A geometrical setting for the newtonian mechanics of robots

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A Geometrical Setting for the Newtonian Mechanics of Robots

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ABSTRACT: *A geometrical setting for the Newtonian mechanics of mechanical manipulators is presented. The configuration space of the mechanical system is modelled by a differentiable manifold. The kinematics of the system is formulated on the tangent and double tangent bundles of the configuration space, and forces are defined as elements of the cotangent bundle. The dynamical properties of the system are introduced by specifying a Riemannian metric on the configuration space. The metric is used in order to generate the generalized momenta and the kinetic energy from the generalized velocities, and the connection it induces makes it possible to formulate a generalization of Newton's second law relating generalized forces and generalized accelerations.*

I. Introduction

Local properties of mechanical systems such as the equations of motion can be described by the classical treatment of mechanics. However, these formulations cannot provide a framework for the discussion of global questions such as the existence and uniqueness of the solutions of the equations of motion. While the classical control theory deals with differential equations in $\mathbf{R}^n$, the motion of a mechanical system takes place (in the holonomic case) on differentiable manifolds.

Comprehensive formulations of Lagrangian and Hamiltonian mechanics on differentiable manifolds have been available for several years, see, for example (**1**, **2**). These general formulations, which are based on the notion of a symplectic manifold, cannot be utilized directly in engineering applications because, as their names indicate, they assume that the motion is given in terms of a Lagrangian function or a Hamiltonian function, and the Newtonian notion of a force is missing.

In this paper, we present a global geometrical formulation of the dynamics of multi-degrees of freedom mechanical systems which incorporates the notion of a force. Such a formulation is suitable for the analysis of the dynamics of robots where (generalized) forces serve as control parameters. In the suggested formulation the equations of motion are written directly on the basis of a generalized invariant version of Newton's second law. For the sake of completeness and easy reference,

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we present in an informal manner all the necessary mathematical terms and we define all the mathematical notions in the context of mechanics.

Section II introduces the notion of a differentiable manifold as the configuration space of a mechanical system and Section III introduces the tangent bundle and the double tangent bundle through the notions of motion and generalized velocity. Section IV introduces the cotangent bundle of the configuration space and presents the statics of a mechanical system as an example. In Section V we introduce the Riemannian metric (kinetic energy), and the structure it induces on the configuration space. In Section VI we use the structure provided by the kinetic energy in order to relate the forces and accelerations by the equation of motion. Geometrical results concerning the case of a free motion of the system are stated in Section VII. It turns out that these results allow us to write the equations of motion in an uncoupled linearized form if the gravity forces are neglected.

II. The Configuration Space

The basic notion of any analytical study of a mechanical system is the configuration space. The configuration space of a mechanical system is the collection of all possible configurations or states of the system. Clearly, the configuration space describes the kinematical properties of the mechanical system completely. The configuration spaces of mechanical systems which are composed of material particles and rigid bodies can be associated with geometric objects. For example, the configuration space of the double planar pendulum can be identified with the surface of a torus. Given a mechanical system, we denote its configuration space by Q.

In order to apply mathematical analysis to the configuration space, generalized coordinates are introduced. The number of generalized coordinates needed to specify any configuration is the number of degrees of freedom of the system. The following assumptions allow us to assign coordinates to the various configurations in a meaningful way

(i) We assume that for every configuration q_0 in Q there exists a collection of neighboring configurations U and a function φ that assigns to any configuration q in U an $n$tuple of real numbers $\varphi(q) = (q^1(q), q^2(q), \ldots, q^n(q))$, such that the image of φ is an open subset of $\mathbf{R}^n$, and if q and q' are two distinct configurations in U, then $\varphi(q) \neq \varphi(q')$.

   The subset U is called a *coordinate neighborhood* and we say that φ is a *chart* on U and that $q^i(q)$, $i = 1, \ldots, n$, are the *coordinates* of q under φ.

(ii) It might happen that two coordinate neighborhoods U and U' with charts φ and φ' intersect and we can have the transformation of coordinates $\varphi' \circ \varphi^{-1}$ on $\varphi(U \cap U')$, which gives the coordinates $q^{i'}$ in the chart φ' in terms of the coordinates q^i in the chart φ in the form of n functions $q^{i'} = q^{i'}(q^i)$. We assume that both $\varphi' \circ \varphi^{-1}$ and its inverse transformation can be continuously differentiated as many times as we wish and it follows that the matrix $\partial q^{i'}/\partial q^i$ is nowhere singular.

The collection of coordinate neighborhoods and the charts defined on them are called an *atlas* on Q. Together with such an atlas Q has the geometric structure of an n-dimensional *differentiable manifold*. It can be shown that the coordinate neighborhoods form a basis for a topology on Q. It can also be shown that if Q is compact in that topology it cannot be covered by a single coordinate neighborhood and it follows that the torus and the sphere, for example, cannot be covered by a single coordinate neighborhood.

Let $h: Q \to Q'$ be a continuous mapping from a manifold Q into a manifold Q'. Then, by the definition of the topology on a manifold, for every coordinate neighborhood V on Q', there exists a coordinate neighborhood U on Q such that $h(U) \subset V$. However, denoting the charts on the two coordinate neighborhoods by φ and ψ, the unique relation between points on manifolds and their coordinates implies that the mapping $\psi \circ h \circ \varphi^{-1}$ associates the coordinates s^j of $h(q)$ with the coordinates q^i of any point q in U, in the form $s^j = s^j(q^i)$. The mapping $\psi \circ h \circ \varphi^{-1}$ is called a *local representative* of h. We say that a mapping $h: Q \to Q'$ is differentiable if every local representative of h is differentiable. Seemingly, this definition depends on the charts used to construct the local representatives. However, the differentiability of the coordinate transformations guarantees that if a local representative of a mapping with respect to some charts on Q and Q' is differentiable, then, any other local representative of h with the same domain on Q will be differentiable so that the notion of differentiability is well defined.

III. The Tangent Bundle

By a *motion* of the mechanical system having a configuration space Q, we mean a differentiable mapping $m: \mathbf{R} \to Q$, where $\mathbf{R}$ is the time axis. Mathematically, such a motion is called a *curve* in Q. Let q be a configuration of the mechanical system and let m and m' be two curves such that $m(0) = m'(0) = q$. If φ is a chart in some neighborhood of q then the two motions induce the motions $\varphi \circ m = (q^1(t), \ldots, q^n(t))$ and $\varphi \circ m' = (q^{1'}(t), \ldots, q^{n'}(t))$ in a neighborhood of $\varphi(q)$ in $\mathbf{R}^n$. We say that m and m' are tangent at q if $\mathrm{d}/\mathrm{d}t\,(\varphi \circ m)|_{t=0} = \mathrm{d}/\mathrm{d}t\,(\varphi \circ m')|_{t=0}$, i.e. their representatives are tangent in $\mathbf{R}^n$ at the coordinates of q. Again, it can be shown that if two motions are tangent when we use the chart φ they are tangent with respect to any other chart.

Let v denote a class of all motions that are tangent to a certain motion m at q. We call v a *tangent vector* at q or a *generalized velocity* at q and we say that m or any other motion in v represents v. The *tangent space* to Q at q is the set of all velocities at q and it is denoted by T_qQ.

We show now that T_qQ has the structure of a vector space. Let v and u be two velocities at Q, let φ be a chart in a neighborhood of q and let m and m' be motions representing v and u. Then, for a and b in $\mathbf{R}$, we define $av+bu$ to be the tangent vector whose representing motion is $\varphi^{-1}(\varphi(q)+t(a\mathbf{v}+b\mathbf{u}))$, where $\mathbf{v} = \mathrm{d}/\mathrm{d}t\,(\varphi \circ m)|_{t=0}$, $\mathbf{u} = \mathrm{d}/\mathrm{d}t\,(\varphi \circ m')|_{t=0}$, i.e. $v^i = \mathrm{d}/\mathrm{d}t\,(q^i(t))|_{t=0}$, $u^i = \mathrm{d}/\mathrm{d}t\,(q^{i'}(t))|_{t=0}$. One can easily verify that this definition satisfies all the axioms of a vector space and that it is independent of the chart φ.

The *tangent bundle* to the configuration space is defined as $TQ = \cup_{q \in Q} T_qQ$. The *tangent bundle projection* $\tau_Q: TQ \to Q$ is the mapping which assigns to every tangent vector v the configuration q where it is tangent. the tangent bundle TQ can be given the structure of a differentiable manifold in the following way. Given any tangent vector v, let φ be a chart in a neighborhood U of $q = \tau_Q(v)$. We can define the chart $\Phi: \tau_Q^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n$ by

$$\Phi(v) = (\varphi(q), \mathrm{d}/\mathrm{d}t\,(\varphi\circ m)|_{t=0}) = (q^1, \ldots, q^n, \mathrm{d}/\mathrm{d}t\,(q^1(t)), \ldots, \mathrm{d}/\mathrm{d}t\,(q^n(t)))|_{t=0},$$

where m is any motion representing v and $q^i(t)$ are the components of $\varphi\circ m$. This chart on TQ induced by the chart φ on Q is called the natural chart and it clearly satisfies the first axiom of manifolds. In addition, by the definition of the linear structure on T_qQ, $\Phi_q: T_qQ \to \mathbf{R}^n$ is an isomorphism of vector spaces. Assuming that φ' is another chart in a neighborhood of q and that Φ' is the chart it induces on the tangent bundle, we relate the coordinates (q^i, v^j) of v with respect to Φ with the coordinates $(q^{i'}, v^{j'})$ of v with respect to Φ'. Let $q^i(t)$ be the components of $\varphi\circ m$, then, by definition, $q^i = q^i(0)$, $v^j = \mathrm{d}/\mathrm{d}t\,(q^i(\mathrm{t}))|_{t=0}$. The components of $\varphi'\circ m$ will be $q^{i'}(t) = q^{i'}(q^i(t))$, and using the chain rule and the summation convention, the coordinates of v will be

$$\begin{aligned}\Phi'(v) &= (q^{1'}, \ldots, q^{n'}, v^{1'}, \ldots, v^{n'})\\ &= (q^{1'}(q^i(0)), \ldots, q^{n'}(q^i(0)), \mathrm{d}/\mathrm{d}t\,(q^{1'}(q^i(t))), \ldots, \mathrm{d}/\mathrm{d}t\,(q^{n'}(q^i(t))))|_{t=0}\\ &= \left(q^{1'}, \ldots, q^{n'}, \frac{\partial q^{1'}}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t}, \ldots, \frac{\partial q^{n'}}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t}\right)\bigg|_{t=0}\end{aligned}$$

Hence, denoting the partial differentiation $\partial q^{i'}/\partial q^i$ by $q^{i'}_{,i}$, we have $v^{i'} = q^{i'}_{,i}v^i$ and we note that $q^{i'}_{,i}$ is a vector space isomorphism.

Given a differentiable mapping $h: Q \to Q'$ there is an induced mapping Th: $TQ \to TQ'$ which is defined as follows. Let v be any tangent vector, let $\tau_Q(v) = q$ and let m be a motion representing v. Then, we define $Th(v)$ to be the tangent vector to $h(q)$ which is represented by the motion $h\circ m$ in Q'. If $q^p(q)$ are the coordinates in a neighborhood of q and $s^j(s)$ are the coordinates in some chart in the neighborhood of $h(q)$, the coordinates (s^r, u^k) of $Th(v)$ in the natural chart are given by

$$s^r = s^r(q^p),\ u^k = \mathrm{d}/\mathrm{d}t\,(s^k(q^i(t)))|_{t=0} = s^k{}_{,i}v^i,$$

where (q^p, v^i) are the coordinates of v in the natural chart on TQ. Thus, for a fixed q, the restriction T_qh: $T_qQ \to T_{f(q)}Q'$ of Th to the tangent space at q is a linear mapping and given charts in neighborhoods of q and $h(q)$, it is locally represented by the derivative of the local representative of h. The mapping Th is called the tangent mapping or the derivative of h.

The process of generating the tangent bundle can be repeated and we can construct the tangent bundle to the tangent bundle $T(TQ)$, or T^2Q. Clearly, elements of T^2Q have coordinates in $\mathbf{R}^{4n}$, and if the local representative of a curve in TQ is of the form $(q^i(t), v^j(t))$, the coordinates in the natural chart of the element of T^2Q that the curve represents are $(q^i, v^j, \mathrm{d}/\mathrm{d}t\,(q^k), \mathrm{d}/\mathrm{d}t\,(v^s))|_{t=0}$. The tangent bundle projection for this second tangent bundle is τ_{TQ}: $T^2Q \to TQ$, and locally, it is given by $(q^i, v^j, u^k, w^s) \to (q^i, v^j)$. Given a motion m: $\mathbf{R} \to Q$ there is an induced curve dm in TQ which assigns to any time t_0 the tangent vector (velocity) to m at time t_0. If $(q^i(t))$ are local representatives of this motion, the local representatives of dm are $(q^i(t), \mathrm{d}/\mathrm{d}t\,(q^i(t)))$. Similarly, m induces a curve d^2m on T^2Q which assigns to any time t_0 the acceleration at that time, i.e. if $(q^i(t))$ are the local representatives of m, the local representatives of d^2m are

$$(q^i(t), \mathrm{d}/\mathrm{d}t\,(q^i(t)), \mathrm{d}/\mathrm{d}t\,(q^i(t)), \mathrm{d}^2/\mathrm{d}t(q^i(t))).$$

The curve dm is called the lifting of m and d^2m is called the second lifting of m. Just as $dm(t)$ represents the velocity at time t, $d^2m(t)$ represents the (generalized) acceleration of the system undergoing the motion m at the time t.

As a special case of the derivative of a mapping between two manifolds we have the mapping $T\tau_Q$: $T^2Q \to TQ$, which is the tangent mapping of the tangent bundle projection. Since the local representatives of τ_Q are of the form $(q^i, v^j) \to (q^i)$, the local representatives of $T\tau_Q$ are of the form $(q^i, v^j, u^k, w^s) \to (q^i, u^k)$. We note that if w is in the image of d^2m for some motion m, then $T\tau_Q(w) = \tau_{TQ}(w)$.

A *vector field* on Q is a mapping X: $Q \to TQ$ such that $\tau_Q \circ X(q) = q$. We note that the derivative of a vector field is a mapping TX: $TQ \to T^2Q$. If the vector field is given locally in the form $(q^i, X^j(q^i))$, then the local representative of TX is of the form $(q^i, v^j) \to (q^i, X^k(q^i), v^j, X^s{}_{,r}(q^i)v^r)$. A vector field on Q induces a *first-order differential equation* on Q in the following way. Let q_0 be a configuration. We say that the motion m on Q, with $m(0) = q_0$, is the solution of the differential equation induced by X if its lifting dm is the restriction of X to the image of m. It follows that if $(q^i, X^j(q^i))$ are the coordinates of X, the coordinates of the solution curve $q^i(t)$ satisfy the equation $\mathrm{d}/\mathrm{d}t\,(q^j)(t) = X^j(q^i(t))$, which justifies the term. Similarly, a first-order equation on TQ is induced by a vector field Y: $TQ \to T^2Q$ on TQ. A vector field Y on TQ is a *second-order differential equation* if $\tau_{TQ} \circ Y = T\tau_Q \circ Y$, or equivalently, if the coordinates of Y are of the form $(q^i, v^j, v^k, Y^s(q^i))$. If $(q^i(t), v^j(t))$ are the coordinates of the solution curve on TQ to a second-order differential equation, they satisfy the relation $\mathrm{d}/\mathrm{d}t\,(q^j) = v^j$, $\mathrm{d}/\mathrm{d}t\,(v^j)(t) = Y^j(q^i(t))$, so that the use of the term is justified.

IV. The Cotangent Bundle and Forces in Statics

Let $q \in Q$ and let T^*Q_q denote the dual vector space to T_qQ. We call T^*Q_q the cotangent space to Q at q and we call $T^*Q = \cup_{q \in Q} T^*Q_q$ the *cotangent bundle* of Q. An element of T^*Q will be referred to as a *covector*, and π_Q: $T^*Q \to Q$ will denote the mapping that assigns to any covector f, the configuration q such that $f \in T^*Q_q$. We endow T^*Q with the structure of a differentiable manifold as follows. Let φ be a chart in a neighborhood U of q, then the chart Φ^*: $\pi_Q^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n$ is defined by $\Phi^*(f) = (q^i, f_j) = (q^i, f(e_j))$, where e_j is the tangent vector whose natural coordinates are $(0, \ldots, 1, \ldots, 0)$, with 1 in the jth place. In other words, if (q^i, f_j) are the coordinates of f, then $f(v) = f_j v^j$, where v^j are the natural coordinates of v. One can easily show that if a coordinate transformation $q^{i'} = q^{i'}(q^i)$ is given in a neighborhood of q, then, $f_{i'} = f_i q^i{}_{,i'}$, i.e. the coordinates of f are related by the transpose of the matrix of the partial derivatives.

Let h: $Q \to Q'$ be a differentiable mapping. Note that for $q \in Q$ we have a mapping

$T^*h_{h(q)}: T^*Q'_{h(q)} \to T^*Q_q$ which is the adjoint of $T_q h$ and therefore its local representative is the transpose of the matrix of $T_q h$.

Convectors represent mathematically the *generalized forces* and the meaning of the real number obtained by the action of a force on a generalized velocity is simply the *virtual work* that the force performs on the virtual displacement or the power supplied by the force. Indeed, in order that the virtual work will be independent of the coordinates by which we choose to represent the velocities, the transformation of coordinates for forces should be the one we obtained for covectors. For example, let B be a rigid body and let Q be its configuration space. Choosing a Cartesian coordinate system which is fixed to the body and a coordinate system which is fixed in space, we can find for each configuration q_0 a neighborhood of q_0 in which the rectangular components of the origin of the body frame and the rotations of the body axes about the space axes can serve as coordinates. In this coordinate system the components of the generalized velocity will be composed of the three components of the linear velocity of the origin of the body frame and the three components of the angular velocity of the body frame. The coordinates of a generalized force will be composed of the three components of the force acting on the body and the three components of the torque (by torque we mean the components of the generalized force that correspond to the rotations). Let A be a point in B. Using the space frame, the configuration space of A can be identified with the Euclidean space $\mathbf{R}^3$ and the coordinates of a generalized force are simply the components of the force acting on A. We can write the mapping $h: Q \to \mathbf{R}^3$ which assigns to any configuration q of the body the corresponding position of the point A in space. Then, for any configuration q, the linear mapping T^*h: $T^*\mathbf{R}^3_{h(q)} \to T^*Q_q$ will assign a force acting on the body to any force acting on A. It turns out that the torque on B due to a force $\mathbf{f}$ acting on A is given by the vector product $\mathbf{r} \times \mathbf{f}$ where $\mathbf{r}$ is the position vector of point A in the body frame.

V. The Riemannian Metric and the Kinetic Energy

Let $\mathbf{g}$: $TQ \to T^*Q$ be a differentiable mapping such that for each $q \in Q$ the restriction of $\mathbf{g}$ to q is an isomorphism $T_qQ \to T^*Q_q$. The mapping $\mathbf{g}$ induces a nonsingular bilinear form $\mathbf{K}$ on TQ by $\mathbf{K}(v,u) = \frac{1}{2}\mathbf{g}(v)(u)$. If the induced bilinear form is symmetric and positive definite $\mathbf{g}$ is called a *Riemannian metric* on Q. The dynamical properties of a mechanical system are represented by a Riemannian metric $\mathbf{g}$ in the following way. For every velocity v, the *kinetic energy* of the system at that velocity is $\mathbf{K}(v, v) = \frac{1}{2}\mathbf{g}(v)(v)$ and the generalized *momentum* is the element $\mathbf{g}(v)$ of T^*Q.

Since the restriction of $\mathbf{g}$ to T_qQ for any $q \in Q$ is a symmetric linear transformation, it is represented locally by a symmetric tensor g_{ks} called the metric tensor such that if $f = \mathbf{g}(v)$, the coordinates f_j of f are given by $g_{js}v^s$, where v^s are the coordinates of v. The local expression for $\mathbf{K}$ is given by the same tensor in the form $\mathbf{K}(v, u) = \frac{1}{2}g_{ij}v^iu^j$. Since $\mathbf{K}$ is a symmetric bilinear form it can be given the interpretation of a scalar product so that $2\mathbf{K}(u, v)$ is the scalar product between u and v, and $(2\mathbf{K}(u, u))^{1/2}$ is the length of the vector u. One can easily verify that if

$q^{i'} = q^{i'}(q^i)$ is a coordinate transformation, then the coordinate transformation for the metric tensor is $g_{i'j'} = g_{ij} q^j_{,j'} q^i_{,i'}$.

As we saw earlier, the derivative of a vector field is a mapping from TQ into T^2Q. Similarly, the acceleration is an element of T^2Q. We will see in the following that one can use the metric in order that the acceleration, and any other member of T^2Q, can be identified with an element of TQ. In such a case, the derivative of a vector field will be a linear mapping from TQ into TQ and the same will hold for a second-order differential equation.

We recall that although all the tangent spaces are isomorphic to $\mathbf{R}^n$, these isomorphisms are induced by the charts and therefore they are not unique. If however, we were given isomorphisms of the tangent spaces in a certain neighborhood of a configuration q to a given vector space, we could associate with any vector v in T_qQ a unique vector $v' \in T_{q'}Q$. Given such isomorphisms this operation is called a parallel translation.

Let m be a motion on TQ and let w denote the element of T^2Q that this motion represents. Using the parallel translation we can associate with w the element $c(w)$ of $T_q q$, $q = \tau_Q(m(0))$, which is given by $c(w) = \mathrm{d}/\mathrm{d}t\,(m'(t))|_{t=0}$, where $m'(t)$ is the result of the parallel translation of $m(t)$ to T_qQ. (We note that the differentiation makes sense because we calculate the derivative of a curve in the vector space T_qQ.) Such a mapping $c\colon T^2Q \to TQ$ is called the connection mapping of the parallel translation. Given a motion m on TQ, we can use the parallel translation to define the curve $\nabla m\colon \mathbf{R} \to TQ$ by $\nabla m(t_0) = \mathrm{d}/\mathrm{d}t\,(u')\,(t_0) = c(dm(t_0))$, where u' is the parallel translation of $m(t)$ from $T_{q'}Q$, $q' = \tau_Q(m(t))$, to T_qQ, $q = \tau_Q(m(t_0))$.

Given a vector field X whose coordinates are (q^i, X^j) and parallel translations in neighborhoods of all points in Q, we can define the *covariant derivative* $\nabla X\colon TQ \to TQ$ as follows. The restriction of ∇X to q is a linear mapping $T_qQ \to T_qQ$ whose representing, matrix is $X^i_{|j} = \partial/\partial q^j(X'^i)$, where X'^i are the coordinates of the parallel translation of $X(q')$ to T_qQ. Given a vector field Y whose coordinates are $(q^i,\ Y^j)$ we denote the evaluation of the covariant derivative of X on Y by $\nabla_Y X$. The coordinates of the vector field $\nabla_Y X$ are $X^i_{|j} Y^j$.

Assuming that a Riemannian metric is given on Q it can be shown that there is a unique parallel translation on Q that satisfies the following properties:

(i) parallel translation preserves the kinetic energy, i.e. $\mathbf{K}_{q'}(u,v) = \mathbf{K}_q(u',v')$, where u' and v' are the parallel translations of u and v from $T_{q'}Q$ to T_qQ;

(ii) if e^i is the vector field in the neighborhood of q whose coordinates satisfy $(X^j) = (0,\ldots,1,\ldots,0)$, where the 1 is in the ith place, then $\partial_{e^i} e^j = \nabla_{e^j} e^i$.

Let $(q^i,\ v^j,\ u^k,\ w^s)$ be the coordinates in the natural chart of the element w of T^2Q. It can be shown that with the unique parallel translation that has the two aforementioned properties, the coordinates of $c(w)$ are $(q^i,\ w^s + \Gamma^s_{jk} v^j u^k)$, where $\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2} g^{si}(g_{sk,j} + g_{js,k} - g_{jk,s})$ (g^{si} denote the components of the inverse matrix of g_{jk}). The coefficients Γ^i_{jk} are called the *Christoffel symbols*.

Let $(q^i(t), v^j(t))$ be the local expression for the curve m in TQ, so that the coordinate representation of dm in the standard chart in T^2Q is given by $(q^i(t),$ $v^j(t)$, $\mathrm{d}/\mathrm{d}t\,(q^k)(t)$, $\mathrm{d}/\mathrm{d}t\,(v^s)(t))$. It follows that the coordinate representation of ∇m is $(q^i(t),\ \mathrm{d}/\mathrm{d}t\,(v^s)(t) + \Gamma^s_{jk} v^k\,\mathrm{d}/\mathrm{d}t\,(q^j)(t))$. In the particular case where $m = dm_0$, where

m_0 is a curve on Q whose coordinates are (q^i), the local representative of $\nabla m = \nabla dm_0$ is $(q^i(t),\ \mathrm{d}^2/\mathrm{d}t^2(q^s)(t)+\Gamma^s_{kj}\ \mathrm{d}/\mathrm{d}t(q^k)\ \mathrm{d}/\mathrm{d}t(q^j)(t))$. Similarly, any second-order differential equation $w\colon TQ \to T^2Q$ whose local representative is $(q^i, v^j, v^k, w^s(q^i, v^j))$, induces a mapping $c\circ w\colon TQ \to TQ$, which is given locally by $(q^i, w^s+\Gamma^i_{jk}v^jv^k)$. If m_0 is the solution of the second-order differential equation w, it satisfies the equation $\nabla dm_0 = c\circ d^2m_0 = c\circ w\circ m_0$, so that second-order differential equations can be written on TQ. In addition, it can be shown that with this parallel translation we have $X^i_{|j} = X^i_{,j}+\Gamma^i_{jk}X^k$. In conclusion, we note that for a given $q\in Q$ the mapping $\mathbf{C}\colon T^2Q_{|q} \to T_qQ \times T_qQ \times T_qQ$, given by $w \to (\tau_{TQ}(w),\ T\tau_Q(w),\ c(w))$ is one to one and onto and it can be used to identify any element of T^2Q with a triplet of elements of TQ.

VI. The Equations of Motion

Consider the tangent bundle to the cotangent bundle $T(T^*Q)$, equipped with the projections $\tau_{T^*Q}\colon T(T^*Q) \to T^*Q$ and $T\pi_Q\colon T(T^*Q) \to TQ$ which are given locally in the form $(q^i, f_j, v^k, g_s) \to (q^i, f_j)$ and $(q^i, f_j, v^k, g_s) \to (q^i, v^k)$, respectively. Let p be a member of $(T^*Q)_q$ for some configuration q and consider curves m in T^*Q with the property $m(t) = q$ for each t. Since these curves do not leave $(T^*Q)_q$ they are of the form $p+f(t)$ where $f(t)\in(T^*Q)_q$ for all t, and each such curve is tangent to the line $p+tf(0)$. It follows that the collection of tangents to these curves form the vector space $T_p(T^*Q)$ and that the elements of $T_p(T^*Q)$ have natural coordinates of the form $(q^i, p_j, 0, f_k)$ where (q^i, f_k) are the coordinates of $f(0)$. Hence, given two elements p and f of $(T^*Q)_q$, an element of $T_p(T^*Q)$ is induced by the line $p+tf$. Given p in T^*Q, we denote by $\mathbf{\imath}_p$ the mapping $(T^*Q)_q \to T_p(T^*Q)$, $q = \pi_Q(p)$, given by $f \to \mathrm{d}/\mathrm{d}t\,(p = 1tf)$.This map is clearly an isomorphism and it follows that any force f can be identified with $\mathbf{\imath}_{\mathbf{0}_q}(f)$, where $\mathbf{0}_q$ is the zero element of $(T^*Q)_q$, and conversely, any element of $T_{\mathbf{0}q}(T^*Q)$ (the coordinates of such an element are in the form $(q^{i,}0, 0, f_j)$),induces a unique force whose natural coordinates are (q^i, f_j).

In general, there is no natural mapping which acts as $\mathbf{\imath}_p^{-1}$ to associate two members of T^*Q with an element of $T(T^*Q)$. However, given a Riemannian metric, the connection mapping c induces a connection mapping $c^*\colon T(T^*Q) \to T^*Q$ given by $c^* = \mathbf{g}\circ c\circ T\mathbf{g}^{-1}$ so that the following diagram is commutative.

$$\begin{tikzcd} T^2Q \arrow[r, "T\mathbf{g}"] \arrow[d, "c"'] & T(T^*Q) \arrow[d, "c^*"] \\ TQ \arrow[r, "\mathbf{g}"] & T^*Q \end{tikzcd}$$

Let (q^i, p_j, v^k, ρ_s) be the coordinates of an element ρ of $T(T^*Q)$, then the coordinates of $T\mathbf{g}^{-1}(\rho)$ are given by $(q^i, g^{jr}p_j, v^k, g^{sr}_{,t}v^tp_r+g^{sr}\rho_r)$ so that the coordinates of $c(T\mathbf{g}^{-1}(\rho))$ are $(q^i, \Gamma^s_{rk}g^{jr}p_jv^k+g^{sr}_{,t}v^tp_r+g^{sr}\rho_r)$. Thus, the coordinates of $c^*(\rho)$ are

$$(q^i, g_{ls}\Gamma^s_{rk}g^{jr}p_jv^k+g_{ls}g^{sr}_{,t}v^tp_r+g_{ls}g^{sr}\rho_r) = (q^i, g_{ls}\Gamma^s_{rk}g^{jr}p_jv^k-g^{sr}g_{ls,t}v^tp_r+\rho_l).$$

Here again c^* induces an identification $\mathbf{C}^*\colon T(T^*Q) \to T^*Q \times TQ \times T^*Q$ which is given by $\rho \to (\tau_{T^*Q}(\rho),\ T\pi(\rho),\ c^*(\rho))$.

Let $v \in T_qQ$ and let $p = \mathbf{g}(v)$ be the momentum associated with the generalized velocity v. Then, given a force $f \in (T^*Q)_g$, the *effective force* acting on the system due to f at the velocity v is $\mathbf{C}^{*-1}(p, v, f) \in T(T^*Q)_{|q}$. Assuming that (q^i, v^j) and (q^i, f_k) are the coordinates of v and f, respectively, the coordinates of the effective force due to f at the velocity v are $(q^i, g_{jr}v^r, v^k, f_l - g_{ls}\Gamma^s_{rt}v^rv^t + g_{ls,t}v^tv^s)$. Using the symmetry of the product v^tv^s and the relation between Γ^s_{rt} and g_{jr}, the last expression reduces to $(q^i, g_{jr}v^r, v^k, f_l + g_{ls}\Gamma^s_{rt}v^rv^t)$.

A *force field* $\mathbf{f}$ is a mapping $TQ \to T^*Q$ such that $\pi_Q(\mathbf{f}(v)) = \tau_Q(v)$. Given a force field $\mathbf{f}$ and a Riemannian metric $\mathbf{g}$ on the configuration space Q, a second-order differential equation $\boldsymbol{\rho}: TQ \to T^2Q$ is induced on Q by

$$\boldsymbol{\rho}(v) = T\mathbf{g}^{-1} \circ \mathbf{C}^{*-1}(\mathbf{g}(v), v, \mathbf{f}(v)).$$

Assuming that the local representatives of $\mathbf{f}$ are given in the form $(q^i, f_j(q^k, v^s))$, one can calculate the components of $T\mathbf{g}^{-1} \circ \mathbf{C}^{*-1}(\mathbf{g}(v), v, \mathbf{f}(v))$ to obtain $(q^i, v^j, v^k, g^{sr}f_r - \Gamma^s_{rt}v^rv^t)$ so that $\boldsymbol{\rho}$ is indeed a second-order differential equation. We say that $\boldsymbol{\rho}$ is the *acceleration field* induced by the force field f, and we note that if the motion m is a solution of this second-order differential equation, its local representatives $(q^i(t))$ satisfy

$$\mathrm{d}^2/\mathrm{d}t^2\,(q^i)(t) = g^{ir}f_r(q^j(t), \mathrm{d}/\mathrm{d}t\,(q^j)(t)) - \Gamma^i_{rs}\,\mathrm{d}/\mathrm{d}t\,(q^r)\,\mathrm{d}/\mathrm{d}t\,(q^s)(t).$$

From this last expression it follows that if m is a solution of the acceleration field induced by f, then m satisfies $\mathbf{g}(\nabla dm(t)) = \mathbf{f}(m(t))$ which is the required generalization of Newton's second low.

VII. Free Motion and Normal Coordinates

By a free motion or a geodesic we mean a motion of the mechanical system under the zero force field. It follows that a free notion satisfies the equation $\nabla dm = 0$, and in coordinates

$$\mathrm{d}^2/\mathrm{d}t^2(q^i)(t) + \Gamma^i_{rs}\,\mathrm{d}/\mathrm{d}t(q^r)\,\mathrm{d}/\mathrm{d}t(q^s)(t) = 0.$$

The following properties of free motion can be proved [see (**2, 3**)] for the case of a compact configuration space.

a. For every configuration q and every velocity $v \in T_qQ$ there exists a unique free motion $m(t)$ such that $m(0) = q$ and $dm(0) = v$.

b. Let $\sigma: \mathbf{R} \to \mathbf{R}$ be a change of scale of the time domain in the form $\sigma(t) = \alpha t + \beta$. Then, if $m: \mathbf{R} \to Q$ is a free motion, i.e. a solution of $\nabla dm = 0$, satisfying the initial condition $m(0) = v$, the motion $m \circ \sigma$ is also a free motion satisfying the same initial condition. In addition, the free motion m' satisfying the initial condition γv can be obtained by a reparametrization of m and it satisfies $m'(t) = m(\gamma t)$, $dm'(t) = \gamma dm(\gamma t)$.

c. The kinetic energy is conserved along a free motion, i.e.

$$\mathrm{d}/\mathrm{d}t\,(\mathbf{K}(dm(t), dm(t))) = 0.$$

d. Consider the mapping $\Phi: \mathbf{R} \times TQ \to TQ$, defined by $\Phi(t, v) = dm(t)$, where m is the unique free motion whose initial condition is v. This mapping, called the geodesic flow, can be shown to be differentiable.

The differentiable mapping $\exp: TQ \rightarrow Q$ defined by $\exp(v) = m(1) = \tau_Q(\Phi(1,v))$ is called the exponential mapping. The restriction $\exp_q: T_qQ \rightarrow Q$ of the exponential mapping to T_qQ induces a one to one and onto differentiable mapping from a neighborhood V of the zero vector in T_qQ onto a neighborhood U of q in Q. Thus, choosing a basis in T_qQ we can construct a chart on U in the following way. For every point q' in U let its coordinates v^i be the components of the vector v in V such that $q' = \exp(v) = m(1)$ (m is the free motion with initial condition v). This chart in the neighborhood U of q is called the normal chart. In particular, we can choose an orthonormal basis in T_qQ so that $g_{ij} = \delta_{ij}$ (the Kronecker δ) in the normal chart. It can be shown that in the normal chart we have $\Gamma^i_{jk} = 0$. Thus, using the compactness assumption, it follows that Q can be covered by a finite number of normal charts in which the equations of motion are of the form

$$\frac{d^2q^i}{dt^2} = f_i.$$

e. The configuration space Q is geodesically complete, i.e. for any two configurations q and q' there exists a free motion which starts at q and ends at q'.
f. Given the configurations q and q', the functional

$$L(m) = \int_q^{q'} (\mathbf{K}(dm(t), dm(t))^{1/2}\, dt$$

which can be interpreted as the length of the curve joining the two configurations, and the functional

$$E(m) = \int_q^{q'} (\mathbf{K}(dm(t), dm(t))\, dt$$

are at a local minimum if m is a free motion between q and q'.

VIII. Conclusions

In the preceding sections, we presented the Newtonian mechanics of mechanical manipulators from a geometric point of view. Generalized forces that serve as the control parameters in the control of robots were introduced as elements of the cotangent bundle of the configuration space. The kinetic energy was modelled by a Riemannian metric and the isomorphism it induces between the tangent and the cotangent spaces was used in order to obtain the generalized momentum corresponding to a given generalized velocity. The generalized accelerations, which are elements of the second tangent bundle, were related to the forces by using the connection induced by the Riemannian metric (kinetic energy). To the best of our knowledge this is the first geometric invariant formulation of the equations of motion for a multi-degrees of freedom mechanical system in the form of Newton's second law. In addition, the standard form of the equations of motion found in the literature, i.e.

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{B}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{f}$$

which is written in a coordinate non-invariant form, can be interpreted now as follows: **M** is the matrix of the metric tensor (kinetic energy) and **B** is composed of the covariant Christoffel symbols $\Gamma_{ijk} = g_{ir}\Gamma^r_{jk}$ of the connection induced by the metric.

The introduction of the normal coordinates provides us with a procedure for the linearization and decoupling of the equations of motion in the case where we do not consider generalized forces due to gravity [*cf.* (4, 5)]. We note that the configuration spaces of most mechanical manipulators are compact differentiable manifolds. For example, the configuration spaces of mechanical systems composed of rigid bodies that are joined together so that one of the bodies rotates about a fixed point and the rest of the degrees of freedom of the system are rotations, are compact manifolds. It follows that the results of the previous section regarding compact manifolds are all applicable.

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