \mathcal{S})$ be a linear mapping. The contraction $S \lrcorner \phi \in L(W_x, \wedge^{k-1} T_x \mathcal{S})$ of the form ϕ with the mapping S is defined by

$$(S \lrcorner \phi)(w) = S(w) \lrcorner \phi,$$

for every $w \in W_x$.

If we regard $S \lrcorner \phi$ as a vector valued form in $\wedge^{k-1}(T_x \mathcal{S}, W_x^*)$, then

$$S \lrcorner \phi(v_1, \dots, v_{k-1})(w) = \phi(S(w), v_1, \dots, v_{k-1}).$$

Proposition 5.2. Given a volume element $\theta \in \wedge^m T \mathcal{S}$ and a stress form $\sigma \in \wedge^{m-1}(T_x \mathcal{S}, W_x^*)$ there is a unique linear mapping

$$S \in L(W_x, T_x \mathcal{S})$$

such that $\sigma = S \lrcorner \theta$.

Proof. The mapping

$$L(W, T \mathcal{S}) \rightarrow L(W, \bigwedge^{m-1} (T \mathcal{S}))$$

given by $S \mapsto S \lrcorner \theta$ is an isomorphism due to the fact that for $S \neq 0$,

$$S \lrcorner \theta(w)(v_1, \dots, v_{m-1}) = \theta(S(w), v_1, \dots, v_{m-1}) \neq 0$$

for some choice of $w, v_1, \dots, v_{m-1}$. □

Thus, when a volume element is given, we obtain the representation of stresses by linear mappings as in traditional continuum mechanics. Given a volume element θ and a stress σ , we will refer to the unique linear mapping S such that $S \lrcorner \theta = \sigma$ as the *stress tensor* representing σ relative to θ .

5.2. The Orthogonal Projection of the Stress Tensor. Consider a body $\mathcal{R}$ and a point $x \in \partial \mathcal{R}$. Let h be the inclined hyperplane tangent to $\partial \mathcal{R}$ at x and let $H = p(h)$ be $T_x \partial \mathcal{R}$. In this paragraph we are concerned with the form that the relation

$$\tau_{\mathcal{R}}(x) = \iota_h^*(\sigma)$$

assumes when a volume element is given.

Using the orientation induced by the volume element θ we can define the mapping

$$\text{or}: G_{m-1}^\perp(T_x \mathcal{S}) \rightarrow \mathfrak{F}((T_x \mathcal{S})^{m-1}, \{+1, -1\}),$$

where $\mathfrak{F}((T_x \mathcal{S})^{m-1}, \{+1, -1\})$ is the set of functions from $(T_x \mathcal{S})^{m-1}$ to $\{+1, -1\}$, by

$$\text{or}(h)(v_1, \dots, v_{m-1}) = \begin{cases} +1 & \text{if } v_1, v_1, \dots, v_{m-1} \text{ are postively oriented,} \\ -1 & \text{if } v_1, v_1, \dots, v_{m-1} \text{ are negatively oriented.} \end{cases}$$

Here, v is any vector representing the inclination of h . Clearly, this mapping is odd.

Next, consider the linear mapping

$$r_H(\theta): T_x \mathcal{S} \rightarrow \bigwedge^{m-1} H,$$

given by

$$r_H(\theta)(u) = \mathcal{I}_H^*(u \lrcorner \theta),$$

where $\mathcal{I}_H^*$ is the restriction of $u \lrcorner \theta$ to H induced by the inclusion $\mathcal{I}_H$.

With this notation, the expression

$$\tau_{\mathcal{R}}(x)(w)(v_1, \dots, v_{m-1}) = \text{or}(h)(v_1, \dots, v_{m-1}) \theta(S(w), v_1, \dots, v_{m-1}),$$

where v is any vector specifying the inclination of $H = T_x \partial \mathcal{R}$, may be written in the form

$$\tau_{\mathcal{R}}(w) = \text{or}(h) r_H(\theta)(S(w)).$$

Remark 5.3. Let u be any vector in H . Clearly, $r_H(\theta)(u) = 0$ and in fact $\text{Kernel}(r_H(\theta)) = H$. In the sequel will use the following notation. For a hyperplane H at $x \in \mathcal{S}$, the one dimensional *space of layers* at x is $\Lambda = T_x \mathcal{S} / H$, i.e., a *layer* $\lambda \in \Lambda$ is an equivalence class of vectors such that the difference between any two belongs to H . We will denote by $\pi_\Lambda: T_x \mathcal{S} \rightarrow \Lambda$ the natural projection onto the quotient space.

Thus, there is a linear mapping $\hat{r}_H: \Lambda \rightarrow \bigwedge^{m-1} H$ such that

$$\hat{r}_H(\theta)(\lambda) = \hat{r}_H(\theta)(\pi_\Lambda(u)) = r_H(\theta)(u),$$

where u is any element of the equivalence class $\lambda = [u]$. Thus, the relation between the surface force and the stress tensor assumes the form

$$\tau_{\mathcal{R}}(w) = \text{or}(h) \hat{r}_H(\theta)(\pi_\Lambda \circ S(w)).$$

Remark 5.4. An annihilator ϕ of H induces an isomorphism $I_\phi: \Lambda \rightarrow \mathbb{R}$ by $\lambda \mapsto \phi(v)$, where v is any vector in the equivalence class $\lambda = [v]$. Thus,

$$I_\phi(\pi_\Lambda \circ S) = \phi \circ S = S^*(\phi),$$

where $S^*: (T_X \mathcal{S})^* \rightarrow W^*$ is the dual (adjoint) mapping to the stress tensor.

Also, I_ϕ induces the mapping

$$\rho_\phi(\theta): \mathbb{R} \rightarrow \bigwedge^{m-1} T_X \mathcal{S}, \quad \rho_\phi(\theta) = \hat{r}_H(\theta) \circ I_\phi^{-1},$$

so we have

$$\tau_{\mathcal{R}}(w) = \text{or}(h) \rho_\phi(\theta)(S^*(\phi)(w)).$$

As $\rho_\phi(\theta)$ is represented by the unique $(m-1)$ -form $\rho_\phi = \rho_\phi(\theta)(1)$ we may write

$$\tau_{\mathcal{R}}(w) = \text{or}(h) \rho_\phi S^*(\phi)(w).$$

Remark 5.5. Consider an additional body $\mathcal{R}'$ such that $x \in \partial \mathcal{R}'$ and the inclined hyperplane tangent to $\mathcal{R}'$ at x is $h' = -h$. Then

$$\tau_{\mathcal{R}'} = \text{or}(h') \rho_\phi S^*(\phi) = -\text{or}(h) \rho_\phi S^*(\phi)$$

If we require that ϕ is compatible with the inclination of h so that $\phi(v) > 0$ for every vector v specifying the inclination of h , then, the sign $\text{or}(h)$ is included in the choice of ϕ and

$$\tau_{\mathcal{R}} = \rho_\phi S^*(\phi)$$

holds for both $\mathcal{R}$ and $\mathcal{R}'$.

6. THE CASE OF RIEMANNIAN MANIFOLDS

We now assume that a Riemannian metric is given on $\mathcal{S}$. Hence, even if the volume element θ is not the one induced by the metric, we have a natural volume element $\mathbf{n} \lrcorner \theta$ on $\partial \mathcal{R}$, where $\mathbf{n}$ is the unit outwards pointing normal vector. Thus, the surface force τ is represented by a W^* -valued field $t_{\mathcal{R}}$, the component with respect to the basis element $\mathbf{n} \lrcorner \theta$, such that

$$\tau_{\mathcal{R}} = t_{\mathcal{R}}(\mathbf{n} \lrcorner \theta).$$

On the other hand, with the metric structure there is a natural form $\phi_{\mathbf{n}}$, $\phi_{\mathbf{n}}(u) = \mathbf{n} \cdot u$, compatible with the inclination of h . Thus we have

$$\begin{aligned}\rho_{\phi_{\mathbf{n}}}(v_1, \dots, v_{m-1}) &= \rho_{\phi_{\mathbf{n}}}(\theta)(1)(v_1, \dots, v_{m-1}) \\ &= \theta(u, v_1, \dots, v_{m-1}), \quad \phi_{\mathbf{n}}(u) = \mathbf{n} \cdot u = 1 \\ &= \theta(\mathbf{n}, v_1, \dots, v_{m-1}), \quad \text{for } u = \mathbf{n} \text{ in particular,} \\ &= \mathbf{n} \lrcorner \theta(v_1, \dots, v_{m-1}).\end{aligned}$$

Hence, in the metric case, choosing the natural normal form, we have

$$\rho_{\phi_{\mathbf{n}}} = \mathbf{n} \lrcorner \theta.$$

In addition,

$$\begin{aligned}S^*(\phi_{\mathbf{n}})(w) &= \phi_{\mathbf{n}}(S(w)) \\ &= \mathbf{n} \cdot (S(w)) \\ &= S^T(\mathbf{n})(w),\end{aligned}$$

where $S^T : T_x \mathcal{S} \rightarrow W_x^*$ is defined by the condition

$$v \cdot (S(w)) = S^T(v)(w).$$

We conclude that in the situation where $\mathcal{S}$ is a metric manifold,

$$t_{\mathcal{R}} = S^T(\mathbf{n})$$

—the classical Cauchy formula.

Remark 6.1. It is noted that in the reduction above it is not required that the volume element is the one induced by the Riemannian metric. Also there is no need for any additional structure on the vector bundle W .

Remark 6.2. The reduction to the classical Cauchy formula indicates another method of proof of Cauchy's theorem. Since the theorem is formulated without any reference to a Riemannian metric and since one can always assign a Riemannian metric to a manifold, Cauchy's theorem in the framework of Riemannian Geometry (e.g., [5] or [9] for the three dimensional scalar valued case) implies the validity of the generalized version of Cauchy's theorem given here.

Remark 6.3. The *First Piola-Kirchhoff Stress Tensor*, used frequently in continuum mechanics, is usually defined in term of the Cauchy stress tensor but is introduced as a different entity. (See for example [9], p. 135, or [6] p. 178.) The foregoing formulation implies that the Cauchy stress and first Piola-Kirchhoff stress are simply two representations of the same mathematical entity under different metric tensors (in space and the body reference configuration, respectively). In other words, the *Piola transformation* of [9] is simply the transformation rule for the stress tensor representing a stress form, under a change of a Riemannian metric.

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REFERENCES

- [1] R. Abraham, J.E. Marsden, & R. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Springer, New York, 1988.
- [2] G. Capriz, *Continua with Microstructure*, Springer, New York, 1989.
- [3] G. Capriz & E. Virga, Interactions in general continua with microstructure, *Archive for Rational Mechanics and Analysis*, **109** (1990), 323–342.
- [4] M. Epstein & R. Segev, Differentiable manifolds and the principle of virtual work in continuum mechanics, *Journal of Mathematical Physics* **21** (1980) 1243–1245.
- [5] R. Fosdick & E. Virga, A variational proof of the stress theorem of Cauchy, *Archive for Rational Mechanics and Analysis*, **105** (1989), 95–103.
- [6] M. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, Orlando, 1981.
- [7] M.E. Gurtin & W.O. Williams, An axiomatic foundation for continuum thermodynamics, *Archive for Rational Mechanics and Analysis* **26** (1967) 83–117.
- [8] M.E. Gurtin & L.C. Martins, Cauchy's theorem in classical physics, *Archive for Rational Mechanics and Analysis*, **60** (1976), 305–324.
- [9] J.E. Marsden & T.J.R. Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, NJ, 1983.
- [10] N.D. Mermin, The topological theory of defects in ordered media, *Reviews of Modern Physics*, **51** (1979), 591–648.
- [11] W. Noll, The foundations of Classical Mechanics in light of recent advances in continuum mechanics, pp. 266–281 of *The Axiomatic Method, with Special Reference to Geometry and Physics* (Symposium at Berkeley, 1957), North-Holland, Amsterdam, 1959.
- [12] W. Noll, Materially uniform simple bodies with inhomogeneities, *Archive for Rational Mechanics and Analysis*, **27** (1967), 1–32.
- [13] W. Noll, A new mathematical theory of simple materials, *Archive for Rational Mechanics and Analysis*, **48** (1972), 1–50.
- [14] J.A. Schouten, & W.v.d. Kulk, *Pfaff's Problem and its Generalizations*, Chelsea, New York, 1969.
- [15] R. Segev, Forces and the existence of stresses in invariant continuum mechanics, *Journal of Mathematical Physics* **27** (1986) 163–170.
- [16] R. Segev, On the geometry of Cauchy's theory for balance of scalar valued properties, *Archive for Rational Mechanics and Analysis*, to appear.
- [17] R. Segev & G. de Botton, On the consistency conditions for force systems, *International Journal of Nonlinear Mechanics* **26** (1991) 47–59.
- [18] R. Segev & G. Rodnay, Interactions on Manifolds and the Construction of Material Structure, *Int. J. of Solids and Structures*, to appear.

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