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Zero curvature representation of non-commutative and quantum Painlevé II equation with its non-vacuum solutions

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Abstract: In this paper, I derive a zero curvature representation of quantum Painlevé II equation and its Riccati form which can be reduced to the classical Painlevé II when $\hbar \to 0$. Further I derive non-vacuum solitonic solutions of the noncommutative Painlevé II equation with the help of its Darboux transformation for which the solution of the noncommutative Painlevé Riccati equation has been taken as a seed solution.

Keywords: Integrable systems; Painlevé II equation; zero curvature condition; Riccati equation; Darboux transformation.

2000 Mathematics Subject Classification: 22E46, 53C35, 57S20

1. Introduction

The Painlevé equations were discovered by Painlevé and his colleagues when they have classified the nonlinear second-order ordinary differential equations with respect to their solutions [1].The study of Painlevé equations is important from mathematical point of view because of their frequent appearance in the various areas of physical sciences including plasma physics, fiber optics, quantum gravity and field theory, statistical mechanics, general relativity and nonlinear optics. The classical Painlevé equations are regarded as completely integrable equations and obeyed the Painlevé test [2,3,4]. These equations admit some properties such as linear representations, hierarchies, they possess Darboux transformations(DTs) and Hamiltonian structure. These equations also arise as ordinay differential equations (ODEs) reduction of some integrable systems, i.e, the ODE reduction of the KdV equation is Painlevé II (PII) equation[5,6].

The noncommutative(NC) and quantum extension of Painlevé equations is quite interesting in order to explore the properties which they possess with respect to usual Painlevé systems on ordinary spaces. NC spaces are characterized by the noncommutativity of the spatial co-ordinates. For example, if x^μ are the space co-ordinates then the noncommutativity is defined by $[x^\mu, x^\nu]_\star = i\theta^\mu$ where parameter $\theta^{\mu\nu}$ is anti-symmetric tensor and Lorentz invariant and $[x^\mu, x^\nu]_\star$ is commutator under the star product. NC field theories on flat spaces are given by the replacement of ordinary products with the Moyal-products and realized as deformed theories from the commutative ones.

[*]

Moyal product for ordinary fields $f(x)$ and $g(x)$ is explicitly defined by

$$f(x)\star g(x) = exp(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x'^{\mu}}\frac{\partial}{\partial x''^{\nu}})f(x^{'})g(x^{''})_{x=x'=x''}$$

$$= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial f}{\partial x'^{\mu}}\frac{\partial g}{\partial x''^{\nu}} + \mathscr{O}(\theta^2).$$

this product obeys associative property $f\star(g\star h) = (f\star g)\star h$, if we apply the commutative limit $\theta^{\mu\nu}\to 0$ then above expression will reduce to ordinary product as $f\star g = f.g$.

We are familiar with Lax equations as a nice representation of integrable systems. The Lax equation and zero curvature condition both have same form on deformed spaces as they possess on ordinary space. These representations involve two linear operators, these operators may be differential operators or matrices [7-12]. If A and B are the linear operators then Lax equation is given by $A_t = [B,A]$ where $[B,A]$ is commutator under the star product or quantum product, this Lax pair formalism is also helpful to construct the DT, Riccati equation and BT of integrable systems. The compatibility condition of inverse scattering problem $\Psi_x = A(x,t)\Psi$ and $\Psi_t = B(x,t)\Psi$ yields $A_t - B_x = [B,A]$ which is called the zero curvature representation of integrable systems [13-16. Further we will denote the commutator and anti-commutator by $[,]_-$ and $[,]_+$ respectively. Now the Lax equation and zero curvature condition can be expressed as $A_t = [B,A]_-$ and $A_t - B_x = [B,A]_-$.

The Painlevé equations can be represented by the Noumi-Yamada systems [22], these systems are discovered by Noumi and Yamada while studying symmetry of Painlevé equations and these systems also possess the affine Weyl group symmetry of type A_l^1. For example Noumi-Yamada system for Painlevé II equation is given by

$$\begin{cases} u_0^{'} = u_0u_2 + u_2u_0 + \alpha_0 \\ u_1^{'} = -u_1u_2 - u_2u_1 + \alpha_1 \\ u_2^{'} = u_1 - u_0 \end{cases} \tag{1.1}$$

where $u_i^{'} = \frac{du_i}{dz}$ and α_0 , α_1 are constant parameters. Above system (1.1) also a unique representation of NC and quantum PII equation, for NC derivation of PII equation [19] the dependent functions u_0, u_1, u_2 obey a kind of star product and in case of quantum derivation these functions are subjected to some quantum commutation relations [23].

In this paper I derive an equivalent zero-curvature representation of quantum PII equation with its associated quantum Riccati equation and also I construct non-vacuum solutions of NC PII equation by using its Darboux transformation. The following section of this paper consists a brief introduction to NC and quantum PII equation. In section 3, I present a linear representation of quantum PII equation whose compatibility yields its associated zero-curvature condition. Further I derive quantum PII Riccati equation with the help of its linear system. The section 4 contains the derivation of eigenvector solution for NC PII linear system and NC PII Riccati equation. At the end of this paper I apply some Darboux transformations to construct the non-vacuum solitonic solutions of NC PII equation by taking the solution NC PII Riccati as a seed solution.

2. Noncommutative and quantum Painlevé II equation

2.1. *Noncommutative Painlevé II equation*

The following NC analogue of classical Painlevé II equation

$$u_2'' = 2u_2^3 - 2[z,u_2]_+ + C \tag{2.1}$$

introduced by V. Retakh and V. Roubtsov [19] where $C = 4(\beta + \frac{1}{2})$ is constant. The NC PII equation (2.1) can be obtained by eliminating u_0 and u_1 from (1.1). The above NC PII equation (2.1) also can be represented by equivalent inverse scattering problems with special case $\beta = -\frac{1}{2}$ [20] and in general with non zero constant $C \neq 0$ [21]. Let consider the following inverse scattering problem [21] for the NC PII equation

$$\Psi_\lambda = A(z;\lambda)\Psi \;\; \Psi_z = B(z;\lambda)\Psi \tag{2.2}$$

with Lax matrices

$$\begin{cases} A = (8i\lambda^2 + iu_2^2 - 2iz)\sigma_3 + u_2'\sigma_2 + (\frac{1}{4}C\lambda^{-1} - 4\lambda u_2)\sigma_1 \\ B = -2i\lambda\sigma_3 + u_2\sigma_1 \end{cases} \tag{2.3}$$

where σ_1, σ_2 and σ_3 are Pauli spin matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

here λ is spectral parameter, the compatibility condition of system (2.2) yields NC PII equation (2.1).

2.2. *Quantum Painlevé II equation*

The quantum extension of classical Painlevé equations involves the symmetrical form of Painlevé equations proposed in [23] to noncommuting objects. For the quantum Painlevé II equation let us replace the function u_0 , u_1, u_2 by f_0 , f_1, f_2 respectively in system (1.1), further parameters α_0 and α_1 belong to the complex number field $\mathbb{C}$. The operators f_0, f_1 and f_2 obey the following commutation rules

$$[f_1,f_0]_- = 2\hbar f_2,\ \ [f_0,f_2]_- = [f_2,f_1]_- = \hbar \tag{2.4}$$

where $\hbar$ is Planck constant, the derivation ∂_z preserves the commutation relations (2.4) [23]. The NC differential system (**??**) admits the affine Weyl group actions of type $\mathscr{A}_l^{(1)}$ and quantum PII equation

$$f_2'' = 2f_2^3 - zf_2 + \alpha_1 - \alpha_0. \tag{2.5}$$

can be obtained by elimination f_0 and f_1 from system (1.1) with the help of commutation relations (2.4) [23]. The above equation (2.5) is called quantum PII equation because after eliminating f_0 and f_2 from same system (1.1) we obtain P_{34} that involves Planck constant $\hbar$ [23]. In next section I construct a linear systems whose compatibility condition yields quantum PII equation with quantum commutation relation between function f_2 and independent variable z, further we show that under the classical limit when $\hbar \to 0$ this system will reduce to classical PII equation.

3. Zero curvature representation of quantum PII equation

Proposition 1.1.
The compatibility condition of following linear system

$$\Psi_\lambda = A(z;\lambda)\Psi, \quad \Psi_z = B(z;\lambda)\Psi \tag{3.1}$$

with Lax matrices

$$\begin{cases} A = (8i\lambda^2 + if_2^2 - 2iz)\sigma_3 + f_2^{'}\sigma_2 + (\frac{1}{4}c\lambda^{-1} - 4\lambda f_2)\sigma_1 + i\hbar\sigma_2 \\ B = -2i\lambda\sigma_3 + f_2\sigma_1 + f_2 I \end{cases} \tag{3.2}$$

yields quantum PII equation, here I is 2×2 identity matrix and λ is spectral parameter and c is constant.

Proof:
The compatibility condition of system (3.1) yields zero curvature condition

$$A_z - B_\lambda = [B,A]_-. \tag{3.3}$$

We can easily evaluate the values for A_z, B_λ and $[B,A]_- = BA - AB$ from the linear system (3.2) as follow

$$A_z = (if_2^{'}f_2 + if_2f_2^{'} - 2i)\sigma_3 + f_2^{''}\sigma_2 - 4\lambda f_2^{'}\sigma_1 \tag{3.4}$$

$$B_\lambda = -2i\sigma_3 \tag{3.5}$$

and

$$[B,A] = \begin{pmatrix} if_2^{'}f_2 + if_2f_2^{'} + [f_2,z]_- - i\hbar & \delta \\ \lambda & -if_2^{'}f_2 - if_2f_2^{'}[z,f_2]_- + i\hbar \end{pmatrix} \tag{3.6}$$

where

$$\delta = -if_2^{''} + 2if_2^3 - 2i[z,f_2]_+ + ic + i[f_2^{'},f_2]_- + 4i\lambda\hbar$$

and

$$\lambda = if_2^{''} - 2if_2^3 + 2i[z,f_2]_+ - ic + i[f_2,f_2^{'}]_- - 4i\lambda\hbar.$$

now after substituting these values from (3.4), (3.5) and (3.6) in equation (3.3) we get

$$\begin{pmatrix} [f_2,z]_- - i\hbar & \delta \\ \lambda & [z,f_2]_- + i\hbar \end{pmatrix} = 0 \tag{3.7}$$

and the above result (3.7) yields the following expressions

$$[f_2,z] = \frac{1}{2} i\hbar f_2 \tag{3.8}$$

and

$$if_2'' - 2if_2^3 + 2i[z,f_2]_+ - ic + i[f_2,f_2']_- - 4i\lambda\hbar = 0 \tag{3.9}$$

equation (3.8) shows quantum relation between the variables z and f_2. In equation (3.9) the term $i[f_2,f_2']_- - 2i\lambda\hbar$ can be eliminated by using equation $f_2' = f_1 - f_0$ from (1.1) and quantum commutation relations (2.4). For this purpose let us replace f_2 by $-\frac{1}{2}\lambda^{-1} f_2$ in (2.4), then commutation relations become

$$[f_0,f_2]_- = [f_2,f_1]_- = -2\lambda\hbar. \tag{3.10}$$

Now let us take the commutator of the both side of the equation $f_2' = f_1 - f_0$ with f_2 from right side, we get

$$[f_2',f_2]_- = [f_1,f_2]_- - [f_0,f_2]_-$$

above equation with the commutation relations (3.10) can be written as

$$[f_2',f_2]_- = -4\lambda\hbar. \tag{3.11}$$

Now after substituting the value of $[f_2',f_2]_-$ from (3.11) in (3.9) we get

$$if_2'' - 2if_2^3 + 2i[z,f_2]_+ - ic = 0.$$

Finally, we can say that the compatibility of condition of linear system (3.1) yields the following expressions

$$\begin{cases} f_2'' = 2f_2^3 - 2[z,f_2]_+ + c \\ zf_2 - f_2 z = i\hbar f_2 \end{cases} \tag{3.12}$$

in above system (3.12) the first equation can be considered as a pure version of quantum Painlevé II equation that is equipped with a quantum commutation relation $[z,f_2]_- = -i\hbar$ and this equation can be reduced to the classical PII equation under the classical limit when $\hbar \to 0$.

Remark 1.1.

The linear system (3.1) with eigenvector $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and setting $\Delta = \psi_1 \psi_2^{-1}$ can be reduced to the following quantum PII Riccati form

$$\Delta' = -4i\Delta + f_2 + [f_2,\Delta]_- - \Delta f_2 \Delta$$

Proof:

Here we apply the method of Konno and Wadati [24] to the linear system (3.12) of quantum PII

equation. For this purpose let us substitute the eigenvector $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in linear systems (3.1) and we get

$$\begin{cases} \frac{d\psi_1}{d\lambda} = (8i\lambda^2 + if_2^2 - 2iz)\psi_1 + (-if_2^{'} + \frac{1}{4}C_0\lambda^{-1} - 4\lambda f_2 + \hbar)\psi_2 \\ \frac{d\psi_2}{d\lambda} = (if_2^{'} + \frac{1}{4}C_0\lambda^{-1} - 4\lambda f_2 - \hbar)\psi_1 + (-8i\lambda^2 - if_2^2 + 2iz)\psi_2 \end{cases} \tag{3.13}$$

and

$$\begin{cases} \psi_1^{'} = (-2i\lambda + f_2)\psi_1 + f_2\psi_2 \\ \psi_2^{'} = f_2\psi_1 + (2i\lambda + f_2)\psi_2 \end{cases} \tag{3.14}$$

where $\psi_1^{'} = \frac{d\psi_1}{dz}$ and now from system (3.14) we can derive the following expressions

$$\psi_1^{'}\psi_2^{-1} = (-2i\lambda + f_2)\psi_1\psi_2^{-1} + f_2 \tag{3.15}$$

$$\psi_2^{'}\psi_2^{-1} = -2i\lambda + f_2 + f_2\psi_1\psi_2^{-1} \tag{3.16}$$

. Let consider the following substitution

$$\Delta = \psi_1\psi_2^{-1} \tag{3.17}$$

now taking the derivation of above equation with respect to z

$$\Delta^{'} = \psi_1^{'}\psi_2^{-1} - \psi_1\psi_2^{-1}\psi_2^{'}\psi_2^{-1}$$

after using the (3.15), (3.16) and (3.17) in above equation we obtain

$$\Delta^{'} = -4i\Delta + f_2 + [f_2, \Delta]_- - \Delta f_2 \Delta \tag{3.18}$$

the above expression (3.18) can be considered as quantum Riccati equation in Δ because it involves commutation $[f_2, \Delta]_- = f_2\Delta - \Delta f_2$ that has been derived from the linear system (3.14).

4. NC PII Riccati equation and Solution to the eigenvector of NC PII system

Proposition 1.2.

The linear system (2.2) with eigenvector $\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ and setting $\Gamma = \chi\Phi^{-1}$ can be reduced to the following NC PII Riccati form

$$\Gamma^{'} = -4i\lambda\Gamma + u - \Gamma u \Gamma$$

Proof:

For this purpose once again we apply the method of [24] to derive NC PII Riccati equation from the linear systems (2.2). Let us consider the eigenvector $\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ in linear systems (2.2) and we obtain

$$\begin{cases} \dfrac{d\chi}{d\lambda} = (8i\lambda^2 + iu^2 - 2iz)\chi + (-iu_z + \dfrac{1}{4}C\lambda^{-1} - 4\lambda u)\Phi \\ \dfrac{d\Phi}{d\lambda} = (iu_z + \dfrac{1}{4}C\lambda^{-1} - 4\lambda u)\chi + (-8i\lambda^2 - iu^2 + 2iz)\Phi \end{cases} \tag{4.1}$$

and

$$\begin{cases} \chi^{'} = -2i\lambda\chi + u\Phi \\ \Phi^{'} = u\chi + 2i\lambda\Phi \end{cases} \tag{4.2}$$

where $\chi^{'} = \frac{d\chi}{dz}$ and from system (4.2) we can evaluate the following expressions

$$\chi^{'}\Phi^{-1} = -2i\lambda\chi\Phi^{-1} + u \tag{4.3}$$

$$\Phi^{'}\Phi^{-1} = -2i\lambda + u\chi\Phi^{-1}. \tag{4.4}$$

Now let consider the following substitution

$$\Gamma = \chi\Phi^{-1} \tag{4.5}$$

and after taking the derivation of above equation with respect to z we get

$$\Gamma^{'} = \chi^{'}\Phi^{-1} - \chi\Phi^{-1}\Phi^{'}\Phi^{-1}.$$

Finally by making use of Γ and $\Gamma^{'}$ in (4.3) and (4.4) and after simplification we obtain the following expression

$$\Gamma^{'} = -4i\lambda\Gamma + u - \Gamma u\Gamma \tag{4.6}$$

the above equation (4.6) is NC PII Riccati form in Γ where u is the solution of NC PII equation (2.1). As Γ has been expressed in terms of χ and Φ, the components of eigenvector of NC PII system (2.2), in next section we construct solution for Γ by solving the system (4.2) for χ and Φ.

Proposition 1.3.

The solution of eigenvector $\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ can be derived from 4.2 in the following form

$$\Psi = e^{-2i\lambda z + a_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2i\lambda z + b_0}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where a_0 and b_0 are constants.

Proof:

In order to derive the solution for eigenvector Ψ of NC PII linear system (2.2) let us multiply first equation by Φ^{-1} and second equation by χ^{-1} from right side of that system and then after

subtracting the both sides of the resulting equations we get

$$\left(\chi^{'}+2i\lambda\chi\ -\Phi^{'}+2i\lambda\Phi\right)\begin{pmatrix}\Phi^{-1}\\ \chi^{-1}\end{pmatrix}=0 \tag{4.7}$$

it implies that

$$\left(\chi^{'}+2i\lambda\chi\ -\Phi^{'}+2i\lambda\Phi\right)=0$$

and finally we obtain

$$\chi^{'}=-2i\lambda\chi \tag{4.8}$$

$$\Phi^{'}=2i\lambda\Phi. \tag{4.9}$$

Now equation (4.8) can be written as

$$\chi^{'}\chi^{-1}=-2i\lambda \tag{4.10}$$

the left hand side $\chi^{'}\chi^{-1}$ of above equation (4.10) represents NC derivation of $\log(\chi)$ with respect to z and now the equation (4.10) can be expressed as

$$(\log\chi)^{'}=-2i\lambda$$

now after integrating above equation with respect to z we get

$$\chi=e^{-2i\lambda z+a_0} \tag{4.11}$$

where a_0 is constant of integration, similarly from (4.9) we can calculate

$$\Phi=e^{2i\lambda z+b_0}. \tag{4.12}$$

finally the eigenvector Ψ of NC PII system can be written as

$$\Psi=e^{-2i\lambda z+a_0}\begin{pmatrix}1\\0\end{pmatrix}+e^{2i\lambda z+b_0}\begin{pmatrix}0\\1\end{pmatrix}.$$

5. Darboux transformations of NC PII equation

In the theory of integrable systems the applications of Darboux transformations (DTs) are quite interesting to construct the multi-soliton solutions of these systems. These transformations consist the particular solutions of corresponding linear systems of the integrable equations and their seed (initial) solutions. For example the NC PII equation (2.1) possesses following N fold DT

$$u[N+1]=\Pi_{k=1}^{N}\Theta_k[k]u[1]\Pi_{j=N}^{1}\Theta_j[j]\quad \text{for } N\geq 1 \tag{5.1}$$

with

$$\Theta_N[N]=\Lambda_N^{\phi}[N]\Lambda_N^{\chi}[N]^{-1}$$

where $u[1]$ is seed solution and $u[N+1]$ are the new solutions of NC PII equation [21]. In above transformations (5.1) $\Lambda_N^{\phi}[N]$ and $\Lambda_N^{\chi}[N]$ are the quasideterminants of the particular solutions of NC

PII linear system (2.2). Here the odd order quasideterminant representationS of $\Lambda^{\phi}_{2N+1}[2N+1]$ and $\Lambda^{\chi}_{2N+1}[2N+1]$ are presented below

$$\Lambda^{\phi}_{2N+1}[2N+1]=\begin{vmatrix} \Phi_{2N} & \Phi_{2N-1} & \cdots & \Phi_1 & \Phi \\ \lambda_{2N}\chi_{2N} & \lambda_{2N-1}\chi_{2N-1} & \cdots & \lambda_1\chi_1 & \lambda\chi \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2N}^{2N-1}\chi_{2N} & \lambda_{2N-1}^{2N-1}\chi_{2N-1} & \cdots & \lambda_1^{2N-1}\chi_1 & \lambda^{2N-1}\chi \\ \lambda_{2N}^{2N}\Phi_{2N} & \lambda_{2N-1}^{2N}\Phi_{2N-1} & \cdots & \lambda_1^{2N}\Phi_1 & \boxed{\lambda^{2N}\Phi} \end{vmatrix}$$

and

$$\Lambda^{\chi}_{2N+1}[2N+1]=\begin{vmatrix} \chi_{2N} & \chi_{2N-1} & \cdots & \chi_1 & \chi \\ \lambda_{2N}\Phi_{2N} & \lambda_{2N-1}\Phi_{2N-1} & \cdots & \lambda_1\Phi_1 & \lambda\Phi \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2N}^{2N-1}\Phi_{2N} & \lambda_{2N-1}^{2N-1}\Phi_{2N-1} & \cdots & \lambda_1^{2N-1}\Phi_1 & \lambda^{2N-1}\Phi \\ \lambda_{2N}^{2N}\chi_{2N} & \lambda_{2N-1}^{2N}\chi_{2N-1} & \cdots & \lambda_1^{2N}\chi_1 & \boxed{\lambda^{2N}\chi} \end{vmatrix}$$

with

$$\Lambda^{\phi}_{1}[1]=\Phi_1, \quad \Lambda^{\chi}_{1}[1]=\chi_1$$

where $\chi_1,\chi_2,\chi_3,...,\chi_N$ and $\Phi_1,\Phi_2,\Phi_3,...,\Phi_N$ are the solutions of system (4.2) at $\lambda_1,\lambda_2,\lambda_3,...,\lambda_N$.

6. Non-vacuum solutions of NC PII equation

Proposition 1.4.
All non-vacuum solitonic solutions of NC PII equation can be derived from DTs (5.1) by taking the solution u of NC PII Riccati equation (4.6) as a seed (initial) solution $u[1]$ of NC PII DTs (5.1) as follow

$$u[1]=-8i\lambda_1(1-e^{-8i\lambda_1 z})^{-1}e^{-4i\lambda_1 z}$$

Proof:
Let us derive the non-vacuum solutions of NC PII equation with the help of its DTs (5.1) taking the solution u of NC PII Riccati equation (4.6) as seed or first soliton solution. For this purpose, we can show that that the following solution

$$\Gamma=e^{4i\lambda_1 z} \tag{6.1}$$

and

$$u[1]=-8i\lambda_1(1-e^{-8i\lambda_1 z})^{-1}e^{-4i\lambda_1 z} \tag{6.2}$$

satisfy NC PII Riccati equation (4.6). The solution $u[1]$ (6.2) can be considered as a first soliton solution of NC PII equation (2.2), with help of this non zero soliton solution we can derive all non-vacuum solutions of the NC PII equation.

Let us first consider the case $N = 1$:
For this case the one fold DT of NC PII equation from (5.1) can be written as

$$u[2] = \Lambda_1^{\phi}[1](\Lambda_1^{\chi}[1])^{-1}u[1]\Lambda_1^{\phi}[1](\Lambda_1^{\chi}[1])^{-1}$$

or

$$u[2] = \Phi_1\chi_1^{-1}u[1]\Phi_1\chi_1^{-1}. \tag{6.3}$$

where $u[2]$ is a new solution of NC P-II equation that is generated by its seed solution $u[1]$ and Φ_1 , χ_1 are of solutions system (4.2) at $\lambda = \lambda_1$ can be derived in the following forms

$$\begin{cases} \chi_1 = e^{-2i\lambda_1 z + a_0} \\ \Phi_1 = e^{2i\lambda_1 z + b_0} \end{cases} \tag{6.4}$$

. here a_0 and b_0 are constant of integration. Now after substituting the values from (6.1), (6.2) and (6.4) into (6.3) we obtain second soliton solution of NC PII equation as follow

$$u[2] = -8i\lambda_1 e^{4i\lambda_1 z}(1 - e^{-8i\lambda_1 z})^{-1}. \tag{6.5}$$

Now for $N = 2$ and with the even quasideterminants $\Lambda_2^{\phi}[2]$, $\Lambda_2^{\chi}[2]$ given in [21], the third soliton solution of NC PII equation can be written as

$$u[3] = \Lambda_2^{\phi}[2](\Lambda_2^{\chi}[2])^{-1}u[2]\Lambda_2^{\phi}[2](\Lambda_2^{\chi}[2])^{-1} \tag{6.6}$$

for the above solution (6.6) the quasideterminants $\Lambda_2^{\phi}[2]$ and $\Lambda_2^{\chi}[2]$ can be derived in following forms

$$\Lambda_2^{\phi}[2] = \lambda e^{-2i\lambda z + b} - \lambda_1 e^{-4i\lambda_1 z + 2i\lambda z + C_1}$$

and

$$\Lambda_2^{\chi}[2] = \lambda e^{2i\lambda z + a} - \lambda_1 e^{4i\lambda_1 z - 2i\lambda z + C_2}$$

from their expressions given in [21], here $C_1 = b_1 - a_1 + a$ and $C_1 = a_1 - b_1 + a$ are constants of integration. Similarly we can construct remaining non-vacuum solutions of NC PII equation for $N = 3$, $N = 4$ and so on by applying its Darboux transformations (5.1) iteratively.

7. Conclusion

In this paper, we have derived a zero curvature representation of quantum Painlevé II equation with its associated Riccati form and also we have derived an explicit expression of NC PII Riccati equation from the linear system of NC PII equation by using the method of Konno and Wadati [24]. At the end of this paper it has been shown that by using the Darboux transformation we can construct all non-vacuum solitonic solutions of NC PII equation by taking solution of NC PII Riccati equation as a initial solution. Further, one can derive Bcklund transformations for NC PII equation with the help of our NC PII linear system with its Riccati form by using the technique described in [24, 25], these transformations may be helpful to construct the nonlinear principle of superposition for NC PII solutions. It also seems interesting to construct the connection of NC PII equation to the known integrable systems such as its connection to NC nonlinear Schrödinger equation and to NC KdV equation as it possesses this property in classical case.

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