Artificial boundary conditions to compute correctors in linear elasticity

V. Bonnaillie-Noël[*], D. Brancherie[†], M. Dambrine[‡] and G. Vial[§]

1 Introduction

This work is motivated by the asymptotic analysis for a boundary singular perturbation in an elliptic boundary value problem. Let Ω_0 be a bounded domain of $\mathbb{R}\times\mathbb{R}^+$, with the origin $\mathbf{0}$ being a regular point on its boundary. In the following, the first coordinate axis coincides with the tangent direction of $\partial\Omega_0$ at point $\mathbf{0}$. We denote by ω another bounded domain containing $\mathbf{0}$. The perturbed domain Ω_ε is obtained from Ω_0 by removing a rescaled version of ω at size ε: $\Omega_\varepsilon = \Omega_0\backslash(\overline{\varepsilon\omega})$. The problem we focus on is the following

$$\begin{cases} -\mu\Delta\mathbf{u}_\varepsilon - (\lambda+\mu)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u}_\varepsilon = \mathbf{f} \text{ in } \Omega_\varepsilon,\\ \mathbf{u}_\varepsilon = \mathbf{u}^d \text{ on } \Gamma_d,\\ \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{g} \text{ on } \Gamma_t, \end{cases} \tag{1}$$

where Γ_d and Γ_t are the Dirichlet and Neumann boundaries, respectively. It concerns Navier equations of *linear elasticity*. The perturbation point $\mathbf{0}$ lies on the Neumann boundary. Besides, the volumic load $\mathbf{f}$ and the traction $\mathbf{g}$ are assumed to vanish near the perturbation. Problem (1) naturally involves two scales: the size of the structure (*scale* 1), and the characteristic length ε of the perturbation (*scale* ε). At scale 1, the domain Ω_ε tends to Ω_0 as $\varepsilon\to 0$, while the limit domain after rescaling (i.e. limit of $\Omega_\varepsilon/\varepsilon$) is the semi-infinite domain $\mathbf{H}_\infty$ defined as $\mathbf{H}_\infty = (\mathbb{R}\times\mathbb{R}^+)\backslash\overline{\omega}$. We briefly present the leading terms in the asymptotic expansion of the solution $\mathbf{u}_\varepsilon$ to Problem (1) obtained by following the methods presented in [5]. This description requests two variables: x (*slow variable*) and x/ε (*fast variable*), corresponding to scale 1 and scale ε, respectively. The behavior of $\mathbf{u}_\varepsilon$ in the fast variable relies on *profiles*, which are normalized functions defined in $\mathbf{H}_\infty$ and contributing to the expansion in variable x/ε. Let us introduce the profile basis $(\mathbf{V}_\ell)_{\ell=1,2}$ as the solutions of

$$\begin{cases} -\mu\Delta\mathbf{V}_\ell - (\lambda+\mu)\,\mathbf{grad}\,\mathrm{div}\mathbf{V}_\ell = \mathbf{0} \quad \text{ in } \mathbf{H}_\infty,\\ \sigma(\mathbf{V}_\ell)\cdot\mathbf{n} = \mathbf{G}_\ell \quad \text{ on } \partial\mathbf{H}_\infty, \end{cases} \tag{2}$$

with $\mathbf{G}_1 = (n_1, 0)$, $\mathbf{G}_2 = (0, n_1)$ and $\mathbf{n} = (n_1, n_2)$ the outer normal on $\partial\mathbf{H}_\infty$. Regularity of the solution of Navier equation has been studied, for example, in [4]. We emphasize that Problem (2) depends exclusively on the geometry. In particular, $\mathbf{V}_\ell$ is independent of the loading of Problem (1).

The expansion of $\mathbf{u}_\varepsilon$ takes the form

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) - \varepsilon\left[\alpha_1\mathbf{V}_1\left(\frac{\mathbf{x}}{\varepsilon}\right) + \alpha_2\mathbf{V}_2\left(\frac{\mathbf{x}}{\varepsilon}\right)\right] + \mathbf{r}_\varepsilon(\mathbf{x}), \tag{3}$$

with

- $\mathbf{u}_0$ solves Problem (1) for $\varepsilon = 0$ (i.e. is solution in the unperturbed domain Ω_0),
- the coefficients α_1 and α_2 are the stress values associated with $\mathbf{u}_0$ at point $\mathbf{0}$, namely

$$\alpha_1 = \sigma_{11}(\mathbf{u}_0)(\mathbf{0}) \quad \text{and} \quad \alpha_2 = \sigma_{12}(\mathbf{u}_0)(\mathbf{0}),$$

[*]IRMAR, ENS Cachan Bretagne, Univ. Rennes 1, CNRS, UEB, av. Robert Schuman, F-35170 Bruz, France
[†]Roberval, Université de Technologie de Compiègne, CNRS, F-60200 Compiègne, France
[‡]LMA, Université de Pau et des Pays de l'Adour, CNRS, F-64000 Pau, France
[§]Institut Camile Jordan, Ecole Centrale de Lyon, CNRS, F-69130 Ecully, France

- the remainder $\mathbf{r}_\varepsilon$ satisfies the following estimate in the energy norm

$$\|\mathbf{r}_\varepsilon\|_{\mathrm{H}^1(\Omega_\varepsilon)} \leq C\varepsilon^2. \tag{4}$$

As regards the computation of $\mathbf{u}_\varepsilon$ for small values of ε, asymptotics expansion (3) leads to an alternative numerical strategy to adaptative mesh refinement. Indeed the knowledge of the unperturbed solution $\mathbf{u}_0$ and of the profiles $\mathbf{V}_\ell$ gives a reasonable approximation of $\mathbf{u}_\varepsilon$, see [1]. To approximate the profile defined by, we compute a solution of a problem posed in a bounded domain. To that end, we introduce an artificial boundary at $\partial B_R = \{|x| = R,\ R > 0\}$ and we need to impose a boundary condition on this boundary in the spirit of [3]. Since, the corrector decreases at infinity, the most natural artificial condition is the Dirichlet one. However, this condition is not precise.

In this note, we present the derivation of a higher order artificial boundary condition, we explain that this condition sparks theoretical questions and finally present some numerical simulations.

2 Seeking an artificial boundary condition

To derive artificial boundary conditions for the linear elasticity, the domain under consideration here is the upper half plane $\mathbb{R} \times \mathbb{R}^+$. Since we aim at writing artificial conditions on a circle of radius R, it is more convenient to work with the polar coordinates and we define $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$. To approximate more accurately the profile, we cancel the leading singular parts at infinity of the solution in polar coordinates.

2.1 Computing the leading terms at infinity

Let $L = \mu\Delta + (\lambda + \mu)\,\mathbf{grad}\,\mathrm{div}$, then

$$\begin{aligned} L\mathbf{u} &= \left((\lambda + 2\mu)\left[\partial_r^2 u_r + \frac{1}{r}\partial_r u_r - \frac{1}{r^2}u_r\right] + \frac{\mu}{r^2}\partial_\theta^2 u_r - \frac{\lambda + 3\mu}{r^2}\partial_\theta u_\theta + \frac{\lambda + \mu}{r}\partial_{r\theta}^2 u_\theta\right)\mathbf{e}_r \\ &\quad + \left(\mu\left[\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{1}{r^2}u_\theta\right] + \frac{\lambda + 2\mu}{r^2}\partial_\theta^2 u_\theta + \frac{\lambda + \mu}{r}\partial_{r\theta}^2 u_r + \frac{\lambda + 3\mu}{r^2}\partial_\theta u_r\right)\mathbf{e}_\theta. \end{aligned}$$

In polar coordinates, the stress tensor takes the form:

$$\sigma(\mathbf{u}) = \begin{bmatrix} (\lambda + 2\mu)\partial_r u_r + \frac{\lambda}{r}(u_r + \partial_\theta u_\theta) & \mu\left(\frac{1}{r}(\partial_\theta u_r - u_\theta) + \partial_r u_\theta\right) \\ \mu\left(\frac{1}{r}(\partial_\theta u_r - u_\theta) + \partial_r u_\theta\right) & (\lambda + 2\mu)\frac{1}{r}(\partial_\theta u_\theta + u_r) + \lambda\partial_r u_r \end{bmatrix}. \tag{5}$$

We seek the displacement under the form $\mathbf{u} = r^s[\varphi_r(\theta), \varphi_\theta(\theta)]^T$. Consequently, we obtain

$$L\mathbf{u} = r^{s-2}\begin{bmatrix} \mu\varphi_r'' + (\lambda + 2\mu)(s^2 - 1)\varphi_r + [(\lambda + \mu)s - (\lambda + 3\mu)]\varphi_\theta' \\ (\lambda + 2\mu)\varphi_\theta'' + \mu(s^2 - 1)\varphi_\theta + [(\lambda + \mu)s + (\lambda + 3\mu)]\varphi_r' \end{bmatrix}, \tag{6}$$

$$\sigma(\mathbf{u}) = r^{s-1}\begin{bmatrix} \lambda\varphi_\theta' + ((\lambda + 2\mu)s + \lambda)\varphi_r & \mu(\varphi_r' + (s - 1)\varphi_\theta) \\ \mu(\varphi_r' + (s - 1)\varphi_\theta) & (\lambda + 2\mu)\varphi_\theta' + (\lambda s + (\lambda + 2\mu))\varphi_r \end{bmatrix}. \tag{7}$$

Using (6), we reduce the second order system $L\mathbf{u} = 0$ to a bigger system of first order: Define $\psi_r = \varphi_r'$ and $\psi_\theta = \varphi_\theta'$, and $\mathbf{U} = (\varphi_r, \varphi_\theta, \psi_r, \psi_\theta)^T$, then

$$\mathbf{U}' = A\mathbf{U},$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(\lambda+2\mu)(1-s^2)}{\mu} & 0 & 0 & \frac{(\lambda+3\mu)-(\lambda+\mu)s}{\mu} \\ 0 & \frac{\mu(1-s^2)}{\lambda+2\mu} & -\frac{(\lambda+3\mu)+(\lambda+\mu)s}{\lambda+2\mu} & 0 \end{bmatrix}.$$

The eigenvalues of A are $\pm i(s\pm 1)$. Hence, the polar functions φ_r, φ_θ belong to the space generated by $\cos((s\pm 1)\theta)$, $\sin((s\pm 1)\theta)$. We look for coefficients $A_r, B_r, A_\theta, B_\theta$ so that

$$\begin{aligned} \varphi_r &= A_r\cos((s-1)\theta) + B_r\sin((s-1)\theta) + C_r\cos((s+1)\theta) + D_r\sin((s+1)\theta), \\ \varphi_\theta &= A_\theta\cos((s-1)\theta) + B_\theta\sin((s-1)\theta) + C_\theta\cos((s+1)\theta) + D_\theta\sin((s+1)\theta). \end{aligned}$$

Writing $L\mathbf{u} = 0$ gives:

$$\left\{ \begin{aligned} \alpha A_r + B_\theta &= 0 \\ \alpha B_r - A_\theta &= 0 \\ C_r + D_\theta &= 0 \\ D_r - C_\theta &= 0 \end{aligned} \right. \qquad \text{with} \qquad \alpha = \frac{(\lambda+\mu)s + (\lambda+3\mu)}{(\lambda+\mu)s - (\lambda+3\mu)}. \tag{8}$$

Consequently, the functions φ_r and φ_θ satisfy:

$$\varphi_r = A_r\cos((s-1)\theta) + B_r\sin((s-1)\theta) + C_r\cos((s+1)\theta) + D_r\sin((s+1)\theta), \tag{9}$$

$$\varphi_\theta = \alpha B_r\cos((s-1)\theta) - \alpha A_r\sin((s-1)\theta) + D_r\cos((s+1)\theta) - C_r\sin((s+1)\theta). \tag{10}$$

The boundary conditions read: $\sigma(\mathbf{u})\cdot\mathbf{e}_\theta = \mathbf{0}$ for $\Theta = 0, \pi$, that is to say

$$\left\{ \begin{aligned} \varphi_r'(\Theta) + (s-1)\varphi_\theta(\Theta) &= 0, \\ (\lambda+2\mu)\varphi_\theta'(\Theta) + ((\lambda+2\mu)s + \lambda)\varphi_r(\Theta) &= 0. \end{aligned} \right.$$

Boundary conditions at $\theta = 0$ reads

$$\left\{ \begin{aligned} (s-1)B_r + (s+1)D_r + (s-1)(A_\theta + C_\theta) &= 0, \\ (\lambda+2\mu)((s-1)B_\theta + (s+1)D_\theta) + (\lambda s + \lambda + 2\mu)(A_r + C_r) &= 0. \end{aligned} \right.$$

Boundary conditions at $\theta = \pi$ reads

$$\left\{ \begin{aligned} \sin s\pi[(s-1)(A_r - B_\theta - D_\theta) + (s+1)C_r] - \cos s\pi[(s-1)(A_\theta + B_r + C_\theta) + (s+1)D_r] &= 0 \\ \sin s\pi\,((\lambda+2\mu)[(s-1)(A_\theta + (s+1)C_\theta] - (\lambda+2\mu+\lambda s)(B_r + D_r)) & \\ -\cos s\pi\,((\lambda+2\mu)[(s-1)B_\theta + (s+1)D_\theta] + (\lambda s + \lambda + 2\mu)(A_r + C_r)) &= 0 \end{aligned} \right.$$

Using (8), we deduce that the coefficients A_r, B_r, C_r, D_r have to satisfied

$$M(A_r, B_r, C_r, D_r)^T = 0,$$

with

$$M = \begin{bmatrix} 0 & \gamma & 0 & 2s \\ \beta & 0 & -2\mu s & 0 \\ \gamma\sin s\pi & -\gamma\cos s\pi & 2s\sin s\pi & -2s\cos s\pi \\ -\beta\cos s\pi & -\beta\sin s\pi & 2\mu s\cos s\pi & 2\mu s\sin s\pi \end{bmatrix},$$

and

$$\beta = (1+\alpha)(\lambda+2\mu) + 2(\lambda - \alpha(\lambda+2\mu)), \qquad \gamma = (s-1)(1+\alpha).$$

We compute

$$\det M = -4(\sin s\pi)^2(\mu\gamma + \beta)^2.$$

Therefore, $s \mapsto \det M$ cancels only for $s \in \mathbb{Z}$ and for s such that $\mu\gamma + \beta = 0$ that is to say

$$s = \frac{\mu - \lambda + \sqrt{\lambda^2 + 6\,\lambda\,\mu + 25\,\mu^2}}{2\mu} > 2 \quad \text{and} \quad s = \frac{\mu - \lambda - \sqrt{\lambda^2 + 6\,\lambda\,\mu + 25\,\mu^2}}{2\mu} < -2.$$

Then the leading term is obtained for $s = -1$. For $s = -1$, we have

$$\alpha = -\frac{\mu}{\lambda+2\mu}, \qquad \beta = 0, \qquad \gamma = -\frac{2(\lambda+\mu)}{\lambda+2\mu}.$$

In this case, system $M(A_r, B_r, C_r, D_r)^T = 0$ and (8) read

$$\gamma B_r - 2D_r = 0, \qquad 2\mu C_r = 0, \qquad C_\theta = D_r.$$

Consequently, setting $A = A_r$ and $B = -B_r$ in (9), the functions φ_r and φ_θ take the form

$$\left\{ \begin{aligned} \varphi_r &= A\cos 2\theta + B\sin 2\theta, \\ \varphi_\theta &= \frac{\mu}{\lambda+2\mu}(B\cos 2\theta - A\sin 2\theta) + B\frac{\lambda+\mu}{\lambda+2\mu}. \end{aligned} \right.$$

2.2 Using the singular profiles to obtain a transparent boundary condition for the half plane

We go back to the expression for $\sigma(\mathbf{u}) \cdot \mathbf{e}_r$ deduced from (7). When $s = -1$, we have

$$\sigma(\mathbf{u}) \cdot \mathbf{e}_r = r^{-2} \begin{bmatrix} \lambda\varphi'_\theta - 2\mu\varphi_r \\ \mu(\varphi'_r - 2\varphi_\theta) \end{bmatrix} = r^{-1} \begin{bmatrix} -\dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0 \\ -\dfrac{2\mu(\lambda+\mu)\sin 2\theta}{\mu+(\lambda+\mu)\cos 2\theta} & \dfrac{2\mu(\lambda+\mu)(1-\cos 2\theta)}{\mu+(\lambda+\mu)\cos 2\theta} \end{bmatrix} \mathbf{u}.$$

This means that the relation between $\mathbf{u}$ and $\sigma(\mathbf{u}) \cdot \mathbf{n}$ on ∂B_R has variable (in the θ variable) coefficients, this fact makes the interpretation of this computation more difficult. Hence, we search a relation with the second order tangential derivative of the displacement: If such a relationship exists, the associated bilinear form still remains symmetric. Notice that

$$\varphi'_r = -\frac{\lambda+2\mu}{2\mu}\,\varphi''_\theta,$$

we can also write

$$\sigma(\mathbf{u}) \cdot \mathbf{e}_r = r^{-2} \begin{bmatrix} -\dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\varphi_r \\ \mu\varphi'_r - 2\mu\varphi_\theta \end{bmatrix} = \frac{1}{R} \begin{bmatrix} -\dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\,u_r \\ -2\mu u_\theta - \dfrac{\lambda+2\mu}{2}\,\partial^2_\theta u_\theta \end{bmatrix}.$$

For the elasticity problem, we observe that the leading decreasing profile at infinity satisfies the following inequality on the circle of radius R:

$$\sigma(\mathbf{u}) \cdot \mathbf{n} + \frac{2\mu}{R} \begin{bmatrix} \frac{2(\lambda+\mu)}{\lambda+2\mu} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} + \frac{\lambda+2\mu}{2R} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial^2_\theta \mathbf{u} = 0.$$

Lamé's coefficients are linked to Young's modulus and Poisson's coefficient throught the relations:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \qquad \text{with} \qquad E > 0, \quad -1 < \nu < 0.5.$$

So that, this boundary condition of Ventcel's type rewrites on $\partial B_R \cap \mathbb{R} \times \mathbb{R}^+$:

$$\sigma(\mathbf{u}) \cdot \mathbf{n} + \frac{1}{R}\frac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} + \frac{1}{R}\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial^2_\theta \mathbf{u} = 0. \tag{11}$$

The main difficulty here is that the coefficient $\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}$ is positive; meaning that the variational formulation of the boundary value problem for the profiles $\mathbf{V}_\ell$

$$\begin{cases} -\mu\Delta\mathbf{V}_\ell - (\lambda+\mu)\,\mathbf{grad}\,\mathrm{div}\mathbf{V}_\ell = \mathbf{0} \quad \text{in } B_R \cap \mathbf{H}_\infty, \\ \sigma(\mathbf{V}_\ell) \cdot \mathbf{n} = \mathbf{G}_\ell \quad \text{on } \partial\mathbf{H}_\infty \cap B_R, \\ \sigma(\mathbf{V}_\ell) \cdot \mathbf{n} + \dfrac{1}{R}\dfrac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{V}_\ell + \dfrac{E(1-\nu)}{2(1+\nu)(1-2\nu)R} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial^2_\theta \mathbf{V}_\ell = 0 \quad \text{on } \partial B_R \cap \mathbf{H}_\infty, \end{cases} \tag{12}$$

is not coercive. Therefore, the question of existence and uniqueness of the solution to (12) is open. For the model scalar case of

$$\begin{cases} -\Delta V = 0 \text{ in } \Omega, \\ \partial_n V + \alpha V + \beta\Delta_\Gamma V = G \text{ on } \partial\Omega, \end{cases}$$

with $\beta > 0$ and Δ_Γ denotes the Laplace-Beltrami operator on $\partial\Omega$, it has been shown in [2] that, for a fixed R, the boundary value problem has a unique solution if α/β avoid a countable set of values, and moreover, that it has a unique solution for R large enough. We expect similar results for the elasticity case and we work in that direction.

3 Numerical results

In order to investigate Problem (12) from a numerical point of view, we have considered the test situation where the domain ω is the top half of a ball: $\mathbf{H}_\infty = \mathbb{R}\times\mathbb{R}^+ \setminus B(0,1)$. We have successively considered the following two questions:

(i) resolubility of the Ventcel-type boundary condition (11),

(ii) order of precision of this condition as an artificial boundary condition for Problem (2).

For the first question, we consider the norm of the inverse operator $(L_R)^{-1}$ corresponding to Problem (12):

$$L_R : \mathbf{V}_\ell \mapsto \mathbf{G}_\ell.$$

In Figure 1, we show the evolution of this norm with respect to the radius R. The computation have been performed with the finite element library MÉLINA (see [6]), the domain (a ring) has been meshed into 128 quadrangular elements of degree 8, and a $\mathbb{Q}_{10}$ interpolation is used for the finite element method.

Figure 1 shows some values of R for which the resolvent is unbounded, but theses values remain close to 1. It seems that Problem (12) is uniquely solvable for R sufficiently large (which is the result shown for the scalar model problem).

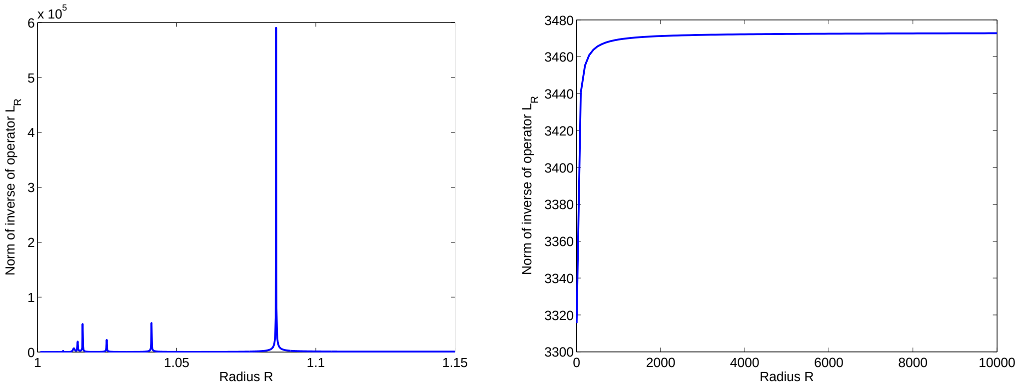

Figure 1: Norm of $(L_R)^{-1}$ with respect to R.

For question *(ii)*, we consider Problem (12) where the data $\mathbf{G}_\ell$ is chosen so that the solution $\mathbf{V}_\ell$ is the sum of two dual singular solutions:

$$\mathbf{V}_\ell = \mathbf{s}^1 + \mathbf{s}^2,$$

where $\mathbf{s}^1$ is given by (2.1) and $\mathbf{s}^2$ has the form

$$\mathbf{s}^2(r,\theta) = r^{-2}\left[\varphi_r^2(\theta)\mathbf{e}_r + \varphi_\theta^2(\theta)\mathbf{e}_\theta\right],$$

with

$$\varphi_r^2(\theta) = A'\cos(3\theta) - B'\sin(3\theta) + \frac{3\alpha\lambda-\lambda-4\mu}{4\mu}A'\cos\theta + \frac{3}{4}(1+\alpha)B'\sin\theta,$$

$$\varphi_\theta^2(\theta) = \alpha B'\cos(3\theta) + \alpha A'\cos(3\theta) - \frac{3}{4}(1+\alpha)B'\cos\theta + \frac{3\alpha\lambda-\lambda-4\mu}{4\mu}A'\sin\theta,$$

and

$$\alpha = \frac{-\lambda+\mu}{-3\lambda-5\mu}.$$

With this choice, the function $\mathbf{V}_\ell$ solves Problem (2) in the infinite domain $\mathbf{H}_\infty$. In Figure 2, we show the H^1-norm of the difference between the exact solution of (2) and its approximation with artificial boundary conditions (Dirichlet or Ventcel). It turns out that – as expected – the Dirichlet condition is of order 1 and the Ventcel condition of order 2.

Remark 1 *Writing the variational formulation of* (12) *makes appear ponctual terms which need a specific treatment for the numerical approach with the finite element method.*

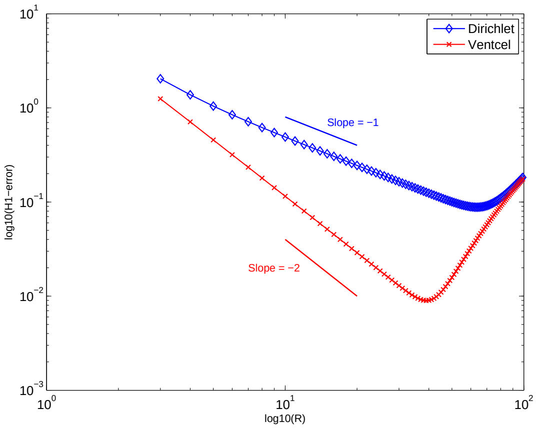

Figure 2: Precision of the artificial conditions with respect to R.

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